SEEMINGLY SHALLOW PRIORS ARE SUFFICIENTLY EXPRESSIVE FOR ASYMPTOTICALLY BAYES OPTIMAL MULTIPLE TESTING UNDER SPARSITY

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ABSTRACT

One-group global-local shrinkage priors, popular alternatives to two-groups spike-and-slab priors, are widely used in Bayesian inference. Regarding the asymptotic optimality of high-dimensional Bayesian multiple testing procedures, the dominant view is that one-group priors should have multiple hyperparameters and heavy tails so that procedures based on these priors are asymptotically Bayes optimal under sparsity (ABOS). Liang et al. (2026) note that virtually all previous procedures based on one-group priors rely solely on the posterior. Instead, by considering both the prior and posterior, Liang et al. (2026) show that relative belief procedures using a simple one-group light-tailed normal prior and a one-group uniform prior, each with a single hyperparameter, are sufficient to be ABOS. All previous ABOS properties of procedures based on one-group priors are established under additive symmetric 0-1 loss. In this paper, we generalize the ABOS properties of relative belief procedures under additive weighted 0-1 loss and broader asymptotic regimes. A rich and flexible scale family of one-group priors, each with a single hyperparameter, can be used in our generalized relative belief procedures, ranging from light-tailed to heavy-tailed priors. The priors can be symmetric or asymmetric, and they can have bounded or unbounded support. Arguably, the ABOS properties of our generalized relative belief procedures are more general and stronger than any existing ABOS properties of procedures based on global-local priors.

1 INTRODUCTION

The multiple testing problem is ubiquitous in various modern scientific studies; the notion of false discovery rate (FDR) and the famous Benjamini–Hochberg (BH) Benjamini & Hochberg (1995) procedure have greatly influenced the development of simultaneous testing. Many multiple testing methods have rapidly emerged, which are primarily distinguished by their inferential objectives and criteria for quantifying error.

In recent years, substantial attention has been devoted to characterizing the fundamental limits and optimality properties of multiple testing procedures in sparse regimes (e.g., Donoho & Jin, 2004; Abramovich et al., 2006; Bogdan et al., 2011). Within this line of literature, Bogdan et al. (2011) study optimality properties of multiple testing procedures from a Bayesian decision-theoretic perspective, in which the objective is to minimize the notion of Bayes risk.

Specifically, under the two-groups formulation and an additive weighted 0–1 loss, Bogdan et al. (2011) introduced the Bayes oracle, defined as the optimal Bayes rule for Bayesian multiple testing, where a two-groups model can serve as a "gold standard" for sparse problems (e.g., Efron, 2008; Carvalho et al., 2009; Scott & Berger, 2010; Efron, 2012). Building on this paradigm, Bogdan et al. (2011) established sufficient conditions under which a multiple testing procedure can asymptotically attain the optimal Bayes risk in sparse regimes as the number of hypotheses tends to infinity. They termed this property the asymptotically Bayes optimal under sparsity (ABOS) property.

In Bayesian multiple testing, priors are broadly classified into two dominant classes: point-mass spike-and-slab two-groups priors (e.g., Mitchell & Beauchamp, 1988; George & McCulloch, 1993; Johnstone & Silverman, 2004; Ishwaran & Rao, 2005) and continuous, heavy-tailed one-group

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global–local shrinkage priors (e.g., Carvalho et al., 2009; 2010; Ghosh et al., 2016; Bhadra et al., 2017). In large-scale and computationally demanding settings, the latter class is often more computationally attractive (e.g., Bhattacharya et al., 2016; Makalic & Schmidt, 2016; Bhadra et al., 2020; Johndrow et al., 2020).

Adopting the asymptotic framework of Bogdan et al. (2011), an important line of research is to understand whether procedures based on appropriately chosen one-group global–local shrinkage priors can asymptotically attain the optimal performance of the two-groups model in large-scale, high-dimensional Bayesian multiple testing problems. Over the years, under an additive symmetric 0-1 loss, Datta & Ghosh (2013), Bhadra et al. (2017), Ghosh et al. (2016), Ghosh & Chakrabarti (2017), and Qin & Ghosh (2025) show that a wide range of one-group global–local priors are near-ABOS or ABOS in univariate and multivariate normal means problems.

For a long time, suitable one-group priors in Bayesian multiple testing problems have been mandatorily required to have heavy tails and multiple hyperparameters. Surprisingly, Liang et al. (2026) show that one-group priors as simple as a light-tailed normal prior or even a uniform prior, each with a single hyperparameter, are sufficient to be used to attain the ABOS property when one adopts relative belief procedures instead of using purely posterior-based procedures.

In this paper, we show the relative belief procedures in Liang et al. (2026) can be greatly generalized. We refer to our generalized procedures as the generalized relative belief (GRB) procedures. Using GRB procedures with a rich scale family of one-group priors, we make three main contributions:

1. Under the general asymptotic framework of Bogdan et al. (2011) and Frommlet & Bogdan (2013), and additive weighted 0-1 loss, we are the first to attain the general ABOS property in both univariate and multivariate normal means problems using a rich scale family of one-group priors. In particular, this is the first time one shows that procedures based on one-group priors can be used to attain the ABOS property under the general asymptotic framework of Frommlet & Bogdan (2013).

2. Our GRB procedures accommodate a broad scale family of one-group priors, each with a single hyperparameter, encompassing light-tailed to heavy-tailed priors, where those priors can be symmetric or asymmetric, with bounded or unbounded support. However, virtually all suitable one-group global–local priors proposed in the existing literature are required to have multiple hyperparameters, heavy tails, and are often symmetric with unbounded support.

3. From the perspective of Bayesian decision theory, adopting the GRB procedures, we greatly unify and generalize the use of one-group priors. Arguably, GRB procedures naturally better reflect the Bayesian spirit than procedures via global-local priors, in the sense that a wide range of priors with proper asymptotic scaling can be used to attain the ABOS property. This also highlights the huge advantages of considering both the prior and posterior in Bayesian multiple testing problems.

2 Related Work

All previous one-group priors (including priors used in the global-local approach and the relative belief approach) are studied within the asymptotic framework of Bogdan et al. (2011) and under an additive symmetric 0-1 loss. Additional details on this section can be found in Appendix B.

Near-ABOS and ABOS properties of procedures via global-local priors. Datta & Ghosh (2013), Bhadra et al. (2017), and Ghosh et al. (2016) established the near-ABOS property for the horseshoe prior, horseshoe+ prior, and a general class of one-group global-local shrinkage priors with heavy tails, respectively. Ghosh & Chakrabarti (2017) sharpened the near-ABOS result in Ghosh et al. (2016) to the exact ABOS result. Qin & Ghosh (2025) extended the results of Ghosh & Chakrabarti (2017) to the multivariate normal means setting, showing that a broad class of multivariate global-local priors remains ABOS. A consistent trend in the literature of global-local priors is toward increasingly deep hierarchical priors with a growing number of hyperparameters and heavy tails. This often results in analytically intractable densities and requires increasingly intricate proof machinery to characterize the internal mechanisms of such multiple testing procedures. Also, it is often impossible to do prior elicitation for global-local priors (Mikkola et al., 2024).

ABOS properties of relative belief procedures. Instead of intentionally constructing complex new priors carried out by pure posterior inference, Liang et al. (2026) think in reverse by considering

a simple single-hyperparameter light-tailed normal prior and even a single-hyperparameter uniform prior to do sparse Bayesian multiple testing. Note that the normal prior and the uniform prior have even better rate of convergence properties than heavy-tailed global-local priors, in terms of parameter estimation. And, both the normal prior and the uniform prior are analytical and easy to do prior elicitation. However, such priors have long been considered insufficient for simultaneously identifying signals and noise. By considering both the prior and the posterior, Liang et al. (2026) rigorously show that the relative belief procedures based on a normal prior and a uniform prior are ABOS. For the continuous setting, the relative belief ratio used in relative belief procedures is defined as the ratio of the posterior density $\pi(\theta \mid X)$ to the prior density $\pi(\theta)$: $\mathrm{RB}(\theta \mid X) = \pi(\theta \mid X)/\pi(\theta)$, where θ denotes the parameter of interest (Evans, 2015). These relative belief procedures are highly analytical and can be implemented easily with fast, stable computations. A comprehensive simulation study in Liang et al. (2026) shows that the empirical Bayes relative belief procedures uniformly outperform the empirical Bayes horseshoe procedure, and often by large margins, both in cases closer to the asymptotic setting of Bogdan et al. (2011) and in cases far away from the asymptotic setting under study. The proof machinery for relative belief procedures is arguably the simplest among those used to establish the ABOS property for procedures based on one-group priors.

3 The ABOS Property of Generalized Relative Belief Procedures in the Univariate Normal Means Problem

3.1 Univariate Model and Bayes Oracle in Bogdan et al. (2011)

Suppose our data $X_i \mid \mu_i \sim N(\mu_i, \sigma_0^2)$ independently for $i = 1, \ldots, m$, where μ_i represent the effects under investigation and $\sigma_0^2 > 0$ is known, as commonly assumed in multiple testing (see, e.g., Abramovich et al., 2006; Bogdan et al., 2008; 2011). We take $\sigma_0^2 = 1$ without loss of generality. We simultaneously test $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$, and assume $\mu_i \overset{\text{i.i.d.}}{\sim} (1-p)\delta_{\{0\}} + pN(0, \sigma^2)$, where $\delta_{\{0\}}$ is the point mass at zero, $p \in (0,1)$ is the population proportion of nonnull hypotheses, and $\sigma^2 > 0$ is the variance of the nonzero means. Marginally, the observations follow the two-groups model

$$X_i \overset{\text{i.i.d.}}{\sim} (1-p)N(0,1) + pN(0, 1+\sigma^2), \qquad i = 1, \ldots, m. \tag{1}$$

Let $\delta_0 > 0$ and $\delta_A > 0$ denote the losses associated with type I and type II errors, respectively, and define $\delta = \delta_0/\delta_A$ and $f = (1-p)/p$. For a multiple testing rule $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_m)$, where $\psi_i = 1$ denotes rejection of H_{0i}, let $t_{1i} = \mathbb{P}(\psi_i = 1 \mid H_{0i})$ and $t_{2i} = \mathbb{P}(\psi_i = 0 \mid H_{Ai})$. Under the additive weighted 0–1 loss, its Bayes risk is

$$R(\boldsymbol{\psi}) = \sum_{i=1}^{m} \left[(1-p)\delta_0 t_{1i} + p\delta_A t_{2i}\right]. \tag{2}$$

For the additive weighted 0–1 loss, Bogdan et al. (2011) show that applying the Bayes classifier (i.e., often referred to the Bayes oracle in the asymptotic framework of Bogdan et al. (2011)) to each hypothesis minimizes the Bayes risk in (2). Let q_0 and q_A denote the densities of $N(0,1)$ and $N(0, 1+\sigma^2)$, respectively. For each $i = 1, \ldots, m$, the resulting procedure rejects H_{0i} when $q_A(X_i)/q_0(X_i) > \delta f$. Simplifying this inequality yields the following rule.

Definition 3.1. *For data generated according to (1) under the framework of Bogdan et al. (2011), for each $i = 1, \ldots, m$, the Bayes oracle rule rejects H_{0i} if and only if $X_i^2 > c_{\mathrm{BO}}^2$, where $c_{\mathrm{BO}}^2 = \frac{1+\sigma^2}{\sigma^2}[2\log(\delta f) + \log(1+\sigma^2)]$.*

The Bayes oracle is not achievable in finite samples because p and σ^2 are unknown. Following the asymptotic framework of Bogdan et al. (2011), let δ_m denote the loss ratio and define $u_m = \sigma_m^2$, $f_m = (1-p_m)/p_m$, and $v_m = u_m(\delta_m f_m)^2$.

Assumption 3.2. *As $m \to \infty$, assume $p_m \to 0$, $u_m \to \infty$, and $\log v_m/u_m \to C \in (0, \infty)$.*

In Assumption 3.2, $p_m \to 0$ makes the population proportion of nonnull hypotheses vanish, so the testing problem becomes increasingly sparse. The condition $u_m \to \infty$ ensures that the signals are sufficiently strong to be distinguished asymptotically. The condition $\log v_m/u_m \to C \in (0, \infty)$ balances signal strength and sparsity so that the limiting power of an individual test is strictly between

zero and one (Bogdan et al., 2011). The Bayes oracle threshold satisfies $c_{\mathrm{BO}}^2/(1+u_m) \to C$. Its type I and type II error probabilities are $t_{1i}^{\mathrm{BO}} = 2\{1-\Phi(c_{\mathrm{BO}})\}$ and $t_{2i}^{\mathrm{BO}} = 2\Phi(c_{\mathrm{BO}}/\sqrt{1+u_m}) - 1$, respectively, where Φ is the standard normal distribution function. By Bogdan et al. (2011, Theorem 3.1), we have $\delta_m f_m t_{1i}^{\mathrm{BO}} = o(1)$ and $t_{2i}^{\mathrm{BO}} = \{2\Phi(\sqrt{C}) - 1\}(1+o(1))$, yielding the asymptotic Bayes risk of the Bayes oracle

$$R_{\mathrm{Opt}}^{\mathrm{AsyBO}} = m p_m \delta_A \{2\Phi(\sqrt{C}) - 1\}(1+o(1)). \tag{3}$$

Definition 3.3. *Under Assumption 3.2, a multiple testing procedure with asymptotic Bayes risk R^{Asy} is asymptotically Bayes optimal under sparsity (ABOS), if $R^{\mathrm{Asy}}/R_{\mathrm{Opt}}^{\mathrm{AsyBO}} \to 1$ as $m \to \infty$.*

The following result characterizes the ABOS property in terms of fixed rejection thresholds.

Theorem 3.4 (Bogdan et al., 2011, Theorem 3.2)**.** *Suppose Assumption 3.2 holds. Consider the fixed threshold rule that, for each $i = 1, \ldots, m$, rejects H_{0i} when $X_i^2 > c_m^2$, where $c_m^2 = \log v_m + z_m > 0$. This rule is ABOS if and only if $z_m = o(\log v_m)$, $z_m + 2\log\log v_m \longrightarrow \infty$.*

3.2 Generalized Relative Belief Procedures

Liang et al. (2026) showed that relative belief procedures induced by normal and uniform priors are ABOS under suitable scales. To investigate whether this extends to a more general class of priors with possibly different tail behaviors, we introduce the following broader prior family and study the rejection thresholds of the induced generalized relative belief procedures, guided by the characterization in Theorem 3.4. Proofs for this subsection are given in Appendix C.

Definition 3.5 (Univariate scale family of one-group priors)**.** *For $\tau > 0$, let*

$$\pi_\tau(\mu) = \tau^{-1/2} h\left(\frac{\mu}{\sqrt{\tau}}\right), \tag{4}$$

where h is a probability density on $\mathbb{R}$. The collection $\{\pi_\tau : \tau > 0\}$ is called a univariate scale family of one-group priors if h is bounded on $\mathbb{R}$, continuous at zero, and satisfies $0 < h(0) < \infty$.

The scale family in Definition 3.5 includes symmetric and asymmetric priors, allows light and heavy tails, and allows bounded or unbounded support. Examples include normal, Student's t, Cauchy, and uniform distributions. In particular, a normal prior and a uniform prior are considered in Liang et al. (2026). The following proposition gives the relative belief ratio for a prior from this family.

Proposition 3.6. *Let $Z \sim N(0,1)$. The relative belief ratio induced by the prior in Definition 3.5 is*

$$\mathrm{RB}(\mu_i = 0 \mid X_i, \tau) = \frac{(2\pi)^{-1/2}\exp(-X_i^2/2)}{\tau^{-1/2}\mathbb{E}_Z\left[h\left(\frac{X_i - Z}{\sqrt{\tau}}\right)\right]}. \tag{5}$$

The generalized relative belief procedure rejects H_{0i} whenever $\mathrm{RB}(\mu_i = 0 \mid X_i, \tau) < 1/\delta_m$.

To show that the rejection criterion in Proposition 3.6 has a unique threshold in X_i^2 for each sign of X_i, we consider positive and negative observations separately. Proposition 3.7 establishes a unique rejection threshold in X_i^2 for each sign of X_i.

Proposition 3.7. *For $X_i \neq 0$, let $u_i = X_i/|X_i| \in \mathbb{S}^0 := \{-1, 1\}$. Suppose Definition 3.5 holds, $\tau_m \to \infty$, and $\delta_m\sqrt{\tau_m} \to \infty$. Then, for every $u \in \mathbb{S}^0$ and all sufficiently large m, there exists a unique $c_{\tau_m}(u) > 0$ satisfying*

$$(2\pi)^{1/2}\tau_m^{-1/2}\exp\left\{\frac{c_{\tau_m}^2(u)}{2}\right\}\mathbb{E}_Z\left[h\left(\frac{c_{\tau_m}(u)u - Z}{\sqrt{\tau_m}}\right)\right] = \delta_m. \tag{6}$$

For sufficiently large m, the generalized relative belief procedure rejects H_{0i} when $X_i^2 > c_{\tau_m}^2(u_i)$.

To show that the rejection thresholds for positive and negative observations have the same asymptotic order, we establish the following local approximation for the expectation in (6).

Lemma 3.8. *Suppose Definition 3.5 holds and $\tau_m \to \infty$. Then*

$$\lim_{\eta \downarrow 0} \limsup_{m\to\infty} \sup_{|y| \le \eta\sqrt{\tau_m}} \left|\mathbb{E}_Z\left[h\left(\frac{y - Z}{\sqrt{\tau_m}}\right)\right] - h(0)\right| = 0. \tag{7}$$

Substituting the local approximation in Lemma 3.8 into (6) shows that the rejection thresholds for positive and negative observations share the same asymptotic order, as stated in Proposition 3.9.

Proposition 3.9. *Suppose Definition 3.5 holds, $\tau_m \to \infty$, $\delta_m\sqrt{\tau_m} \to \infty$, and $\log(\delta_m\sqrt{\tau_m})/\tau_m \to 0$. Then*

$$\sup_{u\in\{-1,1\}} \left| c^2_{\tau_m}(u) - \left[\log\tau_m + 2\log\delta_m - 2\log\{(2\pi)^{1/2}h(0)\}\right]\right| \longrightarrow 0. \tag{8}$$

Thus, although the rejection threshold may depend on the sign of X_i in finite samples, its asymptotic order depends on the density h in Definition 3.5 only through $h(0)$. We use (8) to select an appropriate scale of the one-group prior depending only on p_m, so that the resulting rejection thresholds match the ABOS characterization for fixed threshold rules in Theorem 3.4. With this choice, the generalized relative belief procedure has the following ABOS property.

Theorem 3.10. *Suppose the data $X_1, \ldots, X_m$ are generated independently according to (1), and consider testing $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$. For each i, apply the generalized relative belief rule induced by π_{τ_m}, that is, rejecting H_{0i} when $\mathrm{RB}(\mu_i = 0 \mid X_i, \tau_m) < 1/\delta_m$. Suppose Assumption 3.2 holds, $\log\delta_m = o(\log p_m)$, π_{τ_m} belongs to the family in Definition 3.5, and $\lim_{m\to\infty} \tau_m/\{p_m^{-2}\log(1/p_m)\} \in (0,\infty)$. Then the resulting generalized relative belief procedure is ABOS in the sense of Definition 3.3.*

Theorem 3.10 follows from a comparison between the generalized relative belief thresholds and fixed thresholds rules which are ABOS. Indeed, under the assumptions of Theorem 3.10, τ_m automatically satisfies the conditions of Lemma 3.8 and Propositions 3.7 and 3.9. In particular, Proposition 3.9 further gives $\sup_{u\in\{-1,1\}} |c^2_{\tau_m}(u) - \log v_m| = O(1)$. This relation allows us to compare the error probabilities with those of fixed threshold rules satisfying the conditions in Theorem 3.4, thereby proving that the generalized relative belief procedure is ABOS.

Since p_m is unknown in practice, we develop a data-driven empirical Bayes approach for the generalized relative belief procedure using the one-group priors in Definition 3.5. Specifically, following the construction of Liang et al. (2026), we use the data to estimate the tuning parameter τ_m by

$$\widehat{\tau}_m^{\mathrm{EB_{BC}}} = \min\left\{ m^2\log m,\ \log m\left[\frac{d_1 m}{\max\{1, \sum_{i=1}^m \mathbf{1}\left(X_i^2 > d_2\log m\right)\}}\right]^2\right\}. \tag{9}$$

where $d_1 \geq 1$ and $d_2 \geq 2$ are prespecified fixed constants. The resulting empirical Bayes generalized relative belief procedure replaces τ_m by $\widehat{\tau}_m^{\mathrm{EB_{BC}}}$ and rejects H_{0i} whenever $\mathrm{RB}(\mu_i = 0 \mid X_i, \widehat{\tau}_m^{\mathrm{EB_{BC}}}) < 1/\delta_m$. To establish the corresponding ABOS property, we adopt the condition used in the empirical Bayes literature (Ghosh et al., 2016; Ghosh & Chakrabarti, 2017), together with a condition on the ratio of loss considered by Bogdan et al. (2011) and Frommlet & Bogdan (2013).

Assumption 3.11. *For some unknown $0 < \beta < 1$, $p_m \propto m^{-\beta}$ and $\log\delta_m = o(\log m)$.*

The following theorem shows that the corresponding empirical Bayes procedure is ABOS.

Theorem 3.12. *Suppose $X_1, \ldots, X_m$ are generated independently according to (1), and consider testing $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$, simultaneously. Let $\widehat{\tau}_m^{\mathrm{EB_{BC}}}$ be defined by (9), and apply the generalized relative belief rule induced by $\pi_{\widehat{\tau}_m^{\mathrm{EB_{BC}}}}$, that is, rejecting H_{0i} when $\mathrm{RB}(\mu_i = 0 \mid X_i, \widehat{\tau}_m^{\mathrm{EB_{BC}}}) < 1/\delta_m$, for each $i = 1, \ldots, m$. Suppose Assumptions 3.2 and 3.11 hold, and $\{\pi_\tau : \tau > 0\}$ belongs to the family in Definition 3.5. Then the resulting empirical Bayes generalized relative belief multiple testing procedure is ABOS in the sense of Definition 3.3.*

3.3 Univariate Normal Means Problem with Increasing Sample Size

Section 3.1 assumes that the variance of the nonzero means increases as $m \to \infty$. Following Frommlet & Bogdan (2013), we now study the ABOS property of generalized relative belief procedures in an asymptotic regime where the number of tests $m \to \infty$ and the sample size for each test also tends to infinity at a slower rate, while the distribution of the nonzero means remains fixed. Proofs for this subsection are given in Appendix D.

Consider m independent populations. For $i = 1, \ldots, m$ and $j = 1, \ldots, n$, let $X_i = (X_{1i}, \ldots, X_{ni})$, where $X_{ji} \mid \mu_i \overset{\text{i.i.d.}}{\sim} N(\mu_i, 1)$. For fixed $\widetilde{\sigma}^2 > 0$, assume $\mu_i \overset{\text{i.i.d.}}{\sim} (1-p)\delta_{\{0\}} + pN(0, \widetilde{\sigma}^2)$. We

simultaneously test $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$ for $i = 1, \ldots, m$. Each X_{ji} has marginal distribution $X_{ji} \sim (1-p)N(0,1) + pN(0, 1+\widetilde{\sigma}^2)$, and

$$X_i \overset{\text{i.i.d.}}{\sim} (1-p)N_n(\mathbf{0}, \boldsymbol{I}_n) + pN_n(\mathbf{0}, \boldsymbol{I}_n + \widetilde{\sigma}^2 \mathbf{1}_n \mathbf{1}_n^\top), \qquad i = 1, \ldots, m, \tag{10}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the vector of n ones. The sample mean $\overline{X}_i = n^{-1} \sum_{j=1}^n X_{ji}$ is a sufficient statistic for μ_i, and its marginal distribution is

$$\overline{X}_i \overset{\text{i.i.d.}}{\sim} (1-p)N(0, n^{-1}) + pN(0, \widetilde{\sigma}^2 + n^{-1}), \qquad i = 1, \ldots, m. \tag{11}$$

We use the additive weighted 0–1 loss and keep $\delta = \delta_0/\delta_A$, $f = (1-p)/p$, and the Bayes risk in (2). Comparing the two marginal densities of $\overline{X}_i$ gives the Bayes oracle rule.

Definition 3.13. *For the data $X_1, \ldots, X_m$ generated according to (10), the Bayes oracle rejects H_{0i} if and only if $n\overline{X}_i^2 > \widetilde{c}_{\text{BO}}^2$, where $\widetilde{c}_{\text{BO}}^2 = \frac{1+n\widetilde{\sigma}^2}{n\widetilde{\sigma}^2}[2\log(\delta f) + \log(1+n\widetilde{\sigma}^2)]$, for $i = 1, \ldots, m$.*

For the asymptotic analysis, index p, n, δ by m and set $f_m = (1-p_m)/p_m$.

Assumption 3.14. *As $m \to \infty$, assume $p_m \to 0$, $n_m \to \infty$, $\limsup_{m\to\infty} \delta_m < \infty$, $\log \delta_m = o(\log p_m)$, $-\frac{2\log p_m}{n_m} \longrightarrow \widetilde{C} \in (0, \infty)$, and $\widetilde{\sigma}^2 \in (0, \infty)$ is fixed.*

Under Assumption 3.14, define $v_m = n_m \delta_m^2 / p_m^2$. The threshold satisfies $\widetilde{c}_{\text{BO}}^2 = \log v_m + \log \widetilde{\sigma}^2 + \widetilde{C}/\widetilde{\sigma}^2 + o(1)$ and $\widetilde{c}_{\text{BO}}^2/(1 + n_m\widetilde{\sigma}^2) \to \widetilde{C}/\widetilde{\sigma}^2$. Its type I and type II error probabilities are $\widetilde{t}_{1i}^{\text{BO}} = 2\{1 - \Phi(\widetilde{c}_{\text{BO}})\}$ and $\widetilde{t}_{2i}^{\text{BO}} = 2\Phi(\widetilde{c}_{\text{BO}}/\sqrt{1+n_m\widetilde{\sigma}^2}) - 1$, respectively. The general asymptotic Bayes risk result of the Bayes oracle of Frommlet & Bogdan (2013, Theorem 2.1) gives the asymptotic Bayes risk of the Bayes oracle in this case (i.e., $\widetilde{R}_{\text{Opt}}^{\text{AsyBO}}$):

$$\widetilde{R}_{\text{Opt}}^{\text{AsyBO}} = m p_m \delta_A \{2\Phi(\sqrt{\widetilde{C}/\widetilde{\sigma}^2}) - 1\}\{1 + o(1)\}. \tag{12}$$

Definition 3.15. *Under Assumption 3.14, a multiple testing procedure with asymptotic Bayes risk $\widetilde{R}^{\text{Asy}}$ is asymptotically Bayes optimal under sparsity (ABOS) if $\widetilde{R}^{\text{Asy}}/\widetilde{R}_{\text{Opt}}^{\text{AsyBO}} \to 1$ as $m \to \infty$.*

To construct the generalized relative belief procedure, assign μ_i a prior π_{τ_m} from the scale family in Definition 3.5. Since $\overline{X}_i \mid \mu_i \sim N(\mu_i, n_m^{-1})$, the relative belief ratio, with $Z \sim N(0,1)$, is

$$\text{RB}(\mu_i = 0 \mid \overline{X}_i, \tau_m) = \frac{\sqrt{n_m/(2\pi)}\exp(-n_m\overline{X}_i^2/2)}{\tau_m^{-1/2}\mathbb{E}_Z\left[h\left(\frac{\overline{X}_i - Z/\sqrt{n_m}}{\sqrt{\tau_m}}\right)\right]}. \tag{13}$$

The generalized relative belief procedure rejects H_{0i} when $\text{RB}(\mu_i = 0 \mid \overline{X}_i, \tau_m) < 1/\delta_m$.

Theorem 3.16. *Suppose the data $X_1, \ldots, X_m$ are generated according to (10), and consider testing $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$. For each i, apply the generalized relative belief rule induced by π_{τ_m} that rejects H_{0i} when $\text{RB}(\mu_i = 0 \mid \overline{X}_i, \tau_m) < 1/\delta_m$. Suppose Assumption 3.14 holds, π_{τ_m} belongs to the family in Definition 3.5, and $\lim_{m\to\infty} \tau_m/p_m^{-2} \in (0, \infty)$. Then the generalized relative belief multiple testing procedure is ABOS in the sense of Definition 3.15.*

To obtain a fully data-driven generalized relative belief procedure in the framework of Assumption 3.14, we use similar constructions in Section 3.2. We now additionally require Assumption 3.11 holds. To have the desired asymptotic order p_m^{-2}, we define

$$\widehat{\tau}_m^{\text{EB}_{\text{FB}}} = \min\left\{m^2, \left[\frac{d_3 m}{\max\left\{1, \sum_{i=1}^m \mathbf{1}\left(n_m\overline{X}_i^2 > d_4 \log m\right)\right\}}\right]^2\right\}, \tag{14}$$

for predetermined fixed constants $d_3 \geq 1$ and $d_4 \geq 2$. The empirical Bayes generalized relative belief procedure replaces τ_m by $\widehat{\tau}_m^{\text{EB}_{\text{FB}}}$ and rejects H_{0i} whenever $\text{RB}(\mu_i = 0 \mid \overline{X}_i, \widehat{\tau}_m^{\text{EB}_{\text{FB}}}) < 1/\delta_m$.

Theorem 3.17. *Suppose the data $X_1, \ldots, X_m$ are generated according to (10), and consider testing $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$, simultaneously. Let $\widehat{\tau}_m^{\text{EB}_{\text{FB}}}$ be defined by (14), and apply the relative belief rule induced by $\pi_{\widehat{\tau}_m^{\text{EB}_{\text{FB}}}}$, that is, rejecting H_{0i} when $\text{RB}(\mu_i = 0 \mid \overline{X}_i, \widehat{\tau}_m^{\text{EB}_{\text{FB}}}) < 1/\delta_m$, for each $i = 1, \ldots, m$. Suppose Assumptions 3.14 and 3.11 hold, and $\{\pi_\tau : \tau > 0\}$ belongs to the family in Definition 3.5. Then the resulting empirical Bayes generalized relative belief multiple testing procedure is ABOS in the sense of Definition 3.15.*

4 THE ABOS PROPERTY OF GENERALIZED RELATIVE BELIEF IN THE MULTIVARIATE NORMAL MEANS PROBLEM

We now extend the univariate results to testing vector-valued means, where the rejection threshold may depend on the direction of the observation and the analysis involves chi-square rather than normal random variables. We show that the generalized relative belief procedure remains ABOS under an appropriate scaling for the prior and develop a fully data-driven empirical Bayes procedure. The results in Sections 4.1, 4.2, and 4.3 are proved in Appendices E, F, and G, respectively.

4.1 MULTIVARIATE MODEL AND BAYES ORACLE

We extend the multivariate framework of Qin & Ghosh (2025) to additive weighted 0–1 loss. Let $k \geq 2$ be fixed and consider m independent observations with $\boldsymbol{X}_i \mid \boldsymbol{\mu}_i \sim N_k(\boldsymbol{\mu}_i, \boldsymbol{\Sigma})$, $i = 1, \ldots, m$, where $\boldsymbol{\Sigma}$ is known and positive definite. We simultaneously test $H_{0i} : \boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$. Assume $\boldsymbol{\mu}_i \overset{\text{i.i.d.}}{\sim} (1-p)\delta_{\{\mathbf{0}\}} + pN_k(\mathbf{0}, g\boldsymbol{\Sigma})$, where $\delta_{\{\mathbf{0}\}}$ denotes the point mass at $\mathbf{0}$, $p \in (0,1)$ is the proportion of nonnull hypotheses, and $g > 0$. Marginally,

$$\boldsymbol{X}_i \overset{\text{i.i.d.}}{\sim} (1-p)N_k(\mathbf{0}, \boldsymbol{\Sigma}) + pN_k(\mathbf{0}, (1+g)\boldsymbol{\Sigma}), \qquad i = 1, \ldots, m. \tag{15}$$

Using $\delta = \delta_0/\delta_A$, $f = (1-p)/p$, and the Bayes risk in (2), we minimize the risk for each hypothesis to obtain the following Bayes oracle rule.

Definition 4.1. *For data generated by (15), the Bayes oracle rule rejects H_{0i} if and only if $\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X}_i > c^2_{\text{MBO}}$, where $c^2_{\text{MBO}} = \frac{1+g}{g}[2\log(\delta f) + k\log(1+g)]$, for $i = 1, \ldots, m$.*

The Bayes oracle is unavailable in finite samples because p and g are unknown. For asymptotic analysis, index p, g, δ by m and set $f_m = (1-p_m)/p_m$ and $\mathcal{L}_m = \log\{(\delta_m f_m)^2 g_m^k\}$. We adopt the following multivariate extension of the framework in Bogdan et al. (2011); Qin & Ghosh (2025).

Assumption 4.2. *As $m \to \infty$, assume $p_m \to 0$, $g_m \to \infty$, and $\mathcal{L}_m/g_m \to C_0 \in (0, \infty)$. The smallest and largest eigenvalues of $\boldsymbol{\Sigma}$ are bounded away from zero and infinity, respectively.*

The following theorem gives the asymptotic Bayes risk of the Bayes oracle in Definition 4.1.

Theorem 4.3. *Under Assumption 4.2, the Bayes oracle in Definition 4.1 has asymptotic Bayes risk $R^{\text{AsyMBO}}_{\text{Opt}} = mp_m\delta_A F_{\chi^2_k}(C_0)\{1 + o(1)\}$, where $F_{\chi^2_k}$ is the distribution function of a chi-squared random variable with k degrees of freedom.*

Definition 4.4. *Under Assumption 4.2, a multiple testing procedure with asymptotic Bayes risk R^{AsyM} is asymptotically Bayes optimal under sparsity (ABOS) if $\lim_{m\to\infty} R^{\text{AsyM}}/R^{\text{AsyMBO}}_{\text{Opt}} = 1$.*

4.2 GENERALIZED RELATIVE BELIEF PROCEDURES AND THE ABOS PROPERTY

To extend the univariate ABOS results for prior from the scale family of one-group priors with different tail behaviours to vector-valued means, we introduce the following multivariate prior family.

Definition 4.5 (Multivariate scale family of one-group priors)**.** *For $\tau > 0$, let*

$$\pi_\tau(\boldsymbol{\mu}) = \tau^{-k/2}|\boldsymbol{\Sigma}|^{-1/2} h\left(\frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}}{\sqrt{\tau}}\right), \tag{16}$$

where h is a probability density on $\mathbb{R}^k$. The collection $\{\pi_\tau : \tau > 0\}$ is called a multivariate scale family of one-group priors if h is bounded on $\mathbb{R}^k$, continuous at $\mathbf{0}$, and satisfies $0 < h(\mathbf{0}) < \infty$.

This family includes light-tailed densities such as multivariate normal densities, heavy-tailed densities such as multivariate Student-t densities, and densities of bounded support containing $\mathbf{0}$.

Proposition 4.6. *The relative belief ratio induced by the prior in Definition 4.5 is*

$$\text{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \boldsymbol{X}_i, \tau) = \frac{(2\pi)^{-k/2}\exp\{-\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i/2\}}{\tau^{-k/2}\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i - \boldsymbol{Z}_k}{\sqrt{\tau}}\right)\right]}, \tag{17}$$

where $\boldsymbol{Z}_k \sim N_k(\mathbf{0}, \boldsymbol{I}_k)$. For each $i = 1, \ldots, m$, the generalized relative belief procedure rejects H_{0i} whenever $\text{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \boldsymbol{X}_i, \tau) < 1/\delta_m$.

For the scale family of one-group priors in Definition 4.5, let $\boldsymbol{u}_i = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i/\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i\| \in \mathbb{S}^{k-1}$ for $\boldsymbol{X}_i \neq \boldsymbol{0}$. Unlike the univariate case, where $u_i \in \mathbb{S}^0 = \{-1, 1\}$ only distinguishes negative and positive observations, $\boldsymbol{u}_i$ ranges over the entire sphere $\mathbb{S}^{k-1}$. Consequently, when $h(\boldsymbol{z})$ is not determined solely by $\|\boldsymbol{z}\|$, the rejection threshold may depend on $\boldsymbol{u}_i$. Following the univariate analysis in Section 3.2, we characterize the asymptotic order of the rejection thresholds uniformly over $\boldsymbol{u} \in \mathbb{S}^{k-1}$. With a suitable choice of the prior scale depending only on p_m, this uniform characterization establishes that the generalized relative belief procedure is ABOS, as stated in Theorem 4.7.

Theorem 4.7. *For fixed $k \geq 2$, suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are generated independently according to (15), and consider testing $H_{0i} : \boldsymbol{\mu}_i = \boldsymbol{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \boldsymbol{0}$, $i = 1, \ldots, m$, simultaneously. For each $i = 1, \ldots, m$, apply the relative belief rule induced by π_{τ_m}, that is, rejecting H_{0i} when $\mathrm{RB}(\boldsymbol{\mu}_i = \boldsymbol{0} \mid \boldsymbol{X}_i, \tau_m) < 1/\delta_m$. Suppose Assumption 4.2 holds, $\{\pi_\tau : \tau > 0\}$ belongs to the family in Definition 4.5, $\log \delta_m = o(\log p_m)$, and $\lim_{m\to\infty} \tau_m/\{p_m^{-2/k}\log(1/p_m)\} \in (0, \infty)$. Then the resulting generalized relative belief multiple testing procedure is ABOS in the sense of Definition 4.4.*

To obtain a fully data-driven empirical Bayes generalized relative belief procedure, we estimate the tuning parameter from the data by adapting (9): for prespecified fixed $d_5 \geq 1$ and $d_6 \geq 2$, define

$$\widehat{\tau}_m^{\mathrm{MEB_{BC}}} = \log(m) \min\left\{ m^{2/k}, \left[\frac{d_5 m}{\max\left\{1, \sum_{i=1}^m \mathbf{1}\left(\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X}_i > d_6 \log m\right)\right\}} \right]^{2/k} \right\}. \tag{18}$$

For each $i = 1, \ldots, m$, the empirical Bayes generalized relative belief procedure rejects H_{0i} if and only if $\mathrm{RB}(\boldsymbol{\mu}_i = \boldsymbol{0} \mid \boldsymbol{X}_i, \widehat{\tau}_m^{\mathrm{MEB_{BC}}}) < 1/\delta_m$. The following theorem establishes its ABOS property.

Theorem 4.8. *For fixed $k \geq 2$, suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are generated independently according to (15), and consider testing $H_{0i} : \boldsymbol{\mu}_i = \boldsymbol{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \boldsymbol{0}$, $i = 1, \ldots, m$, simultaneously. Let $\widehat{\tau}_m^{\mathrm{MEB_{BC}}}$ be defined by (18), and apply the generalized relative belief rule induced by $\pi_{\widehat{\tau}_m^{\mathrm{MEB_{BC}}}}$, that is, rejecting H_{0i} when $\mathrm{RB}(\boldsymbol{\mu}_i = \boldsymbol{0} \mid \boldsymbol{X}_i, \widehat{\tau}_m^{\mathrm{MEB_{BC}}}) < 1/\delta_m$, for each $i = 1, \ldots, m$. Suppose Assumptions 4.2 and 3.11 hold, and $\{\pi_\tau : \tau > 0\}$ belongs to the family in Definition 4.5. Then the resulting empirical Bayes generalized relative belief multiple testing procedure is ABOS in the sense of Definition 4.4.*

4.3 Multivariate Normal Means with Increasing Sample Size

We also consider the multivariate counterpart of the setting in Frommlet & Bogdan (2013): the signal distribution remains fixed while the sample size in each population increases.

For fixed $k \geq 2$, consider m independent populations. Let $\boldsymbol{\Sigma}$ be known and positive definite, and fix $\tilde{g} > 0$. For $i = 1, \ldots, m$, assume $\boldsymbol{\mu}_i \overset{\text{i.i.d.}}{\sim} (1-p)\delta_{\{\boldsymbol{0}\}} + pN_k(\boldsymbol{0}, \tilde{g}\boldsymbol{\Sigma})$. For $j = 1, \ldots, n$, let $\boldsymbol{X}_{ji} \mid \boldsymbol{\mu}_i \overset{\text{i.i.d.}}{\sim} N_k(\boldsymbol{\mu}_i, \boldsymbol{\Sigma})$. Writing $\boldsymbol{X}_i = (\boldsymbol{X}_{1i}^\top, \ldots, \boldsymbol{X}_{ni}^\top)^\top \in \mathbb{R}^{nk}$, its marginal distribution is

$$\boldsymbol{X}_i \overset{\text{i.i.d.}}{\sim} (1-p)N_{nk}(\boldsymbol{0}, \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}) + pN_{nk}\big(\boldsymbol{0}, (\boldsymbol{I}_n + \tilde{g}\,\mathbf{1}_n\mathbf{1}_n^\top) \otimes \boldsymbol{\Sigma}\big), \qquad i = 1, \ldots, m. \tag{19}$$

Here, $\boldsymbol{I}_n$ is the $n \times n$ identity matrix, $\mathbf{1}_n$ is the n-dimensional vector of ones, and $\otimes$ denotes the Kronecker product. We simultaneously test $H_{0i} : \boldsymbol{\mu}_i = \boldsymbol{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \boldsymbol{0}$ for $i = 1, \ldots, m$. The sample mean $\overline{\boldsymbol{X}}_i = n^{-1}\sum_{j=1}^n \boldsymbol{X}_{ji}$ is a sufficient statistic for $\boldsymbol{\mu}_i$, with marginal distribution

$$\overline{\boldsymbol{X}}_i \overset{\text{i.i.d.}}{\sim} (1-p)N_k(\boldsymbol{0}, n^{-1}\boldsymbol{\Sigma}) + pN_k(\boldsymbol{0}, (\tilde{g} + n^{-1})\boldsymbol{\Sigma}). \tag{20}$$

We assume the additive weighted 0–1 loss, $\delta = \delta_0/\delta_A$, $f = (1-p)/p$, and the Bayes risk in (2). Comparing the two marginal densities of $\overline{\boldsymbol{X}}_i$ gives the Bayes oracle.

Definition 4.9. *For the data $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ generated according to (19), the Bayes oracle rejects H_{0i} if and only if $n\overline{\boldsymbol{X}}_i^\top \boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{X}}_i > \widetilde{c}^2_{\mathrm{MBO}}$, where $\widetilde{c}^2_{\mathrm{MBO}} = \frac{1+n\tilde{g}}{n\tilde{g}}\left[2\log(\delta f) + k\log(1+n\tilde{g})\right]$.*

For asymptotic analysis, index p, n, δ by m and set $f_m = (1-p_m)/p_m$.

Assumption 4.10. *As $m \to \infty$, assume $p_m \to 0$, $n_m \to \infty$, $\limsup_{m\to\infty} \delta_m < \infty$, $\log \delta_m = o(\log p_m)$, and $-2\log p_m/n_m \to \widetilde{C}_0 \in (0, \infty)$. The smallest and largest eigenvalues of $\boldsymbol{\Sigma}$ are bounded away from zero and infinity, respectively.*

The following theorem gives the asymptotic Bayes risk of the Bayes oracle in Definition 4.9.

Theorem 4.11. *Under Assumption 4.10, the Bayes oracle has asymptotic Bayes risk*

$$\widetilde{R}^{\mathrm{AsyMBO}}_{\mathrm{Opt}} = mp_m\delta_A F_{\chi^2_k}(\widetilde{C}_0/\tilde{g})\{1+o(1)\}. \tag{21}$$

Definition 4.12. *Under Assumption 4.10, a multiple testing procedure with asymptotic Bayes risk $\widetilde{R}^{\mathrm{AsyM}}$ is asymptotically Bayes optimal under sparsity (ABOS) if* $\lim_{m\to\infty} \widetilde{R}^{\mathrm{AsyM}}/\widetilde{R}^{\mathrm{AsyMBO}}_{\mathrm{Opt}} = 1$.

The following proposition gives the relative belief ratio based on $\overline{\boldsymbol{X}}_i$ in (20).

Proposition 4.13. *The relative belief ratio induced by π_{τ_m} from the scale family in Definition 4.5 is*

$$\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i, \tau_m) = \frac{\{n_m/(2\pi)\}^{k/2}\exp\{-n_m\overline{\boldsymbol{X}}_i^\top\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{X}}_i/2\}}{\tau_m^{-k/2}\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\boldsymbol{\Sigma}^{-1/2}\overline{\boldsymbol{X}}_i - \boldsymbol{Z}_k/\sqrt{n_m}}{\sqrt{\tau_m}}\right)\right]}, \qquad \boldsymbol{Z}_k \sim N_k(\mathbf{0}, \boldsymbol{I}_k).$$

The generalized relative belief procedure rejects H_{0i} when $\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i, \tau_m) < 1/\delta_m$.

Theorem 4.14. *For fixed $k \geq 2$, suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are generated independently according to (19), and consider testing $H_{0i} : \boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$, simultaneously. For each $i = 1, \ldots, m$, apply the generalized relative belief rule induced by π_{τ_m}, that is, rejecting H_{0i} when $\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i, \tau_m) < 1/\delta_m$. Suppose Assumption 4.10 holds, $\{\pi_\tau : \tau > 0\}$ belongs to the family in Definition 4.5, and $\lim_{m\to\infty} \tau_m/p_m^{-2/k} \in (0, \infty)$. Then the resulting generalized relative belief multiple testing procedure is ABOS in the sense of Definition 4.12.*

To obtain a fully data-driven empirical Bayes procedure under Assumption 3.11, we adapt (18). For predetermined fixed $d_7 \geq 1$ and $d_8 \geq 2$, define

$$\widehat{\tau}_m^{\mathrm{MEB_{FB}}} = \min\left\{m^{2/k}, \left[\frac{d_7 m}{\max\left\{1, \sum_{i=1}^m \mathbf{1}\left(n_m\overline{\boldsymbol{X}}_i^\top\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{X}}_i > d_8\log m\right)\right\}}\right]^{2/k}\right\}. \tag{22}$$

For each $i = 1, \ldots, m$, the empirical Bayes generalized relative belief procedure rejects H_{0i} if and only if $\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i, \widehat{\tau}_m^{\mathrm{MEB_{FB}}}) < 1/\delta_m$. The following theorem establishes its ABOS property.

Theorem 4.15. *For predetermined fixed $k \geq 2$, suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are generated independently according to (19), and consider testing $H_{0i} : \boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$, simultaneously. Let $\widehat{\tau}_m^{\mathrm{MEB_{FB}}}$ be defined by (22). For each $i = 1, \ldots, m$, apply the generalized relative belief rule induced by $\pi_{\widehat{\tau}_m^{\mathrm{MEB_{FB}}}}$, rejecting H_{0i} when $\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i, \widehat{\tau}_m^{\mathrm{MEB_{FB}}}) < 1/\delta_m$. Suppose Assumptions 4.10 and 3.11 hold, and $\{\pi_\tau : \tau > 0\}$ belongs to the family in Definition 4.5. Then the resulting empirical Bayes generalized relative belief multiple testing procedure is ABOS in the sense of Definition 4.12.*

5 Discussions

As we have shown, the relative belief approach based on one-group priors is highly amenable to extensions. Thus, it would be interesting to see whether it can also offer simpler and more powerful procedures to achieve sharp asymptotic minimaxity than global-local priors (Paul et al., 2026; Banerjee et al., 2026).

The two-groups model is also often used nonparametrically (e.g., Sun & Cai, 2007; Efron, 2012; Scott et al., 2015; Roquain & Verzelen, 2022). It would be intriguing to see whether the relative belief approach based on a one-group prior can still closely approximate the optimal two-groups performance in nonparametric settings. In the meantime, it would be worth investigating possible computational and modeling advantages that the one-group relative belief approach brings.

The relative belief ratio is also closely related to rapidly emerging e-value literature due to its relation to the Savage-Dickey ratio (Dickey, 1971; Grünwald, 2023). Thus, it is natural to wonder about the interplay between the relative belief ratio and the e-value. This link may open the door to developing various relative belief procedures for more error metrics being studied (such as family-wise error rate (FWER) and false discovery rate (FDR)) or even a general principle of multiple testing. A recent work by Xu et al. (2026) is a good example of developing a general principle of multiple testing based on the e-value.

AI USE STATEMENT

In this work, we used generative AI tools for none of the required disclosure tasks. We did not use generative AI tools for research ideation or execution, drafting sections of the paper, generating synthetic datasets, proving mathematical claims, or identifying relevant literature. Additionally, we used generative AI tools for aiding and polishing writing. We did not use generative AI tools for other tasks with recommended disclosure. We have reviewed all AI-assisted work. All LLM-assisted text, particularly technical terminology regarding the testing procedures, was thoroughly reviewed, verified, and refined by the authors to ensure factual accuracy and proper phrasing. We take responsibility for the final content of this work, including text, claims or artifacts produced with the aid of generative AI.

REPRODUCIBILITY STATEMENT

This work is theoretical. All statistical models, asymptotic regimes and assumptions, testing procedures, and corresponding ABOS results are stated explicitly in the main text. The literature review and complete proofs of the theoretical results are provided in the appendix.

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A Organization and Notation

Section B reviews the related literature. In Section C, we characterize the rejection thresholds induced by the priors in Definition 3.5 and prove that the resulting generalized relative belief procedures with fixed and empirical Bayes scales are ABOS. The setting with an increasing sample size described in Assumption 3.14 is treated in Section D. For the multivariate normal means problem under the additive weighted 0–1 loss, Section E derives the properties of the multivariate Bayes oracle, including its rejection threshold, type I and type II error probabilities, and asymptotic Bayes risk. Taking this risk as the benchmark, we prove in Section F that the generalized relative belief procedures based on the priors in Definition 4.5 are ABOS by controlling their thresholds uniformly over $\mathbb{S}^{k-1}$. Finally, the multivariate results under Assumption 4.10 are established in Section G.

Throughout the paper, we use the following notation for nonnegative sequences $\{A_m\}$ and $\{B_m\}$, with $B_m > 0$ for all sufficiently large m.

1. $A_m \sim B_m$ means $A_m/B_m \to 1$.
2. $A_m \propto B_m$ means $A_m = cB_m$ for every $m \geq 1$, where $c > 0$ is independent of m.
3. $A_m = O(B_m)$ means $|A_m| \leq c|B_m|$ for all sufficiently large m, where $c > 0$ is independent of m.
4. $A_m = o(B_m)$ means $A_m/B_m \to 0$. In particular, $A_m = o(1)$ means $A_m \to 0$.

B LITERATURE REVIEW

This appendix reviews existing results for ABOS and near-ABOS procedures in normal means problems. Section B.1 summarizes the univariate and multivariate asymptotic frameworks and the Bayes risks of the corresponding Bayes oracle rules. Section B.2 reviews the conditions under which half-thresholding procedures induced by global–local shrinkage priors are near-ABOS or ABOS, including corresponding empirical Bayes procedures. Section B.3 reviews existing relative belief procedures under normal and uniform priors and explains how these results motivate the subsequent developments in this paper.

B.1 ASYMPTOTIC BAYES OPTIMALITY UNDER SPARSITY IN NORMAL MEANS PROBLEMS

Under the univariate two-groups model in Section 3.1 and an additive 0–1 loss, Bogdan et al. (2011) introduced asymptotic Bayes optimality under sparsity. The Bayes oracle minimizes the Bayes risk, and its asymptotic risk serves as a benchmark for evaluating multiple testing procedures. Subsequent work compared testing rules induced by one-group shrinkage priors with this benchmark and established near-ABOS or ABOS results under different priors and conditions (Datta & Ghosh, 2013; Ghosh et al., 2016; Bhadra et al., 2017; Ghosh & Chakrabarti, 2017). Qin & Ghosh (2025) later extended this framework to multivariate normal means. We next summarize the asymptotic assumptions and Bayes oracle risks in the univariate and multivariate settings. Note that all previous near-ABOS and ABOS results are studied under the additive symmetric 0–1 loss (i.e., $\delta_0 = \delta_A = 1$ and hence $\delta_m = 1$).

Univariate normal means. As described in Section 3.1, suppose $X_i \mid \mu_i \sim N(\mu_i, 1)$, independently for $i = 1, \ldots, m$, with marginal distribution given by (1). We simultaneously test $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$, under the symmetric 0–1 loss in (2). Following Bogdan et al. (2011), let $u_m = \sigma_m^2$, $f_m = (1 - p_m)/p_m$, and $v_m = u_m f_m^2$.

Assumption B.1. *As $m \to \infty$, assume $p_m \to 0$, $u_m \to \infty$, and*

$$\frac{\log v_m}{u_m} \longrightarrow C \in (0, \infty).$$

Under Assumption B.1, the asymptotic Bayes risk of the Bayes oracle satisfies (Bogdan et al., 2011, Theorem 3.1)

$$R_{\mathrm{Opt}}^{\mathrm{AsyBO}} = m p_m \{2\Phi(\sqrt{C}) - 1\}\{1 + o(1)\}, \tag{23}$$

where Φ is the standard normal distribution function.

Multivariate normal means. As described in Section 4.1, let $k \geq 2$ be fixed and suppose $\boldsymbol{X}_i \mid \boldsymbol{\mu}_i \sim N_k(\boldsymbol{\mu}_i, \boldsymbol{\Sigma})$, independently for $i = 1, \ldots, m$, where $\boldsymbol{\Sigma}$ is known and positive definite. The marginal distribution is given by (15). We simultaneously test $H_{0i} : \boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$, under the symmetric 0–1 loss in (2). Following Qin & Ghosh (2025), let $f_m = (1 - p_m)/p_m$ and

$$\mathcal{L}_m = \log\{f_m^2 g_m^k\}.$$

Assumption B.2. *As $m \to \infty$, assume $p_m \to 0$, $g_m \to \infty$, and*

$$\frac{\mathcal{L}_m}{g_m} \longrightarrow C_0 \in (0, \infty).$$

The smallest and largest eigenvalues of $\boldsymbol{\Sigma}$ are bounded away from zero and infinity.

Under Assumption B.2, the asymptotic Bayes risk of the multivariate Bayes oracle satisfies

$$R_{\mathrm{Opt}}^{\mathrm{AsyMBO}} = m p_m F_{\chi_k^2}(C_0)\{1 + o(1)\}, \tag{24}$$

where $F_{\chi_k^2}$ is the distribution function of a chi-squared random variable with k degrees of freedom. In either setting, a multiple testing procedure is ABOS when the ratio of its asymptotic Bayes risk to that of the corresponding Bayes oracle converges to one. The procedure is near-ABOS when its Bayes risk is asymptotically within a constant factor of that of the corresponding Bayes oracle, where the constant can be close to one.

B.2 Existing Near-ABOS and ABOS Results for One-Group Shrinkage Priors

Univariate results. We first review existing ABOS results for procedures using global–local shrinkage priors under the univariate framework in Section B.1. With independent observations $X_i \mid \mu_i \sim N(\mu_i, 1)$ and the marginal model in (1), we simultaneously test $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \dots, m$, under the symmetric additive 0–1 loss specified above. The results below assume Assumption B.1 and use the Bayes risk in (23) as a benchmark. One-group shrinkage priors can be expressed as global–local scale mixtures of normals (Carvalho et al., 2009; 2010; Polson & Scott, 2011):

$$\mu_i \mid \lambda_i^2, \xi^2 \sim N(0, \lambda_i^2 \xi^2), \qquad \lambda_i^2 \overset{\text{i.i.d.}}{\sim} \pi_1, \qquad \xi^2 \sim \pi_2, \qquad i = 1, \dots, m. \tag{25}$$

The global parameter ξ controls the overall shrinkage, whereas the local parameters λ_i allow different amounts of shrinkage across the means. The results of Datta & Ghosh (2013), Bhadra et al. (2017), and Ghosh et al. (2016) reviewed below treat ξ_m as a tuning parameter and assume $\lim_{m\to\infty} \xi_m / p_m \in (0, \infty)$. With suitably chosen heavy-tailed local priors, (25) gives a continuous prior concentrated near zero while accommodating large signals. Defining $\kappa_i = (1 + \lambda_i^2 \xi^2)^{-1}$, the posterior mean satisfies

$$\mathbb{E}(\mu_i \mid X_i, \xi) = \{1 - \mathbb{E}(\kappa_i \mid X_i, \xi)\} X_i. \tag{26}$$

We next introduce the half-thresholding rule associated with (25) and review several global–local priors for which the resulting procedures are near-ABOS or ABOS.

Under the symmetric 0–1 loss, the Bayes oracle for the two-groups model in (1) rejects H_{0i} when the posterior probability of H_{Ai} exceeds $1/2$. Carvalho et al. (2010) observed a connection between the posterior probabilities of the alternative hypotheses H_{Ai} and the shrinkage coefficient $1 - \mathbb{E}(\kappa_i \mid X_i, \xi)$, suggesting that the same threshold could be applied to the latter. This motivates the half-thresholding rule

$$\text{reject } H_{0i} \quad \text{if} \quad 1 - \mathbb{E}(\kappa_i \mid X_i, \xi_m) > \frac{1}{2}, \qquad i = 1, \dots, m. \tag{27}$$

Building on the rule in (27), Datta & Ghosh (2013), Bhadra et al. (2017), and Ghosh et al. (2016) investigated asymptotic Bayes risks under different global–local priors.

For the horseshoe prior (Carvalho et al., 2010), the hierarchical structure for the testing procedure is

$$\begin{aligned} X_i \mid \mu_i &\sim N(\mu_i, 1), \\ \mu_i \mid \lambda_i^2, \xi_m &\sim N(0, \lambda_i^2 \xi_m^2), \\ \lambda_i &\overset{\text{i.i.d.}}{\sim} \mathrm{C}^+(0, 1), \end{aligned} \tag{28}$$

independently for $i = 1, \dots, m$, where $\mathrm{C}^+(0, s)$ denotes the half-Cauchy distribution with scale s, and ξ_m is treated as a tuning parameter. If $\lim_{m\to\infty} \xi_m / p_m \in (0, \infty)$, Datta & Ghosh (2013) studied the asymptotic Bayes risk R^{AsyHS} of the horseshoe multiple testing procedure based on (27). By Ghosh et al. (2016, Section 3.3), their argument gives

$$R^{\mathrm{AsyHS}} \leq m p_m \{2\Phi(3\sqrt{C}) - 1\}\{1 + o(1)\}. \tag{29}$$

Comparison with (23) shows that the horseshoe procedure is near-ABOS.

The horseshoe+ prior of Bhadra et al. (2017) adds another layer of local parameters η_i:

$$\begin{aligned} X_i \mid \mu_i &\sim N(\mu_i, 1), \\ \mu_i \mid \lambda_i^2, \xi_m &\sim N(0, \lambda_i^2 \xi_m^2), \\ \lambda_i \mid \eta_i &\sim \mathrm{C}^+(0, \eta_i), \\ \eta_i &\overset{\text{i.i.d.}}{\sim} \mathrm{C}^+(0, 1), \end{aligned} \tag{30}$$

independently for $i = 1, \dots, m$. The type I and type II error bounds in Bhadra et al. (2017, Theorems 3 and 5) imply that the asymptotic Bayes risk $R^{\mathrm{AsyHS+}}$ of the horseshoe+ multiple testing procedure based on (27) satisfies

$$R^{\mathrm{AsyHS+}} \leq m p_m \left\{ 2\Phi\left(\sqrt{\frac{2C}{\zeta_{\mathrm{HS+}}(1 - \nu_{\mathrm{HS+}})}} \right) - 1 \right\} \{1 + o(1)\}, \tag{31}$$

for fixed $\zeta_{\mathrm{HS}+} \in (0, 1/2)$ and $\nu_{\mathrm{HS}+} \in (0, 1)$. Comparison with (23) shows that the horseshoe+ procedure is near-ABOS.

Ghosh et al. (2016) extended the analysis to a general collection of one-group shrinkage priors with the hierarchy

$$\begin{aligned} X_i \mid \mu_i &\sim N(\mu_i, 1), \\ \mu_i \mid \lambda_i^2, \xi_m &\sim N(0, \lambda_i^2 \xi_m^2), \\ \lambda_i^2 &\overset{\text{i.i.d.}}{\sim} \pi_1, \\ \pi_1(t) &= K t^{-a-1} L(t), \qquad t > 0, \end{aligned} \tag{32}$$

independently for $i = 1, \ldots, m$. Here, $a > 0$ is a positive real number and $K > 0$ is the normalizing constant. The function $L : (0, \infty) \to (0, \infty)$ is measurable and nonconstant, and is slowly varying at infinity in the sense that $\lim_{t\to\infty} L(s_* t)/L(t) = 1$ for every fixed $s_* > 0$; see Bingham et al. (1989, Chapter 1) and Ghosh et al. (2016, Sections 1.1 and 2). This collection includes three-parameter beta normal mixtures (Armagan et al., 2011) and generalized double Pareto priors (Armagan et al., 2013; Ghosh et al., 2016). In the parameterization of Ghosh et al. (2016), the former have $\pi_1(t) \propto t^{\alpha_0 - 1}(1 + t)^{-(\alpha_0 + \beta_0)}$, where $\alpha_0, \beta_0 > 0$ and $a = \beta_0$. The choice $(\alpha_0, \beta_0) = (1/2, 1/2)$ gives the horseshoe prior, while $(\alpha_0, \beta_0) = (1, 1/2)$ and $\xi = 1$ give the Strawderman–Berger prior (Strawderman, 1971; Berger, 1980).

Ghosh et al. (2016, Theorem 1) assumed either $1/2 < a < 1$, or $a = 1/2$ and $L(t)/\sqrt{\log t} \to 0$ as $t \to \infty$. The asymptotic Bayes risk R^{AsyOG} of the half-thresholding procedure induced by (32) then satisfies, for every fixed $\zeta_{\mathrm{OG}} \in (0, 1/2)$ and $\nu_{\mathrm{OG}} \in (0, 1)$,

$$\begin{aligned} R^{\mathrm{AsyOG}} &\geq m p_m \{2\Phi(\sqrt{2aC}) - 1\}\{1 + o(1)\}, \\ R^{\mathrm{AsyOG}} &\leq m p_m \left\{ 2\Phi\left(\sqrt{\frac{2aC}{\zeta_{\mathrm{OG}}(1 - \nu_{\mathrm{OG}})}} \right) - 1 \right\} \{1 + o(1)\}. \end{aligned} \tag{33}$$

These bounds in (33) establish that the half-thresholding rule induced by the general class of one-group priors in (32) is near-ABOS. For the horseshoe prior, taking $a = 1/2$ and suitable values of ζ_{OG} and ν_{OG} gives a sharper upper bound than (29).

Ghosh & Chakrabarti (2017, Assumption 1) further assumed that the nonconstant slowly varying component L in (32) satisfies

$$\begin{aligned} &\text{(i)} \quad \sup_{t \in (0,\infty)} L(t) \leq M_0 \qquad \text{for some constant } 0 < M_0 < \infty, \\ &\text{(ii)} \quad \lim_{t \to \infty} L(t) \in (0, \infty). \end{aligned} \tag{34}$$

Under these conditions, Ghosh & Chakrabarti (2017, Theorem 6) assumed $a \in [1/2, 1)$ and $\lim_{m\to\infty} \xi_m / p_m^{\gamma} \in (0, \infty)$ for some fixed $\gamma \geq 1$. They showed that the asymptotic Bayes risk $R^{\mathrm{AsyOG(GC)}}$ of the half-thresholding rule induced by the priors in (32) satisfies

$$R^{\mathrm{AsyOG(GC)}} = m p_m \{2\Phi(\sqrt{2a\gamma C}) - 1\}\{1 + o(1)\}. \tag{35}$$

Comparison with (23) shows that the procedure is near-ABOS, with

$$\frac{R^{\mathrm{AsyOG(GC)}}}{R^{\mathrm{AsyBO}}_{\mathrm{Opt}}} \longrightarrow \frac{2\Phi(\sqrt{2a\gamma C}) - 1}{2\Phi(\sqrt{C}) - 1}. \tag{36}$$

Within this collection, priors with $a = 1/2$ are called horseshoe-type priors. Taking $a = 0.5, \gamma = 1$ makes this limit equal to one, so the corresponding half-thresholding procedure induced by the priors in (32) attains the ABOS property.

The preceding procedures based on (27) depend on ξ_m, which is treated as a tuning parameter depending on p_m. Since p_m is unknown, to implement the half-thresholding rule (27), Ghosh et al. (2016) followed the empirical Bayes construction of van der Pas et al. (2014) for minimax estimation of sparse normal means and estimated ξ_m by

$$\widehat{\xi}_m = \max\left\{ \frac{1}{m}, \frac{1}{\widetilde{d}_1 m} \sum_{i=1}^{m} \mathbf{1}(X_i^2 > \widetilde{d}_2 \log m) \right\}, \tag{37}$$

where $\widetilde{d}_1 \geq 1$ and $\widetilde{d}_2 \geq 2$ are prespecified fixed constants. For the general class of one-group priors in (32), they replaced ξ_m by $\widehat{\xi}_m$ in (27), giving the empirical Bayes procedure

$$\text{reject } H_{0i} \quad \text{if} \quad 1 - \mathbb{E}(\kappa_i \mid X_i, \widehat{\xi}_m) > \frac{1}{2}, \qquad i = 1, \ldots, m. \tag{38}$$

Under Assumption B.1 and condition $p_m \propto m^{-\beta}$ for some $\beta \in (0,1)$ (see Assumption 3.11), Ghosh et al. (2016, Theorem 2) showed that the empirical Bayes procedure in (38) is near-ABOS under the same conditions on a and L as in their Theorem 1. For horseshoe-type priors, namely the priors in (32) with $a = 1/2$ and L satisfying (34), Ghosh & Chakrabarti (2017, Theorem 7) further showed, under the same assumptions, that the corresponding empirical Bayes procedure in (38) is ABOS.

Multivariate results. Under Assumption B.2, suppose the observations are generated independently from the two-groups model in (15): $\boldsymbol{X}_i \sim (1-p_m)N_k(\mathbf{0}, \boldsymbol{\Sigma}) + p_m N_k(\mathbf{0}, (1+g_m)\boldsymbol{\Sigma}), \quad i = 1, \ldots, m$. We simultaneously test $H_{0i} : \boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$, under the symmetric additive 0–1 loss specified above. For this testing problem, Qin & Ghosh (2025) considered the following hierarchy:

$$\begin{aligned} \boldsymbol{X}_i \mid \boldsymbol{\mu}_i &\sim N_k(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}), \\ \boldsymbol{\mu}_i \mid \lambda_i^2, \widetilde{\tau}_m &\sim N_k(\mathbf{0}, \lambda_i^2 \widetilde{\tau}_m \boldsymbol{\Sigma}), \\ \lambda_i^2 &\overset{\text{i.i.d.}}{\sim} \pi_1, \qquad \pi_1(t) = K t^{-a-1} L(t), \quad t > 0, \end{aligned} \tag{39}$$

independently for $i = 1, \ldots, m$, where $\boldsymbol{\Sigma}$ is known and positive definite. The global parameter $\widetilde{\tau}_m$ controls the overall shrinkage, whereas the local parameters λ_i^2 allow different amounts of shrinkage across the mean vectors. Here, $\widetilde{\tau}_m$ is treated as a tuning parameter chosen as

$$\widetilde{\tau}_m = p_m^{(1+\zeta_{\mathrm{QG}})/a}, \qquad \zeta_{\mathrm{QG}} = \frac{1}{\log\log(1/p_m)}. \tag{40}$$

The density π_1 is specified in (32), with $a > 0$. The function L satisfies (34) and is additionally assumed to be nondecreasing, meaning that $L(t_1) \leq L(t_2)$ whenever $0 < t_1 < t_2$.

Defining $\widetilde{\kappa}_i = (1 + \lambda_i^2 \widetilde{\tau}_m)^{-1}$, the posterior mean is $\mathbb{E}(\boldsymbol{\mu}_i \mid \boldsymbol{X}_i) = \mathbb{E}(1 - \widetilde{\kappa}_i \mid \boldsymbol{X}_i)\boldsymbol{X}_i$. The corresponding half-thresholding rule is

$$\text{reject } H_{0i} \quad \text{if} \quad 1 - \mathbb{E}(\widetilde{\kappa}_i \mid \boldsymbol{X}_i) > \frac{1}{2}, \qquad i = 1, \ldots, m. \tag{41}$$

Qin & Ghosh (2025, Theorem 5) showed that the asymptotic Bayes risk R^{AsyMGL} of the half-thresholding procedure in (41) satisfies

$$R^{\mathrm{AsyMGL}} = m p_m F_{\chi_k^2}(C_0)\{1 + o(1)\}. \tag{42}$$

Comparison with (24) shows that this procedure is ABOS.

Since p_m is unknown, Qin & Ghosh (2025) proposed the following estimate of $\widetilde{\tau}_m$ to match the desired order $p_m^{(1+\zeta_{\mathrm{QG}})/a}$ in (40):

$$\widehat{\tau}^{\mathrm{MEB_{GL}}} = \max\left\{ \left[\frac{1}{m} \sum_{i=1}^{m} \mathbf{1}\left(\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X}_i > 2\log m \right) \right]^{(1+\widetilde{h}_m)/a}, \, m^{-(1+\widetilde{h}_m)/a} \right\}, \tag{43}$$

where $\widetilde{h}_m = 1/\log\log(m)$. Replacing $\widetilde{\tau}_m$ by $\widehat{\tau}^{\mathrm{MEB_{GL}}}$ in (41) gives the empirical Bayes procedure

$$\text{reject } H_{0i} \quad \text{if} \quad 1 - \mathbb{E}\big(\widetilde{\kappa}_i \mid \boldsymbol{X}_i, \widehat{\tau}^{\mathrm{MEB_{GL}}}\big) > \frac{1}{2}, \qquad i = 1, \ldots, m. \tag{44}$$

Under Assumption B.2 and the condition $p_m \propto m^{-\beta}$ for some $\beta \in (0,1)$ (see Assumption 3.11), Qin & Ghosh (2025) showed that the empirical Bayes procedure in (44) attains the ABOS property.

B.3 Existing ABOS Results of the Relative Belief Procedures

The procedures reviewed in Section B.2 make testing decisions based on the posterior shrinkage. Liang et al. (2026) investigated whether the ABOS property could be attained using simpler prior structures by measuring statistical evidence through both the prior and posterior. Their relative belief approach compares posterior and prior densities and yields analytically tractable testing rules under a one-group light-tailed normal prior with a single hyperparameter.

Univariate results. For independent observations generated from the two-groups model in (1), Liang et al. (2026) studied simultaneous testing of $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$, under the symmetric additive 0–1 loss specified above. They used the normal prior

$$X_i \mid \mu_i \sim N(\mu_i, 1), \qquad \mu_i \mid \tau_{\mathrm{RB}} \sim N(0, \tau_{\mathrm{RB}}^2), \tag{45}$$

independently for $i = 1, \ldots, m$, where $\tau_{\mathrm{RB}} > 0$ is the prior standard deviation and may depend on m. The RB rule rejects H_{0i} when

$$\mathrm{RB}(0 \mid X_i, \tau_{\mathrm{RB}}) = \sqrt{1 + \tau_{\mathrm{RB}}^2} \exp\left\{ -\frac{\tau_{\mathrm{RB}}^2 X_i^2}{2(1 + \tau_{\mathrm{RB}}^2)} \right\} < 1. \tag{46}$$

Under Assumption B.1 and

$$\lim_{m \to \infty} \frac{\tau_{\mathrm{RB}}}{p_m^{-1} \sqrt{\log(1/p_m)}} \in (0, \infty), \tag{47}$$

Liang et al. (2026, Theorem 5.7) showed that the asymptotic Bayes risk R^{AsyRB} of the procedure in (46) satisfies

$$R^{\mathrm{AsyRB}} = m p_m \{2\Phi(\sqrt{C}) - 1\}\{1 + o(1)\}. \tag{48}$$

Thus, the procedure is ABOS by (23).

Since p_m is unknown, they proposed an empirical Bayes procedure that replaces τ_{RB} with the following estimate to match the order in (47):

$$\widehat{\tau}_{\mathrm{EB_{RB}}} = \min\left\{ m\sqrt{\log m}, \frac{\widetilde{d}_3 m \sqrt{\log m}}{\sum_{i=1}^m \mathbf{1}(X_i^2 > \widetilde{d}_4 \log m)} \right\}, \tag{49}$$

where $\widetilde{d}_3 \geq 1$ and $\widetilde{d}_4 \geq 2$ are predetermined fixed constants. Replacing τ_{RB} in (46) by $\widehat{\tau}_{\mathrm{EB_{RB}}}$ gives the empirical Bayes procedure, which rejects H_{0i} when $\mathrm{RB}(0 \mid X_i, \widehat{\tau}_{\mathrm{EB_{RB}}}) < 1$. Under Assumption B.1 and the condition $p_m \propto m^{-\beta}$ for some unknown $\beta \in (0, 1)$, as in Assumption 3.11, Liang et al. (2026, Theorem 5.8) proved that this empirical Bayes procedure is ABOS.

Liang et al. (2026, Appendix B) also established the ABOS property of the relative belief procedure induced by the uniform prior $\mu_i \sim U[-\tau_U, \tau_U]$, where $\tau_U > 0$. They considered the same univariate model $X_i \mid \mu_i \sim N(\mu_i, 1)$, independently for $i = 1, \ldots, m$, and rejected H_{0i} when $\mathrm{RB}(0 \mid X_i, \tau_U) < 1$. After some simplification, their Propositions B.1 and B.2 give the corresponding rule

$$\text{reject } H_{0i} \quad \text{if} \quad \frac{2\tau_U \phi(X_i)}{\Phi(X_i + \tau_U) - \Phi(X_i - \tau_U)} < 1, \qquad i = 1, \ldots, m, \tag{50}$$

where ϕ and Φ denote the standard normal density and distribution function, respectively. Under Assumption B.1, with τ_U satisfying $\lim_{m \to \infty} \tau_U / \{p_m^{-1} \sqrt{\log(1/p_m)}\} \in (0, \infty)$, the same order condition as in (47), their Theorem B.3 showed that this procedure is ABOS, with the same asymptotic Bayes risk as in (48). Replacing τ_U in (50) by $\widehat{\tau}_{\mathrm{EB_{RB}}}$ gives their empirical Bayes procedure under the uniform prior. With the additional condition $p_m \propto m^{-\beta}$, $\beta \in (0, 1)$, their Theorem B.4 showed that the corresponding empirical Bayes procedure is also ABOS.

Multivariate results. For independent observations generated from the two-groups model in (15), Liang et al. (2026) considered simultaneous testing of $H_{0i} : \boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$, under the same symmetric additive 0–1 loss. They used the multivariate normal prior

$$\boldsymbol{X}_i \mid \boldsymbol{\mu}_i \sim N_k(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}), \qquad \boldsymbol{\mu}_i \mid \tau_{\mathrm{MRB}} \sim N_k(\mathbf{0}, \tau_{\mathrm{MRB}} \boldsymbol{\Sigma}), \tag{51}$$

independently for $i = 1, \ldots, m$, with fixed $k \geq 2$, known positive definite $\boldsymbol{\Sigma}$, and $\tau_{\mathrm{MRB}} > 0$, which may depend on m. By their Proposition 6.5, the relative belief rule rejects H_{0i} when

$$\mathrm{RB}(\mathbf{0} \mid \boldsymbol{X}_i, \tau_{\mathrm{MRB}}) = (1 + \tau_{\mathrm{MRB}})^{k/2} \exp\left\{ -\frac{\tau_{\mathrm{MRB}} \boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X}_i}{2(1 + \tau_{\mathrm{MRB}})} \right\} < 1. \tag{52}$$

Under Assumption B.2 and

$$\lim_{m \to \infty} \frac{\tau_{\mathrm{MRB}}}{p_m^{-2/k} \log(1/p_m)} \in (0, \infty), \tag{53}$$

Liang et al. (2026, Theorem 6.11) showed that the asymptotic Bayes risk R^{AsyMRB} of the procedure in (52) satisfies

$$R^{\mathrm{AsyMRB}} = mp_m F_{\chi^2_k}(C_0)\{1 + o(1)\}. \tag{54}$$

Thus, this procedure is ABOS by comparison with (24).

Since p_m is unknown, Liang et al. (2026, Section 7) also suggested an empirical Bayes procedure that replaces τ_{MRB} in (52) with the following estimate to match the order in (53):

$$\widehat{\tau}_{\mathrm{EB_{MRB}}} = \min\left\{ m^{2/k}\log m, \left[\frac{m}{\sum_{i=1}^m \mathbf{1}\left(\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > 2\log m\right)}\right]^{2/k} \log m \right\}. \tag{55}$$

The resulting empirical Bayes procedure rejects H_{0i} when $\mathrm{RB}(\mathbf{0} \mid \boldsymbol{X}_i, \widehat{\tau}_{\mathrm{EB_{MRB}}}) < 1$. They conjectured that this procedure is ABOS.

The results of Liang et al. (2026) show that the relative belief procedures induced by simple normal and uniform priors can be ABOS under symmetric 0–1 loss. These examples motivate the subsequent study of more general one-group priors and additive weighted 0–1 loss, together with empirical Bayes procedures for unknown p_m, in both univariate and multivariate settings.

C Proofs for Section 3.2

This section proves the results stated in Section 3.2. The arguments proceed as follows.

- We first derive the relative belief ratio in Proposition 3.6 and rewrite its rejection rule in a form suitable for threshold analysis.
- We then prove Proposition 3.7, showing that the rejection region is determined by a unique threshold for each $u \in \{-1, 1\}$.
- Next, Lemma 3.8 and Proposition 3.9 establish a common asymptotic expansion for the two thresholds.
- Finally, we compare these thresholds with the characterization of fixed threshold procedures that are ABOS in Theorem 3.4. This proves Theorem 3.10. For the empirical Bayes procedure, we establish sufficient conditions under which random threshold procedures are ABOS (Theorem C.2) and apply them to prove Theorem 3.12.

C.1 Proof of Proposition 3.6

Proof. Fix $i \in \{1, \ldots, m\}$. We derive the relative belief ratio by expressing it as the ratio of the null sampling density to the prior predictive density.

Write the standard normal density as $\varphi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$. Since $X_i \mid \mu_i \sim N(\mu_i, 1)$, its conditional density is

$$\varphi(X_i - \mu_i) = (2\pi)^{-1/2}\exp\{-(X_i - \mu_i)^2/2\}. \tag{56}$$

For a prior from the scale family of one-group priors in Definition 3.5, the density of μ_i is

$$\pi_\tau(\mu_i) = \tau^{-1/2} h(\mu_i/\sqrt{\tau}). \tag{57}$$

The condition $h(0) > 0$ implies that $\pi_\tau(0) = \tau^{-1/2}h(0) > 0$. Hence, the relative belief ratio at the null is well defined.

By the definition of relative belief,

$$\mathrm{RB}(\mu_i = 0 \mid X_i, \tau) = \frac{\pi_\tau(0 \mid X_i)}{\pi_\tau(0)}. \tag{58}$$

Let $\mu_i^\star$ denote the integration variable in the prior predictive density. Bayes' formula yields

$$\pi_\tau(0 \mid X_i) = \frac{\varphi(X_i)\pi_\tau(0)}{\int_{\mathbb{R}} \varphi(X_i - \mu_i^\star)\pi_\tau(\mu_i^\star)\, d\mu_i^\star}. \tag{59}$$

The denominator is finite and strictly positive because φ is bounded and strictly positive and π_τ is a probability density. Substitution of (59) into (58) cancels the common factor $\pi_\tau(0)$, giving

$$\mathrm{RB}(\mu_i = 0 \mid X_i, \tau) = \frac{\varphi(X_i)}{\int_{\mathbb{R}} \varphi(X_i - \mu_i^\star)\pi_\tau(\mu_i^\star)\, d\mu_i^\star}. \tag{60}$$

We next evaluate the prior predictive density in the denominator. Substituting (57) yields

$$\int_{\mathbb{R}} \varphi(X_i - \mu_i^\star)\pi_\tau(\mu_i^\star)\, d\mu_i^\star = \tau^{-1/2} \int_{\mathbb{R}} \varphi(X_i - \mu_i^\star) h(\mu_i^\star/\sqrt{\tau})\, d\mu_i^\star. \tag{61}$$

Apply the change of variables $z_i^\star = X_i - \mu_i^\star$, or equivalently $\mu_i^\star = X_i - z_i^\star$. Since the absolute value of the Jacobian is one, (61) becomes

$$\int_{\mathbb{R}} \varphi(X_i - \mu_i^\star)\pi_\tau(\mu_i^\star)\, d\mu_i^\star = \tau^{-1/2} \int_{\mathbb{R}} \varphi(z_i^\star) h\left(\frac{X_i - z_i^\star}{\sqrt{\tau}}\right) dz_i^\star. \tag{62}$$

The factor $\varphi(z_i^\star)$ is the density of a standard normal random variable. Therefore, for $Z \sim N(0,1)$,

$$\int_{\mathbb{R}} \varphi(X_i - \mu_i^\star)\pi_\tau(\mu_i^\star)\, d\mu_i^\star = \tau^{-1/2}\mathbb{E}_Z\left[h\left(\frac{X_i - Z}{\sqrt{\tau}}\right)\right]. \tag{63}$$

Combining (60) and (63) gives

$$\mathrm{RB}(\mu_i = 0 \mid X_i, \tau) = \frac{(2\pi)^{-1/2}\exp(-X_i^2/2)}{\tau^{-1/2}\mathbb{E}_Z[h((X_i - Z)/\sqrt{\tau})]}.$$

This proves (5).

The generalized relative belief procedure rejects H_{0i} when $\mathrm{RB}(\mu_i = 0 \mid X_i, \tau) < 1/\delta_m$. Since the prior predictive density is strictly positive, this rejection condition is equivalent to

$$(2\pi)^{1/2}\tau^{-1/2}e^{X_i^2/2}\mathbb{E}_Z\left[h\left(\frac{X_i - Z}{\sqrt{\tau}}\right)\right] > \delta_m. \tag{64}$$

This is the form of the rejection criterion used in the proof of Proposition 3.7, completing the proof. □

C.2 Proof of Proposition 3.7

Proof. Fix $u \in \{-1, 1\}$. We first express the generalized relative belief rejection criterion at $X_i = ru$, $r \geq 0$, and then establish the existence and uniqueness of its crossing with the cutoff δ_m.

For $r \geq 0$, evaluate the left-hand side of (64) at $X_i = ru$ and $\tau = \tau_m$, and define

$$\mathcal{G}_{\tau_m,u}(r) = (2\pi)^{1/2}\tau_m^{-1/2}e^{r^2/2}\mathbb{E}_Z\left[h\left(\frac{ru - Z}{\sqrt{\tau_m}}\right)\right]. \tag{65}$$

Writing the expectation as an integral and applying the change of variables $s = ru - z$ gives

$$\mathcal{G}_{\tau_m,u}(r) = \tau_m^{-1/2}\int_{\mathbb{R}} e^{rus - s^2/2} h(s/\sqrt{\tau_m})\, ds. \tag{66}$$

Here, we used $u^2 = 1$ to obtain $r^2/2 - (ru - s)^2/2 = rus - s^2/2$.

The integral in (66) is finite for every $r \geq 0$. In fact, boundedness of h and completion of the square give

$$\int_{\mathbb{R}} e^{rus - s^2/2} h(s/\sqrt{\tau_m})\, ds \leq \|h\|_\infty e^{r^2/2}\int_{\mathbb{R}} e^{-(s-ru)^2/2}\, ds < \infty.$$

When r is restricted to a compact interval, the same calculation, with additional factors $|s|$ and s^2, provides integrable dominating functions independent of r. Consequently, differentiation under the integral sign is valid up to second order.

We next examine $\mathcal{G}_{\tau_m,u}$ at the origin and at infinity. Evaluation of (65) at the origin gives

$$\mathcal{G}_{\tau_m,u}(0) = (2\pi)^{1/2}\tau_m^{-1/2}\mathbb{E}_Z[h(-Z/\sqrt{\tau_m})].$$

Boundedness of h therefore implies

$$\frac{\mathcal{G}_{\tau_m,u}(0)}{\delta_m} \le \frac{(2\pi)^{1/2}\|h\|_\infty}{\delta_m\sqrt{\tau_m}} \longrightarrow 0. \tag{67}$$

The upper bound does not depend on u. Hence, for both $u \in \{-1, 1\}$ and all sufficiently large m,

$$\mathcal{G}_{\tau_m,u}(0) < \delta_m. \tag{68}$$

We next show that $\mathcal{G}_{\tau_m,u}(r)$ eventually exceeds the cutoff. Since h is continuous at zero and $h(0) > 0$, there exists $\eta > 0$ such that

$$h(x) \ge h(0)/2 \qquad \text{whenever } |x| \le \eta. \tag{69}$$

For each m, set $a_m = \eta\sqrt{\tau_m}/4$. Whenever $a_m u - a_m/2 \le s \le a_m u + a_m/2$, the relations $u^2 = 1$ and the triangle inequality give $us \ge a_m/2$ and $|s| \le 3a_m/2 = 3\eta\sqrt{\tau_m}/8$. It follows that $|s|/\sqrt{\tau_m} < \eta$, so (69) applies throughout this interval.

Restricting the integral in (66) to this interval yields

$$\mathcal{G}_{\tau_m,u}(r) \ge \tau_m^{-1/2}\frac{h(0)}{2}a_m \exp\{a_m r/2 - 9a_m^2/8\}. \tag{70}$$

For each fixed m, all factors except $\exp(a_m r/2)$ are positive and independent of r. Therefore,

$$\mathcal{G}_{\tau_m,u}(r) \longrightarrow \infty \qquad \text{as } r \to \infty. \tag{71}$$

We now establish the existence and uniqueness of the crossing. Define

$$\mathcal{I}_{\tau_m,u}(r) = \int_{\mathbb{R}} e^{rus - s^2/2} h(s/\sqrt{\tau_m})\, ds. \tag{72}$$

The integral representation in (66) can then be written as $\mathcal{G}_{\tau_m,u}(r) = \tau_m^{-1/2}\mathcal{I}_{\tau_m,u}(r)$. The local lower bound in (69) ensures that $\mathcal{I}_{\tau_m,u}(r) > 0$. Differentiation under the integral sign gives

$$\mathcal{I}'_{\tau_m,u}(r) = \int_{\mathbb{R}} us\, e^{rus - s^2/2} h(s/\sqrt{\tau_m})\, ds, \tag{73}$$

$$\mathcal{I}''_{\tau_m,u}(r) = \int_{\mathbb{R}} s^2 e^{rus - s^2/2} h(s/\sqrt{\tau_m})\, ds. \tag{74}$$

Since the factor $\tau_m^{-1/2}$ does not depend on r, the second derivative of $\log \mathcal{G}_{\tau_m,u}$ is

$$\frac{d^2}{dr^2}\log \mathcal{G}_{\tau_m,u}(r) = \frac{\mathcal{I}''_{\tau_m,u}(r)\mathcal{I}_{\tau_m,u}(r) - [\mathcal{I}'_{\tau_m,u}(r)]^2}{[\mathcal{I}_{\tau_m,u}(r)]^2}. \tag{75}$$

The Cauchy–Schwarz inequality applied to (73) gives

$$[\mathcal{I}'_{\tau_m,u}(r)]^2 < \mathcal{I}''_{\tau_m,u}(r)\mathcal{I}_{\tau_m,u}(r). \tag{76}$$

The inequality is strict because equality could hold only if us were constant almost everywhere with respect to the measure having density $e^{rus - s^2/2}h(s/\sqrt{\tau_m})$. By (69), this density is strictly positive throughout the interval $(-\eta\sqrt{\tau_m}, \eta\sqrt{\tau_m})$. The function $s \mapsto us$ is not Lebesgue-almost-everywhere constant on any interval of positive length, so the equality condition cannot hold.

Combining (75) and (76) shows that

$$\frac{d^2}{dr^2}\log \mathcal{G}_{\tau_m,u}(r) > 0, \qquad r \ge 0. \tag{77}$$

Thus, $\log \mathcal{G}_{\tau_m,u}$ is strictly convex on $[0, \infty)$.

For all sufficiently large m, (68) and (71) imply that the sublevel set

$$\{r \ge 0 : \log \mathcal{G}_{\tau_m,u}(r) \le \log \delta_m\}$$

is nonempty, closed, bounded, and contains a neighborhood of zero in $[0, \infty)$. Strict convexity implies that this sublevel set is an interval of the form $[0, c_{\tau_m}(u)]$ for some $c_{\tau_m}(u) > 0$. Continuity of $\mathcal{G}_{\tau_m,u}$ then gives

$$\mathcal{G}_{\tau_m,u}(c_{\tau_m}(u)) = \delta_m. \tag{78}$$

This positive solution is unique. Indeed, for every $0 < r < c_{\tau_m}(u)$, strict convexity gives

$$\begin{aligned}\log \mathcal{G}_{\tau_m,u}(r) &< \left(1 - \frac{r}{c_{\tau_m}(u)}\right) \log \mathcal{G}_{\tau_m,u}(0) + \frac{r}{c_{\tau_m}(u)} \log \delta_m \\ &< \log \delta_m.\end{aligned}$$

For $r > c_{\tau_m}(u)$, the definition of the sublevel set implies $\log \mathcal{G}_{\tau_m,u}(r) > \log \delta_m$. Consequently,

$$\begin{aligned}&\mathcal{G}_{\tau_m,u}(r) < \delta_m, \quad 0 \le r < c_{\tau_m}(u), \\ &\mathcal{G}_{\tau_m,u}(r) > \delta_m, \quad r > c_{\tau_m}(u).\end{aligned} \tag{79}$$

The crossing equation in (78) is exactly (6).

Finally, we express the rejection rule in terms of these thresholds. For $X_i \neq 0$, write $r_i = |X_i|$ and $u_i = X_i/|X_i| \in \{-1, 1\}$. Then $X_i = r_i u_i$ and $r_i^2 = X_i^2$. Applying the rejection criterion in (64) with $\tau = \tau_m$ shows that the generalized relative belief procedure rejects H_{0i} if and only if $\mathcal{G}_{\tau_m,u_i}(r_i) > \delta_m$. For all sufficiently large m, (79) shows that this is equivalent to $r_i > c_{\tau_m}(u_i)$, and hence to $X_i^2 > c_{\tau_m}^2(u_i)$. This proves the claimed threshold representation. □

C.3 Proof of Lemma 3.8

Proof. We prove the uniform approximation by separating the event on which the Gaussian perturbation remains in a neighborhood of zero from its complement.

We first control the integrand near zero. Fix $\varepsilon > 0$. Since h is continuous at zero, there exists $\eta_0 > 0$ such that

$$|h(x) - h(0)| < \varepsilon \qquad \text{whenever } |x| \le \eta_0. \tag{80}$$

Fix $0 < \eta \le \eta_0/2$, and consider any $y \in \mathbb{R}$ satisfying $|y| \le \eta\sqrt{\tau_m}$. On the event $|Z| \le \eta_0\sqrt{\tau_m}/2$, the triangle inequality gives

$$\left|\frac{y - Z}{\sqrt{\tau_m}}\right| \le \frac{|y| + |Z|}{\sqrt{\tau_m}} \le \eta + \eta_0/2 \le \eta_0. \tag{81}$$

It follows from (80) that

$$\left|h\left(\frac{y - Z}{\sqrt{\tau_m}}\right) - h(0)\right| < \varepsilon \tag{82}$$

on this event.

Next, we bound the expectation uniformly over the allowed values of y. The triangle inequality for expectations gives

$$\left|\mathbb{E}_Z\left[h\left(\frac{y - Z}{\sqrt{\tau_m}}\right)\right] - h(0)\right| \le \mathbb{E}_Z\left[\left|h\left(\frac{y - Z}{\sqrt{\tau_m}}\right) - h(0)\right|\right]. \tag{83}$$

On the event considered above, the integrand on the right-hand side is bounded by ε according to (82). On its complement, boundedness of h and $h(0) \le \|h\|_\infty$ imply $|h((y - Z)/\sqrt{\tau_m}) - h(0)| \le 2\|h\|_\infty$. Splitting the expectation over these two events yields

$$\left|\mathbb{E}_Z\left[h\left(\frac{y - Z}{\sqrt{\tau_m}}\right)\right] - h(0)\right| \le \varepsilon + 2\|h\|_\infty \mathbb{P}(|Z| > \eta_0\sqrt{\tau_m}/2). \tag{84}$$

The right-hand side does not depend on y. Taking the supremum over $|y| \le \eta\sqrt{\tau_m}$ therefore gives

$$\sup_{|y| \le \eta\sqrt{\tau_m}} \left|\mathbb{E}_Z\left[h\left(\frac{y - Z}{\sqrt{\tau_m}}\right)\right] - h(0)\right| \le \varepsilon + 2\|h\|_\infty \mathbb{P}(|Z| > \eta_0\sqrt{\tau_m}/2). \tag{85}$$

Finally, we pass to the limit. The assumption $\tau_m \to \infty$ implies $\eta_0\sqrt{\tau_m}/2 \to \infty$. Since the distribution of Z does not depend on m, the tail probability in (85) converges to zero. Hence, for every fixed $0 < \eta \le \eta_0/2$,

$$\limsup_{m\to\infty} \sup_{|y|\le\eta\sqrt{\tau_m}} \left| \mathbb{E}_Z\left[h\left(\frac{y-Z}{\sqrt{\tau_m}}\right)\right] - h(0)\right| \le \varepsilon. \tag{86}$$

The expression on the left-hand side is nondecreasing as η increases. Consequently, its limit as $\eta \downarrow 0$ exists. Because (86) holds for every sufficiently small $\eta > 0$, it follows that

$$\lim_{\eta\downarrow 0} \limsup_{m\to\infty} \sup_{|y|\le\eta\sqrt{\tau_m}} \left| \mathbb{E}_Z\left[h\left(\frac{y-Z}{\sqrt{\tau_m}}\right)\right] - h(0)\right| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and the expression is nonnegative, the limit equals zero. This proves (7). □

C.4 Proof of Proposition 3.9

Proof. We first construct two candidate values and apply Lemma 3.8 at these values. We then use the inequalities in (79) to obtain a uniform bound for $c^2_{\tau_m}(u)$, $u \in \{-1, 1\}$.

Define the centering term

$$b_m^{\mathrm{RB}} = \log\tau_m + 2\log\delta_m - 2\log\{(2\pi)^{1/2}h(0)\}. \tag{87}$$

This quantity can equivalently be written as $b_m^{\mathrm{RB}} = 2\log(\delta_m\sqrt{\tau_m}) - 2\log\{(2\pi)^{1/2}h(0)\}$. The assumption $\delta_m\sqrt{\tau_m} \to \infty$ therefore implies $b_m^{\mathrm{RB}} \to \infty$. Fix $\varepsilon > 0$. Both $b_m^{\mathrm{RB}} - \varepsilon$ and $b_m^{\mathrm{RB}} + \varepsilon$ are positive for all sufficiently large m.

Since $\tau_m \to \infty$ and $\log(\delta_m\sqrt{\tau_m})/\tau_m \to 0$, we have

$$\frac{b_m^{\mathrm{RB}} \pm \varepsilon}{\tau_m} = \frac{2\log(\delta_m\sqrt{\tau_m}) - 2\log\{(2\pi)^{1/2}h(0)\} \pm \varepsilon}{\tau_m} \longrightarrow 0. \tag{88}$$

Consequently, $\sqrt{b_m^{\mathrm{RB}} \pm \varepsilon}/\sqrt{\tau_m} \to 0$. For every fixed $\eta > 0$, both candidate values are therefore at most $\eta\sqrt{\tau_m}$ for all sufficiently large m. Applying Lemma 3.8 and then letting $\eta \downarrow 0$ gives

$$\sup_{u\in\{-1,1\}} \left| \mathbb{E}_Z\left[h\left(\frac{\sqrt{b_m^{\mathrm{RB}} \pm \varepsilon}\,u - Z}{\sqrt{\tau_m}}\right)\right] - h(0)\right| \longrightarrow 0. \tag{89}$$

Since $h(0) > 0$, the same conclusion can be written uniformly over $u \in \{-1, 1\}$ as

$$\mathbb{E}_Z\left[h\left(\frac{\sqrt{b_m^{\mathrm{RB}} \pm \varepsilon}\,u - Z}{\sqrt{\tau_m}}\right)\right] = h(0)\{1 + o(1)\}. \tag{90}$$

Next, we evaluate $\mathcal{G}_{\tau_m,u}$, defined in (65), at the two candidate values. The definition of b_m^{RB} implies the identity

$$(2\pi)^{1/2}\tau_m^{-1/2}\exp\{b_m^{\mathrm{RB}}/2\}h(0) = \delta_m. \tag{91}$$

At the lower candidate value, combining (65), (90), and (91) gives, uniformly over $u \in \{-1, 1\}$,

$$\mathcal{G}_{\tau_m,u}(\sqrt{b_m^{\mathrm{RB}} - \varepsilon}) = \delta_m e^{-\varepsilon/2}\{1 + o(1)\}. \tag{92}$$

Since $e^{-\varepsilon/2} < 1$, the uniform remainder in (92) implies that, for all sufficiently large m,

$$\mathcal{G}_{\tau_m,u}(\sqrt{b_m^{\mathrm{RB}} - \varepsilon}) < \delta_m \tag{93}$$

simultaneously for $u \in \{-1, 1\}$.

Evaluation at the upper candidate value similarly gives

$$\mathcal{G}_{\tau_m,u}(\sqrt{b_m^{\mathrm{RB}} + \varepsilon}) = \delta_m e^{\varepsilon/2}\{1 + o(1)\}, \tag{94}$$

uniformly over $u \in \{-1, 1\}$. Since $e^{\varepsilon/2} > 1$, this expansion implies that, for all sufficiently large m,

$$\mathcal{G}_{\tau_m,u}(\sqrt{b_m^{\mathrm{RB}} + \varepsilon}) > \delta_m \tag{95}$$

simultaneously for $u \in \{-1, 1\}$.

Finally, we use these inequalities to bound $c_{\tau_m}(u)$. The assumptions of the present proposition include those of Proposition 3.7. Hence, for both $u \in \{-1, 1\}$ and all sufficiently large m, the crossing $c_{\tau_m}(u)$ exists and is unique. Moreover, (79) shows that $\mathcal{G}_{\tau_m,u}(r)$ is below δ_m before the crossing and above δ_m after the crossing.

Combining these inequalities with (93) and (95) yields

$$\sqrt{b_m^{\mathrm{RB}} - \varepsilon} < c_{\tau_m}(u) < \sqrt{b_m^{\mathrm{RB}} + \varepsilon}$$

simultaneously for $u \in \{-1, 1\}$. All three quantities are positive for sufficiently large m, so squaring the inequalities gives

$$b_m^{\mathrm{RB}} - \varepsilon < c_{\tau_m}^2(u) < b_m^{\mathrm{RB}} + \varepsilon, \qquad u \in \{-1, 1\}. \tag{96}$$

Therefore, $\sup_{u\in\{-1,1\}} |c_{\tau_m}^2(u) - b_m^{\mathrm{RB}}| \le \varepsilon$ for all sufficiently large m. Since $\varepsilon > 0$ is arbitrary, substituting (87) proves (8). □

C.5 Proof of Theorem 3.10

We first record the fixed threshold result used in the proof.

Theorem C.1 (Bogdan et al., 2011, Theorem 3.2)**.** *Suppose Assumption 3.2 holds. Consider the fixed threshold rule that, for each $i = 1, \dots, m$, rejects H_{0i} when*

$$X_i^2 > \log v_m + z_m, \tag{97}$$

where the threshold is positive for all sufficiently large m. This rule is ABOS if and only if

$$z_m = o(\log v_m), \qquad z_m + 2\log\log v_m \longrightarrow \infty. \tag{98}$$

In particular, every sequence $z_m = O(1)$ gives a fixed threshold rule that is ABOS.

By Bogdan et al. (2011, Remark 3.1 and Theorems 3.1–3.2), for every $z_m = O(1)$,

$$\delta_m f_m \mathbb{P}_{H_{0i}}\left(X_i^2 > \log v_m + z_m\right) = o(1), \tag{99}$$

$$\mathbb{P}_{H_{Ai}}\left(X_i^2 < \log v_m + z_m\right) = 2\Phi(\sqrt{C}) - 1 + o(1). \tag{100}$$

These conclusions will be applied to the two fixed threshold rules bounding the generalized relative belief rejection region.

Proof. We first verify that the scale sequence τ_m satisfies the conditions required for the uniform threshold expansion. We then compare the generalized relative belief thresholds for $u \in \{-1, 1\}$ with nearby fixed thresholds covered by Theorem C.1.

Assumption 3.2 gives $p_m \to 0$, $u_m \to \infty$, and $\log v_m/u_m \to C \in (0, \infty)$. Consequently, $f_m \to \infty$, $\log v_m \to \infty$, and $v_m \to \infty$. Moreover, since $\log u_m/u_m \to 0$,

$$\frac{\log(\delta_m f_m)}{u_m} = \frac{\log v_m - \log u_m}{2u_m} \longrightarrow \frac{C}{2}.$$

The condition $\log \delta_m = o(\log p_m)$ and $f_m = (1 - p_m)/p_m$ imply $\log(\delta_m f_m)/\log(1/p_m) \to 1$. Thus $u_m/\log(1/p_m) \to 2/C$.

Let $\ell_1 = \lim_{m\to\infty} \tau_m/\{p_m^{-2}\log(1/p_m)\} \in (0, \infty)$. Using also $f_m = p_m^{-1}\{1 + o(1)\}$, the scale condition in Theorem 3.10 therefore gives

$$\tau_m = \frac{C\ell_1}{2} u_m f_m^2 \{1 + o(1)\}. \tag{101}$$

Since $u_m \to \infty$ and $f_m \to \infty$, this relation implies that $\tau_m \to \infty$.

Using $v_m = u_m(\delta_m f_m)^2$, the same scale relation gives

$$\delta_m\sqrt{\tau_m} = \sqrt{\frac{C\ell_1}{2}v_m}\{1+o(1)\} \longrightarrow \infty. \tag{102}$$

Taking logarithms in (102) yields

$$\log(\delta_m\sqrt{\tau_m}) = \frac{1}{2}\log v_m + \frac{1}{2}\log(C\ell_1/2) + o(1).$$

Combining this expansion with (101) gives

$$\frac{\log(\delta_m\sqrt{\tau_m})}{\tau_m} = \frac{C+o(1)}{C\ell_1 f_m^2} \longrightarrow 0. \tag{103}$$

All the conditions of Proposition 3.9 are therefore satisfied.

Next, we match the generalized relative belief thresholds with fixed thresholds. Taking logarithms in (101) gives

$$\log\tau_m = \log u_m + 2\log f_m + \log(C\ell_1/2) + o(1).$$

Since $\log v_m = \log u_m + 2\log f_m + 2\log\delta_m$, we obtain

$$\log\tau_m + 2\log\delta_m = \log v_m + \log(C\ell_1/2) + o(1). \tag{104}$$

Proposition 3.9 and (104) now imply

$$\sup_{u\in\{-1,1\}}\left|c^2_{\tau_m}(u) - \left[\log v_m + \log(C\ell_1/2) - 2\log\{(2\pi)^{1/2}h(0)\}\right]\right| \longrightarrow 0. \tag{105}$$

For notational convenience, define the constant $b_{\ell_1,h} = \log(C\ell_1/2) - 2\log\{(2\pi)^{1/2}h(0)\}$. Fix $\varepsilon > 0$. The uniform convergence in (105) implies that, for all sufficiently large m,

$$\log v_m + b_{\ell_1,h} - \varepsilon \le c^2_{\tau_m}(u) \le \log v_m + b_{\ell_1,h} + \varepsilon \tag{106}$$

simultaneously for both $u \in \{-1,1\}$.

For $X_i \neq 0$, let $u_i = X_i/|X_i|$. The threshold representation in Proposition 3.7, together with (106), gives

$$\begin{aligned}\{X_i^2 > \log v_m + b_{\ell_1,h} + \varepsilon\} &\subseteq \{X_i^2 > c^2_{\tau_m}(u_i)\} \\ &\subseteq \{X_i^2 > \log v_m + b_{\ell_1,h} - \varepsilon\}.\end{aligned} \tag{107}$$

Under both H_{0i} and H_{Ai}, the distribution of X_i is continuous, so the event $X_i = 0$ has probability zero and does not affect the corresponding error probabilities.

The two fixed thresholds in (107) have bounded perturbations $b_{\ell_1,h} - \varepsilon$ and $b_{\ell_1,h} + \varepsilon$. Hence, both corresponding fixed threshold rules are ABOS by Theorem C.1.

We now bound the error probabilities. Define the RB type I and type II errors by $t_{1i}^{\mathrm{RB}} = \mathbb{P}_{H_{0i}}\{\mathrm{RB}(\mu_i = 0 \mid X_i, \tau_m) < 1/\delta_m\}$ and $t_{2i}^{\mathrm{RB}} = \mathbb{P}_{H_{Ai}}\{\mathrm{RB}(\mu_i = 0 \mid X_i, \tau_m) > 1/\delta_m\}$. The right-hand inclusion in (107) implies

$$t_{1i}^{\mathrm{RB}} \le \mathbb{P}_{H_{0i}}\left(X_i^2 > \log v_m + b_{\ell_1,h} - \varepsilon\right). \tag{108}$$

The perturbation $b_{\ell_1,h} - \varepsilon$ is bounded, so the corresponding fixed threshold rule is covered by Theorem C.1. Applying (99) gives

$$\delta_m f_m t_{1i}^{\mathrm{RB}} = o(1). \tag{109}$$

The two inclusions in (107) also bound the type II error probability from both sides:

$$\begin{aligned}\mathbb{P}_{H_{Ai}}\left(X_i^2 < \log v_m + b_{\ell_1,h} - \varepsilon\right) &\le t_{2i}^{\mathrm{RB}} \\ &\le \mathbb{P}_{H_{Ai}}\left(X_i^2 < \log v_m + b_{\ell_1,h} + \varepsilon\right).\end{aligned} \tag{110}$$

Both perturbations are bounded. Applying (100) to the lower and upper bounds therefore gives

$$t_{2i}^{\mathrm{RB}} = 2\Phi(\sqrt{C}) - 1 + o(1). \tag{111}$$

Finally, we compare the Bayes risks. Since all coordinates have the same distribution and use the same generalized relative belief rule, their error probabilities do not depend on i. The additive Bayes risk decomposition in (2), together with (109) and (111), yields

$$R^{\mathrm{AsyRB}} = mp_m\delta_A\left[\delta_m f_m t_{1i}^{\mathrm{RB}} + t_{2i}^{\mathrm{RB}}\right] = mp_m\delta_A\left[2\Phi(\sqrt{C}) - 1 + o(1)\right]. \tag{112}$$

By (3), the asymptotic Bayes risk of the Bayes oracle satisfies

$$R_{\mathrm{Opt}}^{\mathrm{AsyBO}} = mp_m\delta_A\{2\Phi(\sqrt{C}) - 1\}\{1 + o(1)\}.$$

Since $C > 0$, the quantity $2\Phi(\sqrt{C}) - 1$ is strictly positive. Comparing these two expressions gives $R^{\mathrm{AsyRB}}/R_{\mathrm{Opt}}^{\mathrm{AsyBO}} \to 1$. Thus, the generalized relative belief procedure is ABOS in the sense of Definition 3.3. □

C.6 The ABOS Property of Random Threshold Rules under Assumption 3.2

The proof of Theorem 3.12 requires a random threshold extension of Theorem C.1. If, on an event, a procedure admits thresholds indexed by $u \in \{-1, 1\}$ that are uniformly close to the threshold of a fixed threshold procedure that is ABOS, then, on this event, its rejection region lies between the rejection regions obtained by subtracting and adding a fixed constant to that threshold. Both corresponding fixed threshold rules remain ABOS. It therefore remains to require that the contribution of the complementary event to the Bayes risk be negligible relative to the asymptotic Bayes risk of the Bayes oracle. The following result extends the random threshold criterion in Liang et al. (2026, Theorem F.2) to thresholds indexed by $u \in \{-1, 1\}$ under additive weighted 0–1 loss.

Theorem C.2. *Suppose $X_1, \ldots, X_m$ are generated independently according to (1) under Assumption 3.2. Consider testing $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$. For each $i = 1, \ldots, m$ with $X_i \neq 0$, let $u_i = X_i/|X_i|$, and suppose the procedure rejects H_{0i} if and only if $X_i^2 > \widehat{c}_m^2(u_i)$. Let*

$$c_{m,\mathrm{fix}}^2 = \log v_m + z_m > 0 \tag{113}$$

satisfy

$$z_m = o(\log v_m), \qquad z_m + 2\log\log v_m \longrightarrow \infty. \tag{114}$$

For every fixed $\varepsilon > 0$, suppose there exists an event $G_m(\varepsilon)$ such that

$$\sup_{u\in\{-1,1\}} \left|\widehat{c}_m^2(u) - c_{m,\mathrm{fix}}^2\right| \le \varepsilon \qquad \text{on } G_m(\varepsilon) \tag{115}$$

for all sufficiently large m. For every fixed $\varepsilon > 0$, if

$$\max\{1, \delta_m\}\mathbb{P}(G_m(\varepsilon)^c) = o(p_m) \tag{116}$$

then the procedure is ABOS in the sense of Definition 3.3.

Proof of Theorem C.2. Fix $\varepsilon > 0$, and take the event $G_m(\varepsilon)$ from the assumptions of the theorem. We first compare the procedure with the two fixed threshold rules having thresholds $c_{m,\mathrm{fix}}^2 - \varepsilon$ and $c_{m,\mathrm{fix}}^2 + \varepsilon$.

By (113), the two comparison thresholds can be written as $c_{m,\mathrm{fix}}^2 \pm \varepsilon = \log v_m + (z_m \pm \varepsilon)$. Since $z_m = o(\log v_m)$ and $\log v_m \to \infty$, both comparison thresholds are positive for all sufficiently large m. Because ε is fixed, (114) implies

$$z_m \pm \varepsilon = o(\log v_m), \qquad z_m \pm \varepsilon + 2\log\log v_m \longrightarrow \infty.$$

Hence, both fixed comparison rules are ABOS by Theorem C.1.

Next, we bound the loss on $G_m(\varepsilon)$. By (115), the thresholds satisfy

$$c_{m,\mathrm{fix}}^2 - \varepsilon \le \widehat{c}_m^2(u) \le c_{m,\mathrm{fix}}^2 + \varepsilon, \qquad u \in \{-1, 1\}. \tag{117}$$

For each coordinate, define the weighted loss of the actual procedure by

$$L_{m,i} = \delta_0\mathbf{1}(\mu_i = 0,\ H_{0i} \text{ is rejected}) + \delta_A\mathbf{1}(\mu_i \neq 0,\ H_{0i} \text{ is not rejected}).$$

This loss is defined for all observations. The threshold representation on $G_m(\varepsilon)$ and the inequalities in (117) therefore imply the following bound almost surely on $G_m(\varepsilon)$:

$$L_{m,i} \le \delta_0 \mathbf{1}\big(\mu_i = 0,\ X_i^2 > c_{m,\mathrm{fix}}^2 - \varepsilon\big) + \delta_A \mathbf{1}\big(\mu_i \neq 0,\ X_i^2 < c_{m,\mathrm{fix}}^2 + \varepsilon\big)\,. \tag{118}$$

On the complementary event, the total loss over all coordinates is at most $m\max\{\delta_0, \delta_A\}$. Multiplying the bound above by $\mathbf{1}(G_m(\varepsilon))$, summing over coordinates, and taking expectations therefore gives

$$\begin{aligned} R_m^{\mathrm{rand}} \le\ & m(1-p_m)\delta_0 \mathbb{P}_{H_{0i}}\big(X_i^2 > c_{m,\mathrm{fix}}^2 - \varepsilon\big) \\ & + m p_m \delta_A \mathbb{P}_{H_{Ai}}\big(X_i^2 < c_{m,\mathrm{fix}}^2 + \varepsilon\big) \\ & + m\max\{\delta_0, \delta_A\}\mathbb{P}(G_m(\varepsilon)^c)\,, \end{aligned} \tag{119}$$

where R_m^{rand} denotes the Bayes risk of the procedure.

By Bogdan et al. (2011, Remark 3.1, Theorems 3.1–3.2, and Remark 3.3), the type I component associated with the lower comparison threshold satisfies

$$\frac{m(1-p_m)\delta_0 \mathbb{P}_{H_{0i}}\Big(X_i^2 > c_{m,\mathrm{fix}}^2 - \varepsilon\Big)}{R_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \longrightarrow 0. \tag{120}$$

The same results show that the type II component associated with the upper comparison threshold satisfies

$$\frac{m p_m \delta_A \mathbb{P}_{H_{Ai}}\Big(X_i^2 < c_{m,\mathrm{fix}}^2 + \varepsilon\Big)}{R_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \longrightarrow 1. \tag{121}$$

Finally, we control the contribution of the complementary event. The expansion of the asymptotic Bayes risk of the Bayes oracle in (3) gives

$$\frac{m\max\{\delta_0, \delta_A\}\mathbb{P}(G_m(\varepsilon)^c)}{R_{\mathrm{Opt}}^{\mathrm{AsyBO}}} = \frac{\max\{1, \delta_m\}\mathbb{P}(G_m(\varepsilon)^c)}{p_m\{2\Phi(\sqrt{C}) - 1\}}\{1 + o(1)\}. \tag{122}$$

The concentration condition in (116) and the strict positivity of $2\Phi(\sqrt{C}) - 1$ imply that

$$\frac{m\max\{\delta_0, \delta_A\}\mathbb{P}(G_m(\varepsilon)^c)}{R_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \longrightarrow 0. \tag{123}$$

Dividing (119) by $R_{\mathrm{Opt}}^{\mathrm{AsyBO}}$, and then using (120), (121), and (123), yields

$$\limsup_{m\to\infty} \frac{R_m^{\mathrm{rand}}}{R_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \le 1.$$

Let R_m^{BO} denote the finite sample Bayes risk of the Bayes oracle procedure in Definition 3.1. The Bayes oracle minimizes the Bayes risk for every m, and hence $R_m^{\mathrm{rand}} \ge R_m^{\mathrm{BO}}$. Moreover, (3) implies $R_m^{\mathrm{BO}}/R_{\mathrm{Opt}}^{\mathrm{AsyBO}} \to 1$. It follows that

$$\liminf_{m\to\infty} \frac{R_m^{\mathrm{rand}}}{R_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \ge 1.$$

Combining the upper and lower limits proves that $R_m^{\mathrm{rand}}/R_{\mathrm{Opt}}^{\mathrm{AsyBO}} \to 1$. Thus, the multiple testing procedure is ABOS. □

C.7 Proof of Theorem 3.12

Proof. We apply the criterion in Theorem C.2. The empirical Bayes procedure is defined through the relative belief ratio for all observations. Define

$$q_{\mathrm{R}} = 2\left\{1 - \Phi\left(\sqrt{\frac{d_2 C}{2\beta}}\right)\right\} \in (0, 1), \tag{124}$$

and consider the fixed threshold

$$c_{\mathrm{R}}^2 = 2\log(\delta_m p_m^{-1}) + \log\log m + 2\log\left(\frac{d_1}{q_{\mathrm{R}}}\right) - 2\log\left\{(2\pi)^{1/2}h(0)\right\}. \tag{125}$$

We establish the following two statements.

(i) There exists a measurable event G_{R}, determined by the observations, satisfying $\max\{1,\delta_m\}\mathbb{P}(G_{\mathrm{R}}^c) = o(p_m)$, such that, for every fixed $\varepsilon > 0$ and all sufficiently large m, the procedure admits positive thresholds indexed by $u \in \{-1,1\}$ on G_{R} satisfying

$$\sup_{u\in\{-1,1\}} \left| c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{BC}}}}(u) - c_{\mathrm{R}}^2 \right| \le \varepsilon \qquad \text{on } G_{\mathrm{R}}. \tag{126}$$

(ii) The fixed threshold rule with threshold c_{R}^2 is ABOS. Equivalently, $c_{\mathrm{R}}^2 = \log v_m + z_{\mathrm{R},m}$, where

$$z_{\mathrm{R},m} = o(\log v_m), \qquad z_{\mathrm{R},m} + 2\log\log v_m \longrightarrow \infty.$$

Once these statements are proved, Theorem C.2 applies with $G_m(\varepsilon) = G_{\mathrm{R}}$, $c^2_{m,\mathrm{fix}} = c_{\mathrm{R}}^2$, and $\widehat{c}_m^2(u) = c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{BC}}}}(u)$ on G_{R}.

We first prove statement (i). Assumption 3.11 implies

$$\log f_m = \beta\log m + O(1), \qquad \log(mp_m) = (1-\beta)\log m + O(1). \tag{127}$$

and also gives $\log\delta_m = o(\log m)$. Consequently, $\log(\delta_m f_m) = \beta\log m + o(\log m)$. Since $v_m = u_m(\delta_m f_m)^2$, the condition $\log v_m/u_m \to C$ can be written as

$$\frac{2\beta\log m + \log u_m + o(\log m)}{u_m} \longrightarrow C. \tag{128}$$

Relation (128) first implies that $u_m/\log m$ is bounded above and bounded away from zero. Indeed, along a subsequence on which this ratio diverged, both $\log m/u_m$ and $\log u_m/u_m$ would converge to zero, contradicting $C > 0$. Along a subsequence on which the ratio converged to zero, the leading term $2\beta\log m/u_m$ would diverge, again contradicting the finite limit in (128). It follows that $\log u_m = o(\log m)$. Substitution into (128) yields

$$u_m = \frac{2\beta}{C}\log m\{1 + o(1)\}. \tag{129}$$

Define

$$N_{\mathrm{R}} = \sum_{i=1}^{m} \mathbf{1}\left(X_i^2 > d_2\log m\right). \tag{130}$$

Under the null, $X_i \sim N(0,1)$, whereas under the alternative, the marginal distribution is $N(0, 1+u_m)$. Therefore,

$$\mathbb{E}(N_{\mathrm{R}}) = m(1-p_m)2\left\{1-\Phi\left(\sqrt{d_2\log m}\right)\right\} + mp_m 2\left\{1-\Phi\left(\sqrt{\frac{d_2\log m}{1+u_m}}\right)\right\}. \tag{131}$$

Using the Gaussian tail bound in Vershynin (2026, Proposition 2.1.2), we obtain

$$2\left\{1-\Phi\left(\sqrt{d_2\log m}\right)\right\} = O\left(\frac{m^{-d_2/2}}{\sqrt{\log m}}\right).$$

Combining this bound with (127) yields

$$\frac{m(1-p_m)2\left\{1-\Phi\left(\sqrt{d_2\log m}\right)\right\}}{mp_m} = O\left(\frac{m^{\beta-d_2/2}}{\sqrt{\log m}}\right) \longrightarrow 0, \tag{132}$$

because $d_2 \ge 2$ and $0 < \beta < 1$.

For the nonnull contribution, (129) gives $d_2 \log m/(1+u_m) \to d_2 C/(2\beta)$. The continuity of Φ and (124) therefore imply that

$$2\left\{1-\Phi\left(\sqrt{\frac{d_2 \log m}{1+u_m}}\right)\right\} \longrightarrow q_{\mathrm{R}}.$$

Combining the null and nonnull contributions in (131) gives

$$\mathbb{E}(N_{\mathrm{R}}) = q_{\mathrm{R}} m p_m \{1+o(1)\}. \tag{133}$$

Consider the event

$$G_{\mathrm{R}} = \left\{|N_{\mathrm{R}} - \mathbb{E}(N_{\mathrm{R}})| \le (\log m)^{-1/4}\mathbb{E}(N_{\mathrm{R}})\right\}. \tag{134}$$

Since N_{R} is a sum of independent Bernoulli random variables, the Chernoff bound in Vershynin (2026, Corollary 2.3.4) gives, for all sufficiently large m,

$$\mathbb{P}(G_{\mathrm{R}}^c) \le 2\exp\left\{-\frac{1}{3}(\log m)^{-1/2}\mathbb{E}(N_{\mathrm{R}})\right\}. \tag{135}$$

We next verify $\max\{1,\delta_m\}\mathbb{P}(G_{\mathrm{R}}^c) = o(p_m)$, as required by Theorem C.2. Assumption 3.11 implies

$$\log(1/p_m) = \beta \log m + O(1), \qquad \log\max\{1,\delta_m\} = o(\log m).$$

On the other hand, (133) and Assumption 3.11 show that $(\log m)^{-1/2}\mathbb{E}(N_{\mathrm{R}})$ grows at the rate $m^{1-\beta}/\sqrt{\log m}$. Consequently,

$$\frac{(\log m)^{-1/2}\mathbb{E}(N_{\mathrm{R}})}{\log(1/p_m) + \log\max\{1,\delta_m\}} \longrightarrow \infty.$$

Combining this relation with (135) proves that

$$\max\{1,\delta_m\}\mathbb{P}(G_{\mathrm{R}}^c) = o(p_m). \tag{136}$$

On G_{R}, (133) implies

$$N_{\mathrm{R}} = q_{\mathrm{R}} m p_m \{1+o(1)\}. \tag{137}$$

The remainder is uniform over the values of N_{R} allowed by G_{R}. Since $mp_m \to \infty$, we have $N_{\mathrm{R}} > 1$ for all sufficiently large m on this event. Moreover, the ratio of the second term in the minimum defining $\widehat{\tau}_m^{\mathrm{EB_{BC}}}$ to the first term is

$$\frac{\log m [d_1 m/N_{\mathrm{R}}]^2}{m^2 \log m} = \left(\frac{d_1}{N_{\mathrm{R}}}\right)^2 \longrightarrow 0 \qquad \text{uniformly on } G_{\mathrm{R}}.$$

Thus, the second term is eventually selected, and

$$\widehat{\tau}_m^{\mathrm{EB_{BC}}} = \log m \left[\frac{d_1 m}{N_{\mathrm{R}}}\right]^2 = \log m \left[\frac{d_1}{q_{\mathrm{R}} p_m}\right]^2 \{1+o(1)\} \qquad \text{on } G_{\mathrm{R}}. \tag{138}$$

The expansion in (138) implies, uniformly over the values permitted by G_{R}, that $\widehat{\tau}_m^{\mathrm{EB_{BC}}} \to \infty$. It also gives

$$\begin{aligned}\log\left(\delta_m\sqrt{\widehat{\tau}_m^{\mathrm{EB_{BC}}}}\right) &= \log(\delta_m p_m^{-1}) + \frac{1}{2}\log\log m + \log\left(\frac{d_1}{q_{\mathrm{R}}}\right) + o(1) \\ &= \beta \log m + o(\log m) \longrightarrow \infty \qquad \text{on } G_{\mathrm{R}}. \end{aligned} \tag{139}$$

Hence, $\delta_m\sqrt{\widehat{\tau}_m^{\mathrm{EB_{BC}}}} \to \infty$ uniformly on G_{R}. In addition,

$$\frac{\log(\delta_m\sqrt{\widehat{\tau}_m^{\mathrm{EB_{BC}}}})}{\widehat{\tau}_m^{\mathrm{EB_{BC}}}} = O\left(\frac{\log m}{\log m\, p_m^{-2}}\right) = O(p_m^2) \longrightarrow 0 \qquad \text{uniformly on } G_{\mathrm{R}}. \tag{140}$$

Propositions 3.7 and 3.9 therefore give the existence of the positive thresholds $c_{\widehat{\tau}_m^{\mathrm{EB_{BC}}}}(u)$, $u \in \{-1,1\}$, and their expansion uniformly over the values of $\widehat{\tau}_m^{\mathrm{EB_{BC}}}$ allowed by G_{R}. To justify this uniform application, suppose, to the contrary, that either conclusion failed to hold uniformly on G_{R}.

Then there would exist a subsequence and, at each index along this subsequence, a deterministic value in the range of $\widehat{\tau}_m^{\mathrm{EB_{BC}}}$ on G_{R} for which the corresponding conclusion fails. The uniform bounds established above imply that the selected deterministic subsequence satisfies all conditions of the corresponding proposition. Applying that proposition along this subsequence gives a contradiction. It follows that

$$\sup_{u\in\{-1,1\}} \left| c^2_{\widehat{\tau}_m^{\mathrm{EB_{BC}}}}(u) - \left[\log \widehat{\tau}_m^{\mathrm{EB_{BC}}} + 2\log\delta_m - 2\log\left\{ (2\pi)^{1/2} h(0) \right\} \right] \right| \longrightarrow 0 \tag{141}$$

uniformly on G_{R}.

On this event, for $X_i \neq 0$, the procedure rejects H_{0i} if and only if $X_i^2 > c^2_{\widehat{\tau}_m^{\mathrm{EB_{BC}}}}(u_i)$, where $u_i = X_i/|X_i|$. Since N_{R} takes only finitely many values, these thresholds are measurable on G_{R}.

Taking logarithms in (138) yields

$$\log \widehat{\tau}_m^{\mathrm{EB_{BC}}} = \log\log m + 2\log(1/p_m) + 2\log\left(\frac{d_1}{q_{\mathrm{R}}}\right) + o(1) \qquad \text{on } G_{\mathrm{R}}. \tag{142}$$

Combining (141), (142), and (125) gives

$$\sup_{u\in\{-1,1\}} \left| c^2_{\widehat{\tau}_m^{\mathrm{EB_{BC}}}}(u) - c^2_{\mathrm{R}} \right| \longrightarrow 0 \qquad \text{uniformly on } G_{\mathrm{R}}. \tag{143}$$

Therefore, for every fixed $\varepsilon > 0$ and all sufficiently large m, (126) holds. Together with (136), this proves statement (i).

We next prove statement (ii). The relations $f_m = p_m^{-1}\{1 + o(1)\}$ and (129) imply that

$$\begin{aligned} \log v_m &= \log u_m + 2\log(\delta_m f_m) \\ &= 2\log(\delta_m p_m^{-1}) + \log\log m + \log\left(\frac{2\beta}{C}\right) + o(1). \end{aligned} \tag{144}$$

Combining (125) and (144) gives $c^2_{\mathrm{R}} = \log v_m + z_{\mathrm{R},m}$, where

$$z_{\mathrm{R},m} = 2\log\left(\frac{d_1}{q_{\mathrm{R}}}\right) - \log\left(\frac{2\beta}{C}\right) - 2\log\left\{ (2\pi)^{1/2} h(0) \right\} + o(1). \tag{145}$$

Thus, $z_{\mathrm{R},m} = O(1)$. Since $\log v_m \to \infty$, the threshold is eventually positive, and

$$z_{\mathrm{R},m} = o(\log v_m), \qquad z_{\mathrm{R},m} + 2\log\log v_m \longrightarrow \infty.$$

The fixed threshold rule with threshold c^2_{R} is therefore ABOS by Theorem C.1. This proves statement (ii).

Statements (i) and (ii) verify all conditions of Theorem C.2 with $G_m(\varepsilon) = G_{\mathrm{R}}$, $c^2_{m,\mathrm{fix}} = c^2_{\mathrm{R}}$, and $\widehat{c}^2_m(u) = c^2_{\widehat{\tau}_m^{\mathrm{EB_{BC}}}}(u)$ on G_{R}. Consequently, $R_{\mathrm{EB}}^{\mathrm{AsyRB}}/R_{\mathrm{Opt}}^{\mathrm{AsyBO}} \to 1$. Thus, the empirical Bayes generalized relative belief procedure is ABOS in the sense of Definition 3.3. In view of (3), its asymptotic Bayes risk also satisfies

$$R_{\mathrm{EB}}^{\mathrm{AsyRB}} = m p_m \delta_A \left\{ 2\Phi(\sqrt{C}) - 1 \right\} \{1 + o(1)\}.$$

This completes the proof. □

D PROOFS FOR SECTION 3.3

For the data $X_1, \ldots, X_m$ generated according to (10), we work directly with their sample means $\overline{X}_i$, whose marginal distribution is given in (11). We derive the threshold of the generalized relative belief procedure based on the sample mean, then prove Theorems 3.16 and 3.17.

D.1 The Threshold of the Generalized Relative Belief Procedure Based on the Sample Mean

In this subsection we write $n = n_m$ and $\delta = \delta_m$. The following proposition gives the threshold of the generalized relative belief procedure based on $\overline{X}_i$.

Proposition D.1. *Suppose $X_1, \ldots, X_m$ are generated according to (10), and consider testing H_{0i} : $\mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \ldots, m$. For each i, apply the generalized relative belief rule induced by π_{τ_m} that rejects H_{0i} when $\mathrm{RB}(\mu_i = 0 \mid \overline{X}_i, \tau_m) < 1/\delta_m$. The marginal distribution of $\overline{X}_i$ is given in (11), and its relative belief ratio is given in (13). Suppose π_{τ_m} belongs to the scale family of one-group priors in Definition 3.5, $n\tau_m \to \infty$ and $\delta\sqrt{n\tau_m} \to \infty$. Then, for each $u \in \{-1, 1\}$ and all sufficiently large m, there is a unique $c_{\tau_m,n}(u) > 0$ satisfying*

$$\sqrt{\frac{2\pi}{n\tau_m}} \exp\left\{\frac{n c^2_{\tau_m,n}(u)}{2}\right\} \mathbb{E}_Z\left[h\left(\frac{c_{\tau_m,n}(u)u - Z/\sqrt{n}}{\sqrt{\tau_m}}\right)\right] = \delta. \tag{146}$$

For $\overline{X}_i \neq 0$, with $u_i = \overline{X}_i/|\overline{X}_i|$, the relative belief rule rejects H_{0i} if and only if $\overline{X}_i^2 > c^2_{\tau_m,n}(u_i)$. If also $\log\{\delta\sqrt{n\tau_m}\}/(n\tau_m) \to 0$, then

$$\sup_{u\in\{-1,1\}} \left| n c^2_{\tau_m,n}(u) - \left[\log(n\tau_m) + 2\log\delta - 2\log\{(2\pi)^{1/2}h(0)\}\right]\right| \longrightarrow 0. \tag{147}$$

Proof. The calculation in the proof of Proposition 3.6, applied directly to $\overline{X}_i$, gives (13). For fixed $u \in \{-1, 1\}$ and $r \geq 0$, let $G_u(r) = 1/\mathrm{RB}(\mu_i = 0 \mid \overline{X}_i = ru, \tau_m) = \int \exp(nru\mu - n\mu^2/2)\pi_{\tau_m}(\mu)\, d\mu$. As in the proof of Proposition 3.7, G_u is strictly convex, $G_u(r) \to \infty$ as $r \to \infty$, and $G_u(0) = \sqrt{2\pi/(n\tau_m)}\{h(0) + o(1)\} < \delta$ eventually. Thus G_u crosses δ exactly once for $r > 0$, giving (146) and the stated rejection condition.

Let $A_m = \log(n\tau_m) + 2\log\delta - 2\log\{(2\pi)^{1/2}h(0)\}$. The condition $\log\{\delta\sqrt{n\tau_m}\}/(n\tau_m) \to 0$ implies $A_m/(n\tau_m) \to 0$. For each fixed $\varepsilon > 0$, the local approximation in Lemma 3.8 applies at $r_\pm = \sqrt{(A_m \pm \varepsilon)/n}$, uniformly over $u \in \{-1, 1\}$. As in the proof of Proposition 3.9, the left-hand side of (146) is smaller than δ at r_- and larger than δ at r_+, uniformly over $u \in \{-1, 1\}$. Hence, $r_- < c_{\tau_m,n}(u) < r_+$, which yields (147). □

D.2 Proof of Theorem 3.16

We use the following fixed-threshold characterization in the proofs below.

Theorem D.2 (Frommlet & Bogdan, 2013, Theorem 2.2)**.** *Suppose Assumption 3.14 holds. Consider the fixed threshold rule that, for each $i = 1, \ldots, m$, rejects H_{0i} when $\overline{X}_i < \widetilde{c}_{a,m}/\sqrt{n_m}$ or $\overline{X}_i > \widetilde{c}_{b,m}/\sqrt{n_m}$, where $\widetilde{c}_{a,m} = -\sqrt{\log v_m + z_{a,m}} < 0$, $\widetilde{c}_{b,m} = \sqrt{\log v_m + z_{b,m}} > 0$. This rule is ABOS if and only if $z_{a,m} = o(\log v_m), z_{b,m} = o(\log v_m), z_{a,m} + 2\log\log v_m \to \infty, z_{b,m} + 2\log\log v_m \to \infty$.*

Proof of Theorem 3.16. By (11), $\overline{X}_i \sim N(0, n_m^{-1})$ under H_{0i} and $\overline{X}_i \sim N(0, \widetilde{\sigma}^2 + n_m^{-1})$ under H_{Ai}. We first verify the conditions of Proposition D.1, and then compare its thresholds with those in Theorem D.2.

Let $\ell_2 = \lim_{m\to\infty} \tau_m p_m^2 \in (0, \infty)$. Recall that $v_m = n_m\delta_m^2/p_m^2$. Assumption 3.14 gives

$$\frac{\log v_m}{n_m} \longrightarrow \widetilde{C}. \tag{148}$$

In particular, $v_m \to \infty$. The scale condition gives

$$\tau_m = \ell_2 p_m^{-2}\{1 + o(1)\}, \qquad n_m\tau_m \longrightarrow \infty,$$

and

$$\delta_m^2 n_m \tau_m = \ell_2 v_m\{1 + o(1)\} \longrightarrow \infty. \tag{149}$$

Moreover,

$$\frac{\log\{\delta_m\sqrt{n_m\tau_m}\}}{n_m\tau_m} = \frac{\widetilde{C} + o(1)}{2\ell_2 p_m^{-2}} \longrightarrow 0. \tag{150}$$

Thus Proposition D.1 applies. Since

$$\log(n_m\tau_m) + 2\log\delta_m = \log v_m + \log\ell_2 + o(1),$$

its threshold expansion gives

$$\sup_{u\in\{-1,1\}} \left| n_m c^2_{\tau_m,n_m}(u) - \log v_m - \widetilde{b}_{\ell_2,h} \right| \to 0, \quad \widetilde{b}_{\ell_2,h} = \log\ell_2 - 2\log\{(2\pi)^{1/2}h(0)\}. \tag{151}$$

Fix $\varepsilon > 0$. For all sufficiently large m, the threshold expansion gives

$$\begin{aligned} \{\overline{X}_i^2 > c^2_{\tau_m,n_m}(u_i)\} &\overset{\text{(a)}}{\subseteq} \{n_m\overline{X}_i^2 > \log v_m + \widetilde{b}_{\ell_2,h} - \varepsilon\}, \\ \{n_m\overline{X}_i^2 > \log v_m + \widetilde{b}_{\ell_2,h} + \varepsilon\} &\overset{\text{(b)}}{\subseteq} \{\overline{X}_i^2 > c^2_{\tau_m,n_m}(u_i)\}, \end{aligned} \tag{152}$$

where $u_i = \overline{X}_i/|\overline{X}_i|$ for $\overline{X}_i \neq 0$. The event $\overline{X}_i = 0$ has probability zero under both hypotheses. The two fixed thresholds in (152) have bounded offsets from $\log v_m$ and therefore satisfy Theorem D.2.

The type I and type II error probabilities of the generalized relative belief procedure are

$$\begin{aligned} \widetilde{t}^{\,\mathrm{RB}}_{1i} &= \mathbb{P}_{H_{0i}}\{\mathrm{RB}(\mu_i = 0 \mid \overline{X}_i, \tau_m) < 1/\delta_m\}, \\ \widetilde{t}^{\,\mathrm{RB}}_{2i} &= \mathbb{P}_{H_{Ai}}\{\mathrm{RB}(\mu_i = 0 \mid \overline{X}_i, \tau_m) > 1/\delta_m\}. \end{aligned} \tag{153}$$

Applying (a) in (152) and using $\sqrt{n_m}\overline{X}_i \sim N(0,1)$ under H_{0i}, we obtain

$$\begin{aligned} \widetilde{t}^{\,\mathrm{RB}}_{1i} &\le 2\left[1 - \Phi\left(\sqrt{\log v_m + \widetilde{b}_{\ell_2,h} - \varepsilon}\right)\right] \\ &\le \sqrt{\frac{2}{\pi}} \frac{\exp\left\{-\left(\log v_m + \widetilde{b}_{\ell_2,h} - \varepsilon\right)/2\right\}}{\sqrt{\log v_m + \widetilde{b}_{\ell_2,h} - \varepsilon}} \\ &= O\left(\frac{v_m^{-1/2}}{\sqrt{\log v_m}}\right), \end{aligned}$$

where the second inequality follows from the Gaussian tail bound in Vershynin (2026, Proposition 2.1.2). Since $f_m = (1-p_m)/p_m$ and $v_m = n_m\delta_m^2/p_m^2$,

$$\frac{\delta_m f_m}{\sqrt{v_m}} = \frac{1-p_m}{\sqrt{n_m}}.$$

Consequently,

$$\delta_m f_m \widetilde{t}^{\,\mathrm{RB}}_{1i} = O\left(\frac{1-p_m}{\sqrt{n_m \log v_m}}\right) = o(1). \tag{154}$$

Applying (b) in (152) and using $n_m\overline{X}_i^2/(1+n_m\widetilde{\sigma}^2) \sim \chi^2_1$ under H_{Ai}, we obtain

$$\begin{aligned} \widetilde{t}^{\,\mathrm{RB}}_{2i} &\le \mathbb{P}_{H_{Ai}}\left\{n_m\overline{X}_i^2 < \log v_m + \widetilde{b}_{\ell_2,h} + \varepsilon\right\} \\ &= 2\Phi\left(\sqrt{\frac{\log v_m + \widetilde{b}_{\ell_2,h} + \varepsilon}{1 + n_m\widetilde{\sigma}^2}}\right) - 1 \\ &\longrightarrow 2\Phi\left(\sqrt{\widetilde{C}/\widetilde{\sigma}^2}\right) - 1. \end{aligned} \tag{155}$$

Combining the Bayes risk decomposition in (2) with (154) and (155) gives

$$\limsup_{m\to\infty} \frac{\widetilde{R}^{\mathrm{AsyRB}}}{\widetilde{R}^{\mathrm{AsyBO}}_{\mathrm{Opt}}} \le 1.$$

The Bayes oracle minimizes the Bayes risk for every m, and (12) gives the matching lower bound. Therefore,

$$\frac{\widetilde{R}^{\mathrm{AsyRB}}}{\widetilde{R}^{\mathrm{AsyBO}}_{\mathrm{Opt}}} \longrightarrow 1,$$

which proves Theorem 3.16. □

D.3 The ABOS Property of Random Threshold Rules

Compared with Theorem C.2, the thresholds here apply to $n_m\overline{X}_i^2$, and the fixed threshold conditions are those of Theorem D.2. The same risk comparison gives the following sufficient conditions for procedures with random thresholds to be ABOS.

Theorem D.3. *Suppose $X_1, \dots, X_m$ are generated according to (10) under Assumption 3.14. Consider testing $H_{0i} : \mu_i = 0$ against $H_{Ai} : \mu_i \neq 0$, $i = 1, \dots, m$. For each $i = 1, \dots, m$ with $\overline{X}_i \neq 0$, let $u_i = \overline{X}_i/|\overline{X}_i|$, and suppose the procedure rejects H_{0i} if and only if $n_m\overline{X}_i^2 > \widehat{c}_m^2(u_i)$. Let*

$$\widetilde{c}_{m,\mathrm{fix}}^2 = \log v_m + \widetilde{z}_m > 0 \tag{156}$$

satisfy

$$\widetilde{z}_m = o(\log v_m), \qquad \widetilde{z}_m + 2\log\log v_m \longrightarrow \infty. \tag{157}$$

For every fixed $\varepsilon > 0$, suppose there exists an event $\widetilde{G}_m(\varepsilon)$ such that

$$\sup_{u\in\{-1,1\}} \left|\widehat{c}_m^2(u) - \widetilde{c}_{m,\mathrm{fix}}^2\right| \le \varepsilon \qquad \text{on } \widetilde{G}_m(\varepsilon) \tag{158}$$

for all sufficiently large m. For every fixed $\varepsilon > 0$, if

$$\max\{1, \delta_m\}\mathbb{P}\Big(\widetilde{G}_m(\varepsilon)^c\Big) = o(p_m), \tag{159}$$

then the procedure is ABOS in the sense of Definition 3.15.

Proof of Theorem D.3. We apply the risk comparison in the proof of Theorem C.2 to the sample means.

Fix $\varepsilon > 0$, and take the event $\widetilde{G}_m(\varepsilon)$ from the assumptions of the theorem. On $\widetilde{G}_m(\varepsilon)$, the thresholds satisfy

$$\widetilde{c}_{m,\mathrm{fix}}^2 - \varepsilon \le \widehat{c}_m^2(u) \le \widetilde{c}_{m,\mathrm{fix}}^2 + \varepsilon \tag{160}$$

simultaneously for $u \in \{-1, 1\}$.

Since ε is fixed, the thresholds $\widetilde{c}_{m,\mathrm{fix}}^2 \pm \varepsilon$ satisfy Theorem D.2 and are eventually positive. Both corresponding fixed threshold rules are therefore ABOS.

Let $\widetilde{R}_m^{\mathrm{rand}}$ denote the Bayes risk of the procedure. The threshold sandwich in (160) and the same pointwise loss comparison used in the proof of Theorem C.2 give

$$\begin{aligned}\widetilde{R}_m^{\mathrm{rand}} \le\ & m(1-p_m)\delta_0\mathbb{P}_{H_{0i}}\left(n_m\overline{X}_i^2 > \widetilde{c}_{m,\mathrm{fix}}^2 - \varepsilon\right)\\ &+ mp_m\delta_A\mathbb{P}_{H_{Ai}}\left(n_m\overline{X}_i^2 < \widetilde{c}_{m,\mathrm{fix}}^2 + \varepsilon\right)\\ &+ m\max\{\delta_0, \delta_A\}\mathbb{P}\left(\widetilde{G}_m(\varepsilon)^c\right).\end{aligned} \tag{161}$$

The fixed threshold error bounds in Frommlet & Bogdan (2013, proof of Theorem 2.2) give

$$\begin{aligned}m(1-p_m)\delta_0\mathbb{P}_{H_{0i}}\left(n_m\overline{X}_i^2 > \widetilde{c}_{m,\mathrm{fix}}^2 - \varepsilon\right) &= o(\widetilde{R}_{\mathrm{Opt}}^{\mathrm{AsyBO}}),\\ mp_m\delta_A\mathbb{P}_{H_{Ai}}\left(n_m\overline{X}_i^2 < \widetilde{c}_{m,\mathrm{fix}}^2 + \varepsilon\right) &\le \widetilde{R}_{\mathrm{Opt}}^{\mathrm{AsyBO}}\{1 + o(1)\}.\end{aligned}$$

For the contribution from the complementary event, (12) gives

$$\frac{m\max\{\delta_0, \delta_A\}\mathbb{P}\left(\widetilde{G}_m(\varepsilon)^c\right)}{\widetilde{R}_{\mathrm{Opt}}^{\mathrm{AsyBO}}} = \frac{\max\{1, \delta_m\}\mathbb{P}\left(\widetilde{G}_m(\varepsilon)^c\right)}{p_m\left\{2\Phi\left(\sqrt{\widetilde{C}/\widetilde{\sigma}^2}\right) - 1\right\}}\{1 + o(1)\} \longrightarrow 0. \tag{162}$$

The convergence follows from (159) and the strict positivity of $2\Phi(\sqrt{\widetilde{C}/\widetilde{\sigma}^2}) - 1$. Combining these bounds with (161) yields

$$\limsup_{m\to\infty} \frac{\widetilde{R}_m^{\mathrm{rand}}}{\widetilde{R}_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \le 1.$$

Let $\widetilde{R}_m^{\mathrm{BO}}$ denote the exact finite sample Bayes risk of the Bayes oracle in Definition 3.13. Since the Bayes oracle minimizes the Bayes risk,

$$\widetilde{R}_m^{\mathrm{rand}} \geq \widetilde{R}_m^{\mathrm{BO}}.$$

Moreover, (12) implies

$$\frac{\widetilde{R}_m^{\mathrm{BO}}}{\widetilde{R}_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \longrightarrow 1.$$

The preceding lower and upper bounds yield

$$\frac{\widetilde{R}_m^{\mathrm{rand}}}{\widetilde{R}_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \longrightarrow 1.$$

Thus, the multiple testing procedure is ABOS in the sense of Definition 3.15. □

D.4 Proof of Theorem 3.17

Proof. We apply Theorem D.3. The empirical Bayes rule uses the sample means of the observations in (10). Define

$$q_{\mathrm{FB}} = 2\left\{1 - \Phi\left(\sqrt{\frac{d_4\widetilde{C}}{2\beta\widetilde{\sigma}^2}}\right)\right\} \in (0,1), \tag{163}$$

and consider the fixed threshold

$$c_{\mathrm{FB}}^2 = \log n_m + 2\log(\delta_m p_m^{-1}) + 2\log\left(\frac{d_3}{q_{\mathrm{FB}}}\right) - 2\log\left\{(2\pi)^{1/2}h(0)\right\}. \tag{164}$$

We establish the following two statements.

(i) There exists a measurable event G_{FB}, determined by the observations, satisfying $\max\{1,\delta_m\}\mathbb{P}(G_{\mathrm{FB}}^c) = o(p_m)$, such that, for every fixed $\varepsilon > 0$ and all sufficiently large m, the procedure admits positive thresholds $c_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}},n_m}(u)$, $u \in \{-1,1\}$, on G_{FB} satisfying

$$\sup_{u\in\{-1,1\}} \left| n_m c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}},n_m}(u) - c_{\mathrm{FB}}^2 \right| \leq \varepsilon \qquad \text{on } G_{\mathrm{FB}}. \tag{165}$$

(ii) The fixed threshold rule with threshold c_{FB}^2 is ABOS. Equivalently, $c_{\mathrm{FB}}^2 = \log v_m + z_{\mathrm{FB},m}$, where

$$z_{\mathrm{FB},m} = o(\log v_m), \qquad z_{\mathrm{FB},m} + 2\log\log v_m \longrightarrow \infty.$$

Once these statements are proved, Theorem D.3 applies with $\widetilde{G}_m(\varepsilon) = G_{\mathrm{FB}}$, $\widetilde{c}^2_{m,\mathrm{fix}} = c_{\mathrm{FB}}^2$, and $\widehat{c}^2_m(u) = n_m c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}},n_m}(u)$ on G_{FB}.

We first prove statement (i). Assumption 3.11 implies

$$\log(p_m^{-1}) = \beta\log m + O(1), \qquad \log(mp_m) = (1-\beta)\log m + O(1). \tag{166}$$

Combining the first relation with $-2\log p_m/n_m \to \widetilde{C}$ from Assumption 3.14 gives

$$n_m = \frac{2\beta}{\widetilde{C}}\log m\{1 + o(1)\}. \tag{167}$$

Define

$$N_{\mathrm{FB}} = \sum_{i=1}^{m} \mathbf{1}\left(n_m\overline{X}_i^2 > d_4\log m\right). \tag{168}$$

Under H_{0i}, $\overline{X}_i \sim N(0, n_m^{-1})$; under H_{Ai}, $\overline{X}_i \sim N(0, \widetilde{\sigma}^2 + n_m^{-1})$. Therefore,

$$\mathbb{E}(N_{\mathrm{FB}}) = m(1-p_m)2\left\{1-\Phi\left(\sqrt{d_4 \log m}\right)\right\} + mp_m 2\left\{1-\Phi\left(\sqrt{\frac{d_4 \log m}{1+n_m\widetilde{\sigma}^2}}\right)\right\}. \tag{169}$$

Using Vershynin (2026, Proposition 2.1.2) gives

$$2\left\{1-\Phi\left(\sqrt{d_4 \log m}\right)\right\} = O\left(\frac{m^{-d_4/2}}{\sqrt{\log m}}\right).$$

Consequently,

$$\frac{m(1-p_m)2\left\{1-\Phi\left(\sqrt{d_4 \log m}\right)\right\}}{mp_m} = O\left(\frac{m^{\beta-d_4/2}}{\sqrt{\log m}}\right) \longrightarrow 0. \tag{170}$$

The last convergence follows from $d_4 \geq 2$ and $0 < \beta < 1$.

For the nonnull contribution, (167) gives

$$\frac{d_4 \log m}{1+n_m\widetilde{\sigma}^2} \longrightarrow \frac{d_4\widetilde{C}}{2\beta\widetilde{\sigma}^2}.$$

The continuity of Φ and (163) therefore imply

$$2\left\{1-\Phi\left(\sqrt{\frac{d_4 \log m}{1+n_m\widetilde{\sigma}^2}}\right)\right\} \longrightarrow q_{\mathrm{FB}}.$$

Combining the null and nonnull contributions gives

$$\mathbb{E}(N_{\mathrm{FB}}) = q_{\mathrm{FB}} mp_m\{1+o(1)\}. \tag{171}$$

In particular, $\mathbb{E}(N_{\mathrm{FB}})$ grows at the rate $m^{1-\beta}$ and tends to infinity.

Consider the concentration event

$$G_{\mathrm{FB}} = \left\{|N_{\mathrm{FB}} - \mathbb{E}(N_{\mathrm{FB}})| \leq (\log m)^{-1/4}\mathbb{E}(N_{\mathrm{FB}})\right\}. \tag{172}$$

Since N_{FB} is a sum of independent Bernoulli random variables, the Chernoff bound in Vershynin (2026, Corollary 2.3.4) gives

$$\mathbb{P}(G_{\mathrm{FB}}^c) \leq 2\exp\left\{-\frac{1}{3}(\log m)^{-1/2}\mathbb{E}(N_{\mathrm{FB}})\right\}. \tag{173}$$

Relations (166) and (171) imply

$$\frac{(\log m)^{-1/2}\mathbb{E}(N_{\mathrm{FB}})}{\log(1/p_m)} \longrightarrow \infty.$$

Moreover, Assumption 3.14 gives $\limsup_{m\to\infty} \delta_m < \infty$. Combining these relations with (173) yields

$$\max\{1, \delta_m\}\mathbb{P}(G_{\mathrm{FB}}^c) = o(p_m). \tag{174}$$

On G_{FB}, (171) gives

$$N_{\mathrm{FB}} = q_{\mathrm{FB}} mp_m\{1+o(1)\}. \tag{175}$$

The remainder is uniform over the values of N_{FB} allowed by G_{FB}. Since $mp_m \to \infty$, we have $N_{\mathrm{FB}} > 1$ for all sufficiently large m on this event.

The ratio of the second term in the minimum defining $\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}$ to the first term is

$$\frac{(d_3 m/N_{\mathrm{FB}})^2}{m^2} = \left(\frac{d_3}{N_{\mathrm{FB}}}\right)^2 \longrightarrow 0 \qquad \text{on } G_{\mathrm{FB}}.$$

Hence, the second term is eventually selected, and

$$\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}} = \left(\frac{d_3 m}{N_{\mathrm{FB}}}\right)^2 = \left(\frac{d_3}{q_{\mathrm{FB}} p_m}\right)^2 \{1 + o(1)\} \qquad \text{on } G_{\mathrm{FB}}. \tag{176}$$

From (176), uniformly on G_{FB}, we have $\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}} \to \infty$, $n_m \widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}} \to \infty$, and

$$n_m \widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}} = n_m \left(\frac{d_3}{q_{\mathrm{FB}} p_m}\right)^2 \{1 + o(1)\}. \tag{177}$$

The relation $\log \delta_m = o(\log p_m)$ gives

$$\log(\delta_m p_m^{-1}) = \beta \log m + o(\log m).$$

Combining this relation with (177) gives

$$\begin{aligned} \log\left(\delta_m \sqrt{n_m \widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}}\right) &= \log(\delta_m p_m^{-1}) + \frac{1}{2}\log n_m + \log\left(\frac{d_3}{q_{\mathrm{FB}}}\right) + o(1) \\ &= \beta \log m + o(\log m) \longrightarrow \infty \qquad \text{on } G_{\mathrm{FB}}. \end{aligned} \tag{178}$$

Thus,

$$\delta_m \sqrt{n_m \widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}} \longrightarrow \infty \qquad \text{uniformly on } G_{\mathrm{FB}}.$$

Relations (167) and (177) also give

$$\frac{\log\left(\delta_m \sqrt{n_m \widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}}\right)}{n_m \widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}} = O\left(\frac{\log m}{n_m p_m^{-2}}\right) = O(p_m^2) \longrightarrow 0 \qquad \text{uniformly on } G_{\mathrm{FB}}. \tag{179}$$

The preceding bounds hold uniformly over all values of $\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}$ allowed by G_{FB}. Proposition D.1 therefore gives positive thresholds $c_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}, n_m}(u)$, $u \in \{-1, 1\}$, and their uniform expansion on G_{FB}. It follows that

$$\sup_{u \in \{-1,1\}} \left| n_m c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}, n_m}(u) - \left[\log(n_m \widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}) + 2\log\delta_m - 2\log\left\{(2\pi)^{1/2} h(0)\right\}\right] \right| \longrightarrow 0 \tag{180}$$

uniformly on G_{FB}.

Taking logarithms in (177) gives

$$\log(n_m \widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}) = \log n_m + 2\log(p_m^{-1}) + 2\log\left(\frac{d_3}{q_{\mathrm{FB}}}\right) + o(1) \qquad \text{on } G_{\mathrm{FB}}. \tag{181}$$

Combining (180), (181), and (164) yields

$$\sup_{u \in \{-1,1\}} \left| n_m c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}, n_m}(u) - c^2_{\mathrm{FB}} \right| \longrightarrow 0 \qquad \text{uniformly on } G_{\mathrm{FB}}. \tag{182}$$

On this event, for $\overline{X}_i \neq 0$, the procedure rejects H_{0i} if and only if $n_m \overline{X}_i^2 > n_m c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}, n_m}(u_i)$, where $u_i = \overline{X}_i / |\overline{X}_i| \in \{-1, 1\}$. Since N_{FB} takes only finitely many values, these thresholds are measurable on G_{FB}. Therefore, for every fixed $\varepsilon > 0$ and all sufficiently large m,

$$\sup_{u \in \{-1,1\}} \left| n_m c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}}, n_m}(u) - c^2_{\mathrm{FB}} \right| \leq \varepsilon \qquad \text{on } G_{\mathrm{FB}}.$$

Together with (174), this proves statement (i).

We next prove statement (ii). Since

$$v_m = \frac{n_m \delta_m^2}{p_m^2},$$

we have

$$\log v_m = \log n_m + 2\log(\delta_m p_m^{-1}).$$

It follows from (164) that

$$c_{\mathrm{FB}}^2 = \log v_m + z_{\mathrm{FB},m}, \tag{183}$$

where

$$z_{\mathrm{FB},m} = 2\log\left(\frac{d_3}{q_{\mathrm{FB}}}\right) - 2\log\left\{(2\pi)^{1/2}h(0)\right\}. \tag{184}$$

Thus, $z_{\mathrm{FB},m} = O(1)$. Moreover, Assumption 3.14 gives $\log v_m/n_m \to \widetilde{C}$, and hence $\log v_m \to \infty$. Therefore, c_{FB}^2 is eventually positive, and

$$z_{\mathrm{FB},m} = o(\log v_m), \qquad z_{\mathrm{FB},m} + 2\log\log v_m \longrightarrow \infty.$$

Taking $z_{a,m} = z_{b,m} = z_{\mathrm{FB},m}$ in Theorem D.2 shows that the fixed threshold rule with threshold c_{FB}^2 is ABOS. This proves statement (ii).

Finally, statements (i) and (ii) verify all conditions of Theorem D.3 with

$$\widetilde{G}_m(\varepsilon) = G_{\mathrm{FB}}, \qquad \widetilde{c}_{m,\mathrm{fix}}^2 = c_{\mathrm{FB}}^2,$$

and

$$\widehat{c}_m^2(u) = n_m c^2_{\widehat{\tau}_m^{\mathrm{EB}_{\mathrm{FB}}},n_m}(u) \qquad \text{on } G_{\mathrm{FB}}.$$

Consequently,

$$\frac{\widetilde{R}_{\mathrm{EB}}^{\mathrm{AsyRB}}}{\widetilde{R}_{\mathrm{Opt}}^{\mathrm{AsyBO}}} \longrightarrow 1.$$

Thus, the empirical Bayes generalized relative belief procedure is ABOS in the sense of Definition 3.15. In view of (12), its asymptotic Bayes risk also satisfies

$$\widetilde{R}_{\mathrm{EB}}^{\mathrm{AsyRB}} = mp_m\delta_A\left\{2\Phi\left(\sqrt{\widetilde{C}/\widetilde{\sigma}^2}\right) - 1\right\}\{1 + o(1)\}.$$

This completes the proof. □

E Proofs for Section 4.1

This section establishes the asymptotic Bayes risk of the multivariate Bayes oracle introduced in Section 4.1. We first establish the asymptotic behavior of its threshold and the corresponding type I and type II error probabilities, and then apply the additive Bayes-risk decomposition.

E.1 Bayes Oracle Threshold and Error Probabilities

We now establish the asymptotic behavior of its threshold and the corresponding type I and type II error probabilities.

Lemma E.1. *Suppose Assumption 4.2 holds. For data generated according to (15), we have*

$$\boldsymbol{X}_i^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i \mid H_{0i} \sim \chi_k^2, \qquad \frac{\boldsymbol{X}_i^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i}{1+g_m}\,\Big|\, H_{Ai} \sim \chi_k^2. \tag{185}$$

The Bayes oracle threshold satisfies

$$c_{\mathrm{MBO}}^2 = \mathcal{L}_m + C_0 + o(1), \qquad \frac{c_{\mathrm{MBO}}^2}{1+g_m} \longrightarrow C_0. \tag{186}$$

Moreover, its type I error probability satisfies

$$t_{1i}^{\mathrm{MBO}} = \frac{2^{1-k/2}}{\Gamma(k/2)}\mathcal{L}_m^{k/2-1}\left[(\delta_m f_m)^2 g_m^k\right]^{-1/2} e^{-C_0/2}\{1 + o(1)\}, \tag{187}$$

and hence

$$\delta_m f_m t_{1i}^{\mathrm{MBO}} = o(1). \tag{188}$$

Its type II error probability satisfies

$$t_{2i}^{\mathrm{MBO}} = F_{\chi_k^2}(C_0)\{1 + o(1)\}. \tag{189}$$

Proof. We first establish (185). Under H_{0i}, we may write

$$\boldsymbol{X}_i \overset{\mathrm{d}}{=} \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}_i, \qquad \boldsymbol{Z}_i \sim N_k(\boldsymbol{0}, \boldsymbol{I}_k),$$

whereas under H_{Ai},

$$\boldsymbol{X}_i \overset{\mathrm{d}}{=} (1+g_m)^{1/2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}_i.$$

Consequently,

$$\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i \mid H_{0i} \sim \chi^2_k, \qquad \frac{\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i}{1+g_m} \,\Big|\, H_{Ai} \sim \chi^2_k,$$

which proves (185). We next study the Bayes oracle threshold. Recall that

$$\mathcal{L}_m = \log\left\{(\delta_m f_m)^2 g_m^k\right\}.$$

Assumption 4.2 gives

$$\frac{\mathcal{L}_m}{g_m} \longrightarrow C_0 \in (0,\infty), \qquad g_m \longrightarrow \infty,$$

and consequently $\mathcal{L}_m \to \infty$.

By the Bayes oracle rule in Definition 4.1,

$$\begin{aligned} c^2_{\mathrm{MBO}} &= \frac{1+g_m}{g_m}\left[2\log(\delta_m f_m) + k\log(1+g_m)\right] \\ &= \left(1+\frac{1}{g_m}\right)\left[\mathcal{L}_m + k\log\left(1+\frac{1}{g_m}\right)\right]. \end{aligned} \tag{190}$$

Since $\mathcal{L}_m/g_m \to C_0$ and $\log(1+g_m^{-1}) = o(1)$, it follows that

$$c^2_{\mathrm{MBO}} = \mathcal{L}_m + C_0 + o(1).$$

Dividing by $\mathcal{L}_m \to \infty$ yields

$$\frac{c^2_{\mathrm{MBO}}}{\mathcal{L}_m} \longrightarrow 1. \tag{191}$$

In particular, $c^2_{\mathrm{MBO}} \to \infty$. Moreover, (190) gives

$$\frac{c^2_{\mathrm{MBO}}}{1+g_m} = \frac{1}{g_m}\left[\mathcal{L}_m + k\log\left(1+\frac{1}{g_m}\right)\right] \longrightarrow C_0.$$

This proves (186).

We will use the standard chi-square upper-tail expansion. Indeed,

$$1 - F_{\chi^2_k}(x) = \frac{\Gamma(k/2, x/2)}{\Gamma(k/2)},$$

where $\Gamma(a,x)$ is the upper incomplete gamma function. The incomplete-gamma expansion in Abramowitz & Stegun (1965, Equation 6.5.32) therefore gives, for fixed k,

$$1 - F_{\chi^2_k}(x) = \frac{2^{1-k/2}}{\Gamma(k/2)} x^{k/2-1} e^{-x/2}\{1+o(1)\}, \qquad x \to \infty. \tag{192}$$

We next consider the type I error probability. Under H_{0i}, (185) gives $\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i \sim \chi^2_k$. The Bayes oracle rule therefore implies

$$t^{\mathrm{MBO}}_{1i} = 1 - F_{\chi^2_k}(c^2_{\mathrm{MBO}}).$$

Because $c^2_{\mathrm{MBO}} \to \infty$, applying (192) gives

$$t^{\mathrm{MBO}}_{1i} = \frac{2^{1-k/2}}{\Gamma(k/2)}(c^2_{\mathrm{MBO}})^{k/2-1}\exp\left\{-\frac{c^2_{\mathrm{MBO}}}{2}\right\}\{1+o(1)\}. \tag{193}$$

The threshold expansion and (191) imply

$$(c^2_{\mathrm{MBO}})^{k/2-1} = \mathcal{L}_m^{k/2-1}\{1+o(1)\}$$

and

$$\exp\left\{-\frac{c_{\mathrm{MBO}}^2}{2}\right\} = e^{-\mathcal{L}_m/2}e^{-C_0/2}\{1+o(1)\}.$$

Substituting these relations into (193) yields

$$t_{1i}^{\mathrm{MBO}} = \frac{2^{1-k/2}}{\Gamma(k/2)}\mathcal{L}_m^{k/2-1}e^{-\mathcal{L}_m/2}e^{-C_0/2}\{1+o(1)\}. \tag{194}$$

By the definition of $\mathcal{L}_m$,

$$e^{-\mathcal{L}_m/2} = \left[(\delta_m f_m)^2 g_m^k\right]^{-1/2}.$$

Therefore,

$$t_{1i}^{\mathrm{MBO}} = \frac{2^{1-k/2}}{\Gamma(k/2)}\mathcal{L}_m^{k/2-1}\left[(\delta_m f_m)^2 g_m^k\right]^{-1/2} e^{-C_0/2}\{1+o(1)\},$$

which proves (187).

To determine its contribution to the Bayes risk, multiply (194) by $\delta_m f_m$. Since

$$\delta_m f_m e^{-\mathcal{L}_m/2} = g_m^{-k/2},$$

we obtain

$$\begin{aligned}\delta_m f_m t_{1i}^{\mathrm{MBO}} &= \frac{2^{1-k/2}}{\Gamma(k/2)}\mathcal{L}_m^{k/2-1}g_m^{-k/2}e^{-C_0/2}\{1+o(1)\} \\ &= \frac{2^{1-k/2}}{\Gamma(k/2)}\left(\frac{\mathcal{L}_m}{g_m}\right)^{k/2-1}\frac{e^{-C_0/2}}{g_m}\{1+o(1)\}. \end{aligned}\tag{195}$$

Using $\mathcal{L}_m/g_m \to C_0$ and $g_m \to \infty$, it follows that

$$\delta_m f_m t_{1i}^{\mathrm{MBO}} = \frac{2^{1-k/2}C_0^{k/2-1}e^{-C_0/2}}{\Gamma(k/2)g_m}\{1+o(1)\} = o(1). \tag{196}$$

This proves (188).

Finally, we consider the type II error probability. Under H_{Ai}, (185) gives

$$\frac{\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i}{1+g_m} \sim \chi_k^2.$$

Hence,

$$t_{2i}^{\mathrm{MBO}} = F_{\chi_k^2}\left(\frac{c_{\mathrm{MBO}}^2}{1+g_m}\right). \tag{197}$$

By (186),

$$\frac{c_{\mathrm{MBO}}^2}{1+g_m} \longrightarrow C_0.$$

The continuity of $F_{\chi_k^2}$ therefore gives

$$t_{2i}^{\mathrm{MBO}} = F_{\chi_k^2}(C_0) + o(1).$$

Since $C_0 > 0$, we have $F_{\chi_k^2}(C_0) > 0$, and thus

$$t_{2i}^{\mathrm{MBO}} = F_{\chi_k^2}(C_0)\{1+o(1)\}.$$

This proves (189).

Neither the Bayes oracle threshold nor the distributions in (185) depend on i. Therefore, the conclusions hold for every $i = 1, \ldots, m$, completing the proof. □

E.2 Asymptotic Bayes Risk of the Bayes Oracle

Proof of Theorem 4.3. Because the m testing problems are identically distributed, the Bayes-risk decomposition in (2) gives

$$R_{\mathrm{Opt}}^{\mathrm{AsyMBO}} = mp_m\delta_A\left[\delta_m f_m t_{1i}^{\mathrm{MBO}} + t_{2i}^{\mathrm{MBO}}\right]. \tag{198}$$

Lemma E.1 gives

$$\delta_m f_m t_{1i}^{\mathrm{MBO}} = o(1), \qquad t_{2i}^{\mathrm{MBO}} = F_{\chi_k^2}(C_0)\{1+o(1)\}.$$

Since $F_{\chi_k^2}(C_0) > 0$, these relations imply

$$\delta_m f_m t_{1i}^{\mathrm{MBO}} + t_{2i}^{\mathrm{MBO}} = F_{\chi_k^2}(C_0)\{1+o(1)\}.$$

Substitution into (198) yields

$$R_{\mathrm{Opt}}^{\mathrm{AsyMBO}} = mp_m\delta_A F_{\chi_k^2}(C_0)\{1+o(1)\}, \tag{199}$$

which proves the theorem. □

F Proofs for Section 4.2

This section proves the generalized relative belief results in Section 4.2. The argument proceeds as follows.

- Sections F.1 and F.2 establish conditions under which fixed and random threshold procedures are ABOS. We first characterize the fixed threshold rules that are ABOS, then extend the argument to random thresholds that admit a uniform fixed threshold approximation on an event whose complement contributes negligibly to the Bayes risk.
- We derive the relative belief ratio in Section F.3 and evaluate its rejection condition at $\boldsymbol{X} = r\boldsymbol{\Sigma}^{1/2}\boldsymbol{u}$, where $r \geq 0$ and $\boldsymbol{u} \in \mathbb{R}^k$ satisfies $\|\boldsymbol{u}\| = 1$. Section F.4 then establishes the existence and uniqueness of a positive rejection threshold for each such $\boldsymbol{u}$.
- The uniform local approximation in Section F.5 yields the threshold expansion in Section F.6. This expansion holds uniformly over the unit sphere and depends on the prior density only through its value at the origin.
- Finally, Section F.7 applies the expansion to show that the procedure is ABOS under a suitable fixed prior scale. Section F.8 proves the empirical Bayes result by controlling the estimated scale and applying the result for random threshold procedures.

F.1 The ABOS Property of Fixed Threshold Rules

We first give the multivariate counterpart of Theorem 3.4, which characterizes the fixed threshold rules that are ABOS.

Proposition F.1. *Suppose Assumption 4.2 holds. For data generated according to (15), consider the rule that, for each $i = 1, \ldots, m$, rejects H_{0i} if and only if $\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > \mathcal{C}_m^2$, where $\mathcal{C}_m^2 = \mathcal{L}_m + \mathcal{Z}_m > 0$ is fixed. This rule is ABOS in the sense of Definition 4.4 if and only if*

$$\mathcal{Z}_m = o(\mathcal{L}_m), \qquad \mathcal{Z}_m + 2\log\mathcal{L}_m \longrightarrow \infty. \tag{200}$$

Proof. We write the squared threshold as its leading term plus an offset:

$$\mathcal{C}_m^2 = \mathcal{L}_m + \mathcal{Z}_m.$$

The leading term is defined by

$$\mathcal{L}_m = \log\left\{(\delta_m f_m)^2 g_m^k\right\}.$$

Assumption 4.2 gives

$$\frac{\mathcal{L}_m}{g_m} \longrightarrow C_0 \in (0,\infty), \qquad g_m \longrightarrow \infty. \tag{201}$$

In particular, $\mathcal{L}_m \to \infty$.

We first record the error probabilities and the corresponding Bayes-risk decomposition. Under H_{0i} and H_{Ai}, respectively, (185) gives

$$\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X}_i \sim \chi_k^2, \qquad \frac{\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X}_i}{1+g_m} \sim \chi_k^2.$$

It follows that the type I and type II error probabilities of the fixed threshold rule are

$$t_{1i} = 1 - F_{\chi_k^2}\left(\mathcal{C}_m^2\right), \tag{202}$$

$$t_{2i} = F_{\chi_k^2}\left(\frac{\mathcal{C}_m^2}{1+g_m}\right). \tag{203}$$

Because the m testing problems are identically distributed, the Bayes risk of the fixed threshold rule can be decomposed as

$$R^{\mathrm{AsyM}} = R_{1,m} + R_{2,m}, \tag{204}$$

where the type I and type II contributions are

$$R_{1,m} = m(1-p_m)\delta_0 t_{1i} = m p_m \delta_A \delta_m f_m t_{1i}, \tag{205}$$

$$R_{2,m} = m p_m \delta_A t_{2i}. \tag{206}$$

Theorem 4.3 gives the asymptotic Bayes risk of the Bayes oracle as

$$R_{\mathrm{Opt}}^{\mathrm{AsyMBO}} = m p_m \delta_A F_{\chi_k^2}(C_0)\{1+o(1)\}. \tag{207}$$

Suppose the two conditions in (200) hold:

$$\mathcal{Z}_m = o(\mathcal{L}_m), \qquad \mathcal{Z}_m + 2\log \mathcal{L}_m \longrightarrow \infty.$$

Using $\mathcal{C}_m^2 = \mathcal{L}_m + \mathcal{Z}_m$, we obtain

$$\frac{\mathcal{C}_m^2}{1+g_m} = \frac{\mathcal{L}_m}{g_m}\frac{g_m}{1+g_m}\left(1+\frac{\mathcal{Z}_m}{\mathcal{L}_m}\right) \longrightarrow C_0. \tag{208}$$

By continuity of the chi-square distribution function, the type II error probability satisfies

$$t_{2i} = F_{\chi_k^2}(C_0)\{1+o(1)\}. \tag{209}$$

Combining (206), (207), and (209) gives

$$\frac{R_{2,m}}{R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}} \longrightarrow 1. \tag{210}$$

We next analyze the contribution of the type I error probability. The first condition gives

$$\mathcal{C}_m^2 = \mathcal{L}_m\left(1+\frac{\mathcal{Z}_m}{\mathcal{L}_m}\right) = \mathcal{L}_m\{1+o(1)\} \longrightarrow \infty,$$

so the chi-square upper-tail expansion in (192) gives

$$\begin{aligned} t_{1i} &= \frac{2^{1-k/2}}{\Gamma(k/2)}\left(\mathcal{C}_m^2\right)^{k/2-1}\exp\left\{-\frac{\mathcal{C}_m^2}{2}\right\}\{1+o(1)\} \\ &= \frac{2^{1-k/2}}{\Gamma(k/2)}\left(\mathcal{C}_m^2\right)^{k/2-1} e^{-\mathcal{L}_m/2} e^{-\mathcal{Z}_m/2}\{1+o(1)\}. \end{aligned} \tag{211}$$

The definition of $\mathcal{L}_m$ implies

$$e^{-\mathcal{L}_m/2} = (\delta_m f_m)^{-1} g_m^{-k/2}. \tag{212}$$

Substituting this identity into the tail expansion and multiplying by the weight gives

$$\begin{aligned} \delta_m f_m t_{1i} &= \frac{2^{1-k/2}}{\Gamma(k/2)}\left(\mathcal{C}_m^2\right)^{k/2-1} g_m^{-k/2} e^{-\mathcal{Z}_m/2}\{1+o(1)\} \\ &= \frac{2^{1-k/2}}{\Gamma(k/2)}\left(\frac{\mathcal{C}_m^2}{g_m}\right)^{k/2}\frac{e^{-\mathcal{Z}_m/2}}{\mathcal{C}_m^2}\{1+o(1)\}. \end{aligned} \tag{213}$$

The first condition also gives the threshold limits

$$\frac{\mathcal{C}_m^2}{g_m} \longrightarrow C_0, \qquad \frac{\mathcal{C}_m^2}{\mathcal{L}_m} \longrightarrow 1,$$

so the weighted type I error probability satisfies

$$\delta_m f_m t_{1i} = \frac{2^{1-k/2} C_0^{k/2}}{\Gamma(k/2)} \frac{e^{-\mathcal{Z}_m/2}}{\mathcal{L}_m} \{1 + o(1)\}. \tag{214}$$

The second condition in (200) is equivalent to

$$\frac{e^{-\mathcal{Z}_m/2}}{\mathcal{L}_m} \longrightarrow 0, \tag{215}$$

as can be seen from the logarithmic identity

$$\log\left(\frac{e^{-\mathcal{Z}_m/2}}{\mathcal{L}_m}\right) = -\frac{1}{2}\left(\mathcal{Z}_m + 2\log \mathcal{L}_m\right).$$

It follows that $\delta_m f_m t_{1i} = o(1)$, and hence

$$\frac{R_{1,m}}{R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}} \longrightarrow 0. \tag{216}$$

Combining (204), (210), and (216) yields

$$\frac{R^{\mathrm{AsyM}}}{R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}} \longrightarrow 1.$$

Thus, the fixed threshold rule is ABOS.

Suppose now that the fixed threshold rule is ABOS. We first show that $\mathcal{Z}_m = o(\mathcal{L}_m)$. If this condition fails, then there exist $\varepsilon > 0$ and a subsequence along which

$$|\mathcal{Z}_m| \geq \varepsilon \mathcal{L}_m.$$

Passing to a further subsequence if necessary, either

$$\mathcal{Z}_m \geq \varepsilon \mathcal{L}_m \tag{217}$$

or

$$\mathcal{Z}_m \leq -\varepsilon \mathcal{L}_m \tag{218}$$

holds throughout that subsequence. The following limits are taken along this subsequence.

The positive deviation in (217) gives the lower bound

$$\mathcal{C}_m^2 \geq (1+\varepsilon)\mathcal{L}_m.$$

Dividing by $1 + g_m$ and applying (201) gives

$$\liminf_{m\to\infty} \frac{\mathcal{C}_m^2}{1+g_m} \geq (1+\varepsilon)C_0,$$

and the monotonicity and continuity of $F_{\chi_k^2}$ imply

$$\liminf_{m\to\infty} \frac{R_{2,m}}{R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}} \geq \frac{F_{\chi_k^2}\{(1+\varepsilon)C_0\}}{F_{\chi_k^2}(C_0)} > 1. \tag{219}$$

Since $R_{1,m} \geq 0$, this contradicts the assumption that the rule is ABOS.

Next, consider (218). We may take $0 < \varepsilon < 1$, and then

$$\mathcal{C}_m^2 \leq (1-\varepsilon)\mathcal{L}_m.$$

Because the chi-square upper-tail probability is decreasing,

$$t_{1i} \geq 1 - F_{\chi_k^2}\left((1-\varepsilon)\mathcal{L}_m\right). \tag{220}$$

Applying (192) and using $\delta_m f_m = e^{\mathcal{L}_m/2} g_m^{-k/2}$, we obtain

$$\begin{aligned}\delta_m f_m t_{1i} &\geq \frac{2^{1-k/2}(1-\varepsilon)^{k/2-1}}{\Gamma(k/2)} \mathcal{L}_m^{k/2-1} g_m^{-k/2} e^{\varepsilon \mathcal{L}_m/2} \{1+o(1)\} \\ &= \frac{2^{1-k/2}(1-\varepsilon)^{k/2-1} C_0^{k/2}}{\Gamma(k/2)} \frac{e^{\varepsilon \mathcal{L}_m/2}}{\mathcal{L}_m} \{1+o(1)\} \longrightarrow \infty. \end{aligned} \tag{221}$$

The corresponding type I risk ratio therefore satisfies

$$\frac{R_{1,m}}{R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}} \longrightarrow \infty,$$

which again contradicts the assumption that the rule is ABOS. Thus, the first condition

$$\mathcal{Z}_m = o(\mathcal{L}_m) \tag{222}$$

is necessary.

Under (222), the type II risk ratio converges to one. The derivation of (214) uses only $\mathcal{Z}_m = o(\mathcal{L}_m)$, so the expansion remains valid under (222). Since the entire rule is ABOS, it follows that

$$\frac{R_{1,m}}{R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}} \longrightarrow 0.$$

The expansion in (214) therefore implies

$$\frac{e^{-\mathcal{Z}_m/2}}{\mathcal{L}_m} \longrightarrow 0.$$

By (215), this is equivalent to

$$\mathcal{Z}_m + 2\log \mathcal{L}_m \longrightarrow \infty. \tag{223}$$

Both conditions in (200) are therefore necessary. □

F.2 The ABOS Property of Random Threshold Rules

We next extend Proposition F.1 to random thresholds indexed by $\boldsymbol{u} \in \mathbb{S}^{k-1}$, where $\mathbb{S}^{k-1} = \{\boldsymbol{u} \in \mathbb{R}^k : \|\boldsymbol{u}\| = 1\}$, as in Theorem C.2. Uniform approximation by the threshold of a fixed threshold rule that is ABOS suffices when the complementary event has a negligible contribution to the Bayes risk.

Proposition F.2. *Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are generated independently according to (15) under Assumption 4.2. Consider testing $H_{0i} : \boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$. For each $i = 1, \ldots, m$ with $\boldsymbol{X}_i \neq \mathbf{0}$, let*

$$\boldsymbol{u}_i = \frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i}{\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i\|} \in \mathbb{S}^{k-1},$$

and suppose the procedure rejects H_{0i} if and only if

$$\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X}_i > \widehat{\mathcal{C}}_m^2(\boldsymbol{u}_i),$$

where $\widehat{\mathcal{C}}_m^2(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{S}^{k-1}$, are nonnegative random thresholds. Let

$$\mathcal{C}_{m,\mathrm{fix}}^2 = \mathcal{L}_m + \mathcal{Z}_m > 0$$

satisfy (200). For every fixed $\varepsilon > 0$, suppose there exists an event $\mathcal{E}_{m,\varepsilon}$ such that

$$\sup_{\boldsymbol{u} \in \mathbb{S}^{k-1}} \left| \widehat{\mathcal{C}}_m^2(\boldsymbol{u}) - \mathcal{C}_{m,\mathrm{fix}}^2 \right| \leq \varepsilon \qquad \text{on } \mathcal{E}_{m,\varepsilon}$$

for all sufficiently large m. For every fixed $\varepsilon > 0$, if

$$\max\{1, \delta_m\} \mathbb{P}(\mathcal{E}_{m,\varepsilon}^c) = o(p_m), \tag{224}$$

then the procedure is ABOS in the sense of Definition 4.4.

Proof. Fix $\varepsilon > 0$, and take the event $\mathcal{E}_{m,\varepsilon}$ from the assumptions. The two comparison thresholds are

$$\mathcal{C}^2_{m,\mathrm{fix}} \pm \varepsilon = \mathcal{L}_m + (\mathcal{Z}_m \pm \varepsilon).$$

Since $\mathcal{L}_m \to \infty$ and $\mathcal{Z}_m = o(\mathcal{L}_m)$, both thresholds are positive for all sufficiently large m. Because ε is fixed, both perturbations satisfy

$$\mathcal{Z}_m \pm \varepsilon = o(\mathcal{L}_m), \qquad \mathcal{Z}_m \pm \varepsilon + 2\log \mathcal{L}_m \longrightarrow \infty.$$

Thus, both fixed comparison rules are ABOS by Proposition F.1.

On $\mathcal{E}_{m,\varepsilon}$, the thresholds satisfy

$$\mathcal{C}^2_{m,\mathrm{fix}} - \varepsilon \le \widehat{\mathcal{C}}^2_m(\boldsymbol{u}) \le \mathcal{C}^2_{m,\mathrm{fix}} + \varepsilon, \qquad \boldsymbol{u} \in \mathbb{S}^{k-1}. \tag{225}$$

Consequently, the contribution of type I error is bounded above by that of the lower comparison threshold, while the contribution of type II error is bounded above by that of the upper comparison threshold. On $\mathcal{E}^c_{m,\varepsilon}$, the total loss is at most $m\max\{\delta_0, \delta_A\}$. Taking unconditional expectations of these pointwise loss bounds gives

$$\begin{aligned} R^{\mathrm{rand}}_m \le\ & m(1-p_m)\delta_0 \mathbb{P}_{H_{0i}}\left(\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > \mathcal{C}^2_{m,\mathrm{fix}} - \varepsilon\right) \\ & + mp_m\delta_A \mathbb{P}_{H_{Ai}}\left(\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i < \mathcal{C}^2_{m,\mathrm{fix}} + \varepsilon\right) \\ & + m\max\{\delta_0, \delta_A\}\mathbb{P}(\mathcal{E}^c_{m,\varepsilon}), \end{aligned} \tag{226}$$

where R^{rand}_m denotes the Bayes risk of the procedure. The type I calculation in the proof of Proposition F.1 gives

$$\frac{m(1-p_m)\delta_0 \mathbb{P}_{H_{0i}}\left(\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > \mathcal{C}^2_{m,\mathrm{fix}} - \varepsilon\right)}{R^{\mathrm{AsyMBO}}_{\mathrm{Opt}}} \longrightarrow 0, \tag{227}$$

while the corresponding type II calculation gives

$$\frac{mp_m\delta_A \mathbb{P}_{H_{Ai}}\left(\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i < \mathcal{C}^2_{m,\mathrm{fix}} + \varepsilon\right)}{R^{\mathrm{AsyMBO}}_{\mathrm{Opt}}} \longrightarrow 1. \tag{228}$$

Moreover, Theorem 4.3 gives

$$\frac{m\max\{\delta_0, \delta_A\}\mathbb{P}(\mathcal{E}^c_{m,\varepsilon})}{R^{\mathrm{AsyMBO}}_{\mathrm{Opt}}} = \frac{\max\{1, \delta_m\}\mathbb{P}(\mathcal{E}^c_{m,\varepsilon})}{p_m F_{\chi^2_k}(C_0)}\{1 + o(1)\} \longrightarrow 0, \tag{229}$$

where the final convergence follows from (224). Dividing (226) by $R^{\mathrm{AsyMBO}}_{\mathrm{Opt}}$ yields

$$\limsup_{m\to\infty} \frac{R^{\mathrm{rand}}_m}{R^{\mathrm{AsyMBO}}_{\mathrm{Opt}}} \le 1.$$

The Bayes oracle minimizes the Bayes risk for every m, so the corresponding limit inferior is at least one. These bounds imply

$$\frac{R^{\mathrm{rand}}_m}{R^{\mathrm{AsyMBO}}_{\mathrm{Opt}}} \longrightarrow 1,$$

and the procedure is ABOS. □

F.3 Proof of Proposition 4.6

We next derive the relative belief ratio and express its rejection condition in a form suitable for threshold analysis.

Proof. Fix $i \in \{1, \ldots, m\}$, and write the null density as

$$\varphi_{\boldsymbol{\Sigma}}(\boldsymbol{x}) = (2\pi)^{-k/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left\{-\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{x}\right\}.$$

By Bayes' formula, the relative belief ratio at the null is

$$\mathrm{RB}\left(\boldsymbol{\mu}_i = \mathbf{0} \mid \boldsymbol{X}_i, \tau\right) = \frac{\varphi_{\boldsymbol{\Sigma}}(\boldsymbol{X}_i)}{\displaystyle\int_{\mathbb{R}^k} \varphi_{\boldsymbol{\Sigma}}(\boldsymbol{X}_i - \boldsymbol{\mu}_i^\star)\pi_\tau(\boldsymbol{\mu}_i^\star)\, d\boldsymbol{\mu}_i^\star}. \tag{230}$$

Indeed, the common factor $\pi_\tau(\mathbf{0}) > 0$ cancels from the posterior-to-prior density ratio.

Under Definition 4.5, the prior predictive density in the denominator is

$$\begin{aligned}&\int_{\mathbb{R}^k} \varphi_{\boldsymbol{\Sigma}}(\boldsymbol{X}_i - \boldsymbol{\mu}_i^\star)\pi_\tau(\boldsymbol{\mu}_i^\star)\, d\boldsymbol{\mu}_i^\star \\ &\quad = (2\pi)^{-k/2}\tau^{-k/2}|\boldsymbol{\Sigma}|^{-1}\int_{\mathbb{R}^k} \exp\left\{-\frac{1}{2}(\boldsymbol{X}_i - \boldsymbol{\mu}_i^\star)^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{X}_i - \boldsymbol{\mu}_i^\star)\right\} h\left(\frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}_i^\star}{\sqrt{\tau}}\right) d\boldsymbol{\mu}_i^\star.\end{aligned} \tag{231}$$

Introduce the change of variables

$$\boldsymbol{z}_i^\star = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X}_i - \boldsymbol{\mu}_i^\star).$$

The inverse transformation and its Jacobian are

$$\boldsymbol{\mu}_i^\star = \boldsymbol{X}_i - \boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_i^\star, \qquad d\boldsymbol{\mu}_i^\star = |\boldsymbol{\Sigma}|^{1/2} d\boldsymbol{z}_i^\star,$$

and substitution into the predictive density yields

$$\begin{aligned}&\int_{\mathbb{R}^k} \varphi_{\boldsymbol{\Sigma}}(\boldsymbol{X}_i - \boldsymbol{\mu}_i^\star)\pi_\tau(\boldsymbol{\mu}_i^\star)\, d\boldsymbol{\mu}_i^\star \\ &\quad = |\boldsymbol{\Sigma}|^{-1/2}\tau^{-k/2}\int_{\mathbb{R}^k} (2\pi)^{-k/2} e^{-\|\boldsymbol{z}_i^\star\|^2/2} h\left(\frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i - \boldsymbol{z}_i^\star}{\sqrt{\tau}}\right) d\boldsymbol{z}_i^\star \\ &\quad = |\boldsymbol{\Sigma}|^{-1/2}\tau^{-k/2}\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i - \boldsymbol{Z}_k}{\sqrt{\tau}}\right)\right].\end{aligned} \tag{232}$$

Here, $\boldsymbol{Z}_k \sim N_k(\mathbf{0}, \boldsymbol{I}_k)$. Substituting this expression and the null density

$$\varphi_{\boldsymbol{\Sigma}}(\boldsymbol{X}_i) = (2\pi)^{-k/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left\{-\frac{1}{2}\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i\right\}$$

into (230) proves (17).

The prior predictive density is strictly positive. Therefore, the generalized relative belief rejection condition can be rearranged as

$$\begin{aligned}&\mathrm{RB}\left(\boldsymbol{\mu}_i = \mathbf{0} \mid \boldsymbol{X}_i, \tau\right) < \frac{1}{\delta_m} \\ &\quad\Longleftrightarrow\quad (2\pi)^{k/2}\tau^{-k/2}\exp\left\{\frac{1}{2}\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i\right\}\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i - \boldsymbol{Z}_k}{\sqrt{\tau}}\right)\right] > \delta_m.\end{aligned} \tag{233}$$

This is the form used below. □

F.4 Threshold Representation

The following proposition extends Proposition 3.7 from $u \in \{-1, 1\}$ to $\boldsymbol{u} \in \mathbb{S}^{k-1}$, where $\mathbb{S}^{k-1} = \{\boldsymbol{u} \in \mathbb{R}^k : \|\boldsymbol{u}\| = 1\}$.

Proposition F.3. *Suppose Definition 4.5 holds and*

$$\tau_m \longrightarrow \infty, \qquad \delta_m \tau_m^{k/2} \longrightarrow \infty. \tag{234}$$

Then, for all sufficiently large m and every $\boldsymbol{u} \in \mathbb{S}^{k-1}$, there exists a unique $\mathcal{C}_{\tau_m}(\boldsymbol{u}) > 0$ satisfying

$$(2\pi)^{k/2}\tau_m^{-k/2}\exp\left\{\frac{\mathcal{C}_{\tau_m}^2(\boldsymbol{u})}{2}\right\}\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\mathcal{C}_{\tau_m}(\boldsymbol{u})\boldsymbol{u} - \boldsymbol{Z}_k}{\sqrt{\tau_m}}\right)\right] = \delta_m. \tag{235}$$

For $\boldsymbol{X}_i \neq \mathbf{0}$, define

$$\boldsymbol{u}_i = \frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i}{\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i\|} \in \mathbb{S}^{k-1}.$$

For all sufficiently large m, the generalized relative belief procedure rejects H_{0i} if and only if

$$\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > \mathcal{C}_{\tau_m}^2(\boldsymbol{u}_i). \tag{236}$$

Proof. Fix $\boldsymbol{u} \in \mathbb{S}^{k-1}$. For $r \geq 0$, evaluate the reciprocal relative belief ratio at $\boldsymbol{X}_i = r\boldsymbol{\Sigma}^{1/2}\boldsymbol{u}$ and define

$$\mathcal{G}_{\tau_m,\boldsymbol{u}}(r) = (2\pi)^{k/2}\tau_m^{-k/2}e^{r^2/2}\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{r\boldsymbol{u}-\boldsymbol{Z}_k}{\sqrt{\tau_m}}\right)\right]. \tag{237}$$

Writing the expectation as an integral and making the change of variables $\boldsymbol{\vartheta} = r\boldsymbol{u} - \boldsymbol{z}$ gives

$$\mathcal{G}_{\tau_m,\boldsymbol{u}}(r) = \tau_m^{-k/2}\int_{\mathbb{R}^k}\exp\left\{r\boldsymbol{u}^\top\boldsymbol{\vartheta} - \frac{\|\boldsymbol{\vartheta}\|^2}{2}\right\}h\left(\frac{\boldsymbol{\vartheta}}{\sqrt{\tau_m}}\right)d\boldsymbol{\vartheta}. \tag{238}$$

Since h is bounded, this integral is finite. The same Gaussian domination with the additional factors $\boldsymbol{u}^\top\boldsymbol{\vartheta}$ and $(\boldsymbol{u}^\top\boldsymbol{\vartheta})^2$ justifies differentiation under the integral sign twice on compact intervals.

At the origin, boundedness of the prior density gives

$$\frac{\mathcal{G}_{\tau_m,\boldsymbol{u}}(0)}{\delta_m} \leq \frac{(2\pi)^{k/2}\|h\|_\infty}{\delta_m\tau_m^{k/2}} \longrightarrow 0.$$

The bound is uniform over $\boldsymbol{u} \in \mathbb{S}^{k-1}$, so

$$\mathcal{G}_{\tau_m,\boldsymbol{u}}(0) < \delta_m \tag{239}$$

for every $\boldsymbol{u} \in \mathbb{S}^{k-1}$ and all sufficiently large m.

Since h is continuous at $\mathbf{0}$ and $h(\mathbf{0}) > 0$, there exists $\eta > 0$ such that

$$h(\boldsymbol{x}) \geq \frac{h(\mathbf{0})}{2} \qquad \text{whenever } \|\boldsymbol{x}\| \leq \eta. \tag{240}$$

Let $a_m = \eta\sqrt{\tau_m}/4$, and let $B(\boldsymbol{a}, b)$ denote the Euclidean ball with center $\boldsymbol{a}$ and radius b. Restricting the integral in (238) to $B(a_m\boldsymbol{u}, a_m/2)$ gives

$$\mathcal{G}_{\tau_m,\boldsymbol{u}}(r) \geq \tau_m^{-k/2}\frac{h(\mathbf{0})}{2}\,\mathrm{Vol}\left\{B\left(\mathbf{0}, \frac{a_m}{2}\right)\right\}\exp\left\{\frac{a_m r}{2} - \frac{9a_m^2}{8}\right\}. \tag{241}$$

This lower bound grows without bound as $r \to \infty$, so

$$\mathcal{G}_{\tau_m,\boldsymbol{u}}(r) \longrightarrow \infty \qquad \text{as } r \to \infty. \tag{242}$$

To establish uniqueness, write the integral in (238) as

$$\mathcal{I}_{\tau_m,\boldsymbol{u}}(r) = \int_{\mathbb{R}^k} e^{r\boldsymbol{u}^\top\boldsymbol{\vartheta} - \|\boldsymbol{\vartheta}\|^2/2}h\left(\frac{\boldsymbol{\vartheta}}{\sqrt{\tau_m}}\right)d\boldsymbol{\vartheta}.$$

Then $\mathcal{G}_{\tau_m,\boldsymbol{u}}(r) = \tau_m^{-k/2}\mathcal{I}_{\tau_m,\boldsymbol{u}}(r)$, and differentiation under the integral gives

$$\mathcal{I}'_{\tau_m,\boldsymbol{u}}(r) = \int_{\mathbb{R}^k}(\boldsymbol{u}^\top\boldsymbol{\vartheta})e^{r\boldsymbol{u}^\top\boldsymbol{\vartheta} - \|\boldsymbol{\vartheta}\|^2/2}h\left(\frac{\boldsymbol{\vartheta}}{\sqrt{\tau_m}}\right)d\boldsymbol{\vartheta},$$

$$\mathcal{I}''_{\tau_m,\boldsymbol{u}}(r) = \int_{\mathbb{R}^k}(\boldsymbol{u}^\top\boldsymbol{\vartheta})^2e^{r\boldsymbol{u}^\top\boldsymbol{\vartheta} - \|\boldsymbol{\vartheta}\|^2/2}h\left(\frac{\boldsymbol{\vartheta}}{\sqrt{\tau_m}}\right)d\boldsymbol{\vartheta}.$$

The Cauchy–Schwarz inequality gives

$$\left\{\mathcal{I}'_{\tau_m,\boldsymbol{u}}(r)\right\}^2 < \mathcal{I}''_{\tau_m,\boldsymbol{u}}(r)\mathcal{I}_{\tau_m,\boldsymbol{u}}(r).$$

The inequality is strict because $\boldsymbol{u}^\top\boldsymbol{\vartheta}$ is not almost everywhere constant on the ball $\|\boldsymbol{\vartheta}\| \leq \eta\sqrt{\tau_m}$, on which the integration weight is strictly positive by (240). The second derivative of the logarithm is therefore strictly positive:

$$\frac{d^2}{dr^2}\log\mathcal{G}_{\tau_m,\boldsymbol{u}}(r) = \frac{\mathcal{I}''_{\tau_m,\boldsymbol{u}}(r)\mathcal{I}_{\tau_m,\boldsymbol{u}}(r) - \{\mathcal{I}'_{\tau_m,\boldsymbol{u}}(r)\}^2}{\{\mathcal{I}_{\tau_m,\boldsymbol{u}}(r)\}^2} > 0. \tag{243}$$

Thus, $\log\mathcal{G}_{\tau_m,\boldsymbol{u}}$ is strictly convex.

By (239) and (242), the sublevel set

$$\{r \geq 0 : \mathcal{G}_{\tau_m,\boldsymbol{u}}(r) \leq \delta_m\}$$

is a nonempty closed bounded interval containing a neighborhood of zero in $[0, \infty)$. Strict convexity and continuity show that it is of the form $[0, \mathcal{C}_{\tau_m}(\boldsymbol{u})]$, where $\mathcal{C}_{\tau_m}(\boldsymbol{u})$ is the unique positive solution of (235). The function lies below the rejection cutoff before this crossing and above it afterwards:

$$\begin{aligned} &\mathcal{G}_{\tau_m,\boldsymbol{u}}(r) < \delta_m, \quad 0 \le r < \mathcal{C}_{\tau_m}(\boldsymbol{u}), \\ &\mathcal{G}_{\tau_m,\boldsymbol{u}}(r) > \delta_m, \quad r > \mathcal{C}_{\tau_m}(\boldsymbol{u}). \end{aligned} \tag{244}$$

For $\boldsymbol{X}_i \neq \mathbf{0}$, write

$$r_i = \|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i\|, \qquad \boldsymbol{u}_i = \frac{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i}{\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i\|} \in \mathbb{S}^{k-1}.$$

Then $\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i = r_i\boldsymbol{u}_i$ and $r_i^2 = \boldsymbol{X}_i^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i$. Combining (233) and (244) proves (236).

For later use, the same domination in (238) gives joint continuity of $\mathcal{G}_{\tau_m,\boldsymbol{u}}(r)$ in $(r, \boldsymbol{u})$. Together with the strict inequalities in (244), this implies that $\boldsymbol{u} \mapsto \mathcal{C}_{\tau_m}(\boldsymbol{u})$ is continuous on $\mathbb{S}^{k-1}$, and hence measurable. □

F.5 A Uniform Local Approximation

To expand the thresholds $\mathcal{C}_{\tau_m}(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{S}^{k-1}$, we first approximate the expectation in the crossing equation uniformly near the origin, as in Lemma 3.8.

Lemma F.4. *Suppose Definition 4.5 holds and $\tau_m \to \infty$. The expectation satisfies the uniform approximation*

$$\lim_{\eta\downarrow 0}\limsup_{m\to\infty}\sup_{\|\boldsymbol{y}\|\le\eta\sqrt{\tau_m}}\left|\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\boldsymbol{y}-\boldsymbol{Z}_k}{\sqrt{\tau_m}}\right)\right] - h(\mathbf{0})\right| = 0. \tag{245}$$

Proof. Fix $\varepsilon > 0$. By continuity at $\mathbf{0}$, there exists $\eta_0 > 0$ such that

$$|h(\boldsymbol{x}) - h(\mathbf{0})| < \varepsilon \qquad \text{whenever } \|\boldsymbol{x}\| \le \eta_0. \tag{246}$$

Fix $0 < \eta \le \eta_0/2$, and suppose $\|\boldsymbol{y}\| \le \eta\sqrt{\tau_m}$. On the event

$$\left\{\|\boldsymbol{Z}_k\| \le \frac{\eta_0\sqrt{\tau_m}}{2}\right\},$$

the triangle inequality gives

$$\left\|\frac{\boldsymbol{y}-\boldsymbol{Z}_k}{\sqrt{\tau_m}}\right\| \le \eta + \frac{\eta_0}{2} \le \eta_0.$$

Using continuity on this event and boundedness on its complement gives

$$\left|\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\boldsymbol{y}-\boldsymbol{Z}_k}{\sqrt{\tau_m}}\right)\right] - h(\mathbf{0})\right| \le \varepsilon + 2\|h\|_\infty\mathbb{P}\left(\|\boldsymbol{Z}_k\| > \frac{\eta_0\sqrt{\tau_m}}{2}\right). \tag{247}$$

The bound is independent of $\boldsymbol{y}$, so taking the supremum gives

$$\sup_{\|\boldsymbol{y}\|\le\eta\sqrt{\tau_m}}\left|\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\boldsymbol{y}-\boldsymbol{Z}_k}{\sqrt{\tau_m}}\right)\right] - h(\mathbf{0})\right| \le \varepsilon + 2\|h\|_\infty\mathbb{P}\left(\|\boldsymbol{Z}_k\| > \frac{\eta_0\sqrt{\tau_m}}{2}\right). \tag{248}$$

Because $\tau_m \to \infty$, the probability on the right-hand side converges to zero. Hence, for every fixed $0 < \eta \le \eta_0/2$,

$$\limsup_{m\to\infty}\sup_{\|\boldsymbol{y}\|\le\eta\sqrt{\tau_m}}\left|\mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\boldsymbol{y}-\boldsymbol{Z}_k}{\sqrt{\tau_m}}\right)\right] - h(\mathbf{0})\right| \le \varepsilon.$$

Letting $\eta \downarrow 0$ and then $\varepsilon \downarrow 0$ proves (245). □

F.6 Uniform Expansion of the Rejection Thresholds

Combining the threshold representation with the local approximation gives the following multivariate counterpart of Proposition 3.9.

Proposition F.5. *Suppose Definition 4.5 holds and the scales satisfy*

$$\tau_m \longrightarrow \infty, \qquad \delta_m \tau_m^{k/2} \longrightarrow \infty, \qquad \frac{\log(\delta_m \tau_m^{k/2})}{\tau_m} \longrightarrow 0. \tag{249}$$

The thresholds $\mathcal{C}_{\tau_m}(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{S}^{k-1}$, then admit the uniform expansion

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{k-1}} \left| \mathcal{C}^2_{\tau_m}(\boldsymbol{u}) - \left[k \log \tau_m + 2 \log \delta_m - 2 \log \left\{ (2\pi)^{k/2} h(\mathbf{0}) \right\} \right] \right| \longrightarrow 0. \tag{250}$$

Proof. Fix $\varepsilon > 0$, and denote the leading term of the proposed expansion by

$$\mathcal{B}_m^{\mathrm{RB}} = k \log \tau_m + 2 \log \delta_m - 2 \log \left\{ (2\pi)^{k/2} h(\mathbf{0}) \right\}. \tag{251}$$

Combining the logarithmic terms gives

$$\mathcal{B}_m^{\mathrm{RB}} = 2 \log(\delta_m \tau_m^{k/2}) - 2 \log \left\{ (2\pi)^{k/2} h(\mathbf{0}) \right\}.$$

The assumptions imply $\mathcal{B}_m^{\mathrm{RB}} \to \infty$ and

$$\frac{\mathcal{B}_m^{\mathrm{RB}} \pm \varepsilon}{\tau_m} \longrightarrow 0.$$

Lemma F.4 therefore gives, uniformly over $\boldsymbol{u} \in \mathbb{S}^{k-1}$,

$$\mathbb{E}_{\boldsymbol{Z}_k} \left[h \left(\frac{\sqrt{\mathcal{B}_m^{\mathrm{RB}} \pm \varepsilon}\, \boldsymbol{u} - \boldsymbol{Z}_k}{\sqrt{\tau_m}} \right) \right] = h(\mathbf{0})\{1 + o(1)\}. \tag{252}$$

Using (237) and (251), we obtain uniformly over $\boldsymbol{u} \in \mathbb{S}^{k-1}$ that

$$\mathcal{G}_{\tau_m, \boldsymbol{u}} \left(\sqrt{\mathcal{B}_m^{\mathrm{RB}} - \varepsilon} \right) = \delta_m e^{-\varepsilon/2} \{1 + o(1)\} < \delta_m, \tag{253}$$

$$\mathcal{G}_{\tau_m, \boldsymbol{u}} \left(\sqrt{\mathcal{B}_m^{\mathrm{RB}} + \varepsilon} \right) = \delta_m e^{\varepsilon/2} \{1 + o(1)\} > \delta_m \tag{254}$$

for all sufficiently large m. By the uniqueness and sign characterization in Proposition F.3,

$$\sqrt{\mathcal{B}_m^{\mathrm{RB}} - \varepsilon} < \mathcal{C}_{\tau_m}(\boldsymbol{u}) < \sqrt{\mathcal{B}_m^{\mathrm{RB}} + \varepsilon}.$$

Squaring these bounds and taking the supremum over $\boldsymbol{u} \in \mathbb{S}^{k-1}$ gives

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{k-1}} \left| \mathcal{C}^2_{\tau_m}(\boldsymbol{u}) - \mathcal{B}_m^{\mathrm{RB}} \right| \le \varepsilon \tag{255}$$

for all sufficiently large m. Since $\varepsilon > 0$ is arbitrary, substituting (251) proves (250). □

F.7 Proof of Theorem 4.7

We now compare the uniform expansion with the fixed threshold characterization in Proposition F.1 to establish that the generalized relative belief procedure is ABOS.

Proof. We first verify the conditions required for the uniform threshold expansion. The definition of $\mathcal{L}_m$ and Assumption 4.2 give

$$\mathcal{L}_m = \log \left\{ (\delta_m f_m)^2 g_m^k \right\}, \qquad \frac{\mathcal{L}_m}{g_m} \longrightarrow C_0.$$

Since $\log \delta_m = o(\log p_m)$, $f_m = p_m^{-1}\{1 + o(1)\}$, and $\log g_m / g_m \to 0$, we have

$$\frac{\log(\delta_m f_m)}{\log(1/p_m)} \longrightarrow 1,$$
$$\frac{2\log(\delta_m f_m)}{g_m} = \frac{\mathcal{L}_m}{g_m} - \frac{k \log g_m}{g_m} \longrightarrow C_0.$$

Consequently, $g_m / \log(1/p_m) \to 2/C_0$. Let $\ell_3 = \lim_{m\to\infty} \tau_m / \{p_m^{-2/k} \log(1/p_m)\} \in (0, \infty)$. The assumed prior scale can therefore be written as

$$\tau_m = \ell_3 p_m^{-2/k} \log(1/p_m)\{1 + o(1)\} = \frac{C_0 \ell_3}{2} g_m f_m^{2/k}\{1 + o(1)\}.$$

Since $f_m \to \infty$ and $g_m \to \infty$, it follows that $\tau_m \to \infty$. The same scale relation gives the second divergence condition:

$$\begin{aligned}\delta_m \tau_m^{k/2} &= \left(\frac{C_0\ell_3}{2}\right)^{k/2} \delta_m f_m g_m^{k/2}\{1 + o(1)\} \\ &= \left(\frac{C_0\ell_3}{2}\right)^{k/2} \exp\left\{\frac{\mathcal{L}_m}{2}\right\}\{1 + o(1)\} \longrightarrow \infty \end{aligned} \tag{256}$$

and the logarithmic term is negligible relative to the scale:

$$\begin{aligned}\frac{\log(\delta_m \tau_m^{k/2})}{\tau_m} &= \frac{\mathcal{L}_m/2 + (k/2)\log(C_0\ell_3/2) + o(1)}{(C_0\ell_3/2) g_m f_m^{2/k}\{1 + o(1)\}} \\ &= \frac{C_0 + o(1)}{C_0 \ell_3 f_m^{2/k}} \longrightarrow 0. \end{aligned} \tag{257}$$

Thus, Proposition F.5 applies.

Taking logarithms of the scale relation above gives

$$\begin{aligned}k \log \tau_m &= k \log g_m + 2 \log f_m + k \log(C_0\ell_3/2) + o(1) \\ &= \mathcal{L}_m - 2\log \delta_m + k \log(C_0\ell_3/2) + o(1).\end{aligned}$$

Substituting this identity into the uniform threshold expansion yields

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{k-1}} \left| \mathcal{C}^2_{\tau_m}(\boldsymbol{u}) - \left[\mathcal{L}_m + k\log(C_0\ell_3/2) - 2\log\left\{(2\pi)^{k/2} h(\mathbf{0})\right\}\right] \right| \longrightarrow 0. \tag{258}$$

Denote the constant offset by

$$\mathcal{B}_{\ell_3,h} = k\log(C_0\ell_3/2) - 2\log\left\{(2\pi)^{k/2} h(\mathbf{0})\right\}.$$

For every fixed $\varepsilon > 0$, the preceding uniform convergence gives

$$\mathcal{L}_m + \mathcal{B}_{\ell_3,h} - \varepsilon \le \mathcal{C}^2_{\tau_m}(\boldsymbol{u}) \le \mathcal{L}_m + \mathcal{B}_{\ell_3,h} + \varepsilon \tag{259}$$

uniformly over $\boldsymbol{u} \in \mathbb{S}^{k-1}$. For $\boldsymbol{X}_i \neq \mathbf{0}$, let $\boldsymbol{u}_i = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i / \|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}_i\| \in \mathbb{S}^{k-1}$. The threshold bounds imply the following inclusions between the rejection regions:

$$\begin{aligned}&\left\{\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > \mathcal{L}_m + \mathcal{B}_{\ell_3,h} + \varepsilon\right\} \\ &\quad\subseteq \left\{\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > \mathcal{C}^2_{\tau_m}(\boldsymbol{u}_i)\right\} \\ &\quad\subseteq \left\{\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > \mathcal{L}_m + \mathcal{B}_{\ell_3,h} - \varepsilon\right\}. \end{aligned} \tag{260}$$

Both bounding thresholds have offsets of order $O(1)$, so they satisfy (200).

Define its RB error probabilities by $t_{1i}^{\mathrm{MRB}} = \mathbb{P}_{H_{0i}}\{\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \boldsymbol{X}_i, \tau_m) < 1/\delta_m\}$ and $t_{2i}^{\mathrm{MRB}} = \mathbb{P}_{H_{Ai}}\{\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \boldsymbol{X}_i, \tau_m) > 1/\delta_m\}$. The right-hand inclusion and the type I calculation in Proposition F.1 give

$$\delta_m f_m t_{1i}^{\mathrm{MRB}} = o(1). \tag{261}$$

Similarly, the left-hand inclusion and the corresponding type II calculation give

$$t_{2i}^{\mathrm{MRB}} \le F_{\chi_k^2}(C_0) + o(1). \tag{262}$$

Substituting these error bounds into the Bayes-risk decomposition gives

$$\begin{aligned} R^{\mathrm{AsyMRB}} &= mp_m\delta_A\left[\delta_m f_m t_{1i}^{\mathrm{MRB}} + t_{2i}^{\mathrm{MRB}}\right] \\ &\le mp_m\delta_A F_{\chi_k^2}(C_0)\{1+o(1)\} = R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}\{1+o(1)\}. \end{aligned} \tag{263}$$

The Bayes oracle minimizes the Bayes risk for every m, so its lower bound and the preceding upper bound imply

$$\frac{R^{\mathrm{AsyMRB}}}{R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}} \longrightarrow 1,$$

which proves that the generalized relative belief procedure is ABOS. □

F.8 Proof of Theorem 4.8

Finally, we estimate the prior scale from the observations and apply Proposition F.2.

Proof. For the fixed comparison rule, define the constant

$$q_{\mathrm{R}} = 1 - F_{\chi_k^2}\left(\frac{d_6 C_0}{2\beta}\right) \in (0,1), \tag{264}$$

and consider the fixed threshold

$$\mathcal{C}_{\mathrm{R}}^2 = 2\log(\delta_m p_m^{-1}) + k\log\log m + 2\log\left(\frac{d_5}{q_{\mathrm{R}}}\right) - 2\log\left\{(2\pi)^{k/2}h(\mathbf{0})\right\}. \tag{265}$$

We construct an event G_{R} satisfying $\max\{1,\delta_m\}\mathbb{P}(G_{\mathrm{R}}^c) = o(p_m)$, on which the positive thresholds indexed by $\boldsymbol{u} \in \mathbb{S}^{k-1}$ exist and, for every fixed $\varepsilon > 0$ and all sufficiently large m,

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{k-1}} \left|\mathcal{C}^2_{\widehat{\tau}_m^{\mathrm{MEB}_{\mathrm{BC}}}}(\boldsymbol{u}) - \mathcal{C}_{\mathrm{R}}^2\right| \le \varepsilon \qquad \text{on } G_{\mathrm{R}}. \tag{266}$$

We then verify that $\mathcal{C}_{\mathrm{R}}^2 = \mathcal{L}_m + \mathcal{Z}_m$, where $\mathcal{Z}_m$ satisfies (200).

Assumption 3.11 gives

$$\log(p_m^{-1}) = \beta\log m + O(1), \qquad mp_m \asymp m^{1-\beta}, \qquad 0 < \beta < 1.$$

The sparsity and loss assumptions give the following relations for the prior odds:

$$f_m = p_m^{-1}\{1+o(1)\}, \qquad \log(\delta_m f_m) = \beta\log m + o(\log m).$$

Assumption 4.2 therefore implies

$$\frac{2\beta\log m + k\log g_m + o(\log m)}{g_m} \longrightarrow C_0. \tag{267}$$

Since $\log g_m / g_m \to 0$, it follows that

$$g_m = \frac{2\beta}{C_0}\log m\{1+o(1)\}. \tag{268}$$

Let N_{R} count the observations whose quadratic form exceeds the threshold:

$$N_{\mathrm{R}} = \sum_{i=1}^{m} \mathbf{1}\left(\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i > d_6\log m\right). \tag{269}$$

The null and alternative distributions in (185) give

$$\mathbb{E}(N_{\mathrm{R}}) = m(1-p_m)\left\{1 - F_{\chi_k^2}(d_6\log m)\right\} + mp_m\left[1 - F_{\chi_k^2}\left(\frac{d_6\log m}{1+g_m}\right)\right].$$

The chi-square upper-tail expansion in (192) gives

$$1 - F_{\chi^2_k}(d_6 \log m) = O\left((\log m)^{k/2-1} m^{-d_6/2}\right).$$

Since $d_6 \geq 2$ and $0 < \beta < 1$, the null contribution is $o(mp_m)$. The signal scale in (268) also gives

$$\frac{d_6 \log m}{1 + g_m} \longrightarrow \frac{d_6 C_0}{2\beta}.$$

Combining the null and alternative contributions yields

$$\mathbb{E}(N_{\mathrm{R}}) = q_{\mathrm{R}} m p_m \{1 + o(1)\}. \tag{270}$$

To control the relative error of this count, consider the event

$$G_{\mathrm{R}} = \left\{ |N_{\mathrm{R}} - \mathbb{E}(N_{\mathrm{R}})| \leq (\log m)^{-1/4} \mathbb{E}(N_{\mathrm{R}}) \right\}. \tag{271}$$

Because N_{R} is a sum of independent Bernoulli random variables, the Chernoff bound in Vershynin (2026, Corollary 2.3.4) gives

$$\mathbb{P}(G_{\mathrm{R}}^c) \leq 2 \exp\left\{ -\frac{(\log m)^{-1/2}}{3} \mathbb{E}(N_{\mathrm{R}}) \right\}. \tag{272}$$

The magnitude of the exponent in the Chernoff bound satisfies

$$(\log m)^{-1/2} \mathbb{E}(N_{\mathrm{R}}) \asymp \frac{m^{1-\beta}}{\sqrt{\log m}},$$

whereas the sparsity and loss assumptions give

$$\log(1/p_m) + \log \max\{1, \delta_m\} = O(\log m),$$

so the Chernoff bound gives the required control of the complementary event:

$$\max\{1, \delta_m\} \mathbb{P}(G_{\mathrm{R}}^c) = o(p_m). \tag{273}$$

On G_{R}, the count has vanishing relative error, giving

$$N_{\mathrm{R}} = q_{\mathrm{R}} m p_m \{1 + o(1)\}. \tag{274}$$

All remainder terms on G_{R} below are uniform over that event. In particular, $N_{\mathrm{R}} > 1$ eventually. Moreover, the ratio of the second term in the minimum defining $\widehat{\tau}_m^{\mathrm{MEB_{BC}}}$ to the first term is

$$\left(\frac{d_5}{N_{\mathrm{R}}} \right)^{2/k} \longrightarrow 0.$$

Hence, the second term is eventually selected and

$$\widehat{\tau}_m^{\mathrm{MEB_{BC}}} = \log m \left(\frac{d_5}{q_{\mathrm{R}} p_m} \right)^{2/k} \{1 + o(1)\} \qquad \text{on } G_{\mathrm{R}}. \tag{275}$$

This expression and $\log \delta_m = o(\log m)$ give the two required divergence conditions

$$\widehat{\tau}_m^{\mathrm{MEB_{BC}}} \longrightarrow \infty, \qquad \delta_m \left(\widehat{\tau}_m^{\mathrm{MEB_{BC}}} \right)^{k/2} \longrightarrow \infty$$

uniformly on G_{R}. The logarithmic term is also negligible relative to the estimated scale:

$$\frac{\log\left[\delta_m \left(\widehat{\tau}_m^{\mathrm{MEB_{BC}}} \right)^{k/2} \right]}{\widehat{\tau}_m^{\mathrm{MEB_{BC}}}} = O(p_m^{2/k}) \longrightarrow 0 \qquad \text{on } G_{\mathrm{R}}. \tag{276}$$

Proposition F.3 therefore gives a unique positive threshold for each $\boldsymbol{u} \in \mathbb{S}^{k-1}$ on G_{R} for all sufficiently large m. The expansion in Proposition F.5 also holds uniformly over the scale values on G_{R}: otherwise, a sequence of violating fixed scale values would contradict that proposition. Applying that expansion on G_{R} gives

$$\sup_{\boldsymbol{u} \in \mathbb{S}^{k-1}} \left| \mathcal{C}^2_{\widehat{\tau}_m^{\mathrm{MEB_{BC}}}}(\boldsymbol{u}) - \left[k \log \widehat{\tau}_m^{\mathrm{MEB_{BC}}} + 2 \log \delta_m - 2 \log \left\{ (2\pi)^{k/2} h(\mathbf{0}) \right\} \right] \right| \longrightarrow 0 \tag{277}$$

uniformly on G_{R}. Taking logarithms in (275) gives

$$k \log \widehat{\tau}_m^{\mathrm{MEB}_{\mathrm{BC}}} = k \log \log m + 2 \log(p_m^{-1}) + 2 \log \left(\frac{d_5}{q_{\mathrm{R}}} \right) + o(1) \tag{278}$$

on G_{R}. Substituting this identity into the threshold expansion yields

$$\sup_{\boldsymbol{u} \in \mathbb{S}^{k-1}} \left| \mathcal{C}^2_{\widehat{\tau}_m^{\mathrm{MEB}_{\mathrm{BC}}}}(\boldsymbol{u}) - \mathcal{C}^2_{\mathrm{R}} \right| \longrightarrow 0 \qquad \text{on } G_{\mathrm{R}}. \tag{279}$$

This proves (266). On G_{R}, for $\boldsymbol{X}_i \neq \boldsymbol{0}$, Proposition F.3 also shows that the empirical Bayes generalized relative belief procedure rejects H_{0i} if and only if $\boldsymbol{X}_i^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X}_i > \mathcal{C}^2_{\widehat{\tau}_m^{\mathrm{MEB}_{\mathrm{BC}}}}(\boldsymbol{u}_i)$, where $\boldsymbol{u}_i = \boldsymbol{\Sigma}^{-1/2} \boldsymbol{X}_i / \|\boldsymbol{\Sigma}^{-1/2} \boldsymbol{X}_i\| \in \mathbb{S}^{k-1}$. Since N_{R} takes only finitely many values and, for each fixed scale, the threshold is continuous in $\boldsymbol{u}$ by Proposition F.3, the random thresholds are measurable on G_{R}. Their supremum over the unit sphere is also measurable, since it equals the supremum over a countable dense subset.

It remains to verify that the fixed rule with threshold $\mathcal{C}^2_{\mathrm{R}}$ is ABOS. Substituting the signal scale in (268) into the definition of $\mathcal{L}_m$ gives

$$\begin{aligned} \mathcal{L}_m &= 2 \log(\delta_m f_m) + k \log g_m \\ &= 2 \log(\delta_m p_m^{-1}) + k \log \log m + k \log \left(\frac{2\beta}{C_0} \right) + o(1). \end{aligned} \tag{280}$$

Hence, $\mathcal{C}^2_{\mathrm{R}} = \mathcal{L}_m + \mathcal{Z}_m$, where

$$\mathcal{Z}_m = 2 \log \left(\frac{d_5}{q_{\mathrm{R}}} \right) - k \log \left(\frac{2\beta}{C_0} \right) - 2 \log \left\{ (2\pi)^{k/2} h(\boldsymbol{0}) \right\} + o(1). \tag{281}$$

Thus, $\mathcal{Z}_m = O(1)$, so the offset satisfies both conditions in (200):

$$\mathcal{Z}_m = o(\mathcal{L}_m), \qquad \mathcal{Z}_m + 2 \log \mathcal{L}_m \longrightarrow \infty.$$

Therefore, the fixed rule with threshold $\mathcal{C}^2_{\mathrm{R}}$ is ABOS by Proposition F.1.

Finally, the empirical Bayes procedure is defined by the relative belief ratio for all observations. Apply Proposition F.2 with $\mathcal{E}_{m,\varepsilon} = G_{\mathrm{R}}$, $\mathcal{C}_{m,\mathrm{fix}} = \mathcal{C}_{\mathrm{R}}$, and, on G_{R},

$$\widehat{\mathcal{C}}_m(\boldsymbol{u}) = \mathcal{C}_{\widehat{\tau}_m^{\mathrm{MEB}_{\mathrm{BC}}}}(\boldsymbol{u}), \qquad \boldsymbol{u} \in \mathbb{S}^{k-1}.$$

The thresholds are uniformly close to the fixed threshold by (266). Moreover, (273) shows that $\max\{1, \delta_m\} \mathbb{P}(G_{\mathrm{R}}^c) = o(p_m)$. Proposition F.2 therefore gives

$$\frac{R_{\mathrm{EB}}^{\mathrm{AsyMRB}}}{R_{\mathrm{Opt}}^{\mathrm{AsyMBO}}} \longrightarrow 1.$$

Thus, the empirical Bayes generalized relative belief procedure is ABOS. □

G Proofs for Section 4.3

For $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ generated according to (19), we use their sample means, whose marginal distribution is given in (20), throughout. We first compute the error probabilities of the Bayes oracle in Definition 4.9 and the risk of fixed threshold rules, then prove the results for the generalized relative belief procedures. Define

$$\widetilde{\mathcal{L}}_m = \log\{(\delta_m f_m)^2 (n_m \tilde{g})^k\}. \tag{282}$$

Let $\mathbb{S}^{k-1} = \{\boldsymbol{u} \in \mathbb{R}^k : \|\boldsymbol{u}\| = 1\}$. For $\overline{\boldsymbol{X}}_i \neq \boldsymbol{0}$, put $\boldsymbol{u}_i = \boldsymbol{\Sigma}^{-1/2} \overline{\boldsymbol{X}}_i / \|\boldsymbol{\Sigma}^{-1/2} \overline{\boldsymbol{X}}_i\| \in \mathbb{S}^{k-1}$. All limits below are taken as $m \to \infty$.

G.1 Proof of Theorem 4.11

We first give the counterpart for the sample means of Lemma E.1, which supplies the threshold and error expansions needed for the risk calculation.

Lemma G.1. *Suppose Assumption 4.10 holds. The Bayes oracle threshold in Definition 4.9 satisfies*

$$\widetilde{c}^2_{\mathrm{MBO}} = \widetilde{\mathcal{L}}_m + \frac{\widetilde{C}_0}{\tilde{g}} + o(1), \tag{283}$$

$$\frac{\widetilde{c}^2_{\mathrm{MBO}}}{1+n_m\tilde{g}} \longrightarrow \frac{\widetilde{C}_0}{\tilde{g}}. \tag{284}$$

Its type I and type II error probabilities satisfy

$$\delta_m f_m \widetilde{t}^{\mathrm{MBO}}_{1i} = \frac{2^{1-k/2}}{\Gamma(k/2)}\left(\frac{\widetilde{C}_0}{\tilde{g}}\right)^{k/2-1}\frac{\exp\{-\widetilde{C}_0/(2\tilde{g})\}}{n_m\tilde{g}}\{1+o(1)\} = o(1), \tag{285}$$

$$\widetilde{t}^{\mathrm{MBO}}_{2i} = F_{\chi^2_k}\left(\frac{\widetilde{C}_0}{\tilde{g}}\right)\{1+o(1)\}. \tag{286}$$

Proof. Under the null and alternative distributions in (20), the quadratic form satisfies

$$n_m\overline{\boldsymbol{X}}_i^{\top}\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{X}}_i \mid H_{0i} \sim \chi^2_k, \qquad \frac{n_m\overline{\boldsymbol{X}}_i^{\top}\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{X}}_i}{1+n_m\tilde{g}} \;\Big|\; H_{Ai} \sim \chi^2_k. \tag{287}$$

Using $f_m = (1-p_m)/p_m$ in (282) gives

$$\widetilde{\mathcal{L}}_m = -2\log p_m + 2\log\delta_m + 2\log(1-p_m) + k\log(n_m\tilde{g}). \tag{288}$$

Assumption 4.10 implies $\log\delta_m = o(n_m)$ and $\log(n_m\tilde{g}) = o(n_m)$. Dividing the preceding identity by the sample size therefore gives

$$\frac{\widetilde{\mathcal{L}}_m}{n_m} \longrightarrow \widetilde{C}_0, \qquad \frac{\widetilde{\mathcal{L}}_m}{n_m\tilde{g}} \longrightarrow \frac{\widetilde{C}_0}{\tilde{g}}, \qquad \widetilde{\mathcal{L}}_m \longrightarrow \infty. \tag{289}$$

The Bayes oracle threshold in Definition 4.9 can be rewritten as

$$\begin{aligned}\widetilde{c}^2_{\mathrm{MBO}} &= \left(1+\frac{1}{n_m\tilde{g}}\right)\left\{\widetilde{\mathcal{L}}_m + k\log\left(1+\frac{1}{n_m\tilde{g}}\right)\right\}\\ &= \widetilde{\mathcal{L}}_m + \frac{\widetilde{C}_0}{\tilde{g}} + o(1).\end{aligned}$$

This proves the two threshold limits. The chi-square distribution in (287) give the type II limit by continuity of $F_{\chi^2_k}$. The tail expansion (192) with $x = \widetilde{c}^2_{\mathrm{MBO}}$, together with (289), gives the weighted type I expansion. □

The risk theorem now follows by combining the two error contributions under the additive weighted 0–1 loss.

Proof of Theorem 4.11. Let $\widetilde{R}^{\mathrm{MBO}}_m$ denote the Bayes risk of the oracle. Since the testing problems are identically distributed, the decomposition in (2) gives

$$\widetilde{R}^{\mathrm{MBO}}_m = mp_m\delta_A\left\{\delta_m f_m\widetilde{t}^{\mathrm{MBO}}_{1i} + \widetilde{t}^{\mathrm{MBO}}_{2i}\right\} = mp_m\delta_A F_{\chi^2_k}\left(\frac{\widetilde{C}_0}{\tilde{g}}\right)\{1+o(1)\}, \tag{290}$$

where the final equality follows from Lemma G.1 and $F_{\chi^2_k}(\widetilde{C}_0/\tilde{g}) > 0$. This proves the asymptotic Bayes risk in (21). □

G.2 The ABOS Property of Fixed Threshold Rules

The following proposition gives the fixed-threshold characterization for the sample means.

Proposition G.2. *Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are generated according to (19), and consider testing H_{0i} : $\boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$. Suppose Assumption 4.10 holds. For each i, reject H_{0i} if and only if $n_m \overline{\boldsymbol{X}}_i^\top \boldsymbol{\Sigma}^{-1} \overline{\boldsymbol{X}}_i > \widetilde{\mathcal{C}}_m^2$, where $\widetilde{\mathcal{C}}_m^2 = \widetilde{\mathcal{L}}_m + \widetilde{\mathcal{Z}}_m > 0$ is fixed. This rule is ABOS in the sense of Definition 4.12 if and only if*

$$\widetilde{\mathcal{Z}}_m = o(\widetilde{\mathcal{L}}_m), \qquad \widetilde{\mathcal{Z}}_m + 2\log \widetilde{\mathcal{L}}_m \longrightarrow \infty. \tag{291}$$

Proof. Under the chi-square distribution (287), the type I and type II contributions have the displayed forms below. The proof of Proposition F.1 applies to these two explicit chi-square tails with logarithmic scale $\widetilde{\mathcal{L}}_m$; it proves sufficiency from their limits and necessity from the nonnegative risk contributions. Thus both conditions are necessary and sufficient here.

We record the error probabilities for use below. The distributions in (287) give

$$\widetilde{t}_{1,m} = 1 - F_{\chi_k^2}(\widetilde{\mathcal{C}}_m^2), \qquad \widetilde{t}_{2,m} = F_{\chi_k^2}\left(\frac{\widetilde{\mathcal{C}}_m^2}{1 + n_m \tilde{g}}\right).$$

When the first condition in (291) holds, the squared threshold has the same leading term as $\widetilde{\mathcal{L}}_m$. Continuity of the chi-square distribution function therefore yields

$$\widetilde{t}_{2,m} \longrightarrow F_{\chi_k^2}\left(\frac{\widetilde{C}_0}{\tilde{g}}\right). \tag{292}$$

The weighted type I calculation in the proof of Proposition F.1 gives

$$\delta_m f_m \widetilde{t}_{1,m} = \frac{2^{1-k/2}}{\Gamma(k/2)} \left(\frac{\widetilde{C}_0}{\tilde{g}}\right)^{k/2} \frac{e^{-\widetilde{\mathcal{Z}}_m/2}}{\widetilde{\mathcal{L}}_m} \{1 + o(1)\}. \tag{293}$$

The second condition in (291) makes this contribution tend to zero. These are the bounds needed for the subsequent risk comparisons. □

G.3 The ABOS Property of Random Threshold Rules

We next give the counterpart for the sample means of Proposition F.2.

Proposition G.3. *Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are generated according to (19), and consider testing H_{0i} : $\boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$. Under Assumption 4.10, let the fixed threshold $\widetilde{\mathcal{C}}_{m,\mathrm{fix}}^2 = \widetilde{\mathcal{L}}_m + \widetilde{\mathcal{Z}}_m > 0$ satisfy (291). For each $i = 1, \ldots, m$ with $\overline{\boldsymbol{X}}_i \neq \mathbf{0}$, let*

$$\boldsymbol{u}_i = \frac{\boldsymbol{\Sigma}^{-1/2} \overline{\boldsymbol{X}}_i}{\|\boldsymbol{\Sigma}^{-1/2} \overline{\boldsymbol{X}}_i\|} \in \mathbb{S}^{k-1}.$$

For each fixed $\varepsilon > 0$, suppose there are events $\mathcal{E}_{m,\varepsilon}$ on which the procedure rejects H_{0i} if and only if

$$n_m \overline{\boldsymbol{X}}_i^\top \boldsymbol{\Sigma}^{-1} \overline{\boldsymbol{X}}_i > \widehat{\mathcal{C}}_m^2(\boldsymbol{u}_i),$$

where the thresholds $\widehat{\mathcal{C}}_m^2(\boldsymbol{u})$ satisfy

$$\sup_{\boldsymbol{u} \in \mathbb{S}^{k-1}} \left| \widehat{\mathcal{C}}_m^2(\boldsymbol{u}) - \widetilde{\mathcal{C}}_{m,\mathrm{fix}}^2 \right| \leq \varepsilon \qquad \text{on } \mathcal{E}_{m,\varepsilon}.$$

If the complementary events satisfy

$$\max\{1, \delta_m\} \mathbb{P}(\mathcal{E}_{m,\varepsilon}^c) = o(p_m) \tag{294}$$

for every fixed $\varepsilon > 0$, then the procedure is ABOS in the sense of Definition 4.12. Here, $\mathbb{P}$ denotes probability under the joint distribution of the observations under the joint model in (20).

Proof. Fix $\varepsilon > 0$. Adding or subtracting ε preserves both conditions in (291), so the fixed rules with thresholds $\widetilde{\mathcal{C}}^2_{m,\mathrm{fix}} \pm \varepsilon$ are ABOS. On $\mathcal{E}_{m,\varepsilon}$, these thresholds bound the random thresholds from below and above, respectively. The type I contribution is therefore bounded by the lower fixed threshold, and the type II contribution by the upper fixed threshold.

On the complementary event, the total loss is at most $m \max\{\delta_0, \delta_A\}$. Taking expectations of these pointwise loss bounds, as in the proof of Proposition F.2, gives

$$\begin{aligned}\widetilde{R}^{\mathrm{rand}}_m \le\ & m(1-p_m)\delta_0 \left\{1 - F_{\chi^2_k}\left(\widetilde{\mathcal{C}}^2_{m,\mathrm{fix}} - \varepsilon\right)\right\} \\ & + m p_m \delta_A F_{\chi^2_k}\left(\frac{\widetilde{\mathcal{C}}^2_{m,\mathrm{fix}} + \varepsilon}{1 + n_m \tilde{g}}\right) \\ & + m \max\{\delta_0, \delta_A\} \mathbb{P}(\mathcal{E}^c_{m,\varepsilon}),\end{aligned} \tag{295}$$

where $\widetilde{R}^{\mathrm{rand}}_m$ denotes the Bayes risk of the procedure. In (295), the first term is the type I risk contribution and is $o(\widetilde{R}^{\mathrm{AsyMBO}}_{\mathrm{Opt}})$ by (293). The second term is the type II contribution and equals $\widetilde{R}^{\mathrm{AsyMBO}}_{\mathrm{Opt}}\{1 + o(1)\}$ by (292). The third term is also negligible because

$$\frac{m \max\{\delta_0, \delta_A\} \mathbb{P}(\mathcal{E}^c_{m,\varepsilon})}{\widetilde{R}^{\mathrm{AsyMBO}}_{\mathrm{Opt}}} = \frac{\max\{1, \delta_m\} \mathbb{P}(\mathcal{E}^c_{m,\varepsilon})}{p_m F_{\chi^2_k}(\widetilde{C}_0/\tilde{g})}\{1 + o(1)\} \longrightarrow 0.$$

Thus, the limit superior of the risk ratio is at most one. The Bayes oracle supplies the matching lower bound, proving that the procedure is ABOS. □

G.4 The Threshold of the Generalized Relative Belief Procedure Based on the Sample Mean

Proof of Proposition 4.13. Under (19), $\overline{\boldsymbol{X}}_i \mid \boldsymbol{\mu}_i \sim N_k(\boldsymbol{\mu}_i, n_m^{-1}\boldsymbol{\Sigma})$. Applying the density ratio calculation in the proof of Proposition 4.6 directly to this sample mean gives the expression in Proposition 4.13. □

The following proposition supplies thresholds for the quadratic form of the sample mean, uniformly over $\boldsymbol{u} \in \mathbb{S}^{k-1}$ and prior scales. Here and below $n = n_m$.

Proposition G.4. *Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are generated according to (19), and consider testing $H_{0i} : \boldsymbol{\mu}_i = \mathbf{0}$ against $H_{Ai} : \boldsymbol{\mu}_i \neq \mathbf{0}$, $i = 1, \ldots, m$. For each i, apply the generalized relative belief rule induced by π_{τ_m} that rejects H_{0i} when $\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i, \tau_m) < 1/\delta_m$. The marginal distribution of $\overline{\boldsymbol{X}}_i$ is given in (20), and its relative belief ratio is given in Proposition 4.13. Suppose π_{τ_m} belongs to the scale family of one-group priors in Definition 4.5 and $n\tau_m \to \infty$, $\delta_m (n\tau_m)^{k/2} \to \infty$. For each $\boldsymbol{u} \in \mathbb{S}^{k-1}$ and all sufficiently large m, there is a unique positive $\widetilde{\mathcal{C}}_{\tau_m,n}(\boldsymbol{u})$ satisfying*

$$\left(\frac{2\pi}{n\tau_m}\right)^{k/2} e^{n\widetilde{\mathcal{C}}^2_{\tau_m,n}(\boldsymbol{u})/2} \mathbb{E}_{\boldsymbol{Z}_k}\left[h\left(\frac{\widetilde{\mathcal{C}}_{\tau_m,n}(\boldsymbol{u})\boldsymbol{u} - \boldsymbol{Z}_k/\sqrt{n}}{\sqrt{\tau_m}}\right)\right] = \delta_m. \tag{296}$$

For $\overline{\boldsymbol{X}}_i \neq \mathbf{0}$, the generalized relative belief rule rejects H_{0i} if and only if $\overline{\boldsymbol{X}}_i^\top \boldsymbol{\Sigma}^{-1} \overline{\boldsymbol{X}}_i > \widetilde{\mathcal{C}}^2_{\tau_m,n}(\boldsymbol{u}_i)$. If additionally $\log\{\delta_m (n\tau_m)^{k/2}\}/(n\tau_m) \to 0$, then

$$\sup_{\boldsymbol{u} \in \mathbb{S}^{k-1}} \left| n\widetilde{\mathcal{C}}^2_{\tau_m,n}(\boldsymbol{u}) - k\log(n\tau_m) - 2\log\delta_m + 2\log\{(2\pi)^{k/2} h(\mathbf{0})\} \right| \to 0. \tag{297}$$

The existence and expansion conclusions also hold uniformly whenever the three preceding conditions hold uniformly over the prior scale values under consideration.

Proof. We prove the positive crossing and then its expansion. At $\overline{\boldsymbol{X}}_i = r\boldsymbol{\Sigma}^{1/2}\boldsymbol{u}$, let $G_{\boldsymbol{u}}(r) = 1/\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i = r\boldsymbol{\Sigma}^{1/2}\boldsymbol{u}, \tau_m) = \int \exp\{nr\boldsymbol{u}^\top \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu} - n\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/2\} \pi_{\tau_m}(\boldsymbol{\mu})\, d\boldsymbol{\mu}$. As in the proof of Proposition F.3, this function is strictly convex, diverges as $r \to \infty$ uniformly over $\boldsymbol{u} \in \mathbb{S}^{k-1}$, and its value at zero equals $\{2\pi/(n\tau_m)\}^{k/2} \mathbb{E}h(-\boldsymbol{Z}_k/\sqrt{n\tau_m}) = \{2\pi/(n\tau_m)\}^{k/2}\{h(\mathbf{0}) +$

$o(1)\} < \delta_m$. Thus each ray has exactly one positive crossing, giving the strict rejection rule and (296).

Let $\mathcal{A}_m = k\log(n\tau_m) + 2\log\delta_m - 2\log\{(2\pi)^{k/2}h(\mathbf{0})\}$. For fixed $\varepsilon > 0$, the radii $r_\pm = \sqrt{(\mathcal{A}_m \pm \varepsilon)/n}$ are eventually positive and satisfy $r_\pm/\sqrt{\tau_m} \to 0$. Boundedness and continuity of h at zero imply that the expectation in (296) equals $h(\mathbf{0})+o(1)$ at these radii uniformly over $\boldsymbol{u} \in \mathbb{S}^{k-1}$. The crossing equation's left-hand side is below δ_m at r_- and above it at r_+, simultaneously for all $\boldsymbol{u} \in \mathbb{S}^{k-1}$. The crossings lie between these radii; letting $\varepsilon \downarrow 0$ gives (297). □

G.5 Proof of Theorem 4.14

We now apply the threshold expansion and the fixed threshold criterion, following the proof of Theorem 4.7.

Proof. By (20), $n_m\overline{\boldsymbol{X}}_i^\top \boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{X}}_i \sim \chi^2_k$ under H_{0i}, while the same quadratic form divided by $1+n_m\tilde{g}$ follows χ^2_k under H_{Ai}. Let $\ell_4 = \lim_{m\to\infty}\tau_m/p_m^{-2/k} \in (0,\infty)$. The assumed prior scale gives

$$\tau_m = \ell_4 p_m^{-2/k}\{1+o(1)\}, \qquad n_m\tau_m = \ell_4 n_m p_m^{-2/k}\{1+o(1)\} \longrightarrow \infty.$$

Using the definition of $\widetilde{\mathcal{L}}_m$ and $f_m = (1-p_m)/p_m$, the other conditions for Proposition G.4 follow from

$$\delta_m(n_m\tau_m)^{k/2} = \left(\frac{\ell_4}{\tilde{g}}\right)^{k/2} e^{\widetilde{\mathcal{L}}_m/2}\{1+o(1)\} \longrightarrow \infty$$

and

$$\frac{\log\{\delta_m(n_m\tau_m)^{k/2}\}}{n_m\tau_m} = \frac{\widetilde{\mathcal{L}}_m/2 + O(1)}{\ell_4 n_m p_m^{-2/k}\{1+o(1)\}} = \frac{\widetilde{C}_0 + o(1)}{2\ell_4} p_m^{2/k} \longrightarrow 0.$$

Moreover,

$$k\log(n_m\tau_m) + 2\log\delta_m = \widetilde{\mathcal{L}}_m + k\log\left(\frac{\ell_4}{\tilde{g}}\right) - 2\log(1-p_m) + o(1).$$

Since $p_m \to 0$, the threshold expansion therefore has the form

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{k-1}}\left|n_m\widetilde{\mathcal{C}}^2_{\tau_m,n_m}(\boldsymbol{u}) - \left[\widetilde{\mathcal{L}}_m + k\log\left(\frac{\ell_4}{\tilde{g}}\right) - 2\log\{(2\pi)^{k/2}h(\mathbf{0})\}\right]\right| \longrightarrow 0. \tag{298}$$

For the risk comparison, use the fixed squared threshold

$$\widetilde{\mathcal{C}}^2_{m,\mathrm{MRB}} = \widetilde{\mathcal{L}}_m + k\log\left(\frac{\ell_4}{\tilde{g}}\right) - 2\log\{(2\pi)^{k/2}h(\mathbf{0})\}.$$

For every fixed $\varepsilon > 0$, the squared thresholds indexed by $\boldsymbol{u} \in \mathbb{S}^{k-1}$ eventually all lie between $\widetilde{\mathcal{C}}^2_{m,\mathrm{MRB}} - \varepsilon$ and $\widetilde{\mathcal{C}}^2_{m,\mathrm{MRB}} + \varepsilon$. Both fixed thresholds satisfy (291) because their offsets are bounded. For the generalized relative belief rule, the type I and type II error probabilities are

$$\begin{aligned}\widetilde{t}^{\mathrm{MRB}}_{1i} &= \mathbb{P}_{H_{0i}}\left\{\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i, \tau_m) < 1/\delta_m\right\},\\ \widetilde{t}^{\mathrm{MRB}}_{2i} &= \mathbb{P}_{H_{Ai}}\left\{\mathrm{RB}(\boldsymbol{\mu}_i = \mathbf{0} \mid \overline{\boldsymbol{X}}_i, \tau_m) > 1/\delta_m\right\}.\end{aligned} \tag{299}$$

The corresponding bounds are

$$\widetilde{t}^{\mathrm{MRB}}_{1i} \le 1 - F_{\chi^2_k}\left(\widetilde{\mathcal{C}}^2_{m,\mathrm{MRB}} - \varepsilon\right), \qquad \widetilde{t}^{\mathrm{MRB}}_{2i} \le F_{\chi^2_k}\left(\frac{\widetilde{\mathcal{C}}^2_{m,\mathrm{MRB}} + \varepsilon}{1+n_m\tilde{g}}\right).$$

By (293), the lower fixed threshold gives $\delta_m f_m \widetilde{t}^{\mathrm{MRB}}_{1i} = o(1)$. By (292), the upper fixed threshold gives $\widetilde{t}^{\mathrm{MRB}}_{2i} \le F_{\chi^2_k}(\widetilde{C}_0/\tilde{g}) + o(1)$. Substituting these bounds into the Bayes-risk decomposition yields

$$\widetilde{R}^{\mathrm{AsyMRB}} \le m p_m \delta_A F_{\chi^2_k}\left(\frac{\widetilde{C}_0}{\tilde{g}}\right)\{1+o(1)\} = \widetilde{R}^{\mathrm{AsyMBO}}_{\mathrm{Opt}}\{1+o(1)\}.$$

Together with the Bayes oracle lower bound, this proves that the generalized relative belief procedure is ABOS in the sense of Definition 4.12. □

G.6 Proof of Theorem 4.15

Proof. We follow the argument in Section F.8 and apply Proposition G.4 to the sample means. Assumptions 3.11 and 4.10 imply

$$\log(p_m^{-1}) = \beta \log m + O(1), \qquad n_m = \frac{2\beta}{\widetilde{C}_0} \log m\{1 + o(1)\}, \qquad \log \delta_m = o(\log m). \tag{300}$$

We count the observations above the threshold and denote the limiting probability under the alternative by $\widetilde{q}_{\mathrm{MFB}}$:

$$\widetilde{N}_m^{\mathrm{MFB}} = \sum_{i=1}^{m} \mathbf{1}\{n_m \overline{\boldsymbol{X}}_i^\top \boldsymbol{\Sigma}^{-1} \overline{\boldsymbol{X}}_i > d_8 \log m\}, \qquad \widetilde{q}_{\mathrm{MFB}} = 1 - F_{\chi_k^2}\left(\frac{d_8 \widetilde{C}_0}{2\beta \tilde{g}}\right) \in (0,1). \tag{301}$$

The chi-square distribution in (287) give

$$\mathbb{E}\widetilde{N}_m^{\mathrm{MFB}} = m(1-p_m)\left\{1 - F_{\chi_k^2}(d_8 \log m)\right\} + mp_m\left\{1 - F_{\chi_k^2}\left(\frac{d_8 \log m}{1 + n_m \tilde{g}}\right)\right\}.$$

The chi-square tail expansion in (192), together with $d_8 \geq 2$ and $0 < \beta < 1$, gives

$$\frac{1 - F_{\chi_k^2}(d_8 \log m)}{p_m} = O\Big((\log m)^{k/2-1} m^{\beta - d_8/2}\Big) = o(1).$$

The nonnull probability converges to $\widetilde{q}_{\mathrm{MFB}}$, so

$$\mathbb{E}\widetilde{N}_m^{\mathrm{MFB}} = \widetilde{q}_{\mathrm{MFB}} m p_m \{1 + o(1)\}. \tag{302}$$

By Assumption 3.11, this expectation grows at the rate $m^{1-\beta}$. For $a_m = (\log m)^{-1/4}$, consider the event

$$\widetilde{G}_m^{\mathrm{MFB}} = \left\{\left|\widetilde{N}_m^{\mathrm{MFB}} - \mathbb{E}\widetilde{N}_m^{\mathrm{MFB}}\right| \leq a_m \mathbb{E}\widetilde{N}_m^{\mathrm{MFB}}\right\}. \tag{303}$$

The Chernoff bound in Vershynin (2026, Corollary 2.3.4) yields

$$\mathbb{P}\{(\widetilde{G}_m^{\mathrm{MFB}})^c\} \leq 2\exp\left\{-\frac{\mathbb{E}\widetilde{N}_m^{\mathrm{MFB}}}{3\sqrt{\log m}}\right\}. \tag{304}$$

As δ_m is eventually bounded above and $m^{1-\beta}/(\log m)^{3/2} \to \infty$, this gives

$$\max\{1, \delta_m\}\mathbb{P}\{(\widetilde{G}_m^{\mathrm{MFB}})^c\} = o(p_m). \tag{305}$$

All remainder terms on $\widetilde{G}_m^{\mathrm{MFB}}$ below are uniform over that event. On $\widetilde{G}_m^{\mathrm{MFB}}$, the count $\widetilde{N}_m^{\mathrm{MFB}} = \widetilde{q}_{\mathrm{MFB}} m p_m \{1 + o(1)\}$ eventually exceeds $\max\{1, d_7\}$. Both truncations in (22) are therefore inactive, and

$$\widehat{\tau}_m^{\mathrm{MEB_{FB}}} = \left(\frac{d_7 m}{\widetilde{N}_m^{\mathrm{MFB}}}\right)^{2/k} = \left(\frac{d_7}{\widetilde{q}_{\mathrm{MFB}} p_m}\right)^{2/k} \{1 + o(1)\}. \tag{306}$$

Together with (300), this implies $n_m \widehat{\tau}_m^{\mathrm{MEB_{FB}}} \to \infty$ and

$$\log\left\{\delta_m \left(n_m \widehat{\tau}_m^{\mathrm{MEB_{FB}}}\right)^{k/2}\right\} = \beta \log m + o(\log m), \tag{307}$$

$$\frac{\log\left\{\delta_m \left(n_m \widehat{\tau}_m^{\mathrm{MEB_{FB}}}\right)^{k/2}\right\}}{n_m \widehat{\tau}_m^{\mathrm{MEB_{FB}}}} = O(p_m^{2/k}) \longrightarrow 0. \tag{308}$$

The preceding bounds hold uniformly over all values of $\widehat{\tau}_m^{\mathrm{MEB_{FB}}}$ allowed by $\widetilde{G}_m^{\mathrm{MFB}}$. Proposition G.4 therefore supplies the positive thresholds indexed by $\boldsymbol{u} \in \mathbb{S}^{k-1}$ and their uniform expansion on $\widetilde{G}_m^{\mathrm{MFB}}$.

We compare the resulting thresholds with the fixed squared threshold

$$\widetilde{\mathcal{C}}_{m,\mathrm{MFB}}^2 = k \log n_m + 2\log(\delta_m p_m^{-1}) + 2\log\left(\frac{d_7}{\widetilde{q}_{\mathrm{MFB}}}\right) - 2\log\{(2\pi)^{k/2} h(\mathbf{0})\}. \tag{309}$$

It is positive eventually, and (306) gives

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{k-1}}\left|n_m\widetilde{\mathcal{C}}^2_{\widehat{\tau}_m^{\mathrm{MEB}_{\mathrm{FB}}},n_m}(\boldsymbol{u})-\widetilde{\mathcal{C}}^2_{m,\mathrm{MFB}}\right|=o(1)\qquad\text{on }\widetilde{G}_m^{\mathrm{MFB}}. \tag{310}$$

The offset of the fixed threshold from $\widetilde{\mathcal{L}}_m$ is

$$\begin{aligned}\widetilde{\mathcal{Z}}_m^{\mathrm{MFB}}&=\widetilde{\mathcal{C}}^2_{m,\mathrm{MFB}}-\widetilde{\mathcal{L}}_m\\&=2\log\left(\frac{d_7}{\widetilde{q}_{\mathrm{MFB}}}\right)-k\log\tilde{g}-2\log\{(2\pi)^{k/2}h(\mathbf{0})\}-2\log(1-p_m).\end{aligned} \tag{311}$$

This converges to a finite constant and hence satisfies (291). The fixed rule is therefore ABOS by Proposition G.2.

For each fixed scale, the threshold is continuous in $\boldsymbol{u}\in\mathbb{S}^{k-1}$ by the proof of Proposition F.3. Since the count takes only finitely many values, the resulting thresholds are jointly measurable in the observations and $\boldsymbol{u}$ on $\widetilde{G}_m^{\mathrm{MFB}}$.

For every fixed $\varepsilon>0$, take $\mathcal{E}_{m,\varepsilon}=\widetilde{G}_m^{\mathrm{MFB}}$ in Proposition G.3. On this event, the empirical Bayes generalized relative belief procedure has the thresholds indexed by $\boldsymbol{u}\in\mathbb{S}^{k-1}$ in (310), whose squared values are uniformly within ε of the fixed squared threshold for all sufficiently large m. By (305), the complementary event satisfies $\max\{1,\delta_m\}\mathbb{P}\{(\widetilde{G}_m^{\mathrm{MFB}})^c\}=o(p_m)$. Thus Proposition G.3 applies directly and gives

$$\frac{\widetilde{R}_{\mathrm{EB}}^{\mathrm{AsyMRB}}}{\widetilde{R}_{\mathrm{Opt}}^{\mathrm{AsyMBO}}}\longrightarrow 1.$$

Thus, the empirical Bayes generalized relative belief procedure is ABOS. □