How Likely Are Voting Rules Equitable?

Lirong Xia
Rutgers University
lirong.xia@rutgers.edu

Abstract

In social choice, *anonymity* (treating all agents equally) and *neutrality* (treating all alternatives equally) are widely regarded as "*minimal demands*" and "*uncontroversial*" axioms of equity and fairness. However, due to the ANR impossibility theorem, no voting rule can simultaneously satisfy anonymity, neutrality, and resoluteness (always choosing a unique winner). While the worst-case ANR impossibility is well understood, its likelihood in probabilistic models remains underexplored.

We address this question through comprehensive likelihood analysis under semi-random models, providing asymptotically tight bounds to quantify the advantage of the optimal tie-breaking mechanisms over other commonly-used tie-breaking mechanisms. Our characterizations reveal an $n^{\Theta(m!)}$ improvement in many cases, where n is the number of agents and m is the number of alternatives. We also prove a quantitative ANR impossibility theorem characterizing the tradeoff between equity (anonymity and neutrality) and an efficiency criterion called δ-unanimity. Our work advances the understanding of trade-offs between equity and efficiency in voting.

1 Introduction

Designing voting rules to aggregate agents' preferences is a central challenge in social choice theory, with wide applications across diverse domains including political elections, meta-search [13], information retrieval [21], recommender systems [2], crowdsourcing [23], and more recently human-AI alignment [9]. The classical paradigm of *axiomatic approach* in social choice theory evaluates and designs voting rules w.r.t. certain desiderata, known as *axioms* [29]. In particular, anonymity (treating all agents equally) and neutrality (treating all alternatives equally) are consistently regarded as "minimal demands" and "uncontroversial" axioms of equity and fairness [41, 27, 3].

Yet, the ANR impossibility theorem shows that no voting rule can simultaneously satisfy **A**nonymity, **N**eutrality, and **R**esoluteness (ANR) for all numbers of agents n and alternatives m, where resolvability means that the rule always selects a single winner. The ANR impossibility stands as "*among the most well-known results in social choice theory*" [28] and raises a natural and important question:

If perfect ANR is impossible, how likely is a voting rule to fail it?

In this paper, we address this question by accurately bounding the ANR failure rate when agents' preferences are generated randomly—a natural and informative measure of axiomatic satisfaction pioneered by Gehrlein and Fishburn [15] whose studies have now evolved into a "*new sub-domain of the theory of social choice*" [11]. In the literature, most voting rules are defined by starting with an irresolute rule $\overline{r}$ that satisfies anonymity and neutrality, then applying a tie-breaking mechanism to select a unique winner when multiple co-winners exist. In fact, technically, any resolute rule can be viewed as a tie-breaking mechanism on the trivial irresolute rule that always chooses all alternatives. Therefore, it is without loss of generality to focus on the ANR failure rate of tie-breaking mechanisms. Previous work has studied the following tie-breaking mechanisms.

40th Conference on Neural Information Processing Systems (NeurIPS 2026).

- **Fixed-agenda (FA) tie-breaking** breaks ties using a predefined ranking over alternatives.
- **Fixed-voter (FV) tie-breaking** breaks ties using a predetermined agent's ranking (e.g., agent 1's ranking).
- **Most-popular-singleton-ranking (MPSR) tie-breaking** [32] generalizes FV by using the ranking with unique and maximum multiplicity in the votes, when such a ranking exists.
- **Most-favorable-permutation (MFP) tie-breaking** [35] provides a polynomial-time computable way that *optimally* preserves ANR.

A quantitative comparison of these tie-breaking mechanisms is crucial, because different tie-breaking mechanisms offer distinct advantages. For example, MFP optimally preserves ANR; FA and FV may be simpler to explain and easier to use in some applications.

Previous research has analyzed the ANR failure rate under a large class of semi-random models [32], establishing that for FA or FV applied to positional scoring rules, the ANR failure rate is $\Omega(n^{-0.5})$, where n is the number of agents. While this probability approaches zero as n increases, the convergence rate can be slow in certain scenarios, for example in a well-connected social network, each agent spreads bad news (ANR failure) and gets negative utility when hearing from a friend [36]. For MPSR, the ANR failure rate was shown to be $O(n^{-m!/4})$, where m is the number of alternatives, and is tight for even n's [32]. However, these bounds are not asymptotically tight for all n's and the ANR failure rate under MFP remains unanswered.

1.1 Our Contributions

We adopt the *semi-random* model [32] to evaluate ANR failure rate. First, define the indicator function of ANR failure of a voting rule r at a profile P as:

$$\neg\text{ANR}(r, P) \triangleq \mathbb{I}[r \text{ fails anonymity, neutrality, or resoluteness at } P]$$

Then, given n and a set Π of distributions over all rankings over the alternatives, the *(semi-random) ANR failure rate* is defined as:

$$\widetilde{\neg\text{ANR}}(r, n) \triangleq \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(\neg\text{ANR}(r, P) = 1) \tag{1}$$

This formulation captures the worst-case probability of ANR failure under independent but arbitrarily chosen distributions from Π, one per agent. A low value is desirable, because it implies that ANR failure rate is low under all distributions $\vec{\pi} \in \Pi^n$. We assume that Π satisfies regularity conditions assumed in previous work (see Assumption 1 in Section 2), which includes many classical models in social choice, especially Impartial Culture. Our contributions are threefold.

First contribution (Table 1) is an accurate quantitative comparison of $\widetilde{\neg\text{ANR}}$ under a wide range of common-studied voting rules with FA, FV, MPSR, and MFP.

Rule	FA or FV (Coro. 1)	MPSR (Thm. 1)	MFP (Thm. 3, 5 & Prop. 2)
PS, STV, MM, RP, Sch	$\Theta(n^{-0.5})$	$\Theta(n^{-m!/4})$, $\Theta(n^{-m!/4-0.5})$, or 0 (only possible for $m = 3$)	$\Theta(n^{(\alpha^+-m!)/2})$ or 0
Copeland	$\Theta(1)$		

Table 1: Semi-random ANR failure rate of commonly-studied rules under FA, FV, MPSR, and MFP: Positional scoring rules (PS), STV, Maximin (MM), Ranked Pairs (RP), Schulze (Sch), for fixed $m \geq 3$.

In Table 1, $\alpha^+ = \frac{m!}{\eta}$ where η is n's smallest non-trivial divisor that is no more than m; if η does not exist, then we define $\alpha^+ = -\infty$ and let $n^{-\infty} = 0$ for notational convenience. For fixed $m \geq 3$, whenever its failure rate is nonzero, MFP achieves an $n^{\Theta(m!)}$ smaller failure rate than FA and FV. Compared with MPSR, the rates coincide for even n and for $m = 3$; for $m \geq 4$ and odd n with a non-trivial divisor at most m, MFP achieves an $n^{\Theta(m!)}$ improvement, and if no such divisor exists, its failure rate is exactly zero. For example, for Borda with $m = 4$ and odd n divisible by 3, the rates under FA/FV, MPSR, and MFP are $\Theta(n^{-1/2})$, $\Theta(n^{-13/2})$, and $\Theta(n^{-8})$, respectively. We note that these are asymptotic comparisons with fixed m. We believe that these results are theoretically

interesting and practically relevant because they provide quantitative guidance for designers to choose application-specific tie-breaking mechanisms. For example, if $\Theta(n^{-0.5})$ is considered low, then FA and FV can be used for their simplicity. If an $n^{\Theta(m!)}$ improvement is desirable and Pareto efficiency is not required, then MFP is a good choice. MPSR_{FV} can be a good compromise w.r.t. satisfaction of ANR and Pareto efficiency.

Second contribution (Theorem 3) consists of general lower and upper bounds on the ANR failure rate for any anonymous and neutral voting rule under MFP:

$$\Omega(n^{(\alpha^- - m!)/2}) \leq \widetilde{\neg\text{ANR}} \leq O(n^{(\alpha^+ - m!)/2}),$$

where α^- is $m!$ divided by the minimum least common multiple of values in every partition of m as a sum of n's non-trivial divisors; if m cannot be represented as a sum of n's non-trivial divisors, then we let $\alpha^- = -\infty$. These bounds illustrate the effectiveness of MFP and are good news as they show that for any anonymous and neutral rule, including the rules in Table 1, the ANR failure rate can be reduced to $O(n^{(\alpha^+ - m!)/2})$ by applying MFP.

Third contribution (Theorem 4 and 5) consists of two mild semi-random impossibility theorems via tightness of the lower and upper bounds in Theorem 3. Informally, tightness of the lower bound (Theorem 4) implies no resolute rule can achieve an ANR failure rate smaller than $\Theta(n^{(\alpha^- - m!)/2})$; and the tightness of the upper bound (Theorem 5) illustrates a tradeoff between equity (measured by ANR satisfaction) and efficiency (measured by a new notion called δ-unanimity).

Technical innovations. We leverage the algebraic characterization of the "problematic profiles" [35] and the semi-random analysis [33] in novel and non-trivial ways to conduct fine-grained analysis on the likelihood of ANR. Such combination is challenging as it is unclear how to translate the algebraic conditions into polyhedra, and how large their dimensions are, which we address by analyzing the sizes of the orbits of the algebraic representation of the problem profiles. We believe that the structural insights and the techniques developed in this paper are interesting and can be useful for future work on the analysis of ANR rules that satisfies other desiderata.

1.2 Related Work and Discussions

ANR impossibility. The study of anonymity and neutrality in voting traces back to May's seminal work [24], which characterized majority rule by anonymity, neutrality, and responsiveness. Moulin [26] proved an ANR impossibility theorem for the setting where preferences are rankings and the decisions are single alternatives—a voting rule satisfying ANR exists if and only if m cannot be expressed as a sum of non-trivial (i.e., greater than one) divisors of n. Moulin [26] also proved that an ANR rule satisfying Pareto efficiency exists if and only if m is smaller than the smallest non-trivial divisor of n. Subsequent research has identified conditions on m and n under which ANR impossibility holds under various combinations of preference and decision spaces [12, 28, 4–7, 35].

ANR failure rate. As discussed above, Xia [32] proved an $\Omega(n^{-0.5})$ lower bound for positional scoring rules with FA or FV, and an upper bound of $O(n^{-m!/4})$ for MPSR applied to any irresolute rule that satisfies anonymity and neutrality, but the tightness of these bounds and the bounds for MFP were left open. These are addressed by our results (see Table 1) for a wide range of voting rules. It was shown in Xia [32, Theorem 2] that there exist voting rules whose ANR failure rates are $O(n^{(\frac{m!}{\ell} - m!)/2})$ for some integer $\ell \geq 2$, and the bound is tight for infinitely many n's. Our Theorems 3 and 4 strengthen this result by characterizing the optimal ANR failure rate as $\Theta(n^{(\alpha^- - m!)/2})$ where $\alpha^- \leq m!/\ell$. See Appendix E.3 and E.4 for formal comparisons.

Probability analysis in social choice. There is a large body of literature in social choice using probability models to evaluate voting rules. This direction was pioneered by Gehrlein and Fishburn in their influential series of works in the 1970's [15, 17, 16]. Some recent progresses can be found in the book by Diss and Merlin [11]. More recently, semi-random analysis has proven particularly fruitful, finding applications in studying likelihood of ties [33], axiomatic satisfaction and impossibility theorems [34, 14, 36], difference between rules [40], complexity of winner determination [38, 39], judgment aggregation [20], and fair division [1].

Tie-breaking for ANR. Zwicker [41] highlighted tie-breaking as a common approach towards circumventing the ANR impossibility. Several mechanisms have been developed with the explicit goal of preserving ANR properties. Doğan and Giritligil [12] and Bubboloni and Gori [6] proposed tie-breaking mechanisms and identified sufficient conditions under which they satisfy ANR. Xia introduced MPSR [32], proposed MFP and proved that it optimally preserves ANR [35], and generalized MFP to handle broader preference and decision spaces [37].

2 Preliminaries

Preferences. Let $\mathcal{A} = [m] = \{1, \dots, m\}$ denote the set of $m \geq 2$ *alternatives*. Let $\mathcal{L}(\mathcal{A})$ denote the set of all linear orders over $\mathcal{A}$. There are $n \in \mathbb{N}$ agents, each using a linear order in $\mathcal{L}(\mathcal{A})$ to represent his or her preferences, called a *vote*. The collection of n agents' votes, denoted by $P \in \mathcal{L}(\mathcal{A})^n$, is called a *(preference) profile*, sometimes called an n-profile. Let $\text{Hist}(P) \in \mathbb{Z}_{\geq 0}^{m!}$ denote the *histogram* of P, which is the anonymized P that contains the multiplicity of each linear order in P. Let $\mathbb{H}_{m,n}$ denote the set of all histograms of n-profiles.

Voting rules. An *irresolute* rule $\overline{r} : \mathcal{L}(\mathcal{A})^n \to (2^{\mathcal{A}} \setminus \{\emptyset\})$ maps a profile to a non-empty subset of $\mathcal{A}$, called the *co-winners*. A *resolute* rule $r : \mathcal{L}(\mathcal{A})^n \to \mathcal{A}$ can be viewed as an irresolute rule that always chooses a single alternative, called the *winner*. We slightly abuse the notation by using $r(P) = a$ and $r(P) = \{a\}$ interchangeably. A voting rule $\overline{r}'$ is a *refinement* of another voting rule $\overline{r}$ if for all profiles P, $\overline{r}'(P) \subseteq \overline{r}(P)$.

For example, an irresolute *positional scoring rule* (PS) $\overline{r}_{\vec{s}}$ is characterized by a scoring vector $\vec{s} = (s_1, \dots, s_m)$ with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$. We consider rational scoring vectors, equivalently integer scoring vectors after positive rescaling. For any alternative a and any linear order $R \in \mathcal{L}(\mathcal{A})$, we let $\vec{s}(R, a) \triangleq s_i$, where i is the rank of a in R. Given a profile P, let $\vec{s}(P, a) \triangleq \sum_{R \in P} \vec{s}(R, a)$ denote the *total score* of a. Then, $\overline{r}_{\vec{s}}$ chooses all alternatives with maximum score. Special cases include *plurality*, whose scoring vector is $(1, 0, \dots, 0)$, *Borda*, whose scoring vector is $(m-1, m-2, \dots, 0)$, and *veto*, whose scoring vector is $(1, \dots, 1, 0)$.

Example 1. *Let $m = 5$, $n = 9$, and P^+ denote a 9-profile whose histogram is shown in Table 2 and whose first vote is* 24135*. In the table,* 12345 *represents the ranking $1 \succ 2 \succ 3 \succ 4 \succ 5$. Let $\overline{r}$ = veto, we have $\overline{r}(P^+) = \{1, 2, 3, 4\}$.*

	Ranking	12345	21435	24135	31425	34125	42315
Example 1	# in Hist(P^+)	3	0	1	0	2	3
Example 3	# in Hist($\sigma_1(P^+)$)	3	1	0	2	0	3

Table 2: Histograms in Example 1 and 3.

STV repeatedly eliminates a candidate with minimum plurality score. The maximin rule, denoted by $\mathcal{M}$, maximizes the minimum pairwise margin. The Copeland maximizes the number of pairwise wins. The Ranked Pairs rule, denoted by RP, locks pairwise comparisons in decreasing order of margin, avoiding cycles. The Schulze rule, denoted by Sch, compares the strengths of the strongest paths, where a path's strength is its smallest edge margin. For STV and ranked pairs, we take all winners obtainable over tie-breaking choices (parallel-universes tie-breaking [8]). Formal definitions can be found in Appendix B.

Tie-breaking mechanisms. In this paper, we model a tie-breaking mechanism f as a mapping from a profile P and a non-empty set $D \subseteq \mathcal{A}$ (representing the co-winners) to a single winner in D. Commonly-used tie-breaking mechanisms are (1) *fixed-agenda (*FA*)*, which breaks ties in favor of alternatives ranked higher w.r.t. a pre-defined ranking over $\mathcal{A}$, e.g., $1 \rhd \cdots \rhd m$; and (2) *fixed-voter (FV)*, which breaks ties using a pre-defined voter's ranking, for example voter 1's ranking. Let $f * \overline{r}$ denote the application of f to $\overline{r}$, such that for any profile P, $(f * \overline{r})(P) = \{f(P, \overline{r}(P))\}$.

A *singleton ranking* in P is a linear order with unique multiplicity in P, and a *most popular singleton ranking (*MPSR*)* is the singleton ranking with the maximum (and unique) multiplicity. The *most-popular-singleton-ranking tie-breaking mechanism* [32] uses an MPSR to break ties if it exists; otherwise it uses a "backup" tie-breaking mechanism f, e.g., FA or FV. An MPSR tie-breaking is denoted by MPSR_f.

Example 2 (FA, FV, MPSR). *In the setting of Example 1, we have* FA $*$ *veto*$(P) = \{1\}$ *and* FV $*$ *veto*$(P) = \{2\}$ *(recall that the first vote in* P^+ *is* 24135*). There are two singleton rankings:* 24135 *with multiplicity* 1 *and* 34125 *with multiplicity* 2*. Therefore, the* MPSR *is* 34125*, which means that* MPSR_f $*$ *veto*$(P) = \{3\}$.

The *most-favorable-permutation* (MFP) tie-breaking [35] is defined by a priority order $\rhd$ over $\mathcal{A}$ and a "backup" tie-breaking mechanism f. Let $\mathcal{S}_\mathcal{A}$ denote the set of all permutations over $\mathcal{A}$. A permutation is represented in its cycle form in this paper. For example, when $m = 5$, $\sigma_1 = (1, 4)$ represents the permutation that only exchanges 1 and 4, and does not change other alternatives. Any permutation $\sigma \in \mathcal{S}_\mathcal{A}$ can be naturally extended to histograms, and any priority order $\rhd$ over $\mathcal{A}$ can be naturally extended to histograms in a lexicographic way. See Definition 5 in Appendix B for the formal definition.

Example 3. *In the setting of Example 1, let* $\sigma_1 = (1, 4)$. σ_1(*Hist*(P^+)) *is shown in Table 2. The six types of linear orders that appear in* P^+ *and* $\sigma_1(P^+)$ *are ordered lexicographically from left to right.*

Given a profile P and a priority order $\rhd$, we define the set of *most favorable permutations (MFPs)* as all permutations with maximum $\sigma(\text{Hist}(P))$ according to $\rhd$. Formally, define

$$\text{MFP}(P) \triangleq \arg\max^{\rhd}_{\sigma \in \mathcal{S}_\mathcal{A}} \text{Hist}(\sigma(P)) \tag{2}$$

Then, define P's *fixed point decisions* as the alternatives that are invariant to all permutations that map $\text{Hist}(P)$ to itself—such permutations are called *stabilizers* and are denoted by $\textsc{Stab}(\text{Hist}(P))$.

$$\text{FPD}(P) \triangleq \{a \in \mathcal{A} : \forall \sigma \in \textsc{Stab}(\text{Hist}(P)), \sigma(a) = a\}$$

Given a "backup" tie-breaking f, a profile P, and a given set of co-winners D, the MFP tie-breaking MFP_f first chooses an arbitrary but fixed MFP σ^*, then chooses a fixed-point co-winner a from $\text{FPD}(P) \cap D$ such that $\sigma^*(a)$ has the highest priority in $\rhd$. If $\text{FPD}(P) \cap D = \emptyset$, then f is used to break ties. Formally, define

$$\text{MFP}_f(P, D) \triangleq \begin{cases} \arg\max^{\rhd}_{a \in \text{FPD}(P) \cap D} \sigma^*(a) & \text{if } \text{FPD}(P) \cap D \neq \emptyset \\ f(P, D) & \text{otherwise} \end{cases} \tag{3}$$

MFP_f is well-defined and can be computed in polynomial time [35].

Example 4 (MFP_f). *In the setting of Example 1 and 2,* $\sigma_1 = (1, 4)$ *is the only MFP,* $\textsc{Stab}$(*Hist*(P^+)) *only consists of the identity permutation, which means that* $\text{FPD}(P^+) = \mathcal{A}$*. Because* $\text{FPD}(P^+) \cap$ *veto*$(P^+) = \{1, 2, 3, 4\}$ *and* $\sigma_1(4) = 1$*, we have* MFP_f $*$ *veto*$(P) = \{4\}$.

Semi-random ANR failure rate. Following [32], for any rule $\overline{r}$ and any profile P, we say that $\overline{r}$ fails anonymity at P if there exists another profile P' with $\text{Hist}(P') = \text{Hist}(P)$ but $\overline{r}(P') \neq \overline{r}(P)$. We say that $\overline{r}$ fails neutrality at P if there exists $\sigma \in \mathcal{S}_\mathcal{A}$ such that $\overline{r}(\sigma(P)) \neq \sigma(\overline{r}(P))$. We say that $\overline{r}$ fails resoluteness at P if $|\overline{r}(P)| > 1$. [1] The (semi-random) ANR failure rate was defined in (1). Throughout this paper, we make the following assumptions on Π as done in previous works [32, 33].

Assumption 1 (**on** Π). *We assume that (i)* Π *is* strictly positive*, i.e.,* $\exists \epsilon > 0$ *such that all probabilities in all distributions in* Π *are larger than* ϵ*; (ii)* Π *is a closed set in* $\mathbb{R}^{m!}$*; and (iii) the convex hull of* Π *contains* π_{uni}*, which is the uniform distribution over* $\mathcal{L}(\mathcal{A})$.

Assumption 1 is satisfied by many commonly studied models, such as single-agent Mallows and single-agent Plackett-Luce with bounded parameters as shown in [32, Example 2 in Appendix A]. It is also satisfied by the *Impartial Culture*, namely $\Pi = \{\pi_{\text{uni}}\}$. Therefore, all results in this paper on ANR failure rate immediately hold for these models. For subclasses of Mallows and Plackett-Luce, the upper bounds naturally applies and the matching lower bounds apply when the subclass also satisfies Assumption 1.

Assumption 1(i) excludes strictly restricted domains such as single-peaked or single-crossing rankings, which assign zero probability to some rankings; (ii) is a technical closedness condition. Condition (iii) requires that the uniform distribution is in the convex hull of allowed distributions. When it fails, the lower bounds may not hold, which means that ANR satisfaction can be higher, which is good news.

[1] Failures of anonymity and neutrality are defined as properties of P (instead of also on the hypothetical permutations among agents and/or the alternatives), because they allow the agents to verify the satisfaction of anonymity and neutrality after the winner is announced. The definition shares the same spirit with that of coalitional manipulability, see, e.g., [31].

3 Likelihood of ANR failure: FA, FV, MPSR

We start with a simple observation, which says that FA and FV almost always fail ANR when co-winners exist.

Proposition 1. *For any irresolute rule $\overline{r}$ that satisfies anonymity and neutrality, and any profile P, $\neg\text{ANR}(\text{FA} * \overline{r}, P) = 1 \iff |\overline{r}(P)| \geq 2$; when all rankings appear in P, $\neg\text{ANR}(\text{FV} * \overline{r}, P) = 1 \iff |\overline{r}(P)| \geq 2$.*

The proof can be found in Appendix C.1. Notice that any anonymous and neutral rule $\overline{r}$ satisfies ANR at every profile without ties. Therefore, FA is the worst tie-breaking mechanism w.r.t. ANR satisfaction and FV is almost as bad when the profile is generated according to Π satisfying Assumption 1. Leveraging results on semi-random likelihood of ties [33], we have:

Corrollary 1. *For any fixed $m \geq 3$, any Π satisfying Assumption 1, any $f \in \{\text{FA}, \text{FV}\}$, and any $\overline{r} \in \{PS, STV, MM, RP, Sch\}$, $\widetilde{\neg\text{ANR}}(f * \overline{r}, n) = \Theta(\frac{1}{\sqrt{n}})$; moreover, $\widetilde{\neg\text{ANR}}(f * Cd_\alpha, n) = \Theta(1)$. For $m = 2$, all these rules, including Copeland, have failure rate $\Theta(n^{-1/2})$ for even n and 0 for odd n, because ties occur exactly when the two rankings have equal multiplicity.*

Next, we characterize ANR failure rate under MPSR.

Theorem 1 (ANR failure rate under MPSR). *For any fixed $m \geq 3$, any Π satisfying Assumption 1, any $f \in \{\text{FA}, \text{FV}\}$, and any $\overline{r} \in \{PS, STV, Cd_\alpha, MM, RP, Sch\}$,*

- *when $m = 3$, $\widetilde{\neg\text{ANR}}(\text{MPSR}_f * \overline{r}, n) = \begin{cases} \Theta(n^{-1.5}) & \text{if } 2 \mid n \\ \Theta(n^{-2}) & \text{if } 2 \nmid n \text{ and } 3 \mid n \\ 0 & \text{otherwise} \end{cases}$*
- *when $m \geq 4$, $\widetilde{\neg\text{ANR}}(\text{MPSR}_f * \overline{r}, n) = \begin{cases} \Theta(n^{-\frac{m!}{4}}) & \text{if } 2 \mid n \\ \Theta(n^{-\frac{m!}{4}-\frac{1}{2}}) & \text{otherwise} \end{cases}$*

Proof sketch. We use PS and MPSR_{FA} to illustrate the proof idea. Let $\overline{r}_{\vec{s}}$ be the positional scoring rule with scoring vector $\vec{s}$. It is not hard to see that $\text{MPSR}_{\text{FA}} * \overline{r}$ fails ANR at a profile P if and only if $|\overline{r}(P)| \geq 2$ and P has no singleton ranking. The rest of the proof characterizes the probability for this to happen in different cases of n and m, by taking the polyhedral approach [33].

The polyhedral approach [33]

Step 1. Represent the event of interest as finitely many polyhedra in histogram space $\mathbb{H}_{m,n}$.
Step 2. Apply [33, Theorem 1] (Theorem 2 below) to each polyhedron and combine the results.

The $2 \mid n$ case. When $m \geq 3$ and $2 \mid n$, to prove the $O(n^{-\frac{m!}{4}})$ upper bound, it suffices to prove

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(P \text{ has no singleton ranking}) = O(n^{-\frac{m!}{4}}) \tag{4}$$

We first define the polyhedra that will be used to represent the event "P has no singleton ranking". Let $\mathbb{T}_m$ denote the set of all partitions of $\mathcal{L}(\mathcal{A})$ without unique ranking component, that is,

$$\mathbb{T}_m \triangleq \{\text{partition } \{L_1, \ldots, L_t\} \text{ of } \mathcal{L}(\mathcal{A}) : \forall i \leq t, |L_i| \geq 2\}$$

For every $\mathcal{T} \in \mathbb{T}_m$, we define a polyhedron $\mathcal{H}^{\mathcal{T}}$ to represent histograms $\vec{x}$ such that the rankings in the same set in $\mathcal{T}$ have the same multiplicity.

Definition 1 (Polyhedron for partition $\mathcal{T}$). *For any partition $\mathcal{T} \in \mathbb{T}_m$, define*

$$\mathcal{H}^{\mathcal{T}} \triangleq \{\vec{x} : \forall L \in \mathcal{T}, \forall \{R_1, R_2\} \subseteq L, [\vec{x}]_{R_1} = [\vec{x}]_{R_2}\},$$

where $[\vec{x}]_{R_1}$ is the R_1 component of $\vec{x}$.

Clearly, P has no singleton ranking iff $\text{Hist}(P) \in \mathcal{H}^{\mathcal{T}}$ for some $\mathcal{T} \in \mathbb{T}_m$. Notice that $|\mathbb{T}_m|$ only depends on m and can be viewed as a constant. Therefore, to prove (4), it suffices to prove

$$\forall \mathcal{T} \in \mathbb{T}_m, \sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^{\mathcal{T}}) = O(n^{-\frac{m!}{4}}) \tag{5}$$

Next, we recall [33, Theorem 1, sup part] as Theorem 2. For any polyhedron $\mathcal{H} = \{\vec{x} : \mathbf{A} \times (\vec{x})^\top \leq (\mathbf{b})^\top\}$, let $\mathcal{H}_{\leq 0} = \{\vec{x} : \mathbf{A} \times (\vec{x})^\top \leq \left(\vec{0}\right)^\top\}$ denote its *recess cone* and let $\mathcal{H}_n^{\mathbb{Z}} \triangleq \{\vec{x} \in \mathcal{H} \cap \mathbb{Z}_{\geq 0}^q : \vec{x} \cdot \vec{1} = n\}$. When $\mathcal{H}_n^{\mathbb{Z}} \neq \emptyset$, we say that $\mathcal{H}$ is *active* at n; otherwise we say that $\mathcal{H}$ is *inactive* at n. Let $\dim(\mathcal{H})$ denote the *dimension* of $\mathcal{H}$, i.e., the dimension of the smallest affine space that contains $\mathcal{H}$. Given a vector $\vec{\pi}$ of n distributions over $[q] = \{1, \ldots, q\}$, we let $\vec{X}_{\vec{\pi}}$ denote the histogram of the n independent random variables distributed as in $\vec{\pi}$. $\mathrm{CH}(\Pi)$ is the convex hull of Π.

Theorem 2 (**Theorem 1, sup part in [33]**). *Given any $q \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, and any polyhedron $\mathcal{H}$ with integer matrix $\mathbf{A}$, for any $n \in \mathbb{N}$,*

$$\sup_{\vec{\pi} \in \Pi^n} \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{H}) = \begin{cases} 0 & \text{if } \mathcal{H}_n^{\mathbb{Z}} = \emptyset \text{ i.e., } \mathcal{H} \text{ is inactive at } n \\ \exp(-\Theta(n)) & \text{if } \mathcal{H}_n^{\mathbb{Z}} \neq \emptyset \text{ and } \mathcal{H}_{\leq 0} \cap \mathit{CH}(\Pi) = \emptyset \\ \Theta(n^{\frac{\dim(\mathcal{H}_{\leq 0}) - q}{2}}) & \text{otherwise} \end{cases}$$

It is not hard to verify that $\pi_{\text{uni}} \in \mathcal{H}^{\mathcal{T}}$ and $\dim(\mathcal{H}^{\mathcal{T}}_{\leq 0}) = |\mathcal{T}|$. Therefore, by Theorem 2, whenever $\mathcal{H}^{\mathcal{T}}$ is active at n, we have

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\mathrm{Hist}(P) \in \mathcal{H}^{\mathcal{T}}) = \Theta(n^{\frac{|\mathcal{T}| - m!}{2}})$$

If $\mathcal{H}^{\mathcal{T}}$ is inactive at n, its probability is zero. Because no $\mathcal{T} \in \mathbb{T}_m$ contains a singleton set, we have $|\mathcal{T}| \leq \frac{m!}{2}$. This proves (5).

The $\Omega(n^{-\frac{m!}{4}})$ lower bound is proved by lower-bounding the likelihood for $\mathrm{Hist}(P)$ to be in a polyhedron $\mathcal{H}^*$ that is obtained from $\mathcal{H}^{\mathcal{T}}$ (Definition 1) for some $\mathcal{T}$, by adding linear constrains to require that all histograms in $\mathcal{H}^*$ are tied. The proof for other cases and other rules are done similarly: the upper bound is proved by a careful analysis of the minimum size of partitions in $\mathbb{T}_m$, and the lower bound is proved by constructing $\mathcal{T}^*$ such that every histogram in $\mathcal{H}^{\mathcal{T}^*}$ has at least two co-winners, yet none of the histograms has a singleton ranking. The rest of the proof is in Appendix C.2. □

4 ANR Failure Rate: MFP

It turns out that the ANR failure rate under MFP is characterized by the two values defined below.

Definition 2 (**α^- and α^+**). *Given m and n, let $\mathrm{Ptt}_{m,n}$ denote the set of partitions $\vec{m}$ of m as n's non-trivial divisors, i.e., $\vec{m} \cdot \vec{1} = m$ and $\mathrm{lcm}(\vec{m}) \mid n$, where $\mathrm{lcm}(\vec{m})$ is the least common multiple of the components in $\vec{m}$. Let $\mathrm{Div}_{m,n}$ denote the set of all non-trivial divisors of n that are no more than m. Define*

$$\alpha^- \triangleq \begin{cases} \max_{\vec{\eta} \in \mathrm{Ptt}_{m,n}} \frac{m!}{\mathrm{lcm}(\vec{\eta})} & \text{if } \mathrm{Ptt}_{m,n} \neq \emptyset \\ -\infty & \text{otherwise} \end{cases} \text{ and } \alpha^+ \triangleq \begin{cases} \max_{\eta \in \mathrm{Div}_{m,n}} \frac{m!}{\eta} & \text{if } \mathrm{Div}_{m,n} \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

The follow table shows some examples of $\mathrm{Ptt}_{m,n}$, $\mathrm{Div}_{m,n}$, α^+, and α^-.

m	n	$\mathrm{Ptt}_{m,n}$	$\mathrm{Div}_{m,n}$	α^-	α^+
3	5	$\emptyset$	$\emptyset$	$-\infty$	$-\infty$
3	8	$\emptyset$	$\{2\}$	$-\infty$	$\frac{3!}{2} = 3$
5	12	$\{(3,2)\}$	$\{2,3,4\}$	$\frac{5!}{\mathrm{lcm}(3,2)} = 20$	$\frac{5!}{2} = 60$
6	12	$\{(6),(4,2),(3,3),(2,2,2)\}$	$\{2,3,4,6\}$	$\frac{6!}{\mathrm{lcm}(2,2,2)} = 360$	$\frac{6!}{2} = 360$

We note that $\mathrm{Ptt}_{m,n} \neq \emptyset$ is equivalent to Moulin's (1983) condition for the ANR impossibility (m can be represented as a sum of n's non-trivial divisors). Moreover, $\alpha^- \leq \alpha^+$, which means that if $\alpha^+ = -\infty$, then $\alpha^- = -\infty$. For any $n \geq 1$, we define $n^{-\infty} = 0$.

Theorem 3. *For any fixed $m \geq 2$ and any Π satisfying Assumption 1, there exist constants $C_1 > 0, C_2 > 0$ such that for any irresolute rule $\overline{r}$ that satisfies anonymity and neutrality, any* MFP *breaking* MFP_f*, and any $n \geq 1$,*

$$C_1(\sqrt{n})^{\alpha^- - m!} \leq \widetilde{\neg \mathrm{ANR}}(\mathrm{MFP}_f * \overline{r}, n) \leq C_2(\sqrt{n})^{\alpha^+ - m!}$$

Proof sketch. We will use the optimality of MFP [35, Theorem 3] to formulate ANR failure under $\text{MFP}_f * \overline{r}$. Let $\mathcal{P}^{\overline{r},n}$ denote all "problematic" profiles P without fixed-point decisions, i.e.

$$\mathcal{P}^{\overline{r},n} \triangleq \{P \in \mathcal{L}(\mathcal{A})^n : \text{FPD}(P) \cap \overline{r}(P) = \emptyset\}$$

We first recall that the optimality of MFP states that all tie-breaking mechanisms (including MFP) fail ANR at all problematic profiles, yet MFP succeeds at all other profiles.

Optimality of MFP [35, Theorem 3]

- **All tie-breaking fails at problematic profiles.** For any *refinement* r of $\overline{r}$, i.e., $\forall P, r(P) \subseteq \overline{r}(P)$, we have $\forall P \in \mathcal{P}^{\overline{r},n}, \neg\text{ANR}(r, P) = 1$;
- MFP **succeeds at non-problematic profiles.** $\forall P \notin \mathcal{P}^{\overline{r},n}, \neg\text{ANR}(\text{MFP}_f * \overline{r}, P) = 0$.

Therefore, for any $\vec{\pi} \in \Pi^n$, $\Pr_{P\sim\vec{\pi}}(\neg\text{ANR}(\text{MFP}_f * \overline{r}, P) = 1) = \Pr_{P\sim\vec{\pi}}(P \in \mathcal{P}^{\overline{r},n})$. It follows that if $\text{FPD}(P) = \emptyset$, then $P \in \mathcal{P}^{\overline{r},n}$, and for every $P \in \mathcal{P}^{\overline{r},n}$, we have $\text{FPD}(P) \neq \mathcal{A}$, which is equivalent to $|\textsc{Stab}(\text{Hist}(P))| > 1$. Consequently, we can bound $\mathcal{P}^{\overline{r},n}$ as

$$\underbrace{\{P : \text{FPD}(P) = \emptyset\}}_{\mathcal{P}_{\text{lower}}} \subseteq \mathcal{P}^{\overline{r},n} \subseteq \underbrace{\{P : |\textsc{Stab}(\text{Hist}(P))| > 1\}}_{\mathcal{P}_{\text{upper}}} \tag{6}$$

The lower bound and upper bound profiles are denoted by $\mathcal{P}_{\text{lower}}$ and $\mathcal{P}_{\text{upper}}$, respectively. Therefore,

$$\sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(P \in \mathcal{P}_{\text{lower}}) \leq \widetilde{\neg\text{ANR}}(\text{MFP}_f * \overline{r}, n) \tag{7}$$

$$\leq \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(P \in \mathcal{P}_{\text{upper}}) \tag{8}$$

The rest of the proof takes the polyhedral approach to characterize (7) and (8) by representing the histograms of $\mathcal{P}_{\text{upper}}$ and $\mathcal{P}_{\text{lower}}$ as unions of multiple polyhedra, respectively.

Upper bound of Theorem 3. It suffices to prove $\sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(P \in \mathcal{P}_{\text{upper}}) = \Theta((\sqrt{n})^{\alpha^+ - m!})$. For any permutation group $G \leqslant \mathcal{S}_\mathcal{A}$, define polyhedron $\mathcal{H}^G$ to represent histograms of profiles that are fixed points of G.

Definition 3 ($\mathcal{H}^G$). *For any permutation group $G \leqslant \mathcal{S}_\mathcal{A}$, let $\mathcal{H}^G \triangleq \{\vec{x} \in \mathbb{R}^{m!} : \mathbf{A}^G \times (\vec{x})^\top \leq \left(\mathbf{b}^G\right)^\top\}$, where for each $\sigma \in G$ and every $R \in \mathcal{L}(\mathcal{A})$, there is a row $\vec{e}_R - \vec{e}_{\sigma(R)}$ in $\mathbf{A}^G$, where $\vec{e}_R$ is the binary vector that takes 1 at the R-th component and takes 0 otherwise; $\mathbf{b}^G \triangleq \vec{0}$.*

The following claim provides properties of $\mathcal{H}^G$ for applying Theorem 2.

Claim 1 (Properties of $\mathcal{H}^G$). *For any $G \leqslant \mathcal{S}_\mathcal{A}$, (i) $\forall \vec{x} \in \mathbb{R}^{m!}$, $\vec{x} \in \mathcal{H}^G$ iff $\vec{x}$ is a fixed point of G; (ii) $\dim(\mathcal{H}^G_{\leq 0}) = \frac{m!}{|G|}$; (iii) $\forall n \in \mathbb{N}$, $\mathcal{H}^G$ is active at n (i.e., $\mathcal{H}^{G,\mathbb{Z}}_n \neq \emptyset$) iff $|G|$ divides n.*

The proof of Claim 1 can be found in Appendix D.1. Notice that for any fixed m, the total number of permutation groups over $\mathcal{A}$ can be viewed as a constant. Therefore, according to Claim 1 (i), it suffices to prove that for every permutation group $G \leqslant \mathcal{S}_\mathcal{A}$ with $|G| > 1$,

$$\sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^G) = O(n^{(\alpha^+ - m!)/2}) \tag{9}$$

When $\alpha^+ = -\infty$, we prove that $\mathcal{H}^G$ is inactive, which means that $\sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^G) = 0$. Suppose for the sake of contradiction that $\mathcal{H}^G$ is active for some $G \leqslant \mathcal{S}_\mathcal{A}$ with $|G| > 1$. Let P^* denote an n-profile such that $\text{Hist}(P^*) \in \mathcal{H}^{G,\mathbb{Z}}_n$. By Claim 1 (iii), $|G|$ is a divisor of n. Because $|G| > 1$ is equivalent to $G \neq \{\text{Id}\}$, where Id is the identity permutation, there exists an orbit O_t in $\mathcal{A}$ under G, as defined in Definition 6 in Appendix D, such that $|O_t| \geq 2$. Therefore, according to the orbit-stabilizer theorem, $|O_t|$ (which is no more than m) is a non-trivial divisor of $|G|$, which is a divisor n. This contradicts the assumption that $\alpha^+ = -\infty$.

When $\alpha^+ \neq -\infty$, notice that if $\mathcal{H}^G$ is active, then $|G| > 1$ and $|G|$ is a divisor of n (Claim 1 (iii)). The upper bound follows after applying Theorem 2 to $\mathcal{H}^G$ by noticing that $\pi_{\text{uni}} \in \text{CH}(\Pi)$ (according to Assumption 1), $\pi_{\text{uni}} \in \mathcal{H}^G$, and $\dim(\mathcal{H}^G_{\leq 0}) = \frac{m!}{|G|} \leq \alpha^+$. Proof of the lower bound can be found in Appendix D.2. □

5 Tightness of Bounds

Let $\overline{r}_{\mathcal{A}}$ denote the trivial irresolute rule that always chooses all alternatives as co-winners. The following theorem states that the lower bound in Theorem 3 is achieved by $\text{MFP}_f * \overline{r}_{\mathcal{A}}$.

Theorem 4. *In the same setting as Theorem 3,* $\widetilde{\neg\text{ANR}}(\text{MFP}_f * \overline{r}_{\mathcal{A}}, n) = \Theta(n^{(\alpha^- - m!)/2})$.

Proof sketch. Due to the optimality of MFP_f, MFP_f only fails ANR at $\mathcal{P}_{\text{lower}}$ (defined in (6)). It suffices to prove

$$\sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(P \in \mathcal{P}_{\text{lower}}) = \Theta(n^{(\alpha^- - m!)/2}) \tag{10}$$

The proof is done by taking the polyhedral approach to a representation of $\mathcal{P}_{\text{lower}}$ as the union of multiple polyhedra $\mathcal{H}^G$ (Definition 3). More precisely, let $\mathcal{G}_m$ denote the set of all permutation groups $G \leqslant \mathcal{S}_{\mathcal{A}}$ that have no fixed point in $\mathcal{A}$. We define $\mathcal{C}_{\mathcal{A}} \triangleq \bigcup_{G\in\mathcal{G}_m} \mathcal{H}^G$. It follows that $P \in \mathcal{P}_{\text{lower}}$ if and only if $\text{Hist}(P) \in \mathcal{C}_{\mathcal{A}}$. The size of permutation groups in $\mathcal{G}_m$ can be bounded as in the following claim, which is proved by analyzing the sizes of orbits of G and can be found in Appendix E.2.

Claim 2. *For any $G \in \mathcal{G}_m$ and any $n \geq 2$, if $\mathcal{H}^G$ is active at n, then $\alpha^- \neq -\infty$ and $|G| \geq \frac{m!}{\alpha^-}$.*

When $\alpha^- = -\infty$, (10) follows after Claim 2, which implies that for every $G \in \mathcal{G}_m$, $\mathcal{H}^G$ is not active at n. When $\alpha^- \neq -\infty$, Claim 2 implies that $\frac{m!}{|G|} \leq \alpha^-$. By Theorem 2 and Claim 1 (ii), we have

$$\sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^G) = O(n^{(\alpha^- - m!)/2})$$

(10) then follows after applying the union bound to all permutations groups in $\mathcal{G}_m$, noticing that $|\mathcal{G}_m|$ can be viewed as a constant that does not depend on n. □

Discussions. Moulin [26] proved that an ANR rule exists if and only if $\text{Ptt}_{m,n} = \emptyset$. This can be viewed as a corollary of Theorem 4 because when $\text{Ptt}_{m,n} = \emptyset$, we have $\widetilde{\neg\text{ANR}}(\text{MFP}_f * \overline{r}_{\mathcal{A}}, n) = 0$, which means that $\text{MFP}_f * \overline{r}_{\mathcal{A}}$ satisfies ANR in the worst case. When $\text{Ptt}_{m,n} \neq \emptyset$, notice that any resolute rule r can be viewed as the result of a tie-breaking mechanism applied to $\overline{r}_{\mathcal{A}}$, which means that its ANR failure rate is at least that of $\text{MFP}_f * \overline{r}_{\mathcal{A}}$ due to the optimality of MFP_f, which is > 0.

The upper bound is achieved by MFP applied to any voting rule that satisfies anonymity, neutrality, and two additional properties called *block invariance* and *δ-unanimity* defined as follows.

Definition 4. *$\overline{r}$ satisfies* block invariance*, if for every profile P, $\overline{r}(P) = \overline{r}(P \cup \mathcal{L}(\mathcal{A}))$. For any $0 \leq \delta < 1$, an irresolute rule $\overline{r}$ satisfies* δ-unanimity*, if for any nonempty set of alternatives $A \subseteq \mathcal{A}$ and any n-profile P such that A are ranked in the top $|A|$ positions in at least $(1-\delta)n$ votes of P, we have $\overline{r}(P) \subseteq A$.*

Block invariance is the standard requirement that adding one copy of every ranking leaves the outcome unchanged [10, 18, 25]. Such a block contributes equally to positional scores and zero to pairwise margins; although it makes the normalized histogram closer to uniform, it need not change score differences or margins. We use this property for its generality in the proof, without advocating it as a universal normative requirement.

δ-unanimity is a quantitative near-unanimity condition related to the weak mutual majority criterion (WMM) of Kondratev and Nesterov [19]: WMM requires $\overline{r}(P) \subseteq A$ when more than $|A|/(|A|+1)$ of voters rank A at the top. Our condition instead uses a common threshold $1-\delta$ for all set sizes. For fixed m, WMM implies δ-unanimity for every $0 \leq \delta < 1/m$; the converse may not hold.

The classical *unanimity*, which requires whenever an alternative a is ranked at the top in all votes, a must be the winner, is implied by 0-unanimity. The larger δ is, the stronger δ-unanimity is as a property on voting rules. Proposition 2 in Appendix E.1 shows that strictly decreasing positional scoring rules, STV, maximin, Copeland, ranked pairs, and Schulze satisfy δ-unanimity for some constant $0 < \delta < 1$ and block invariance. A direct argument in the same appendix establishes the MFP rate in Table 1 for all positional scoring rules considered here, including plurality and veto.

Theorem 5. *In the same setting as Theorem 3, for any $\overline{r}$ that satisfies anonymity, neutrality, block invariance, and δ-unanimity for some constant $\delta > 0$,* $\widetilde{\neg\text{ANR}}(\text{MFP}_f * \overline{r}, n) = \Theta(n^{(\alpha^+ - m!)/2})$.

Proof sketch. Let $\overline{r}$ be a rule that satisfies the conditions in the theorem statement. When $\alpha^+ = -\infty$, the theorem trivially follows after the upper bound of Theorem 3. When $\alpha^+ \neq -\infty$, we define

$d_{\min} \triangleq \frac{m!}{\alpha^+}$, i.e., $d_{\min}$ is the n's minimum non-trivial divisor that is no more than m. Then, we define a polyhedron $\mathcal{H}^*$ such that every profile P whose histogram is in $\mathcal{H}^*$ has no fixed-point co-winner, which means that $P \in \mathcal{P}^{\bar{r},n}$. Then, we apply the polyhedral approach to prove that when $\alpha^+ \neq -\infty$,

$$\sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^*) = \Omega(n^{(\alpha^+ - m!)/2}), \tag{11}$$

which proves Theorem 5. The full proof can be found in Appendix E.5. □

Discussions. The tightness of the upper bound in Theorem 5 implies that many commonly studied voting rules have the largest failure-rate order permitted by Theorem 3, even after optimal tie-breaking. On the positive side, this upper bound still converges to 0 at a reasonably fast rate. This can be viewed as a semi-random efficiency-equity tradeoff, because δ-unanimity can be viewed as a form of efficiency.

6 Conclusion and Future Work

We highlight four directions for future work: (1) Extensions to broader social choice settings, such as when preferences and decisions are subsets or rankings over subsets of alternatives, or classifiers [22]. (2) Studies of ANR in conjunction with other properties, such as Pareto efficiency and monotonicity; (3) Relaxation of Assumption 1. Relaxing (ii) is a technical question; removing (iii) requires distinguishing recession-cone intersections from disjoint cases, while relaxing (i) to allow restricted domains requires further investigation. (4)Random rules. This work and most previous work on ANR focused on deterministic rules. The efficiency-equity/fairness tradeoff for randomized rules is another interesting and important direction.

Acknowledgements and Disclosures

We thank all anonymous reviewers for their helpful comments and feedback. This work is supported by NSF #2450124, #2517733, and #2518373.

References

[1] Yushi Bai, Uriel Feige, Paul Gölz, and Ariel D. Procaccia. Fair Allocations for Smoothed Utilities. In *Proceedings of ACM EC*, 2022.

[2] Linas Baltrunas, Tadas Makcinskas, and Francesco Ricci. Group recommendations with rank aggregation and collaborative filtering. In *Proceedings of the fourth ACM conference on Recommender systems*, pages 119–126. ACM, 2010.

[3] Felix Brandt. Collective Choice Lotteries. In Jean-François Laslier, Hervé Moulin, Remzi Sanver, and William S. Zwicker, editors, *The Future of Economic Design*, pages 51–56. Berlin: Springer, 2019.

[4] Daniela Bubboloni and Michele Gori. Anonymous and neutral majority rules. *Social Choice and Welfare*, 43(2):377–401, 2014.

[5] Daniela Bubboloni and Michele Gori. Symmetric majority rules. *Mathematical Social Sciences*, 76(73–86), 2015.

[6] Daniela Bubboloni and Michele Gori. Resolute refinements of social choice correspondences. *Mathematical Social Sciences*, 84:37–49, 2016.

[7] Daniela Bubboloni and Michele Gori. Breaking ties in collective decision-making. *Decisions in Economics and Finance*, 44:411–457, 2021.

[8] Vincent Conitzer, Matthew Rognlie, and Lirong Xia. Preference functions that score rankings and maximum likelihood estimation. pages 109–115, 2009.

[9] Vincent Conitzer, Rachel Freedman, Jobst Heitzig, Wesley H. Holliday, Bob M. Jacobs, Nathan Lambert, Milan Mossé, Eric Pacuit, Stuart Russell, Hailey Schoelkopf, Emanuel Tewolde, and William S. Zwicker. Social Choice Should Guide AI Alignment in Dealing with Diverse Human Feedback. In *Proceedings of ICML*, 2024.

[10] Yifeng Ding, Wesley H. Holliday, and Eric Pacuit. Characterizations of voting rules based on majority margins. arXiv:2501.08595, 2025.

[11] Mostapha Diss and Vincent Merlin, editors. *Evaluating Voting Systems with Probability Models*. Studies in Choice and Welfare. Springer International Publishing, 2021.

[12] Onur Doğan and Ayça Ebru Giritligil. Anonymous and Neutral Social Choice: Existence Results on Resoluteness. Murat Sertel Center for Advanced Economic Studies Working Paper Series:2015-01, 2015.

[13] Cynthia Dwork, Ravi Kumar, Moni Naor, and D. Sivakumar. Rank aggregation methods for the web. In *Proceedings of the 10th World Wide Web Conference*, pages 613–622, 2001.

[14] Bailey Flanigan, Daniel Halpern, and Alexandros Psomas. Smoothed Analysis of Social Choice, Revisited. In *Proceedings of WINE*, 2023.

[15] William V. Gehrlein and Peter C. Fishburn. The probability of the paradox of voting: A computable solution. *Journal of Economic Theory*, 13(1):14–25, 1976.

[16] William V. Gehrlein and Peter C. Fishburn. Coincidence probabilities for simple majority and positional voting rules. *Social Science Research*, 7(3):272–283, 1978.

[17] William V. Gehrlein and Peter C. Fishburn. Probabilities of election outcomes for large electorates. *Journal of Economic Theory*, 19(1):38–49, 1978.

[18] Wesley H. Holliday and Eric Pacuit. An extension of May's Theorem to three alternatives: axiomatizing Minimax voting. *Social Choice and Welfare*, 66:395–422, 2026. doi: 10.1007/s00355-025-01598-7.

[19] Aleksei Yu. Kondratev and Alexander S. Nesterov. Weak mutual majority criterion for voting rules. In *Proceedings of COMSOC*, 2018.

[20] Ao Liu and Lirong Xia. The Semi-Random Likelihood of Doctrinal Paradoxes. In *Proceedings of AAAI*, 2022.

[21] Tie-Yan Liu. *Learning to Rank for Information Retrieval*. Springer, 2011.

[22] François Maniquet and Philippe Mongin. A theorem on aggregating classifications. *Mathematical Social Sciences*, 79:6–10, 2016.

[23] Andrew Mao, Ariel D. Procaccia, and Yiling Chen. Better human computation through principled voting. 2013.

[24] K. May. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica*, 20:680–684, 1952.

[25] Vincent Merlin. The axiomatic characterizations of majority voting and scoring rules. *Mathématiques et sciences humaines*, 163, 2003. doi: 10.4000/msh.2919.

[26] Hervé Moulin. *The Strategy of Social Choice*. Elsevier, 1983.

[27] Roger B. Myerson. Fundamentals of Social Choice Theory. *Quarterly Journal of Political Science*, 8(3):305–337, 2013.

[28] Ali Ozkes and M. Remzi Sanver. Anonymous, Neutral, and Resolute Social Choice Revisited. *Social Choice and Welfare*, 57:97–113, 2021.

[29] Charles R. Plott. Axiomatic Social Choice Theory: An Overview and Interpretation. *American Journal of Political Science*, 20(3):511–596, 1976.

[30] Markus Schulze. A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent single-winner election method. *Social Choice and Welfare*, 36(2):267—303, 2011.

[31] Arkadii Slinko. Asymptotic strategy-proofness of the plurality and the runoff rules. *Social Choice and Welfare*, 19:313–324, 2002.

[32] Lirong Xia. The Smoothed Possibility of Social Choice. In *Proceedings of NeurIPS*, 2020.

[33] Lirong Xia. How Likely Are Large Elections Tied? In *Proceedings of ACM EC*, 2021.

[34] Lirong Xia. The Semi-Random Satisfaction of Voting Axioms. In *Proceedings of NeurIPS*, 2021.

[35] Lirong Xia. Most Equitable Voting Rules. In *Proceedings of WINE*, 2023.

[36] Lirong Xia. Semi-Random Impossibilities of Condorcet Criterion. In *Proceedings of AAAI*, 2023.

[37] Lirong Xia. Computing Most Equitable Voting Rules. In *Proceedings of WINE*, 2024.

[38] Lirong Xia and Weiqiang Zheng. The smoothed complexity of computing kemeny and slater rankings. In *Proceedings of AAAI*, 2021.

[39] Lirong Xia and Weiqiang Zheng. Beyond the Worst Case: Semi-Random Complexity Analysis of Winner Determination. In *Proceedings of WINE*, 2022.

[40] Ziqi Yu, Qishen Han, Chengkai Zhang, and Lirong Xia. How Likely Are Two Voting Rules Different? In *Proceedings of UAI*, 2025.

[41] William S. Zwicker. Introduction to the Theory of Voting. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel Procaccia, editors, *Handbook of Computational Social Choice*, chapter 2. Cambridge University Press, 2016.

NeurIPS Paper Checklist

1. **Claims**

   Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

   Answer: [Yes]

   Justification: This is a theoretical paper and the theorems, context, and significance are discussed in the abstract and introduction.

2. **Limitations**

   Question: Does the paper discuss the limitations of the work performed by the authors?

   Answer: [Yes]

   Justification: In various places, for example by explicitly formulating Assumption 1, and discussions in Future work.

3. **Theory assumptions and proofs**

   Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

   Answer: [Yes]

   Justification: The assumptions are formulated in Assumption 1. Full proofs are in the appendix while proof sketches are presented in the main text.

4. **Experimental result reproducibility**

   Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

   Answer: [N/A]

   Justification: N/A

5. **Open access to data and code**

   Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

   Answer: [N/A]

   Justification: N/A

6. **Experimental setting/details**

   Question: Does the paper specify all the training and test details (e.g., data splits, hyperparameters, how they were chosen, type of optimizer) necessary to understand the results?

   Answer: [N/A]

   Justification: N/A

7. **Experiment statistical significance**

   Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

   Answer: [N/A]

   Justification: N/A

8. **Experiments compute resources**

   Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

   Answer: [N/A]

   Justification: N/A

9. **Code of ethics**

Question: Does the research conducted in the paper conform, in every respect, with the NeurIPS Code of Ethics `https://neurips.cc/public/EthicsGuidelines`?

Answer: [Yes]

Justification: The authors read and followed the NeurIPS Code of Ethics, to the best of their knowledge.

10. **Broader impacts**

    Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

    Answer: [Yes]

    Justification: In Appendix F.

11. **Safeguards**

    Question: Does the paper describe safeguards that have been put in place for responsible release of data or models that have a high risk for misuse (e.g., pre-trained language models, image generators, or scraped datasets)?

    Answer: [N/A]

    Justification: N/A

12. **Licenses for existing assets**

    Question: Are the creators or original owners of assets (e.g., code, data, models), used in the paper, properly credited and are the license and terms of use explicitly mentioned and properly respected?

    Answer: [N/A]

    Justification: N/A

13. **New assets**

    Question: Are new assets introduced in the paper well documented and is the documentation provided alongside the assets?

    Answer: [N/A]

    Justification: N/A

14. **Crowdsourcing and research with human subjects**

    Question: For crowdsourcing experiments and research with human subjects, does the paper include the full text of instructions given to participants and screenshots, if applicable, as well as details about compensation (if any)?

    Answer: [N/A]

    Justification: N/A

15. **Institutional review board (IRB) approvals or equivalent for research with human subjects**

    Question: Does the paper describe potential risks incurred by study participants, whether such risks were disclosed to the subjects, and whether Institutional Review Board (IRB) approvals (or an equivalent approval/review based on the requirements of your country or institution) were obtained?

    Answer: [N/A]

    Justification: N/A

16. **Declaration of LLM usage**

    Question: Does the paper describe the usage of LLMs if it is an important, original, or non-standard component of the core methods in this research? Note that if the LLM is used only for writing, editing, or formatting purposes and does *not* impact the core methodology, scientific rigor, or originality of the research, declaration is not required.

    Answer: [N/A]

    Justification: LLMs assisted with manuscript revision and checking proof details, and the free NeurIPS AI review provided feedback. The research topic, all proofs, and initial writing of this paper were completely done by human.

A MFP May Fail Pareto Efficiency

Example 5. *If $\overline{r}(P)$ is closed under unanimous dominance, then* FV $* \overline{r}(P)$ *and* $\text{MPSR}_{\text{FV}} * \overline{r}(P)$ *are Pareto efficient: whenever an observed ranking is used, it ranks every unanimously dominating co-winner above the dominated one. If the MPSR is an absent ranking with unique zero multiplicity, every other ranking appears, so no unanimous dominance is possible. An arbitrary backup need not preserve this property: with two copies each of* 213*,* 231*, and* 321*, no singleton ranking exists, and* MPSR_{FA} *applied to the rule choosing all alternatives selects* 1 *under agenda* $1 \rhd 2 \rhd 3$*, although every voter prefers* 2 *to* 1*. This does not always hold for* MFP*. For example, consider $\overline{r}_{\mathcal{A}}$ that always chooses all alternatives as the co-winners. Consider m and n that satisfy Moulin [26]'s condition for the existence of ANR rules (m cannot be expressed as a sum of n's non-trivial (i.e., greater than one) divisors) but does not satisfy Moulin [26]'s condition for the existence of ANR+Pareto efficient rules (m is smaller than the smallest non-trivial divisor of n). One example is $m = 3$ and $n = 4$. In such cases* MFP_f *satisfies ANR but does not satisfy Pareto efficiency.*

B Additional Preliminaries

Weighted Majority Graphs. For any profile P and any pair of alternatives a, b, let $P[a \succ b]$ denote the total weight of votes in P where a is preferred to b. Let $\text{WMG}(P)$ denote the *weighted majority graph* of P, whose vertices are $\mathcal{A}$ and whose weight on edge $a \to b$ is $w_P(a, b) = P[a \succ b] - P[b \succ a]$.

A voting rule is said to be *weighted-majority-graph-based (WMG-based)* if its winners only depend on the WMG of the input profile. We consider the following commonly studied WMG-based rules.

- **Copeland (Cd_α).** The Copeland rule is parameterized by a number $0 \leq \alpha \leq 1$, and is therefore denoted by Copeland_α, or Cd_α for short. For any fractional profile P, an alternative a gets 1 point for each other alternative it beats in their head-to-head competition, and gets α points for each tie. Copeland_α chooses all alternatives with maximum Copeland scores as the winners.
- **Maximin (MM).** For each alternative a, its min-score is defined to be $\min_{b \in \mathcal{A} \setminus \{a\}} w_P(a, b)$. Maximin for $\mathcal{D} = \mathcal{L}_k$ chooses all alternatives with maximum min-score as the winners.
- **Ranked pairs (RP).** Given a profile P, we fix edges in $\text{WMG}(P)$ one by one in a non-increasing order w.r.t. their weights (and sometimes break ties), unless it creates a cycle with previously fixed edges. After all edges (with positive, 0, or negative weights) are considered, the fixed edges represent a linear order over $\mathcal{A}$ and the ranked top-k alternatives are chosen. We consider all tie-breaking methods to define the top alternatives, and choose of them as winners. This is known as the *parallel-universes tie-breaking (PUT)* [8].
- **Schulze (Sch).** For any directed path in the WMG, its strength is defined to be the minimum weight on any single edge along the path. For any pair of alternatives a, b, let $s[a, b]$ be the highest weight among all paths from a to b. Then, we write $a \succeq b$ if and only if $s[a, b] \geq s[b, a]$, and the strict version of this binary relation, denoted by $\succ$, is transitive [30]. The Schulze rule for $\mathcal{D} = \mathcal{L}_k$ chooses all undominated alternatives as the winners.

STV. The single transferable vote (STV) rule has $m - 1$ rounds. In each round, the alternative with the lowest plurality score is eliminated. STV chooses the last alternative as the winner, and like ranked pairs, PUT is used to decide the elimination orders.

Definition 5 (Extensions of σ and $\rhd$ to $\mathbb{H}_{m,n}$). *For any histogram $\vec{h} \in \mathbb{H}_{m,n}$, let $\sigma(\vec{h})$ be the histogram such that for every ranking R, $[\sigma(\vec{h})]_R = [\vec{h}]_{\sigma^{-1}(R)}$, where $[\sigma(\vec{h})]_R$ is the value of R-component in $\sigma(\vec{h})$.*

$\rhd$ can be naturally extended to all linear orders $\mathcal{L}(\mathcal{A})$ and then to all histograms $\mathbb{H}_{m,n}$ in a lexicographic way: when comparing two linear orders, we first compare the top-ranked alternatives in them according to $\rhd$, and if they are the same, move on to the second-ranked alternative, etc. When comparing two histograms, we compare the numbers of the linear order with the highest priority in $\rhd$ in the two histograms, and if they are the same, move on to the next linear order, etc.

C Materials for Section 3

C.1 Proof of Proposition 1

Proposition 1. *For any irresolute rule $\overline{r}$ that satisfies anonymity and neutrality, for any profile P,*

- $\neg\text{ANR}(\text{FA} * \overline{r}, P) = 1 \iff |\overline{r}(P)| \geq 2$;
- *assuming P contains $\mathcal{L}(\mathcal{A})$,* $\neg\text{ANR}(\text{FV} * \overline{r}, P) = 1 \iff |\overline{r}(P)| \geq 2$.

Proof. For $\text{FA} * \overline{r}$, suppose $\text{FA} * \overline{r}(P) = a$ and $\{a, b\} \subseteq \overline{r}(P)$. Let σ denote the permutation between a and b. Then, $\text{FA} * \overline{r}(\sigma(P)) = a = \text{FA} * \overline{r}(P)$, which means that neutrality fails at P. For $\text{FV} * \overline{r}$, suppose $\text{FV} * \overline{r}(P) = a$ and $\{a, b\} \subseteq \overline{r}(P)$. Let M denote the permutation over the votes that maps any $[b \succ a \succ \text{others}]$ vote to be the first vote (which is used to break ties under FV). Then, we have $\text{FV} * \overline{r}(M(P)) = b \neq a = \text{FV} * \overline{r}(P)$, which means that anonymity fails at P. □

C.2 Remaining Proof of Theorem 1

THEOREM 1 (**ANR failure rate under** MPSR). *For any fixed $m \geq 3$, any Π satisfying Assumption 1, any $f \in \{\text{FA}, \text{FV}\}$, and any $\overline{r} \in \{PS, STV, Cd_\alpha, MM, RP, Sch\}$,*

- *when $m = 3$,* $\widetilde{\neg\text{ANR}}(\text{MPSR}_f * \overline{r}, n) = \begin{cases} \Theta(n^{-1.5}) & \text{if } 2 \mid n \\ \Theta(n^{-2}) & \text{if } 2 \nmid n \text{ and } 3 \mid n \\ 0 & \text{otherwise} \end{cases}$
- *when $m \geq 4$,* $\widetilde{\neg\text{ANR}}(\text{MPSR}_f * \overline{r}, n) = \begin{cases} \Theta(n^{-\frac{m!}{4}}) & \text{if } 2 \mid n \\ \Theta(n^{-\frac{m!}{4}-\frac{1}{2}}) & \text{otherwise} \end{cases}$

Proof. **(continued from the main text, which proved the upper bound of the $2 \mid n$ case.)**

$2 \mid n$: The $\Omega(n^{-\frac{m!}{4}})$ lower bound for positional scoring rules. The lower bound is proved by lower-bounding the likelihood for $\text{Hist}(P)$ to be in a polyhedron $\mathcal{H}^*$ that is obtained from $\mathcal{H}^{\mathcal{T}}$ (Definition 1) for some $\mathcal{T}$, by adding linear constrains to require that in all histograms in $\mathcal{H}^*$, alternatives 1 and 2 are co-winners. Then, according to Proposition 1, the ANR failure rate is at least the likelihood that $\text{Hist}(P)$ is in $\mathcal{H}^*$. We then take the polyhedral approach to prove that it is $\Omega(n^{-\frac{m!}{4}})$. At a high-level, this matches the upper bound because by carefully choosing $\mathcal{T}$, histograms in $\mathcal{H}^{\mathcal{T}}$ already guarantee that the scores of 1 and 2 are the same. The additional linear constraints are effectively only used to guarantee that their scores are strictly higher that the scores of other alternatives, which does not decrease $\dim(\mathcal{H}^*_{\leq 0})$ (compared to $\dim(\mathcal{H}^{\mathcal{T}}_{\leq 0})$) when applying Theorem 2.

Formally, let σ denote the permutation over $\mathcal{A}$ that only exchanges 1 and 2, and let $\mathcal{T}_\sigma$ denote the partition such that for every $R \in \mathcal{L}(\mathcal{A})$, $\{R, \sigma(R)\} \in \mathcal{T}_\sigma$. This can be done because $2 \mid n$. Then, for any $a \in \mathcal{A}$, define $\vec{w}_a \in \mathbb{R}^{m!}$ to be the $m!$-dimensional vector indexed by $\mathcal{L}(\mathcal{A})$ where for every $R \in \mathcal{L}(\mathcal{A})$, the R-coordinate of $\vec{w}_a$, denoted by $[\vec{w}_a]_R$, is the score of a in R according to $\vec{s}$. Then, define $\mathcal{H}^*$ to be the polyhedron satisfying the following linear constrains

$$\begin{aligned} \forall \{R_1, R_2\} \in \mathcal{T}_\sigma, x_{R_1} - x_{R_2} = 0 &\qquad \text{from } \mathcal{H}^{\mathcal{T}_\sigma} \\ \vec{w}_1 \cdot \vec{x} - \vec{w}_2 \cdot \vec{x} = 0 &\qquad 1 \text{ and } 2 \text{ have the same score} \\ \forall a \in \mathcal{A} \setminus \{1, 2\}, \vec{w}_a \cdot \vec{x} - \vec{w}_1 \cdot \vec{x} \leq -1 &\qquad 1 \text{ is a co-winner} \end{aligned}$$

We then prove that $\dim(\mathcal{H}^*_{\leq 0}) = \frac{m!}{2}$. Because $\mathcal{H}^*_{\leq 0} \subseteq \mathcal{H}^{\mathcal{T}_\sigma}_{\leq 0}$ and the dimension of the latter is $\frac{m!}{2}$, we have $\dim(\mathcal{H}^*_{\leq 0}) \leq \frac{m!}{2}$. To prove $\dim(\mathcal{H}^*_{\leq 0}) \geq \frac{m!}{2}$, let $\vec{x} \in \mathbb{R}^{m!}$ be the vector that takes 1 on $R \in \mathcal{L}(\mathcal{A})$ if and only if 1 and 2 are the top-2 alternatives in R. Then, 1 and 2 are the only co-winners in $\vec{x}$, which means that it has an $m!$-dimensional neighborhood whose intersection with $\mathcal{H}^{\mathcal{T}_\sigma}_{\leq 0}$ is in $\mathcal{H}^*_{\leq 0}$. This means that a $\frac{m!}{2}$-dimensional neighborhood of $\vec{x}$ is contained in $\mathcal{H}^*_{\leq 0}$, which proves that $\dim(\mathcal{H}^*_{\leq 0}) \geq \frac{m!}{2}$. Finally, for every sufficiently large even number n, there exists an n-profile P

whose histogram is in $\mathcal{H}^*$. In fact, we can explicitly define P as follows: start with on copy of all linear orders where 1 and 2 are ranked at top-2, and divide the remaining votes equally between $[1 \succ 2 \succ 3 \succ \cdots \succ m]$ and $[2 \succ 1 \succ 3 \succ \cdots \succ m]$. It follows from Theorem 2 that

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^*) = \Theta(n^{-\frac{m!}{4}}),$$

which proves the $\Omega(n^{-\frac{m!}{4}})$ lower bound and concludes the $2 \mid n$ case.

$m = 3$, $2 \nmid n$, and $3 \mid n$. In this case, every partition $\mathcal{T} \in \mathbb{T}_m$ whose polyhedron is active at n must have set sizes $\{3, 3\}$: all other possible size patterns, $\{2, 2, 2\}$, $\{2, 4\}$, and $\{6\}$, force even n. This proves a $\Theta(n^{\frac{2-3!}{2}}) = \Theta(n^{-2})$ upper bound after applying Claim 1 and Theorem 2. To prove the lower bound, let $\mathcal{T}^* = \{\{123, 231, 312\}, \{321, 213, 132\}\}$. It follows that all alternatives are co-winners of every histogram $\mathcal{H}^{\mathcal{T}^*}$ under every positional scoring rule. We have $\dim(\mathcal{H}^{\mathcal{T}^*}_{\leq 0}) = 2$ and for every n with $3 \mid n$, $\mathcal{H}^{\mathcal{T}^*} \neq \emptyset$. This proves the $\Theta(n^{\frac{2-3!}{2}}) = \Theta(n^{-2})$ lower bound.

$m = 3$, the "otherwise" case ($2 \nmid n$ and $3 \nmid n$). In this case, every $\mathcal{H}^{\mathcal{T}}$ with $\mathcal{T} \in \mathbb{T}_m$ is inactive at n, which means that an MPSR always exists for all n-profiles. Therefore, $\widetilde{\neg \text{ANR}}(\text{MPSR}_f * \overline{r}, n) = 0$.

$m \geq 4$, the "otherwise" case ($2 \nmid n$). In this case, for every partition $\mathcal{T} \in \mathbb{T}_m$, at least one set in $\mathcal{T}$ must contain an odd number of linear orders. Because the sum of all sizes of sets in $\mathbb{T}_m$ is $|\mathcal{L}(\mathcal{A})|$, which is an even number, at least two sets in $\mathcal{T}$ must contain odd numbers of linear orders. Therefore, we have $|\mathcal{T}| \leq (m! - 2)/2$ and the upper bound can be achieved by the partition that contains sets of the following sizes: $(3, 3, \underbrace{2, \ldots, 2}_{(m!-6)/2})$. This proves the $\Theta(n^{\frac{(m!-2)/2-m!}{2}}) = \Theta(n^{-\frac{m!}{4}-\frac{1}{2}})$ upper bound.

To prove a matching lower bound for positional scoring rules, we first define a partition $\mathcal{T}^* \in \mathbb{T}_m$. Let $\sigma_{1 \leftrightarrow 2}$ denote the permutation that only exchanges 1 and 2. Define

$$\begin{aligned}\mathcal{T}^* =& \{\{1234 \cdots m, 2314 \cdots m, 3124 \cdots m\}, \{3214 \cdots m, 2134 \cdots m, 1324 \cdots m\}\} \\ & \cup \{\{R, \sigma_{1 \leftrightarrow 2}(R)\} : \text{ all other } R\}\end{aligned}$$

Then, we define $\mathcal{H}^*$ from $\mathcal{H}^{\mathcal{T}^*}$ by adding constraints to guarantee that 1 and 2 are co-winners that are the same as in the proof of the $2 \mid n$ case.

Overview of proof for other rules. Notice that the proof of the upper bounds for positional scoring rules is based on upper bounding the likelihood that MPSR does not exist. Therefore, all upper bounds immediately hold for all rules, including the rules in the statement of the theorem. The proofs for the matching lower bounds follow the same idea in the proof for the lower bounds under positional scoring rules: define a partition $\mathcal{T}' \in \mathbb{T}_m$ whose sizes achieve the upper bound, add linear constraints to force multiple alternatives to be co-winners, and finally apply Theorem 2 to characterize the likelihood. The hardness lies in the choice of $\mathcal{T}'$ so that the additional constraints do not decrease the dimension of the recess cone. At a high level this is achieved by choosing $\mathcal{T}'$ such that some alternatives are already "tied" but are not necessary the co-winners. Then the additional constraints are used to make them the co-winners, like in the proof for positional scoring rules.

Notice that when $m = 3, 2 \nmid n$, and $3 \mid n$ the construction for positional scoring rules already works for other rules. Also in the "otherwise" case for $m = 3$, i.e., $2 \nmid n$ and $3 \nmid n$, every such polyhedron is inactive at n, so the proof works for other rules as well. Therefore, it suffices to show the proofs for other rules for the $2 \mid n$ case and the "otherwise" case for $m \geq 4$, i.e., $2 \nmid n$.

The $2 \mid n$ case for other rules. Let σ and $\mathcal{T}_\sigma$ be the same as in the proof for the $2 \mid n$ case of positional scoring rules. For **STV**, we define $\mathcal{H}^*$ from $\mathcal{H}^{\mathcal{T}_\sigma}$ by adding constraints to guarantee that the elimination order is $m, m-1, \ldots, 3$, then 1 and 2 are tied for the last round—in fact the plurality scores of 1 and 2 are the same in all rounds. Formally, for every round $t \leq m-2$ and every alternative $i \in \mathcal{A}$, define a vector $\vec{w}_{t,i} \in \{0, 1\}^{m!}$ such that for every $R \in \mathcal{L}(\mathcal{A})$, if i is ranked at the top of R after $\{m, m-1, \ldots, m-t\}$ are removed, then $[\vec{w}_{t,i}]_R = 1$; otherwise $[\vec{w}_{t,i}]_R = 0$. Then, define $\mathcal{H}^*$ to be the polyhedron satisfying the following linear constraints

$$\begin{aligned}\forall \{R_1, R_2\} \in \mathcal{T}_\sigma, x_{R_1} - x_{R_2} =& 0 && \text{from } \mathcal{H}^{\mathcal{T}_\sigma} \\ \forall t \leq m-2, i \leq m-t, \vec{w}_{t,m-t+1} \cdot \vec{x} - \vec{w}_{t,i} \cdot \vec{x} \leq& -1 && m-t+1 \text{ is eliminated in round } t\end{aligned}$$

For **other rules**, we add constraints to guarantee that alternatives 1 and 2 are weak Condorcet winners. Formally, for any pair of different alternatives i_1, i_2, define $\vec{w}_{i_1,i_2} \in \{-1,1\}^{m!}$ such that for all $R \in \mathcal{L}(\mathcal{A})$, if $i_1 \succ_R i_2$ then $[\vec{w}_{i_1,i_2}]_R = 1$; otherwise $[\vec{w}_{i_1,i_2}]_R = -1$. Define $\mathcal{H}^*$ to be the polyhedron satisfying the following linear constrains

$$\begin{aligned} \forall \{R_1, R_2\} \in \mathcal{T}_\sigma, x_{R_1} - x_{R_2} =& 0 && \text{from } \mathcal{H}^{\mathcal{T}_\sigma} \\ \forall a \in \mathcal{A} \setminus \{1,2\}, \vec{w}_{a,1} \cdot \vec{x} \leq& -1 && \text{more voters prefer 1 to } a \end{aligned}$$

The $m \geq 4$, $2 \nmid n$ case for other rules. For **STV**, we will define $\mathcal{H}'$ to model histograms of profiles where the elimination order is $m, m-1, \ldots, 4$, and then 1 and 2 are tied to be eliminated in the next round, and then in the following round 3 is eliminated. This will guarantee that the co-winners are 1 and 2. Let σ denote the permutation that only exchanges 1 and 2. Define

$$\begin{aligned} \mathcal{T}' = \{\{1234\cdots m, 2314\cdots m, 3124\cdots m\}, &\{3214\cdots m, 2134\cdots m, 1324\cdots m\}\} \\ &\cup \{\{R, \sigma(R)\} : \text{ all other } R\} \end{aligned}$$

Then, define $\mathcal{H}'$ by adding constraints to guarantee the desired elimination order in a similar way with the proof for the $2 \mid n$ case. It follows that $\pi_{\text{uni}} \in \mathcal{H}'_{\leq 0}$ (because all alternatives are tied in every round). It is also not hard to find an interior point of $\mathcal{H}'$, which means that $\mathcal{H}'$ is active for any sufficiently large n. The tight lower bound follows after taking the polyhedral approach.

For **other rules in the statement of the theorem, i.e.,** $\overline{r} \in \{\mathbf{Cd}_\alpha, \mathbf{MM}, \mathbf{RP}, \mathbf{Sch}\}$, we define a partition $\mathcal{T}'$ as follows. For any linear order R, let Rev (R) denote its reverse order.

$$\begin{aligned} \mathcal{T}' = \{ &\underbrace{\{1234\cdots m, 2314\cdots m, 3124\cdots m\}}_{T_1}, \underbrace{\{3214\cdots m, 2134\cdots m, 1324\cdots m\}}_{T_2}, \\ &\underbrace{\{m\cdots 4123, m\cdots 4321\}}_{T_3}, \underbrace{\{m\cdots 4231, m\cdots 4132\}}_{T_4}, \underbrace{\{m\cdots 4312, m\cdots 4213\}}_{T_5}\} \\ &\cup \{\{R, \text{Rev}\,(R)\} : \text{ all other } R\} \end{aligned}$$

Then, we define $\mathcal{H}'$ from $\mathcal{H}^{\mathcal{T}'}$ by adding constraints so that the WMGs of all vectors in $\mathcal{H}'$ are like the one in Figure 1 (a), where the weights on $1 \to 2$, $2 \to 3$, and $3 \to 1$ (the dashed edges) equal to a, the weights on edges from nodes in $\{1,2,3\}$ to nodes in $\{4, \ldots, m\}$ (the solid edges) equal to b, and $b > a$.

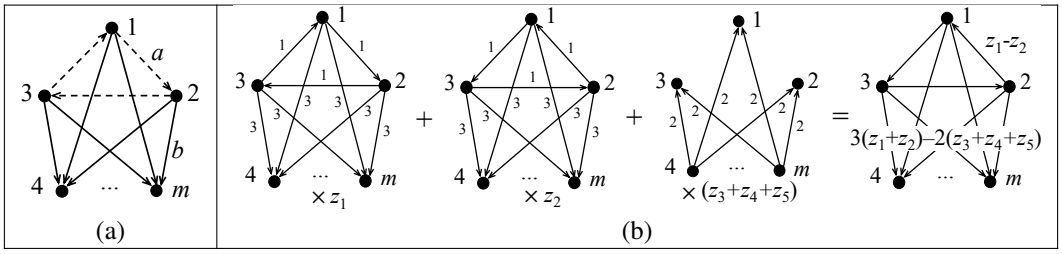


Figure 1: WMGs

More precisely, we define five linear combinations of variables, one for each T_i in $\mathcal{T}'$, to represent a and b:

$$\begin{aligned} (\text{For } T_1) :& z_1 \triangleq x_{1234\cdots m} + x_{2314\cdots m} + x_{3124\cdots m} \\ (\text{For } T_2) :& z_2 \triangleq x_{3214\cdots m} + x_{2134\cdots m} + x_{1324\cdots m} \\ (\text{For } T_3) :& z_3 \triangleq x_{m\cdots 4123} + x_{m\cdots 4321} \\ (\text{For } T_4) :& z_4 \triangleq x_{m\cdots 4231} + x_{m\cdots 4132} \\ (\text{For } T_5) :& z_5 \triangleq x_{m\cdots 4312} + x_{m\cdots 4213} \end{aligned}$$

Then, the WMG of any vector in $\mathcal{H}^{\mathcal{T}'}$ is the same as Figure 1 (b), which means that $a = z_1 - z_2$ and $b = 3(z_1 + z_2) - 2(z_3 + z_4 + z_5)$. Formally, we define $\mathcal{H}'$ as follows:

$$\begin{aligned} \forall L \in \mathcal{T}', \forall R_1, R_2 \in L, x_{R_1} - x_{R_2} = 0 \qquad & \text{from } \mathcal{H}^{\mathcal{T}'} \\ z_1 + z_2 - z_3 - z_4 - z_5 \geq 2 \qquad & b > 0 \\ 3(z_1 + z_2) - 2(z_3 + z_4 + z_5) - (z_1 - z_2) \geq 2 \qquad & b > a \end{aligned}$$

The second inequality can be simplified to $2z_1 + 4z_2 - 2z_3 - 2z_4 - 2z_5 \geq 2$. We prove the following properties about $\mathcal{H}'$.

Claim 3. *For any $m \geq 4$, $n \geq 9$, $f \in \{\text{FA}, \text{FV}\}$, and $\overline{r} \in \{MM, RP, Cd_\alpha, Sch\}$, we have*

(i) for any profile P with Hist$(P) \in \mathcal{H}' \cap \mathbb{R}^{m!}_{>0}$, $\text{ANR}(\text{MPSR}_f * \overline{r}, P) = 0$,

(ii) $\mathcal{H}^{\mathbb{Z}'}_n \neq \emptyset$,

(iii) $\pi_{uni} \in \mathcal{H}'_{\leq 0}$,

(iv) $\dim(\mathcal{H}'_{\leq 0}) = \frac{m!}{2} - 1$.

Proof. Part (i): recall that $\mathcal{H}'$ is defined from $\mathcal{H}^{\mathcal{T}'}$ with additional constraints to guarantee that the WMG of any vector in $\mathcal{H}'$ is like the one in Figure 1. Therefore, for any profile P with $\text{Hist}(P) \in \mathcal{H}' \cap \mathbb{R}^{m!}_{>0}$, there is no MPSR at P because there is no singleton ranking in P. Moreover, it is not hard to verify that for any $\overline{r}$ in the statement of the claim, $\overline{r}(P) = \{1, 2, 3\}$. When $f = \text{FV}$, $\text{MPSR}_{\text{FV}} * \overline{r}(P)$ uses the first vote in P to break the ties among $\{1, 2, 3\}$, and because $\text{Hist}(P) \in \mathbb{R}^{m!}_{>0}$, there exists a permutation μ among voters such that $\text{MPSR}_{\text{FV}} * \overline{r}(P) \neq \text{MPSR}_{\text{FV}} * \overline{r}(\mu(P))$, which means that $\text{MPSR}_{\text{FV}} * \overline{r}$ fails anonymity at P. When $f = \text{FA}$, let $\sigma_{1 \leftrightarrow 2}$ denote the permutation that exchanges 1 and 2. It follows that $\sigma_{1 \leftrightarrow 2}(P)$ has no singleton ranking and $\overline{r}(\sigma_{1 \leftrightarrow 2}(P)) = \{1, 2, 3\}$. Therefore, $\text{MPSR}_{\text{FA}} * \overline{r}(P) = \text{MPSR}_{\text{FA}} * \overline{r}(\sigma_{1 \leftrightarrow 2}(P)) = 1$, which means that $\text{MPSR}_{\text{FA}} * \overline{r}$ fails neutrality at P.

Part (ii): let

$$P \triangleq \begin{cases} \lfloor \frac{n}{3} \rfloor \times T_1 + (n - \lfloor \frac{n}{3} \rfloor \times 3)/2 \times T_3 & \text{if } 2 \mid n - \lfloor \frac{n}{3} \rfloor \times 3 \\ (\lfloor \frac{n}{3} \rfloor - 1) \times T_1 + (n - \lfloor \frac{n}{3} \rfloor \times 3 + 3)/2 \times T_3 & \text{otherwise} \end{cases}$$

It is not hard to verify that $\text{Hist}(P) \in \mathcal{H}^{\mathbb{Z}'}_n$.

Part (iii): the constraints in $\mathcal{H}^{\mathcal{T}'}$ are satisfied because all components in π_{uni} are equal. For the additional constraints, notice that under π_{uni}, we have $z_1 = z_2 = \frac{3}{m!}$ and $z_3 = z_4 = z_5 = \frac{2}{m!}$. Therefore,

$$z_1 + z_2 - z_3 - z_4 - z_5 = 3(z_1 + z_2) - 2(z_3 + z_4 + z_5) - (z_1 - z_2) = 0,$$

which proves that $\pi_{\text{uni}} \in \mathcal{H}'_{\leq 0}$.

Part (iv): it suffices to prove that there exists a vector $\vec{h} \in \mathcal{H}^{\mathcal{T}'}$ that strictly satisfies the two additional constraints. In fact, for the profile P defined in the proof for Part (ii), $\text{Hist}(P)$ strictly satisfies the two additional constraints. □

The lower bound follows after Theorem 2 and Claim 3. □

D Materials for Section 4

D.1 Proof of Claim 1

CLAIM 1. **(Properties of $\mathcal{H}^G$).** *For any permutation group $G \leqslant \mathcal{S}_{\mathcal{A}}$,*

(i) $\forall \vec{x} \in \mathbb{R}^{m!}$, $\vec{x} \in \mathcal{H}^G$ iff $\vec{x}$ is a fixed point of G;
(ii) $\dim(\mathcal{H}^G_{\leq 0}) = \frac{m!}{|G|}$;

(iii) $\forall n \in \mathbb{N}$, $\mathcal{H}^G$ *is active at* n *iff* $|G|$ *divides* n.

Proof. (i) follows after the definition of $\mathcal{H}^G$. To prove (ii), we note that $\mathbf{b}^G = \vec{0}$ and for any pair of rankings R_1, R_2, if $R_2 = \sigma(R_1)$ for some $\sigma \in G$, then $R_1 = \sigma^{-1}(R_2)$ and $\sigma^{-1} \in G$, which means that the implicit equalities of $\mathbf{A}^G$ are themselves.

Definition 6. *Let* $\equiv_G$ *denote the equivalence relationship over* $\mathcal{L}(\mathcal{A})$ *induced by orbits under* G*, i.e.,* $R_1 \equiv_G R_2$ *if and only if there exists* $g \in G$ *such that* $g(R_1) = R_2$. $\equiv_G$ *partitions* $\mathcal{L}(\mathcal{A})$ *into* $\frac{m!}{|G|}$ *orbits* $O_1, \ldots, O_{\frac{m!}{|G|}}$*, each of which has* $|G|$ *linear orders.*

In other words, for any pair of linear orders R_1, R_2, $R_1 \equiv_G R_2$ if and only if there exists $\sigma \in U$ such that $R_1 = \sigma(R_2)$. It is not hard to verify that $\vec{x} \in \mathcal{H}^G_{\leq 0}$ if and only if for every $1 \leq t \leq \frac{m!}{|G|}$ and every pair of rankings $\{R_1, R_2\} \subseteq O_t$, $[\vec{x}]_{R_1} = [\vec{x}]_{R_2}$. Therefore, $\dim(\mathcal{H}^G_{\leq 0}) = m! - \text{Rank}(\mathbf{A}^G) = \frac{m!}{|G|}$, because $\mathcal{H}^G_{\leq 0}$ is an affine space that is spanned by $\frac{m!}{|G|}$ independent vectors, one for each orbit.

For (iii), notice that each orbit O_t has the same number of rankings, i.e., $|G|$. Therefore, $|G|$ divides n. Then, let P denote the profile where there are $\frac{n}{|G|}$ copies of each ranking in O_1. It is not hard to verify that $\text{Hist}(P) \in \mathcal{H}^G$. This proves (iii) and completes the proof of Claim 1. □

D.2 Rest of the Proof for Theorem 3

Proof. **Lower bound of Theorem 3.** The $\alpha^- = -\infty$ case trivially holds. When $\alpha^- \neq -\infty$, let $\vec{m} \in \text{Ptt}_{m,n}$ denote a vector such that $\text{lcm}(\vec{m})$ is minimized among all vectors in $\text{Ptt}_{m,n}$. That is, $\vec{m} \in \arg\min_{\vec{\eta} \in \text{Ptt}_{m,n}} \text{lcm}(\vec{\eta})$. We define a permutation group $G_{\vec{m}}$ such that

$$\{P \in \mathcal{L}(\mathcal{A})^n : \text{Hist}(P) \in \mathcal{H}^{G_{\vec{m}}}\} \subseteq \mathcal{P}_{\text{lower}}$$

Then, we take the polyhedral approach to prove that

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^{G_{\vec{m}}}) = \Omega(n^{(\alpha^- - m!)/2}),$$

which is the desired lower bound. More precisely, we define the permutation group $G_{\vec{m}}$ and its fixed points as follows.

Definition 7. *Given any vector of positive integer* $\vec{m} = (m_1, \ldots, m_T) \in \mathbb{N}^T$ *such that* $\vec{m} \cdot 1 \leq m$*, define*

$$\sigma_{\vec{m}} \triangleq (\underbrace{1, \ldots, m_1}_{m_1})(\underbrace{m_1 + 1, \ldots, m_1 + m_2}_{m_2}) \cdots (\underbrace{m - m_T + 1, \ldots, m}_{m_T}),$$

$$G_{\vec{m}} \triangleq \{(\sigma_{\vec{m}})^K : K \in \mathbb{N}\}$$

Definition 8 (Fixed points). *For any* $G \leqslant \mathcal{S}_{\mathcal{A}}$*, define* $\text{FIXED}_G(\mathcal{A}) \triangleq \{a \in \mathcal{A} : \forall \sigma \in G, \sigma(a) = a\}$.

That is, $G_{\vec{m}}$ is the permutation group generated from $\sigma_{\vec{m}}$. Because for all $t \leq T$, $m_t \geq 2$ (because $\vec{m}$ consists of n's non-trivial divisors), $\text{FIXED}_{G_{\vec{m}}}(\mathcal{A}) = \emptyset$. Therefore, for any n-profile $P \in \mathcal{L}(\mathcal{A})^n$, if $\text{Hist}(P)$ is a fixed point of $G_{\vec{m}}$, then all permutations in $G_{\vec{m}}$ are $\text{Hist}(P)$'s stabilizers. This means that $G_{\vec{m}} \subseteq \text{STAB}(\text{Hist}(P))$, which means that $\text{FIXED}_{\text{STAB}(\text{Hist}(P))} = \emptyset$. Following the definition of $\mathcal{P}_{\text{lower}}$, we have $P \in \mathcal{P}_{\text{lower}}$. In other words,

$$\{P \in \mathcal{L}(\mathcal{A})^n \text{ that is a fixed point of } G_{\vec{m}}\} \subseteq \mathcal{P}_{\text{lower}}$$

Because of Claim 1 (i), it suffices to prove

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^{G_{\vec{m}}}) = \Omega(n^{(\alpha^- - m!)/2}) \tag{12}$$

Because elements in $\vec{m}$ are n's divisors, $|G_{\vec{m}}| = \text{lcm}(\vec{m})$ is a divisor of n. Therefore, by Claim 1 (iii), $\mathcal{H}^{G_{\vec{m}}}$ is active at n. Then, (12) follows after applying Theorem 2 to $\mathcal{H}^{G_{\vec{m}}}$ and noticing that $\dim(\mathcal{H}^{G_{\vec{m}}}_{\leq 0}) = \frac{m!}{\text{lcm}(\vec{m})}$ (by Claim 1 (ii)).

Because the proof is done by bounding the semi-random likelihood for the histogram of a randomly-generated profile to be in $\mathcal{P}_{\text{lower}}$ and $\mathcal{P}_{\text{upper}}$, respectively, which do not depend on the voting rule $\overline{r}$, there exist constants C_1 and C_2 as in the statement of the theorem. □

E Materials for Section 5

E.1 Proposition 2 and Proof

Proposition 2. *For any fixed m, any positional scoring rules with $s_1 > s_2 > \cdots > s_m$, STV, maximin, Copeland, ranked pairs, and Schulze satisfy block invariance and δ-unanimity for some $0 < \delta < 1$.*

Proof. The proof for block invariance follows straightforwardly after the definitions of the rules. For δ-unanimity, let $A \subseteq \mathcal{A}$ denote a set of alternatives and let P denote an n-profile where alternatives in A are ranked at the top positions in at least $(1-\delta)n$ votes.

Proof for positional scoring rules with $s_1 > \cdots > s_m$. Let η denote the smallest difference between adjacent scores, i.e., $\eta = \min_{1 \le i \le m-1}(s_i - s_{i+1})$. For any $\delta < \frac{\eta}{s_1 - s_m + \eta}$, we have $\eta(1-\delta) > \delta(s_1 - s_m)$, which means that for any $a \in A$ and any $b \in \mathcal{A} \setminus A$, the score difference between a and b is at least $\eta(1-\delta)n - \delta(s_1 - s_m)n > 0$, which implies that the rule satisfies δ-unanimity.

Proof for STV. For any $\delta < \frac{1}{m+1}$, as long as an alternative outside A remains, the last remaining alternative in A has plurality score at least $(1-\delta)n$, whereas every alternative outside A has score at most δn. Thus the last alternative in A cannot be eliminated while an outsider remains, so every possible STV winner belongs to A.

Proof for maximin. We may assume $A \neq \mathcal{A}$, and write $k = |A|$. For $\delta < 1/m$, every outsider b has a pairwise margin below $-n(m-2)/m$ against each member of A. If $k \ge 2$, some $a \in A$ is ranked highest within A in at least n/k votes, so $w_P(a, a') \ge 2n/k - n > -n(m-2)/m$ for every $a' \in A \setminus \{a\}$. Its margins against outsiders are positive, so its min-score exceeds every outsider's min-score. The case $k = 1$ follows directly from the positive margins against outsiders. Hence no outsider is a co-winner.

Proof for Copeland$_\alpha$. For any $\delta < \frac{1}{3}$, the weight on the edge in WMG from any alternative $a \in A$ to any alternative $b \in \mathcal{A} \setminus A$ is positive. Therefore, the Copeland score of a is at least $m - |A|$ and the Copeland score of b is no more than $m - |A| - 1$, which means that the winners must be a subset of A.

Proof for ranked pairs. The proof is similar to the proof for Copeland. For any $\delta < \frac{1}{3}$, the edge $a \to b$ from any $a \in A$ to any $b \in \mathcal{A} \setminus A$ has strictly positive weight in WMG. These edges will be fixed during the execution of ranked pairs, because if this is not true, then there exists $c \in \mathcal{A} \setminus A$ such that $c \to a$ is fixed earlier, which means that it has positive weight, a contradiction.

Proof for Schulze. The proof is similar to the proof for Copeland and ranked pairs. For any $\delta < \frac{1}{3}$, the edge $a \to b$ from any $a \in A$ to any $b \in \mathcal{A} \setminus A$ has strictly positive weight in WMG. Therefore, $s[a, b] > 0$. We also have $s[b, a] < 0$, because any path from b to a must contain an edge from an alternative in $\mathcal{A} \setminus A$ to an alternative in A. Therefore, $s[a, b] > s[b, a]$, which means that b is not a co-winner. □

The MFP **bound for general positional scoring rules in Table 1.** Let $\overline{r} = \overline{r}_{\vec{s}}$ be any positional scoring rule considered here. The upper bound follows from Theorem 3, including the zero case. Otherwise, put $d = m!/\alpha^+$, let G be generated by the cycle $(1, \ldots, d)$, and let $A = [d]$. Every histogram in $\mathcal{H}^G$ gives all members of A equal scores. If $d = m$, this makes every alternative a co-winner, so Claim 1 and Theorem 2 give the matching lower bound directly. If $d < m$, intersect $\mathcal{H}^G$ with the constraints that every outsider's score is at least 1 below alternative 1's score, after rescaling $\vec{s}$ to integers; call the resulting polyhedron $\mathcal{H}$.

To see that these constraints do not reduce the recession-cone dimension, let Q contain one copy of every ranking with A in the top d positions. Each member of A has average score $(s_1 + \cdots + s_d)/d$, strictly above the outsiders' average $(s_{d+1} + \cdots + s_m)/(m-d)$ because $s_1 > s_m$. Thus Hist(Q) satisfies every added homogeneous inequality strictly, and $\dim(\mathcal{H}_{\le 0}) = m!/d$. Also $\pi_{\text{uni}} \in \mathcal{H}_{\le 0} \cap$

CH(Π). The size of Q is divisible by d; after taking enough copies of Q to obtain score gaps at least 1, any sufficiently large multiple n of d is reached by adding copies of the d-ranking orbit of $1 \succ \cdots \succ m$, which cannot decrease these gaps. Hence $\mathcal{H}$ is active at every such n. Every profile represented by $\mathcal{H}$ has co-winner set A and no fixed-point co-winner under G, so MFP fails ANR there. Theorem 2 gives $\Omega(n^{(m!/d-m!)/2})$, proving the claimed $\Theta(n^{(\alpha^+-m!)/2})$ rate without assuming δ-unanimity.

E.2 Proof of Claim 2

Claim 2. *For any $G \in \mathcal{G}_m$ and any $n \geq 2$, if $\mathcal{H}^G$ is active at n, then $\alpha^- \neq -\infty$ and $|G| \geq \frac{m!}{\alpha^-}$.*

Proof. Recall that Sizes($\mathcal{A}/G$) is the vector that consists of the sizes of orbits in $\mathcal{A}$ under G. For any $G \in \mathcal{G}_m$, because each alternative is mapped to a different alternative under some permutation in G, each component of Sizes($\mathcal{A}/G$) is at least 2 and Sizes($\mathcal{A}/G$) $\cdot \vec{1} = m$. Because $\mathcal{H}^G$ is active, we have that $|G|$ is a divisor of n according to Claim 1 (iii). Therefore, Sizes($\mathcal{A}/G$) represents a way to write m as the sum of n's non-trivial divisors, which means that $\alpha^- \neq -\infty$. This also means that lcm(Sizes($\mathcal{A}/G$)) $\geq \frac{m!}{\alpha^-}$ due to the maximality of α^-. By applying the orbits-stabilizer theorem to every alternative, we have that the size of every orbit divides $|G|$, which means that lcm(Sizes($\mathcal{A}/G$)) $\leq |G|$. Therefore, for any $G \in \mathcal{G}_m$ such that $\mathcal{H}^G$ is active at n, we have $|G| \geq$ lcm(Sizes($\mathcal{A}/G$)) $\geq \frac{m!}{\alpha^-}$. □

E.3 Theorem 4 is stronger than [32, Theorem 2, upper bound] when Π satisfies Assumption 1

When Π satisfies Assumption 1, [32, Theorem 2 upper bound and Lemma 2] imply that, there exist a rule whose $\widetilde{\neg\text{ANR}}$ is $O(n^{(\frac{m!}{\ell}-m!)/2})$, where $\ell = 2$ if m is even; $\ell = 3$ if m is odd and $3 \mid m$; $\ell = 5$ if m is odd, $3 \nmid m$, and $5 \mid m$; and $\ell = 6$ for all other m. To see that Theorem 4 is stronger, first, when m cannot be represented as a sum of n's non-trivial devisors, then according to Theorem 4,

$$\widetilde{\neg\text{ANR}}(\text{MFP}_f * \overline{r}_{\mathcal{A}}, n) = 0,$$

which improves on the earlier upper bound. When $\text{Ptt}_{m,n} \neq \emptyset$, choose $\vec{m} \in \text{Ptt}_{m,n}$ minimizing lcm($\vec{m}$), so $\alpha^- = m!/\text{lcm}(\vec{m})$; we first prove lcm($\vec{m}$) $\geq \ell$ via case-by-case analysis.

- When $2 \mid m$, there must exists a component of $\vec{m}$ that is at least 2, meaning that lcm($\vec{m}$) $\geq 2 = \ell$.
- When m is odd and $3 \mid m$, there must exists an odd component of $\vec{m}$ that is strictly larger than 1, meaning that lcm($\vec{m}$) $\geq 3 = \ell$.
- When m is odd, $3 \nmid m$, and $5 \mid m$, if $\vec{m}$ contains 3, then it must contain another different component, making lcm($\vec{m}$) $\geq 2 \times 3 > 5 = \ell$. If $\vec{m}$ does not contain 3, then it must contain at least one odd component, meaning that lcm($\vec{m}$) ≥ 5.
- For all other cases of m, suppose for the sake of contradiction that lcm($\vec{m}$) < 6, then the set of components of $\vec{m}$ must belong to one of the following cases: only even numbers, only $\{3\}$, and only $\{5\}$. These correspond to $2 \mid m$, $3 \mid m$, and $5 \mid m$, respectively, which are covered in at least one of the previous case, which is a contradiction.

Then, according to Theorem 4,

$$\widetilde{\neg\text{ANR}}(\text{MFP}_f * \overline{r}_{\mathcal{A}}, n) = \Theta(n^{(\frac{m!}{\text{lcm}(\vec{m})}-m!)/2}) = O(n^{(\frac{m!}{\ell}-m!)/2}).$$

The decay exponent is strictly improved whenever $\alpha^- < m!/\ell$.

E.4 Theorem 1 is stronger than [32, Theorem 2, lower bound] when Π satisfies Assumption 1

[32, Theorem 2, lower bound] and [32, Lemma 2] imply that for any anonymous and neutral rule $\overline{r}$, after MPSR tie-breaking, an $\Omega(n^{(\frac{m!}{\ell}-m!)/2})$ lower bound holds for infinitely many n's. This is implied by the lower bound part of our Theorem 1. To see this, we let n be $\ell\times$ any prime number that is larger than m. For these infinitely many n, Theorem 1 gives $\Theta(n^{-m!/4})$ if ℓ is even, and $\Theta(n^{-m!/4-1/2})$ if ℓ is odd (with $\Theta(n^{-2})$ for $m = 3$ and $\ell = 3$). Each is $\Omega(n^{(m!/\ell-m!)/2})$, recovering the earlier lower bound for $\text{MPSR}_f * \overline{r}$.

E.5 Proof of Theorem 5

THEOREM 5. *In the same setting as Theorem 3, for any $\overline{r}$ that satisfies anonymity, neutrality, block invariance, and δ-unanimity for some constant $\delta > 0$,*

$$\widetilde{\neg\text{ANR}}(\text{MFP}_f * \overline{r}, n) = \Theta(n^{(\alpha^+ - m!)/2})$$

Proof. When $\alpha^+ = -\infty$, Theorem 3 gives zero failure. Otherwise let $d_{\min} = m!/\alpha^+$. If $d_{\min} = m$, the cyclic group $G_{(m)}$ is transitive, so every profile fixed by it has co-winner set $\mathcal{A}$ by neutrality and is problematic; Claim 1 and Theorem 2 give the required lower bound. We henceforth assume $2 \le d_{\min} < m$, so the ranking R^* below exists. For any $R \in \mathcal{L}(\mathcal{A})$, define $\text{Surplus}_R \in \mathbb{R}^{m!}$ as

$$\forall R' \in \mathcal{L}(\mathcal{A}), [\text{Surplus}_R]_{R'} \triangleq \begin{cases} 1 - \frac{1}{m!} & \text{if } R' = R \\ -\frac{1}{m!} & \text{otherwise} \end{cases}$$

It follows that for any n-profile P, $\text{Surplus}_R \cdot \text{Hist}(P) = [\text{Hist}(P)]_R - \frac{n}{m!}$, or in other words, $\text{Surplus}_R \cdot \text{Hist}(P)$ is surplus of number of R votes in P relative to the average $\frac{n}{m!}$. Notice that $\text{Surplus}_R \cdot \text{Hist}(P)$ may not be an integer, and it may be negative.

Let $(d_{\min})$ denote the one-dimensional vector that consists of a single value $d_{\min}$. Recall from Definition 7 that $G_{(d_{\min})}$ is the permutation group generated by $\sigma_{(d_{\min})}$, which is the cyclic permutation $1 \to 2 \to \cdots \to d_{\min} \to 1$. Let $\mathbb{L}_1 \subseteq \mathcal{L}(\mathcal{A})$ denote the set of rankings where $[d_{\min}]$ are ranked at the top-$|d_{\min}|$ positions; let R^* denote an arbitrary but fixed ranking that is not in $\mathbb{L}_1$; and let $\mathbb{L}_2$ denote rankings that are not in $\mathbb{L}_1$ and cannot be obtained from R^* by applying any permutation in $G_{(d_{\min})}$. Because $d_{\min} < m$, the set of rankings outside $\mathbb{L}_1$ is nonempty, so R^* is well-defined. By definition, we have

$$\mathcal{L}(\mathcal{A}) = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \{\sigma(R^*) : \sigma \in G_{(d_{\min})}\}$$

Example 6. *Let $m = 3$ and $n = 4$. Then $d_{\min} = 2$, $(d_{\min}) = (2)$, $\sigma_{(d_{\min})} = (1, 2)$, $G_{(d_{\min})} = \{Id, \sigma_{(d_{\min})}\}$. $\mathbb{L}_1 = \{123, 213\}$. As an example, we can let $R^* = 132$. Then, $\mathbb{L}_2 = \{312, 321\}$.*

We are now ready to define $\mathcal{H}^*$ using the notation introduced above. Let $\mathcal{H}^*$ be polyhedron obtained from $\mathcal{H}^{G_{(d_{\min})}}$ (Definition 3) by adding the following linear constraints.

$$\forall R \in \mathbb{L}_1 \cup \mathbb{L}_2, (\text{Surplus}_{R^*} - \text{Surplus}_R) \cdot \vec{x} \le 0 \quad (13)$$

$$\left[\frac{1-\delta}{2\delta} \times \frac{m!}{d_{\min}!} \times \sum_{R \in \mathbb{L}_2} (\text{Surplus}_R - \text{Surplus}_{R^*}) - \sum_{R \in \mathbb{L}_1} (\text{Surplus}_R - \text{Surplus}_{R^*})\right] \cdot \vec{x} \le 0 \quad (14)$$

(13) is equivalent to requiring that $[\vec{x}]_{R^*}$ is a smallest component in $\vec{x}$, which implies that $\text{Surplus}_{R^*} \cdot \vec{x} \le 0$. (14) implies that, after subtracting $[\vec{x}]_{R^*}$ from every component of $\vec{x}$, rankings in $\mathbb{L}_1$ constitute at least a $(1 - \delta)$ fraction of the remaining total weight.

Therefore, for every n-profile P^*, if $\text{Hist}(P^*) \in \mathcal{H}^*$, then $[d_{\min}] \cap \text{FIXED}_{\text{STAB}(\text{Hist}(P^*))}(\mathcal{A}) = \emptyset$, because constraints in $\mathcal{H}^{G_{(d_{\min})}}$ require that $\sigma_{(d_{\min})}$ is a stabilizer of $\text{Hist}(P^*)$, yet no alternative in $[d_{\min}]$ is a fixed point under $\sigma_{(d_{\min})}$. Moreover, block invariance and δ-unanimity imply $\overline{r}(P^*) \subseteq [d_{\min}]$ after removing $[\text{Hist}(P^*)]_{R^*}$ uniform blocks whenever the residual profile is nonempty. Nonemptiness of the co-winner set and neutrality under the cycle then give $\overline{r}(P^*) = [d_{\min}]$. If the residual profile is empty, the original histogram is uniform, so neutrality gives $\overline{r}(P^*) = \mathcal{A}$ and again no co-winner is a fixed point. This means that $P^* \in \mathcal{P}^{\overline{r},n}$. Consequently,

$$\widetilde{\neg\text{ANR}}(\text{MFP}_f * \overline{r}, n) \ge \sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^*)$$

(11) then follows after the application of Theorem 2 to $\mathcal{H}^*$, based on the following observations.

- $\mathcal{H}^*$ is active at n. To see this, we construct a profile P^* that consists of $\frac{n}{d_{\min}}$ copies of $\{\sigma^t_{(d_{\min})}(1 \succ \cdots \succ m) : 1 \le t \le d_{\min}\}$. It is not hard to verify that $\text{Hist}(P) \in \mathcal{H}^*$.
- $\text{CH}(\Pi) \cap \mathcal{H}^*_{\le 0} \neq \emptyset$. This follows after observing that $\pi_{\text{uni}} \in \text{CH}(\Pi) \cap \mathcal{H}^*_{\le 0} \neq \emptyset$ (notice that $\text{Surplus}_R \cdot \pi_{\text{uni}} = \text{Surplus}_R \cdot \vec{1} = 0$).

- $\dim(\mathcal{H}^*_{\leq 0}) = \frac{m!}{d_{\min}}$. Recall that Claim 1 (ii) implies that $\dim(\mathcal{H}^{G_{(d_{\min})}}_{\leq 0}) = \frac{m!}{|G_{(d_{\min})}|} = \frac{m!}{d_{\min}}$. To exhibit a point strict in all added inequalities, let $x_R = 1$ on $\mathbb{L}_1$, $x_R = \varepsilon$ on $\mathbb{L}_2$, and $x_R = 0$ on the orbit of R^*, for sufficiently small $\varepsilon > 0$. This vector is fixed by $G_{(d_{\min})}$ and satisfies (13) and (14) strictly. Therefore, $\dim(\mathcal{H}^*_{\leq 0}) = \dim(\mathcal{H}^{G_{(d_{\min})}}_{\leq 0}) = \frac{m!}{d_{\min}}$.

This completes the proof of Theorem 5. □

F Broader Impacts Discussions

This paper provides theoretical analysis of equity for voting. On the positive side, the results provide a quantitative comparison and can help decision makers choose better rules that meets their need of equity. On the negative side, blindly using the voting rules with lowest ANR failure may sacrifice other desirable properties. We included some discussions guidance below Table 1.