UNDERSTANDING EDGE OF STABILITY IN RANK-1 LINEAR MODELS FOR BINARY CLASSIFICATION

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ABSTRACT

Recent research in deep learning optimization reveals that many neural network architectures trained using gradient descent with practical step sizes, η, exhibit an interesting phenomenon where the top eigenvalue of the Hessian of the loss function, λ_1^H, increases to and oscillates about the stability threshold, $\frac{2}{\eta}$. The two parts of the trajectory are referred to as Progressive Sharpening and Edge of Stability. The oscillation in λ_1^H is accompanied by a non-monotonically decreasing training loss. In this work, we study the Edge of Stability phenomenon in a two-layer rank-1 linear model for the binary classification task with linearly separable data to minimize logistic loss. By capturing the core training dynamics of our model as a low-dimensional system, we rigorously prove that Edge of Stability behavior is not possible in the simplest one datapoint setting. With two datapoints, empirically we observe that, when the margin between the two points is small, Edge of Stability may occur. The oscillation in loss and sharpness becomes perpetual in the limit where the margin converges to 0. Using bifurcation and normal form analysis of the dynamical system in this asymptotic setting, we provide a sufficient condition under which the dynamical system exhibits perpetual oscillation in both loss and sharpness.

1 INTRODUCTION

To optimize modern deep neural networks, algorithms such as stochastic gradient descent (SGD) and adaptive optimizers, such as Adam (Kingma & Ba, 2017), have become the go-to choice. However, the foundation for both algorithms, gradient descent, is not fully understood. In scenarios with a small learning rate η, gradient descent is well-understood via the descent lemma (Nesterov et al.):

Lemma 1.1. *For some function $f(\theta)$, if $\lambda_{max}\left(\nabla^2 f(\theta)\right) \leq \beta$ and $\theta_{t+1} = \theta_t - \eta\nabla f(\theta_t)$, then*

$$f\left(\theta_{t+1}\right) \leq f(\theta_t) - \eta\left(1 - \frac{\eta}{2}\beta\right)||\nabla f(\theta_t)||^2$$

The descent lemma suggests that one should choose learning rate η to be close to $1/\beta$, where β is an upperbound on the "sharpness" (largest eigenvalue of the Loss Hessian). However, recently efforts to understand neural network training with gradient descent with practical learning rates have revealed some interesting behavior. In particular, work by Cohen et al. (2021) reveals that many standard neural network architectures exhibit trajectories that can be broken down into two phases (see Figure 1(b)). Phase 1, denoted as Progressive Sharpening, is the regime where the sharpness increases until it reaches past the stability threshold $\frac{2}{\eta}$. Then, it enters phase 2, denoted as Edge of Stability (EOS), where the sharpness hovers around the stability threshold $\frac{2}{\eta}$ and the loss function in question decreases non-monotonically. This common training behavior has attracted much attention from the research community and has inspired theoretical research aimed at studying simple, low dimensional models with similar behavior as well as more general characterizations for this sort of training dynamics. However, since much of the analysis on these models is done using regression-type losses, our understanding of Edge of Stability in classification losses, such as logistic loss, is still limited.

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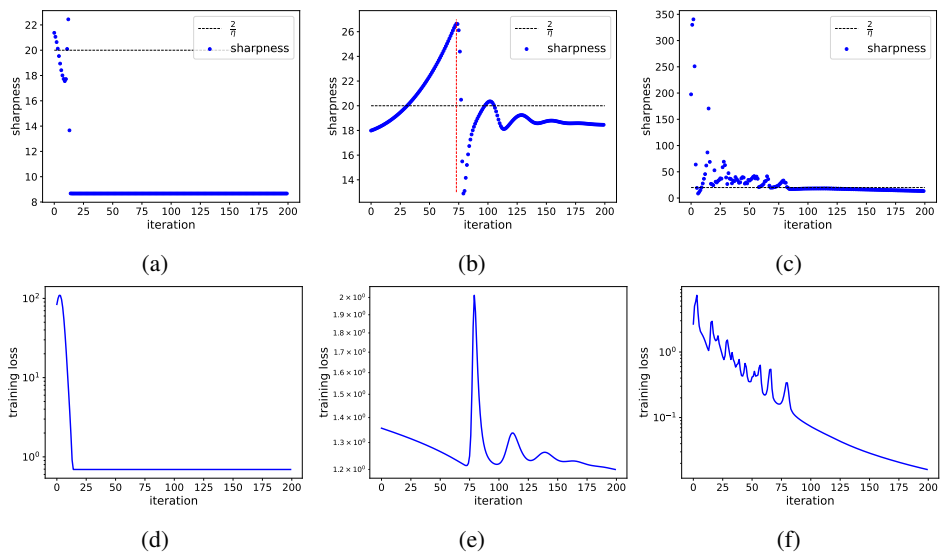


Figure 1: Sharpness and loss trajectories for 3 different models. Top row represents sharpness; bottom row represents training loss. The 3 models from left to right are our model with 1 datapoint (a)(d), 2 datapoints (b)(e) and a 4-layer feedforward ReLU network trained on a 1000 datapoint, linearly separable subset of the MNIST dataset (c)(f).

1.1 Our Contributions

In this work, we build upon the contributions of Wu et al. (2023) and Kalra et al. (2025) by studying Edge of Stability behavior in a two-layer rank-1 linear neural network trained to minimize logistic loss with gradient descent under some initialization assumptions. By distilling model dynamics to a low-dimensional system, we partition the system's phase space and prove that, under the simplest setting of one training datapoint, our model cannot exhibit Edge of Stability.

However, by adding another datapoint to our training dataset, we empirically show that our model can exhibit Edge of Stability behavior. We provide an example of the loss and sharpness dynamics of our model in both dataset settings compared to more realistic neural network in Figure 1. The behavior becomes more prominent (both loss and sharpness have more oscillations) as the two datapoints become more aligned with each other and the dataset has smaller margin.

In our analysis of the two-datapoint setting, we determine the source of the Edge of Stability behavior and explain why it does not exist in the single datapoint setting. Furthermore, we investigate an asymptotic case in the two datapoint setting, where we consider the limit as the margin for the dataset approaches 0 and prove that a sufficient condition exists under which the corresponding asymptotic dynamical system exhibits perpetual stable oscillation in both the sharpness and loss. We use bifurcation analysis to determine the asymptotic system undergoes a Neimark-Sacker Bifurcation and normal form analysis to study the loss and sharpness behavior our system using a canonical norm form, whose dynamics mirror that of our system.

Our results reveal that the condition under which Edge of Stability occurs in our model is very different from the condition for the regression setting (Kalra et al., 2025) – the parameters move between two nullclines of low-dimensional systems instead of between an unstable axis. See Appendix B.3 for detailed discussions.

2 Related Work

Oscillations in overall convergent loss during training have been observed in a number of prior works [(Lewkowycz et al., 2020), Jastrzebski et al. (2020), (Wu et al., 2018), (Xing et al., 2018), (Arora et al., 2018), (Li et al., 2022)]. Cohen et al. (2021) formalized this observation in their empirical study of many neural network architectures and training objectives, denoting it as Edge of Stability.

We first note a number of works that analyze Edge of Stability for general functions. Ma et al. (2022) use subquadratic approximations of loss functions to explain the mechanism behind Edge of Stability training dynamics. Arora et al. (2022) prove Edge of Stability behavior for normalized gradient descent on a smooth loss function with time-dependent learning rate and gradient descent on the square root of a smooth loss function with constant learning rate. Damian et al. (2023) identify the third-order term in the Taylor expansion of a loss function as a source of implicit bias for gradient descent that prevents divergence in the Edge of Stability regime. We also note research by Rosca et al. (2023) and Cohen et al. (2025) that develops continuous time tools to study the oscillatory dynamics in sharpness and loss observed with gradient descent training in the Edge of Stability regime. We note that Cohen et al. (2025) also show how their approach can be extended to adaptive optimizers, such as RMSProp.

Next, we discuss works that study Edge of Stability for particular models. We first note some works [(Wang et al., 2022), (Song & Yun, 2023), (Kalra et al., 2025)] that focus on studying two-layer linear networks in the regression setting. Wang et al. (2022) focus on studying sharpness dynamics by assuming direct correlation of the sharpness with the norm of the output layer. Song & Yun (2023) prove that the gradient descent trajectories observed to minimize a convex, 1-lipschitz, even loss align with the bifurcation diagram of a one-dimensional iterated map. Kalra et al. (2025) conduct fixed point analysis on the two-layer linear neural network and show that it can exhibit Edge of Stability behavior and period-doubling route to chaos. Besides two-layer linear networks, we also note that Zhu et al. (2023b) theoretically study the training dynamics of a 4-layer linear scalar network under a coupled initialization and prove the Edge of Stability behavior in a large local region.

For logistic loss, Wu et al. (2023) develop a new technique to study the implicit bias of logistic regression in the Edge of Stability regime and prove that it is guaranteed to converge, unlike exponential loss. Liu et al. (2023) prove the existence of the "catapult phase" in two-layer linear models on non-separable data with logistic loss. In comparison, we note that our work focuses on the separable data setting, points out the source of the Edge of Stability behavior, and studies an asymptotic case.

We defer further discussion of prior works to Appendix A.

3 Preliminaries and Notation

In this section, we introduce our two-layer linear model, some definitions, and the training dataset settings that we study.

3.1 Two-Layer Rank-1 Linear Model

A standard two-layer linear network would map $\mathbf{x}$ to $q^T M\mathbf{x}$. However, in such cases the only relevant component in M would be a rank-1 component that is aligned with q. Motivated by the results in Ji & Telgarsky (2019), where they prove the asymptotic convergence of the weight matrices in a deep linear network (trained to minimize logistic loss) to rank-1 matrices with specific alignment properties, we break M into a rank 1 matrix uv^T to have a clearer understanding of its alignment behavior.

The model we study, $f : \mathbb{R}^d \to \mathbb{R}$, is parameterized by $u \in \mathbb{R}^h$, $q \in \mathbb{R}^h$, and $v \in \mathbb{R}^d$, where $f_{q,u,v}(\mathbf{x}) = q^T uv^T\mathbf{x}$. For clarity, we denote d as the dimension of our input $\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^d$ is the feature vector and h as the width of our model's hidden linear layer, uv^T. For a binary label $y \in \{\pm 1\}$, we consider the standard logistic loss $L : \mathbb{R}^{2h+d} \to \mathbb{R}$ on our training dataset $(\mathbf{x}_i, y_i)_{i=1}^N$ using gradient descent with constant learning rate η. We use a standard way to express the logistic loss for a datapoint $(\mathbf{x}_i, y_i)$ as $L = \log(1 + \exp(-f(y_i\mathbf{x}_i))$.

At time-step t, the loss for our model is as follows:

$$L(q_t, u_t, v_t) = \sum_{i=1}^{N} \log\left(1 + \exp(-q_t^T u_t v_t^T y_i \mathbf{x}_i)\right)$$

The iterated map for our parameters u, q, and v are also shown below.

$$q_{t+1} = q_t + \eta\, u_t \sum_{i=1}^{N} \left(1 + \exp(q_t^T u_t v_t^T y_i x_i)\right)^{-1} v_t^T y_i \mathbf{x}_i,$$

$$u_{t+1} = u_t + \eta\, q_t \sum_{i=1}^{N} \left(1 + \exp(q_t^T u_t v_t^T y_i x_i)\right)^{-1} v_t^T y_i \mathbf{x}_i,$$

$$v_{t+1} = v_t + \eta\, q_t^T u_t \sum_{i=1}^{N} \left(1 + \exp(q_t^T u_t v_t^T y_i x_i)\right)^{-1} y_i \mathbf{x}_i$$

To further simplify the model, we assume that model parameters q and u start from the same initialization, similar to the setup of Zhu et al. (2023b). With this initialization assumption, we make the following claim.

Claim 3.1. *If q and u start from the same initialization (i.e., $u_0 = q_0$), then, for any time $t > 0$, $u_t = q_t$.*

We defer the proof of the claim to Appendix B.

3.2 Definitions

Now we give the formal definition for sharpness:

Definition 1 (Sharpness)**.** Consider $\theta = \begin{bmatrix} q^T & u^T & v^T \end{bmatrix}^T$. We define sharpness, λ_1^H, as the largest algebraic eigenvalue of the Hessian of loss function L, $\lambda_{max}(\nabla_\theta^2 L)$.

In order to define Edge of Stability, we rely on the property that the loss value oscillates in the Edge of Stability regime. We start by defining the local maximum of the loss.

Definition 2 (Local Maximum)**.** For a discrete-time one-dimensional dynamical system n, we say the n_t (n a time $t \in \mathbb{N}, t > 0$) is a local maximum if $n_{t-1} < n_t$ and $n_{t+1} < n_t$.

With the local maximum, we argue a necessary condition for Edge of Stability regime if the loss has multiple local maxima.

Definition 3 (Necessary Condition for Edge of Stability)**.** Consider training a neural network using gradient descent with step size η to minimize loss L. If there exists some local region during training (for discrete time-steps $k, l \in \mathbb{N}, l \geq k + 4$) where the loss L trajectory has $n \geq 2$ local maxima at times t_i for $i \in \{t_1, ..., t_n\}$ such that $k < t_1 < ... < t_n < l$, then we say the model satisfies a necessary condition for Edge of Stability behavior.

We also note that the non-monotonic behavior of the loss function we describe in our Edge of Stability definition is often accompanied by the sharpness oscillating or hovering about stability threshold $\frac{2}{\eta}$, all these behaviors have been observed in the original work of Cohen et al. (2021).

In our analysis, the main tool is to reduce the model to a lower dimensional equivalent system, and consider *nullclines* of system's components – boundaries where a component stays the same after a single iteration.

Definition 4 (Nullcline)**.** Consider an r-dimensional discrete-time dynamical system n, such that such that the iterated map for it is $n_{t+1} = f(n_t)$, where $f(n_t) = (f_1(n_t), ..., f_r(n_t))$ and $n_t = (n_{1,t}, ..., n_{r,t})$. For component $i \in \{1, .., r\}$, we define the nullcline of $n_{i,t}$ as the set of points $\{n_t\}$ such that $f_i(n_t) - n_{i,t} = 0$.

3.3 Data setup

One Training Datapoint ($N = 1$) In this setting, our training dataset $(\mathbf{x}_i, y_i)_{i=1}^N$ is simply $(\mathbf{x}, y)$, where label $y \in \{\pm 1\}$.

Two Training Datapoints ($N = 2$) In this setting, our training dataset $(\mathbf{x}_i, y_i)_{i=1}^N$ is $(\mathbf{x}_1, 1), (\mathbf{x}_2, -1)$. We focus on the following range of parameters for this setting which often leads to Edge of Stability behavior in our experiments.

1. $||\mathbf{x}_1||_2 = ||\mathbf{x}_2||_2 = 1$
2. $0.99 \leq \mathbf{x}_1^T \mathbf{x}_2 < 1$
3. For $w^* = \frac{x_1 - x_2}{\|x_1 - x_2\|}$, $w_*^T y_1 \mathbf{x}_1 = w_*^T y_2 \mathbf{x}_2 = \gamma$ is the margin.
4. $0.10 \leq \eta \leq 0.20$

3.4 Fixed Point/Direction Analysis

After constructing the low-dimensional systems for each dataset setting, we find and determine the stability of the fixed points. To find the fixed points, we first find the nullclines with respect to each component in the system. After finding the nullclines for each component, we find the points where the nulllclines for all the components of the system intersect. Those are our fixed points. If a fixed point has a non-finite component (i.e., for $n^* = (n_1^*, .., n_r^*)$, $\exists i \in \{1, .., r\}$. $n_i^* \to \infty$), we refer to it as a fixed direction. We then determine the stability properties of the fixed points/directions by forming the Jacobian of the system J and analyzing its eigenvalues λ_i^J. If all the eigenvalues at a particular fixed point satisfy $|\lambda_i^J| < 1$, then we say that fixed point/direction is stable. If all the eigenvalues at a particular fixed point satisfy $|\lambda_i^J| > 1$, then we say that fixed point/direction is unstable. Lastly, if all the eigenvalues at a particular fixed point satisfy $|\lambda_i^J| = 1$, then we say that fixed point/direction is Lyapunov stable.

3.5 Bifurcation Analysis

After determining the fixed points/directions of the low-dimensional system, we determine if/how their stability properties change as parameters of the system are varied. If a fixed point/direction changes its stability (i.e. from stable/unstable to unstable/stable) or is created/destroyed, then we say that the fixed point/direction has undergone a bifurcation. We find the Neimark-Sacker bifurcation (Kuznetsov, 1998) most relevant to our low-dimensional systems. The Neimark-Sacker bifurcation, unique to multi-dimensional discrete-time dynamical systems, is characterized by the Jacobian of the system evaluated at a fixed point/direction having complex-valued eigenvalues, existing as a conjugate pair, that cross the unit circle (i.e., the magnitude of both eigenvalues cross 1) as some parameter of the system passes some critical value. Once the parameter passes the critical value, the fixed point/direction changes its stability and an invariant curve is generated around the fixed point/direction. On this invariant curve, trajectories can be periodic or aperiodic.

4 Analysis of Gradient Descent Dynamics: One Training Datapoint

Motivated by the Edge of Stability results by Kalra et al. (2025) in the one datapoint setting, we analyze the dynamics of our model in the one datapoint setting based on the low-dimensional system that we derive. Using fixed point stability analysis and a phase space partitioning scheme for the low-dimensional system, we rigorously prove that our model does not exhibit Edge of Stability.

To capture the dynamics of our model as a low-dimensional system, we first observe that the dynamics can be completely captured by two quantities, $\|q_t\|_2^2$ and $v_t^T y\mathbf{x}$. This is because, under our initialization assumption, $u_t = q_t$, which we know from Claim 3.1. So, we get that $q_t^T u_t = \|q_t\|_2^2$. Since we are also dealing with only one datapoint, the component of v_t that is orthogonal to $\mathbf{x}$ does not affect the updates of the parameters in the model. This allows us to study the training dynamics of the model as a 2-dimensional system with variables $\|q_t\|_2^2$ and $v_t^T y\mathbf{x}$.

In our analysis, we focus on the part of the phase space where $\|q_t\|_2^2 > 0$ since $\|q_t\|_2^2 = 0$ is a fixed point. We first show that $v_t^T y\mathbf{x}$ is strictly increasing in Lemma 4.1.

Lemma 4.1. *For any $k, l \in \mathbb{N}$ such that $l > k$ and $||q_k||_2^2 > 0$, $v_l^T y\boldsymbol{x} > v_k^T y\boldsymbol{x}$.*

For $\|q_t\|_2^2$, our analysis shows two important nullclines, which are $v_t^T y\mathbf{x} = 0$ and $\left(1+\exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x} = -\frac{2}{\eta}$. The derivation of the nullclines and fixed points can be found in Appendix B.1.1. We also provide a visual representation of these nullclines in Figure 2. Using these nullclines, we partition the phase space into 3 disjoint regions, which we denote as regions 1, 2, and 3. Since $\|q_t\|_2^2 v_t^T y\mathbf{x}$ is the primary component in the loss for this setting, and we know that $\|q_t\|_2^2 v_t^T y\mathbf{x}$ grows (i.e., the loss shrinks) in the next time step if the system is in regions 1 and 2 according to corollaries 4.1.1 and 4.1.2, we further divide region 3 into regions 3.1 and 3.2. We define regions 3.1 and 3.2 such that $\|q_t\|_2^2 v_t^T y\mathbf{x}$ grows and shrinks in the next time step, respectively.

Corollary 4.1.1. *For any $k, l \in \mathbb{N}$ such that $l > k$, if $||q_k||_2^2 > 0, v_k^T y\boldsymbol{x} \geq 0$, then $||q_l||_2^2 v_l^T y\boldsymbol{x} > ||q_k||_2^2 v_k^T y\boldsymbol{x}$.*

Corollary 4.1.2. *For any $k \in \mathbb{N}$, if $||q_k||_2^2 > 0, v_k^T y\boldsymbol{x}$ satisfies*

$-\frac{2}{\eta} < \left(1+\exp\left(||q_k||_2^2 v_k^T y\boldsymbol{x}\right)\right)^{-1} v_k^T y\boldsymbol{x} < 0$, *then $||q_{k+1}||_2^2 v_{k+1}^T y\boldsymbol{x} > ||q_k||_2^2 v_k^T y\boldsymbol{x}$.*

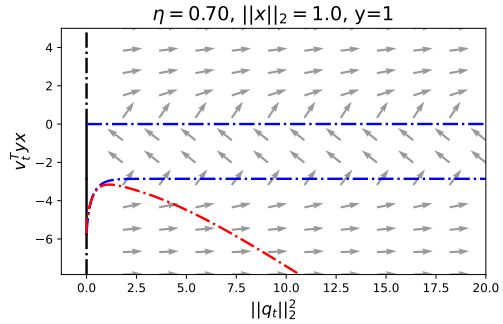


Figure 2: This is the phase space diagram for our one datapoint system for $\eta = 0.70$, $\|\mathbf{x}\|_2^2 = 1$, and $y = 1$. The blue dashed curves represent the nullclines for $\|q_t\|_2^2$, where region 1 is above the top blue curve, region 2 is between both blue curves, and regions 3.1 and 3.2 are below the bottom blue curve. The red dashed curve represents the boundary where $\|q_t\|_2^2 v_t^T y\mathbf{x}$ shrinks inside the curve (region 3.2) and grows outside the curve (region 3.1). Since $\|q_t\|_2^2 = 0$ is also a fixed point of the system, we denote the solid black line portion as $\|q_t\|_2^2 = 0$ being stable, and the dashed black line portion as $\|q_t\|_2^2 = 0$ being unstable. We also include normalized vectors of the gradient field, shown as grey arrows, to show possible trajectories.

By analyzing the dynamics of our system in each region, we prove the following theorem.

Theorem 4.2 (No Edge of Stability in the One Datapoint Setting)**.** *Consider our two-layer rank-1 linear model trained to minimize logistic loss on a single datapoint $(\boldsymbol{x}, y)$, where $y = \pm 1$, using gradient descent with learning rate η. Under any initialization such that $\|q_0\|_2^2 > 0$, our loss trajectory will contain at most one local maximum.*

We defer the proof of the lemma, corollaries, and theorem in this section to Appendix B.1.3. Note that, in the proof of the theorem, we show that trajectory iterates cannot oscillate about region 3.2 (i.e., no multiple loss oscillations) due the monotonic behavior of $v_k^T y\mathbf{x}$.

5 Analysis of Gradient Descent Dynamics: Two Training Datapoints

In this section, we study the dynamics of our model in the two-datapoint setting based on a low-dimensional system that we derive. Based on our fixed point stability analysis, we first empirically show that there exists a trajectory where our model displays Edge of Stability behavior. We also provide intuition as to why we can observe Edge of Stability with two training datapoints and not with one. Next, we study an asymptotic case for the two datapoint setting, where $\mathbf{x}_1^T \mathbf{x}_2 \to 1$. We provide theoretical results on the source of the empirically observed perpetual oscillation in loss and sharpness and a sufficient condition that exhibits this perpetual oscillation.

In the two datapoint setting, we note that the dynamics can be captured by three quantities, $\|q_t\|_2^2$, $v_t^T(\mathbf{x}_1 - \mathbf{x}_2)$, and $v_t^T(\mathbf{x}_1 + \mathbf{x}_2)$. As in the single datapoint case, with our initialization, we know that $u_t = q_t$ from Claim 3.1. So, $q_t^T u_t = \|q_t\|_2^2$. However, since we have two datapoints, we observe the updates to the parameters of our model are affected by $v_t^T\mathbf{x}_1$ and $v_t^T\mathbf{x}_2$. However, we find $v_t^T(\mathbf{x}_1 - \mathbf{x}_2)$ and $v_t^T(\mathbf{x}_1 + \mathbf{x}_2)$ are more meaningful than $v_t^T\mathbf{x}_1$ and $v_t^T\mathbf{x}_2$, since $v_t^T(\mathbf{x}_1 - \mathbf{x}_2)$ and $v_t^T(\mathbf{x}_1 + \mathbf{x}_2)$ represent the alignment of parameter v_t in the max-margin $(\mathbf{x}_1 - \mathbf{x}_2)$ and max-margin complement $(\mathbf{x}_1 + \mathbf{x}_2)$ directions, respectively. Furthermore, we can reconstruct $v_t^T\mathbf{x}_1$ and $v_t^T\mathbf{x}_2$ from $v_t^T(\mathbf{x}_1 - \mathbf{x}_2)$ and $v_t^T(\mathbf{x}_1 + \mathbf{x}_2)$. For conciseness, we define $m_t = v_t^T(\mathbf{x}_1 - \mathbf{x}_2)$ and $c_t = v_t^T(\mathbf{x}_1 + \mathbf{x}_2)$. So, we can study our model's training dynamics as the 3-dimensional system $\left(\|q_t\|_2^2, c_t, m_t\right)$. Similar to the single datapoint case, we find that $\|q_t\|_2^2 = 0$ is a fixed point of our system, so we focus on the system's behavior for $\|q_t\|_2^2 > 0$. And, similar to $v_t^T y\mathbf{x}$ in the single datapoint setting, we prove that m_t is a strictly increasing component in Claim 5.1.

Claim 5.1. *For any $k, l \in \mathbb{N}$ such that $l > k$ and $||q_k||_2^2 > 0$, $m_l > m_k$.*

We defer the proof of this claim to Appendix B.2.

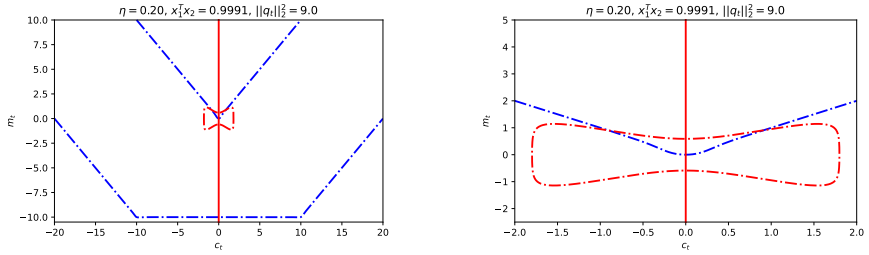
From our analysis of $\|q\|_2^2$, we consider Equation 1 to determine the nullclines of $\|q\|_2^2$.

$$\left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m_t + c_t\right)\right)\right)^{-1}\left(m_t + c_t\right) + \left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m_t - c_t\right)\right)\right)^{-1}\left(m_t - c_t\right) \quad (1)$$

The two nullclines for $\|q\|_2^2$ are the boundaries defined by Equation 1 equals b for $b \in \left\{0, -\frac{4}{\eta}\right\}$. For convenience, we denote $\|q_t\|_2^2$ nullcline 1 for $b = 0$ and $\|q_t\|_2^2$ nullcline 2 for $b = -\frac{4}{\eta}$. However, unlike the single datapoint setting, we have the additional component, c_t. In our analysis, we find that the system can move away from the $c_t = 0$ plane for $\|q_t\|_2^2 > \sqrt{\frac{8}{\eta\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)}}$ and move towards the $c_t = 0$ plane otherwise. To understand how "far" this repelling behavior reaches, we consider Equation 2.

$$|c_t| - \tfrac{\eta\|q_t\|_2^2\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)}{2}\left[\left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m_t - |c_t|\right)\right)\right)^{-1} - \left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m_t + |c_t|\right)\right)\right)^{-1}\right] \quad (2)$$

If the system is in the interior of the boundary defined where Equation 2 equals 0, then $|c_t|$ grows. We provide visuals for $\|q_t\|_2^2$ nullcline 1 and 2 and the $|c_t|$ growth boundary in Figure 3. The derivation for the nullclines, $|c_t|$ growth boundary, and fixed points can be found in Appendix B.2.1.

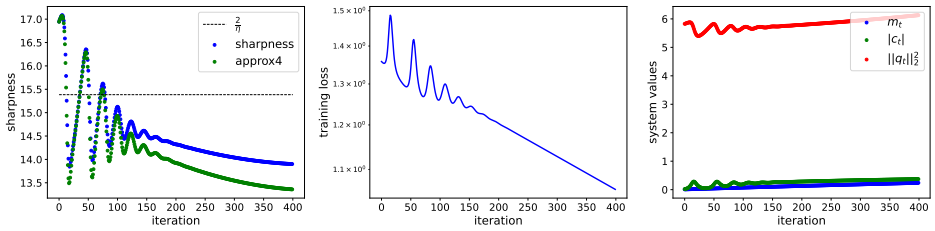


(a) (b)

Figure 3: For a slice of of the 3-dimensional phase at $\|q_t\|_2^2 = 9.0$ for the case where $\eta = 0.20$ and $\mathbf{x}_1^T\mathbf{x}_2 = 0.9991$, we show the nullclines of $\|q_t\|$ and the $|c_t|$ growth boundary. In (a), the blue dashed lines represent the nullclines of $\|q_t\|_2^2$, where the bottom blue curve is $\|q_t\|_2^2$ nullcline 2 (Equation 1 equals $-\frac{4}{\eta}$) and the blue curve above it is $\|q_t\|_2^2$ nullcline 1 (Equation 1 equals 0). In (b), we provide zoomed-in view for the red dashed curve shown in (a) that represents the $|c_t|$ growth boundary (Equation 2 equals 0).

5.1 Edge of Stability Trajectory

Compared to the single datapoint case, where we proved Edge of Stability was not possible, we empirically show Edge of Stability behavior in the two datapoint setting, as seen in Figure 4. We also explain how the trajectory taken by the $\|q_t\|_2^2, c_t, m_t$ system in Figure 4 produces Edge of Stability behavior.

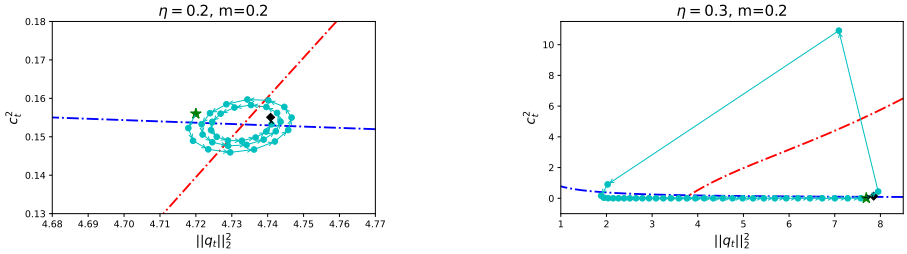


(a) (b) (c)

Figure 4: We show the sharpness (a), training loss (b), system values (c) for $\eta = 0.13$, $\mathbf{x}_1^T\mathbf{x}_2 = 0.9991$, $\|q_0\|_2^2 = 1.05\sqrt{\frac{8}{0.13 \cdot 1.9991}}$, $c_0 = 0.02$, $m_0 = 0.01$. We also show the similarity between our sharpness approximation and the true sharpness.

We attribute the Edge of Stability behavior observed above to c_t. Compared to our single datapoint setting, the c_t component in the two-datapoint setting provides a means for the system to oscillate between $\|q_t\|_2^2$ nullcline 1 (i.e., nC_q1 changes signs), causing oscillations in $\|q_t\|_2^2$, which causes oscillations in the loss. For intuition on how the trajectory taken by the system in Figure 4 exhibits Edge of Stability, please refer to Appendix B.2.3. We provide additional experiments for the two datapoint setting in Appendix B.4.

5.2 Asymptotic Case ($\mathbf{x}_1^T\mathbf{x}_2 \to 1$)



(a) (b)

Figure 5: For the asymptotic case at $m = 0.20$, we show the first 35 steps of a trajectory starting within the vicinity of the fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$, lying at the intersection of $\|q_t\|_2^2$ nullcline 1 (blue curve) and c_t^2 growth boundary (red curve). We denote the starting point as the green star, the 35th step as the black diamond, with cyan arrows pointing to subsequent iterates and a black arrow from iterate 34 to 35. In (a), we show that, for $\eta = 0.2$, the trajectory seems to be converging to $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$. In (b), we show that, for $\eta = 0.3$, the trajectory seems to be orbiting around $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$.

We further study an asymptotic case, where we consider the limit as $\mathbf{x}_1^T\mathbf{x}_2 \to 1$ (or equivalently, the dataset margin approaches 0). From simulations conducted for the asymptotic case, we find multiple instances where the asymptotic low-dimensional system remains localized, jumping between a specific set of values to produce a trajectory "band" and seemingly perpetual oscillations in loss and sharpness. Results from the simulations can be found in Appendix B.5.6. In this subsection, we provide some explanation for how perpertual oscillation in loss and sharpness could be possible.

We first consider how our 3-dimensional system for the two-datapoint setting reduces to 2 dimensions in terms of $\|q_t\|_2^2, c_t$ for the asymptotic case. In the limit $\mathbf{x}_1^T\mathbf{x}_2 \to 1$, m_t becomes a constant based on initialization, which we treat as parameter m of the system. We focus on a reparameterized version of the $\|q_t\|_2^2, c_t$ dynamical system, $\|q_t\|_2^2, c_t^2$. In the reparameterized system, we consider the parameter range $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$ and focus on the intersection of $\|q_t\|_2^2$ nullcline 1 (defined where Equation 3 equals 0) and the c_t^2 growth boundary (defined where Equation 4 equals 0), which we denote as $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$.

$$\left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m + \sqrt{c_t^2}\right)\right)\right)^{-1}\left(m + \sqrt{c_t^2}\right) + \left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m - \sqrt{c_t^2}\right)\right)\right)^{-1}\left(m - \sqrt{c_t^2}\right) \quad (3)$$

$$\sqrt{c_t^2} - \eta\|q_t\|_2^2 \left[\left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2 \left(m - \sqrt{c_t^2}\right)\right)\right)^{-1} - \left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2 \left(m + \sqrt{c_t^2}\right)\right)\right)^{-1} \right] \quad (4)$$

We know $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$ exists for $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$ and is a fixed point for our reparameterized asymptotic system by the following theorem:

Theorem 5.2. *For asymptotic dynamical system defined by Equations 5 and 6 and parameters $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$, the dynamical system has fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$, such that $\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944 < \|\tilde{q}\|_2^2 < \frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15$ and $2m\eta + 0.25m + 0.02 < \tilde{c}^2 < 2m\eta + 0.25m + 0.025$, that lies at an intersection of $\|q_t\|_2^2$ nullcline 1 and the c_t^2 growth boundary.*

We defer the proof of this theorem to Appendix B.5.1.

From our simulations on the asymptotic system for the specified parameter range, we observe a bifurcation occur at the fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$. As we increase η, we observe that $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$ becomes unstable, which is accompanied by trajectories initialized close to the fixed point orbiting around it, strongly suggesting the occurrence of a Neimark-Sacker Bifurcation. We illustrate this transition in Figure 5 for $m = 0.20$ and $\eta = 0.2, 0.3$.

Motivated by our empirical results, we prove for parameter range $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$ that the eigenvalues of the asymptotic system's Jacobian are complex-valued and cross the unit-circle in the following theorems, which is a defining characteristic of the Niemark-Sacker Bifurcation.

Theorem 5.3. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$, the eigenvalues of the Jacobian of the asymptotic dynamical system defined by Equations 5 and 6 evaluated at fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$ are complex-valued and cross the unit-circle at a unique $\eta_{c,m} \in [0.2, 0.3]$.*

We defer the proof of the above theorem to Appendices B.5.2 and B.5.3.

Furthermore, under small perturbation from $\eta_{c,m}$ such that an invariant curve projects from the fixed point by the Neimark-Sacker Bifurcation, we show that a trajectory initialized sufficiently close to the fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$ exhibits perpetual oscillation in loss and sharpness in the theorem below.

Theorem 5.4 (Informal version of Theorem B.148)**.** *Consider some $m \in [0.15, 0.25]$ and the corresponding critical η value, $\eta_{c,m} \in [0.2, 0.3]$, where the Neimark-Sacker Bifurcation occurs. For $\epsilon > 0$ sufficiently small, where $\eta = \eta_{c,m} + \epsilon$, and intialization sufficiently close to the fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$, the trajectory produced exhibits perpetual oscillation in loss and sharpness.*

To prove the sufficient condition, note that a dynamical system undergoing a Neimark-Sacker Bifurcation can be converted into a canonical normal if it satisfies certain non-resonance, criticality, and transversality conditions (Kuznetsov, 1998). We study the loss and sharpness behavior exhibited by the normal form, since its dynamics mirror that of our asymptotic system. Further details regarding the formal version of the theorem and the corresponding proof can be found in Appendix B.5.4.

6 Conclusion and Future Directions

In this work, we study the behavior of a two-layer rank-1 linear neural network to minimize logistic loss on linearly separable data. In this setting, we prove that with one datapoint the model does not display Edge of Stability behavior. We also show that with two datapoints, our model exhibits Edge of Stability. Furthermore, we extend our two datapoint setting to an asymptotic case where the margin approaches 0 and provide evidence that supports the possibility for perpetual stable oscillation in the loss and sharpness. We prove that in the asymptotic setting there exists parameter settings that incurs a Neimark-Sacker Bifurcation and use related techniques to prove the trajectory will exhibit perpetual oscillation in loss and sharpness.

The main open problem is to generalize our asymptotic two datapoint result to more general settings, such as two general datapoints or multiple linearly separable datapoints. We hope our analysis, together with the approximation of sharpness in Appendix B.2.2, provides a starting point for understanding Edge of Stability behavior in logistic losses for more complicated models.

AI USE STATEMENT

In this work, we used we used a Large Language Model to provide critical ingredients for proving mathematical claims. Specifically, we used a language model to assist in the derivations of our sharpness approximations to help us realize how similarity transforms could be applied to the Loss Hessian and the impact of the rank-2 perturbation in the Loss Hessian for the two datapoint setting. We have not used generative AI tools to formulate mathematical claims, help develop theoretical models or conceptual frameworks, assist in the writing of proofs, propose or refine hypotheses, design or provide feedback on research methodology or experiments, implement methods, or interpret results. The other required disclose tasks generate synthetic data sets, assist with translation, clean and reformat dataset, support qualitative and thematic data analysis are not applicable to this work. Additionally, we used generative AI tools for sourcing/searching for information. Specifically, we used a Large Language Model to look for resources related to eigenvalue approximation under perturbation, where the model recommended a book (Stewart & Sun, 1990) from which we learned about Cauchy Interlacing Theorem. We have reviewed all AI-assisted work. We reviewed and verified the similarity transforms proposed by the model for the derivation of our sharpness approximations. We take responsibility for the final content of this work, including text, claims or artifacts produced with the aid of generative AI.

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A Additional Related Works

We continue our discussion of related works in this section. First, we discuss some additional works on general functions. Kong & Tao (2020) show that, under large learning rates and certain loss functions, gradient descent exhibits chaotic behavior leading it to converge to a distribution rather than a local minimizer. Lyu et al. (2022) show that a model with some form of normalization (batch normalization, layer normalization, etc.) and weight decay can enter the Edge of Stability regime, where the implicit bias induced drives the model to reduce sharpness. Ahn et al. (2022) discuss the causes and main features of unstable convergence (Edge of Stability) based on the evolution of gradient descent iterates, loss, and sharpness. Kreisler et al. (2023) identify the gradient-flow sharpness (GFS) as a monotonically decreasing quantity in gradient descent training and study how, in scalar neural networks, the decrease of the GFS below the stability threshold causes the loss to decrease at an exponential rate. Wang et al. (2023) attribute the occurrence of non-trivial training dynamics, such as Edge of Stability, to loss functions that satisfy certain regularity conditions along with gradient descent and sufficiently large learning rates. Bartlett et al. (2023) study sharpness-aware minimization (SAM) for convex quadratic and non-quadratic objectives. In the latter case, they show that the oscillations observed during training are essentially gradient descent steps on the spectral norm of the Hessian, where SAM represents the derivative of the sharpness. Cohen et al. show that the Edge of Stability phenomenon can carry over to adaptive optimizers, such as Adam, by considering the maximum eigenvalue of a preconditioned Loss Hessian. Long & Bartlett (2024) derive the Edge of Stability threshold for sharpness-aware minimization (SAM) and experimentally observe SAM operating in the Edge of Stability regime. Chen et al. (2024a) prove that, for twice-differentiable loss functions, gradient descent is forward-invariant. The authors also show that for initialization outside the forward-invariant set will oscillate for several iterations until entering the forward invariant set containing a local minimum.

Next, we continue the discussion of works focused on specific models. Pedregosa et al. (2022) show that quadratic regression models trained to minimize a quartic loss function exhibit both Progressive Sharpening and Edge of Stability. Chen & Bruna (2023) use two-step gradient updates to analyze matrix factorization and single-neuron ReLU networks for a range of learning rates where gradient descent dynamics hover around minimizers. Ahn et al. (2023) show that a 2-layer ReLU network can exhibit Edge of Stability behavior and prove that, with learning rates not sufficiently large enough, the network can fail to learn a nonzero first-layer bias, which is essential for useful implicit biases that can lead to better generalization. Even et al. (2023) study the implicit bias of 2-layer diagonal linear neural networks trained with gradient descent (GD) and stochastic gradient descent (SGD) using large step sizes. They show that the implicit bias discrepancy between SGD and GD is amplified with learning rates that induce Edge of Stability behavior. Noci et al. (2024) study sharpness dynamics in the context of learning rate transfer for neural networks in the rich regime and in the Neural Tangent Kernel to indicate feature learning. Zhu et al. (2023a) show that quadratic models used the approximate shallow feedforward ReLU networks exhibits a "catapult phase" when trained with sufficiently large learning rates and demonstrate improved generalization following the "catapult phase". Chen et al. (2024b) show that phase retrieval and a cubic iterated map representing a two-layer neural network with quadratic activations, fixed output layer, and orthogonal data exhibit five phases of training, including Edge of Stability. Liu et al. (2025) examine a minimalist model consisting of a two-layer neural network with two-dimensional input and connect their model to prior works related to more general phenomenon, such as the existence of a well-behaved "stable set" and gradient flow sharpness Kreisler et al. (2023).

B Model Analysis

In this section, we provide additional information for our study of the single and two datapoint settings including sharpness derivations, fixed point analysis, and additional experiment results.

First, we prove Claim 3.1, which is essential for our analysis in both dataset settings.

Proof of Claim 3.1. Assume $u_0 = q_0$. We prove the claim by induction. First, we prove the base case for u_1 and q_1.

$$u_1 - q_1 = u_0 + \eta\, q_0 \sum_{i=1}^{N} \left(1 + \exp(q_0^T u_0 v_0^T y_i \mathbf{x}_i)\right)^{-1} v_0^T y_i \mathbf{x}_i$$
$$- \left(q_0 + \eta\, u_0 \sum_{i=1}^{N} \left(1 + \exp(q_0^T u_0 v_0^T y_i \mathbf{x}_i)\right)^{-1} v_0^T y_i \mathbf{x}_i \right)$$

$$= u_0 - q_0 - \eta\,(u_0 - q_0) \sum_{i=1}^{N} \left(1 + \exp(q_0^T u_0 v_0^T y_i \mathbf{x}_i)\right)^{-1} v_0^T y_i \mathbf{x}_i$$
$$= 0 \qquad \text{since } u_0 = q_0$$

Next, we prove the inductive step. Assume $u_k = q_k$. Then, for u_{k+1} and q_{k+1}, we see the following.

$$u_{k+1} - q_{k+1} = u_k + \eta\, q_k \sum_{i=1}^{N} \left(1 + \exp(q_k^T u_k v_k^T y_i \mathbf{x}_i)\right)^{-1} v_k^T y_i \mathbf{x}_i$$
$$- \left(q_k + \eta\, u_k \sum_{i=1}^{N} \left(1 + \exp(q_k^T u_k v_k^T y_i \mathbf{x}_i)\right)^{-1} v_k^T y_i \mathbf{x}_i \right)$$

$$= u_k - q_k - \eta\,(u_k - q_k) \sum_{i=1}^{N} \left(1 + \exp(q_k^T u_k v_k^T y_i \mathbf{x}_i)\right)^{-1} v_k^T y_i \mathbf{x}_i$$
$$= 0 \qquad \text{since } u_k = q_k$$

□

B.1 Single Datapoint Setting ($N = 1$)

B.1.1 Fixed Point/Direction Analysis

Recall from Section 4 that can study the dynamics of our model in the single datapoint setting as the system, $(\|q_t\|_2^2, v_t^T y\mathbf{x})$. In this section, we derive the fixed points/directions for our system, $(\|q_t\|_2^2, v_t^T y\mathbf{x})$. We make the following claims.

Claim B.1 (Nullclines of $\|q_t\|_2^2$)**.** *The nullclines of $\|q_t\|_2^2$ are as follows: $\|q_t\|_2^2 = 0$, $v_t^T y\boldsymbol{x} = 0$, $\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\boldsymbol{x}\right)\right)^{-1} v_t^T y\boldsymbol{x} = -\frac{2}{\eta}$, and $\|q_t\|_2^2 v_t^T y\boldsymbol{x} \to \infty$.*

Claim B.2 (Fixed points/directions for $(\|q_t\|_2^2, v_t^T y\mathbf{x})$ system)**.** *The fixed points/directions for the system of two variables, $\|q_t\|_2^2$ and $v_t^T y\boldsymbol{x}$ are $\|q_t\|_2^2 v_t^T y\boldsymbol{x} \to \infty$ and $\|q_t\|_2^2 = 0$, where $\|q_t\|_2^2 v_t^T y\boldsymbol{x} \to \infty$ is Lyapunov stable and $\|q_t\|_2^2 = 0$ is stable only if $\frac{-4}{\eta} < v_t^T y\boldsymbol{x} < 0$.*

First, we prove Claim B.1.

Proof of Claim B.1. First, we define the iterated map of $\|q_t\|_2^2$.

$$\|q_{t+1}\|_2^2 = \|q_t\|_2^2 + 2\eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} \|q_t\|_2^2 v_t^T y\mathbf{x} + \eta^2 \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-2} \|q_t\|_2^2 \left(v_t^T y\mathbf{x}\right)^2$$
$$= \|q_t\|_2^2 \left(1 + \eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x}\right)^2$$

To solve for the nullclines, we solve for $\|q_{t+1}\|_2^2 - \|q_t\|_2^2 = 0$, which is equivalent to solving $\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} \left(2 + \eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x}\right) \|q_t\|_2^2 v_t^T y\mathbf{x} = 0$. From the factors, we see that the nullclines are $\|q_t\|_2^2 = 0$, $v_t^T y\mathbf{x} = 0$, $\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x} = -\frac{2}{\eta}$, and $\|q_t\|_2^2 v_t^T y\mathbf{x} \to \infty$ to get $\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} = 0$. □

Next, we proceed with the fixed point/direction stability analysis.

Proof of Claim B.2. From Claim B.1, we already have the nullclines for $\|q_t\|_2^2$. So, we find the nullclines of $v_t^T y\mathbf{x}$. First, we define the iterated map for $v_t^T y\mathbf{x}$.

$$v_{t+1}^T y\mathbf{x} = v_t^T y\mathbf{x} + \eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} \|q_t\|_2^2 \|y\mathbf{x}\|_2^2$$

We solve for the nullclines as follows.

$$\begin{aligned} 0 = v_{t+1}^T y\mathbf{x} - v_t^T y\mathbf{x} &= \eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} \|q_t\|_2^2 \|y\mathbf{x}\|_2^2 \\ 0 &= \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} \|q_t\|_2^2 \qquad \text{since } \|y\mathbf{x}\|_2^2 \text{ is a constant} \end{aligned}$$

So, we get that the nullclines are $\|q_t\|_2^2 = 0$ and $\|q_t\|_2^2 v_t^T y\mathbf{x} \to \infty$. Then, we see that the intersection of the nullclines for $\|q_t\|_2^2$ and $v_t^T y\mathbf{x}$ shows that the fixed points/directions are $\|q_t\|_2^2 = 0$ and $\|q_t\|_2^2 v_t^T y\mathbf{x} \to \infty$.

To analyze the stability properties of the fixed points/directions, we find the eigenvalues for the Jacobian of the system $(\|q_t\|_2^2, v_t^T y\mathbf{x})$. First, we define the Jacobian as follows.

$$J(\|q_{t+1}\|_2^2, v_{t+1}^T y\mathbf{x}) = \begin{bmatrix} \frac{\partial \|q_{t+1}\|_2^2}{\partial \|q_t\|_2^2} & \frac{\partial \|q_{t+1}\|_2^2}{\partial v_t^T y\mathbf{x}} \\ \frac{\partial v_{t+1}^T y\mathbf{x}}{\partial \|q_t\|_2^2} & \frac{\partial v_{t+1}^T y\mathbf{x}}{\partial v_t^T y\mathbf{x}} \end{bmatrix}$$

Then, we solve for the partial derivatives.

$$\frac{\partial v_{t+1}^T y\mathbf{x}}{\partial v_t^T y\mathbf{x}} = 1 - \eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-2} \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right) \|q_t\|_2^4 \|y\mathbf{x}\|_2^2$$

$$\frac{\partial v_{t+1}^T y\mathbf{x}}{\partial \|q_t\|_2^2} = \eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} \|y\mathbf{x}\|_2^2 - \eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-2} \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right) \|q_t\|_2^2 v_t^T y\mathbf{x} \|y\mathbf{x}\|_2^2$$

$$\begin{aligned} \frac{\partial \|q_{t+1}\|_2^2}{\partial v_t^T y\mathbf{x}} &= 2\eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} \|q_t\|_2^2 + 2\eta^2 \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-2} \|q_t\|_2^2 v_t^T y\mathbf{x} \\ &\quad - 2\eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-2} \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right) \|q_t\|_2^4 v_t^T y\mathbf{x} \\ &\quad - 2\eta^2 \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-3} \exp(\|q_t\|_2^2 v_t^T y\mathbf{x}) \|q_t\|_2^4 \left(v_t^T y\mathbf{x}\right)^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial \|q_{t+1}\|_2^2}{\partial \|q_t\|_2^2} &= 1 + 2\eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x} + \eta^2 \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-2} \left(v_t^T y\mathbf{x}\right)^2 \\ &\quad - 2\eta \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-2} \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right) \|q_t\|_2^2 \left(v_t^T y\mathbf{x}\right)^2 \\ &\quad - 2\eta^2 \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-3} \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right) \|q_t\|_2^2 \left(v_t^T y\mathbf{x}\right)^3 \end{aligned}$$

We first consider the fixed direction $\|q_t\|_2^2 v_t^T y\mathbf{x} \to \infty$. We see that the Jacobian reduces to the following.

$$J(\|q_{t+1}\|_2^2, v_{t+1}^T y\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since the both eigenvalues for the Jacobian are 1, $\|q_t\|_2^2 v_t^T y\mathbf{x} \to \infty$ is Lyapunov stable.

Next, we consider the fixed point $\|q_t\|_2^2 = 0$. The Jacobian reduces to the following.

$$J(\|q_{t+1}\|_2^2, v_{t+1}^T y\mathbf{x}) = \begin{bmatrix} 1 + \eta v_t^T y\mathbf{x} + \frac{\eta^2 \left(v_t^T y\mathbf{x}\right)^2}{4} & 0 \\ \frac{\eta}{2} \|y\mathbf{x}\|_2^2 & 1 \end{bmatrix}$$

The eigenvalues for the Jacobian are 1 and $1 + \eta v_t^T y\mathbf{x} + \frac{\eta^2 (v_t^T y\mathbf{x})^2}{4}$. This means that a segment of the $v_t^T y\mathbf{x}$ axis is stable. We solve for the segment below.

$$-1 < 1 + \eta v_t^T y\mathbf{x} + \frac{\eta^2 (v_t^T y\mathbf{x})^2}{4} < 1$$
$$\frac{-8}{\eta^2} < \left(\frac{4}{\eta} + v_t^T y\mathbf{x}\right) v_t^T y\mathbf{x} < 0$$

Since the $\frac{-8}{\eta^2}$ lower bound is not tight, we get that the condition is satisfied for $\frac{-4}{\eta} < v_t^T y\mathbf{x} < 0$. □

B.1.2 Sharpness Derivation

In this section, we derive a closed-form formula for the sharpness and the other eigenvalues of the Loss Hessian. We first make the following claim about the sharpness.

Claim B.3 (Closed-Form for Sharpness in One Datapoint Setting)**.** *The sharpness for the model in the one datapoint setting is*

$$\frac{1}{2}\psi_t \|q_t\|_2^4 \|y\boldsymbol{x}\|_2^2 - \frac{1}{2}\phi_t v_t^T y\boldsymbol{x} + \psi_t \|q_t\|_2^2 \left(v_t^T y\boldsymbol{x}\right)^2 + \sqrt{\begin{array}{l} \frac{1}{4}\psi_t^2 \|q_t\|_2^8 \|y\boldsymbol{x}\|_2^4 - \frac{7}{2}\phi_t\psi_t \|q_t\|_2^4 v_t^T y\boldsymbol{x} \|y\boldsymbol{x}\|_2^2 + \psi_t^2 \|q_t\|_2^6 \left(v_t^T y\boldsymbol{x}\right)^2 \|y\boldsymbol{x}\|_2^2 \\ + \frac{1}{4}\phi_t^2 \left(v_t^T y\boldsymbol{x}\right)^2 - \phi_t\psi_t \|q_t\|_2^2 \left(v_t^T y\boldsymbol{x}\right)^3 + \psi_t^2 \|q_t\|_2^4 \left(v_t^T y\boldsymbol{x}\right)^4 + 2\phi_t^2 \|q_t\|_2^2 \|y\boldsymbol{x}\|_2^2 \end{array}}$$

where $\phi_t = \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\boldsymbol{x}\right)\right)^{-1}$ *and* $\psi_t = \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\boldsymbol{x}\right)\right)^{-2} \exp\left(\|q_t\|_2^2 v_t^T y\boldsymbol{x}\right)$.

Proof. To solve for the sharpness, we first define the Loss Hessian. For $\theta_t = (q_t, u_t, v_t)$, we have

$$\nabla_{\theta_t} L(q_t, u_t, v_t) = \begin{bmatrix} \frac{\partial^2 L(q_t,u_t,v_t)}{\partial q_t \partial q_t^T} & \frac{\partial^2 L(q_t,u_t,v_t)}{\partial q_t \partial u_t^T} & \frac{\partial^2 L(q_t,u_t,v_t)}{\partial q_t \partial v_t^T} \\ \frac{\partial^2 L(q_t,u_t,v_t)}{\partial u_t \partial q_t^T} & \frac{\partial^2 L(q_t,u_t,v_t)}{\partial u_t \partial u_t^T} & \frac{\partial^2 L(q_t,u_t,v_t)}{\partial u_t \partial v_t^T} \\ \frac{\partial^2 L(q_t,u_t,v_t)}{\partial v_t \partial q_t^T} & \frac{\partial^2 L(q_t,u_t,v_t)}{\partial v_t \partial u_t^T} & \frac{\partial^2 L(q_t,u_t,v_t)}{\partial v_t \partial v_t^T} \end{bmatrix}$$

We solve for the eigenvalues using the technique developed by Singh & Hofmann (2024). First, we solve for $\det\left(\nabla_{\theta_t} L(q_t, u_t, v_t) - \lambda I_{2h+d}\right) = 0$. We form the matrix $\nabla_{\theta_t} L(q_t, u_t, v_t) - \lambda I_{2h+d}$ below.

$$\nabla_{\theta_t} L(q_t, u_t, v_t) - \lambda I_{2h+d}$$
$$= \begin{bmatrix} \psi_t \left(v_t^T y\mathbf{x}\right)^2 u_t u_t^T - \lambda I_h & -\phi_t v_t^T y\mathbf{x} I_h + \psi_t \left(v_t^T y\mathbf{x}\right)^2 u_t q_t^T & \left(-\phi_t + \psi_t q_t^T u_t v_t^T y\mathbf{x}\right) u_t y\mathbf{x}^T \\ -\phi_t v_t^T y\mathbf{x} I_h + \psi_t \left(v_t^T y\mathbf{x}\right)^2 q_t u_t^T & \psi_t \left(v_t^T y\mathbf{x}\right)^2 q_t q_t^T - \lambda I_h & \left(-\phi_t + \psi_t q_t^T u_t v_t^T y\mathbf{x}\right) q_t y\mathbf{x}^T \\ \left(-\phi_t + \psi_t q_t^T u_t v_t^T y\mathbf{x}\right) y\mathbf{x} u_t^T & \left(-\phi_t + \psi_t q_t^T u_t v_t^T y\mathbf{x}\right) y\mathbf{x} q_t^T & \psi_t \left(q_t^T u_t\right)^2 xx^T - \lambda I_d \end{bmatrix}$$

With our initialization assumption (i.e., $u_0 = q_0$, then $u_t = q_t$ by Claim 3.1) $\nabla_{\theta_t} L(q_t, u_t, v_t) - \lambda I_{2h+d}$ reduces to

$$= \begin{bmatrix} \psi_t \left(v_t^T y\mathbf{x}\right)^2 q_t q_t^T - \lambda I_h & -\phi_t v_t^T y\mathbf{x} I_h + \psi_t \left(v_t^T y\mathbf{x}\right)^2 q_t q_t^T & \left(-\phi_t + \psi_t \|q_t\|_2^2 v_t^T y\mathbf{x}\right) q_t y\mathbf{x}^T \\ -\phi_t v_t^T y\mathbf{x} I_h + \psi_t \left(v_t^T y\mathbf{x}\right)^2 q_t q_t^T & \psi_t \left(v_t^T y\mathbf{x}\right)^2 q_t q_t^T - \lambda I_h & \left(-\phi_t + \psi_t \|q_t\|_2^2 v_t^T y\mathbf{x}\right) q_t y\mathbf{x}^T \\ \left(-\phi_t + \psi_t \|q_t\|_2^2 v_t^T y\mathbf{x}\right) y\mathbf{x} q_t^T & \left(-\phi_t + \psi_t \|q_t\|_2^2 v_t^T y\mathbf{x}\right) y\mathbf{x} q_t^T & \psi_t \|q_t\|_2^4 xx^T - \lambda I_d \end{bmatrix}$$

Based on the pattern introduced in the $\nabla_{\theta_t} L(q_t, u_t, v_t) - \lambda I_{2h+d}$ by the initialization assumption, we perform two similarity transforms. We first define $A = \psi_t \left(v_t^T y\mathbf{x}\right)^2 q_t q_t^T - \lambda I_h$, $B = -\phi_t v_t^T y\mathbf{x} I_h + \psi_t \left(v_t^T y\mathbf{x}\right)^2 q_t q_t^T$, $C = \left(-\phi_t + \psi_t \|q_t\|_2^2 v_t^T y\mathbf{x}\right) q_t y\mathbf{x}^T$, and $D = \psi_t \|q_t\|_2^4 xx^T - \lambda I_d$. Then, we can rewrite $\nabla_{\theta_t} L(q_t, u_t, v_t) - \lambda I_{2h+d}$ as

$$\begin{bmatrix} A & B & C \\ B & A & C \\ C^T & C^T & D \end{bmatrix}$$

We proceed with the following similarity transforms.

$$\frac{1}{2}\begin{bmatrix} I_h & I_h & 0 \\ -I_h & I_h & 0 \\ 0 & 0 & \sqrt{2}I_d \end{bmatrix}\begin{bmatrix} A & B & C \\ B & A & C \\ C^T & C^T & D \end{bmatrix}\begin{bmatrix} I_h & -I_h & 0 \\ I_h & I_h & 0 \\ 0 & 0 & \sqrt{2}I_d \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} I_h & I_h & 0 \\ -I_h & I_h & 0 \\ 0 & 0 & \sqrt{2}I_d \end{bmatrix}\begin{bmatrix} A+B & B-A & \sqrt{2}C \\ A+B & A-B & \sqrt{2}C \\ 2C^T & 0 & \sqrt{2}D \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2(A+B) & 0 & 2\sqrt{2}C \\ 0 & 2(A-B) & 0 \\ 2\sqrt{2}C^T & 0 & 2D \end{bmatrix}$$

$$= \begin{bmatrix} A+B & 0 & \sqrt{2}C \\ 0 & A-B & 0 \\ \sqrt{2}C^T & 0 & D \end{bmatrix}$$

$$\begin{bmatrix} I_h & 0 & 0 \\ 0 & 0 & I_d \\ 0 & I_h & 0 \end{bmatrix}\begin{bmatrix} A+B & 0 & \sqrt{2}C \\ 0 & A-B & 0 \\ \sqrt{2}C^T & 0 & D \end{bmatrix}\begin{bmatrix} I_h & 0 & 0 \\ 0 & 0 & I_h \\ 0 & I_d & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_h & 0 & 0 \\ 0 & 0 & I_d \\ 0 & I_h & 0 \end{bmatrix}\begin{bmatrix} A+B & \sqrt{2}C & 0 \\ 0 & 0 & A-B \\ \sqrt{2}C^T & D & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A+B & \sqrt{2}C & 0 \\ \sqrt{2}C^T & D & 0 \\ 0 & 0 & A-B \end{bmatrix}$$

After applying the similarity transforms, we solve $\det\left(\begin{bmatrix} A+B & \sqrt{2}C \\ \sqrt{2}C^T & D \end{bmatrix}\right) = 0$ to get the sharpness. To do so, we assume that D is invertible. This means that $\lambda \neq \psi_t \|q_t\|_2^4 \|y\mathbf{x}\|_2^2, 0$.

Then, $\det\left(\begin{bmatrix} A+B & \sqrt{2}C \\ \sqrt{2}C^T & D \end{bmatrix}\right) = \det(D)\det\left(A+B-2CD^{-1}C^T\right)$. So, we solve for $\det\left(A+B-2CD^{-1}C^T\right) = 0$.

First, we solve for D^{-1} using the Sherman-Morrison Formula:

$$D^{-1} = -\frac{1}{\lambda}I_d - \frac{\psi_t \|q_t\|_2^4 x x^T}{\lambda\left(\lambda - \psi_t \|q_t\|_2^4 \|y\mathbf{x}\|_2^2\right)}$$

Then, we find that

$$\begin{aligned} A+B-2CD^{-1}C^T = 2\Bigg[&\psi_t\left(v_t^T y\mathbf{x}\right)^2 + \frac{\psi_t^2\|q_t\|_2^4\left(v_t^T y\mathbf{x}\right)^2\|y\mathbf{x}\|_2^2}{\lambda} + \frac{\phi_t^2\|y\mathbf{x}\|_2^2}{\lambda} - \frac{2\phi_t\psi_t\|q_t\|_2^2 v_t^T y\mathbf{x}\|y\mathbf{x}\|_2^2}{\lambda} \\ &+ \frac{\psi_t^3\|q_t\|_2^8\|y\mathbf{x}\|_2^4\left(v_t^T y\mathbf{x}\right)^2}{\lambda\left(\lambda-\psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right)} + \frac{\psi_t\phi_t^2\|q_t\|_2^4\|y\mathbf{x}\|_2^4}{\lambda\left(\lambda-\psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right)} - \frac{2\psi_t^2\phi_t\|q_t\|_2^6\|y\mathbf{x}\|_2^4 v_t^T y\mathbf{x}}{\lambda\left(\lambda-\psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right)}\Bigg] q_t q_t^T \\ &- \left(\phi_t v_t^T y\mathbf{x} + \lambda\right) I_h \end{aligned}$$

To solve for $\det\left(A + B - 2CD^{-1}C^T\right) = 0$, we first multiply by $\lambda\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right)$, which yields the following.

$$\Bigg(2\Bigg[\psi_t\left(v_t^T y\mathbf{x}\right)^2\lambda\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right) + \psi_t^2\|q_t\|_2^4\left(v_t^T y\mathbf{x}\right)^2\|y\mathbf{x}\|_2^2\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right)$$
$$+ \phi_t^2\|y\mathbf{x}\|_2^2\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right) - 2\phi_t\psi_t\|q_t\|_2^2 v_t^T y\mathbf{x}\|y\mathbf{x}\|_2^2\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right) + \psi_t^3\|q_t\|_2^8\|y\mathbf{x}\|_2^4\left(v_t^T y\mathbf{x}\right)^2$$
$$+ \psi_t\phi_t^2\|q_t\|_2^4\|y\mathbf{x}\|_2^4 - 2\psi_t^2\phi_t\|q_t\|_2^6\|y\mathbf{x}\|_2^4 v_t^T y\mathbf{x}\Bigg]\|q_t\|_2^2 - \left(\phi_t v_t^T y\mathbf{x} + \lambda\right)\lambda\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right)\Bigg)$$
$$\cdot\left(-\lambda\left(\phi_t v_t^T y\mathbf{x} + \lambda\right)\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right)\right)^{h-1} = 0$$

We focus on the following factor from above.

$$2\Bigg[\psi_t\left(v_t^T y\mathbf{x}\right)^2\lambda\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right) + \psi_t^2\|q_t\|_2^4\left(v_t^T y\mathbf{x}\right)^2\|y\mathbf{x}\|_2^2\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right)$$
$$+ \phi_t^2\|y\mathbf{x}\|_2^2\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right) - 2\phi_t\psi_t\|q_t\|_2^2 v_t^T y\mathbf{x}\|y\mathbf{x}\|_2^2\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right) + \psi_t^3\|q_t\|_2^8\|y\mathbf{x}\|_2^4\left(v_t^T y\mathbf{x}\right)^2$$
$$+ \psi_t\phi_t^2\|q_t\|_2^4\|y\mathbf{x}\|_2^4 - 2\psi_t^2\phi_t\|q_t\|_2^6\|y\mathbf{x}\|_2^4 v_t^T y\mathbf{x}\Bigg]\|q_t\|_2^2 - \lambda\left(\phi_t v_t^T y\mathbf{x} + \lambda\right)\left(\lambda - \psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\right) = 0$$

We see that the above reduces down to the following.

$$-\lambda\Bigg(-2\lambda\psi_t\left(v_t^T y\mathbf{x}\right)^2\|q_t\|_2^2 - 2\phi_t^2\|y\mathbf{x}\|_2^2\|q_t\|_2^2 + 4\phi_t\psi_t\|q_t\|_2^4 v_t^T y\mathbf{x}\|y\mathbf{x}\|_2^2$$
$$+ \lambda\phi_t v_t^T y\mathbf{x} - \psi_t\phi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2 v_t^T y\mathbf{x} + \lambda^2 - \lambda\psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2\Bigg) = 0$$

Since we assumed earlier that $\lambda \neq 0$, we solve the quadratic. From solving the quadratic, we get the following symmetric pair of eigenvalues.

$$\lambda = \frac{1}{2}\psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2 - \frac{1}{2}\phi_t v_t^T y\mathbf{x} + \psi_t\left(v_t^T y\mathbf{x}\right)^2\|q_t\|_2^2$$
$$\pm\sqrt{\begin{array}{l}\frac{1}{4}\psi_t^2\|q_t\|_2^8\|y\mathbf{x}\|_2^4 - \frac{7}{2}\phi_t\psi_t\|q_t\|_2^4 v_t^T y\mathbf{x}\|y\mathbf{x}\|_2^2 + \psi_t^2\|q_t\|_2^6\|y\mathbf{x}\|_2^2\left(v_t^T y\mathbf{x}\right)^2 \\ + \frac{1}{4}\phi_t^2\left(v_t^T y\mathbf{x}\right)^2 - \phi_t\psi_t\left(v_t^T y\mathbf{x}\right)^3\|q_t\|_2^2 + \psi_t^2\left(v_t^T y\mathbf{x}\right)^4\|q_t\|_2^4 + 2\phi_t^2\|y\mathbf{x}\|_2^2\|q_t\|_2^2\end{array}}$$

where the plus direction is our sharpness. □

In addition to the sharpness, we also have closed-forms for remaining eigenvalues and their multiplicity. We have h copies of $\phi_t v_t^T y\mathbf{x}$, $h-1$ copies of $-\phi_t v_t^T y\mathbf{x}$, $d-1$ copies of 0, and the following eigenvalues from our calculation above.

$$\lambda = \frac{1}{2}\psi_t\|q_t\|_2^4\|y\mathbf{x}\|_2^2 - \frac{1}{2}\phi_t v_t^T y\mathbf{x} + \psi_t\left(v_t^T y\mathbf{x}\right)^2\|q_t\|_2^2$$
$$-\sqrt{\begin{array}{l}\frac{1}{4}\psi_t^2\|q_t\|_2^8\|y\mathbf{x}\|_2^4 - \frac{7}{2}\phi_t\psi_t\|q_t\|_2^4 v_t^T y\mathbf{x}\|y\mathbf{x}\|_2^2 + \psi_t^2\|q_t\|_2^6\|y\mathbf{x}\|_2^2\left(v_t^T y\mathbf{x}\right)^2 \\ + \frac{1}{4}\phi_t^2\left(v_t^T y\mathbf{x}\right)^2 - \phi_t\psi_t\left(v_t^T y\mathbf{x}\right)^3\|q_t\|_2^2 + \psi_t^2\left(v_t^T y\mathbf{x}\right)^4\|q_t\|_2^4 + 2\phi_t^2\|y\mathbf{x}\|_2^2\|q_t\|_2^2\end{array}}$$

We note that $\phi_t v_t^T y\mathbf{x}$ comes from solving $\det(A - B)$, $-\phi_t v_t^T y\mathbf{x}$ comes from our approach above, and 0 also comes from as similar approach as above, but by assuming the $A + B$ matrix is invertible instead of D matrix.

B.1.3 Proof of Theorem 1

In this section we give the proof of Theorem 4.2. Recall that according to our setup, we always have $y = \pm 1$ is the label, $\|\mathbf{x}\|_2^2 > 0$ (to avoid degenerate cases) and $\eta > 0$.

As we mentioned in Section 5, our problem can be simplified to a 2-dimensional system with parameters $\|q_t\|_2^2$ and $v_t^T y\mathbf{x}$. We partition the space for this two dimensional system based on two nullclines of $\|q_t\|_2^2$ and the boundary splits the space based on whether $\|q_t\|_2^2 v_t^T y\mathbf{x}$ grows or shrinks in the next time step. We also give supporting lemmas that illustrate the properties of the 4 regions, shown below in Figure 6.

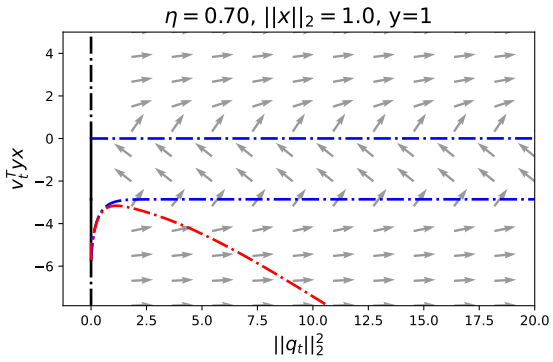


Figure 6: This is the phase space diagram for our one datapoint system for the case where $\eta = 0.70$, $\|\mathbf{x}\|_2^2 = 1$, and $y = 1$. The blue dashed curves represent the nullclines for $\|q_t\|_2^2$, where region 1 is above the top blue curve, region 2 is between both blue curves, and regions 3.1 and 3.2 are below the bottom blue curve. The red dashed curve represents the boundary where $\|q_t\|_2^2 v_t^T y\mathbf{x}$ shrinks inside the curve (region 3.2) and grows outside the curve (region 3.1). Since $\|q_t\|_2^2 = 0$ is also a fixed point of the system, we denote the solid black line portion as $\|q_t\|_2^2 = 0$ being stable, and the dashed black line portion as $\|q_t\|_2^2 = 0$ being unstable. We also include normalized vectors of the gradient field, shown as grey arrows, to provide some intuition for possible trajectories.

First, we provide the proof for Theorem 4.2. Then, we give the supporting lemmas.

For Theorem 4.2, our proof mainly focuses on the behavior of the loss in each region, since our definition for Edge of Stability focuses on specific attributes of the loss (i.e., it must have at least 2 local maxima). The intuition behind the proof is based on our understanding that the loss can only increase in region 3.2 (see Figure 6) and could potentially contribute to the Edge of Stability if we had some form of movement in and out of region 3.2. However, as seen in Figure 6, region 3.2 is upper bounded in the $v_t^T y\mathbf{x}$ direction. So, if for some η, the system "escapes" region 3.2 to regions 2 or 1, we cannot move back into region 3.2, since $v_t^T y\mathbf{x}$ is strictly increasing according to Lemma 4.1. So, our only potential candidate could be movement between regions 3.2 and 3.1. However, this is also not possible either, since region 3.2 does not appear to have an holes or significant indentations, and the strict growth of $\|q_t\|_2^2$ in regions 3.1 and 3.2 makes it that return into region 3.2 from region 3.1 is not possible. So, there would be no significant non-monotonic behavior in the loss. We now show the formal proof below.

Proof of Theorem 4.2. To prove the theorem, we first consider the phase space of our 2-dimensional system for this setting, consisting of $\|q_t\|_2^2$ and $v_t^T y\mathbf{x}$.

Based on our fixed point analysis, we divide up our phase space into four regions. Region 1 is defined as the area where $\|q_t\|_2^2 > 0$ and $v_t^T y\mathbf{x} \geq 0$. Region 2 is the area where $\|q_t\|_2^2 > 0, v_t^T y\mathbf{x}$ satisfies $-\frac{2}{\eta} < \left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x} < 0$. Region 3.1 is the area where $\|q_t\|_2^2 > 0, v_t^T y\mathbf{x}$ satisfies $\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x} < -\frac{2}{\eta}$ and

$$\left(v_t^T y\mathbf{x}\right)^2 \left(2 + \eta\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x}\right) + \|q_t\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x}\right)^2 \geq 0.$$

Region 3.2 is the area where $\|q_t\|_2^2 > 0, v_t^T y\mathbf{x}$ satisfies $\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x} < -\frac{2}{\eta}$ and

$$\left(v_t^T y\mathbf{x}\right)^2 \left(2 + \eta\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x}\right) + \|q_t\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta\left(1 + \exp\left(\|q_t\|_2^2 v_t^T y\mathbf{x}\right)\right)^{-1} v_t^T y\mathbf{x}\right)^2 < 0.$$

We proceed by proving that every possible training loss trajectory in the single datapoint setting does not meet the EOS conditions in our definition by considering the initialization $\|q_0\|_2^2 > 0, v_0^T y\mathbf{x}$ in each region. We don't consider any initialization with $\|q_0\|_2^2 = 0$, since that is a fixed point (see Claim B.2).

Case 1: $\|q_0\|_2^2 > 0, v_0^T y\mathbf{x}$ in region 1. We proceed by induction. For the base case, we know, by corollary 4.1.1, that $\|q_1\|_2^2 v_1^T y\mathbf{x} > \|q_0\|_2^2 v_0^T y\mathbf{x}$. This means that $\log\left(1 + \left(-\|q_1\|_2^2 v_1^T y\mathbf{x}\right)\right) < \log\left(1 + \exp\left(\|q_0\|_2^2 v_0^T y\mathbf{x}\right)\right)$.

For the inductive case, we assume $\|q_k\|_2^2 > 0, v_k^T y\mathbf{x}$ in region 1. By corollary 4.1.1, we know that $\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x} > \|q_k\|_2^2 v_k^T y\mathbf{x}$, meaning $\log\left(1 + \left(-\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right) < \log\left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)$

Since our loss is strictly decreasing in this case, it does not meet the EOS condition.

Case 2: $\|q_0\|_2^2 > 0, v_0^T y\mathbf{x}$ in region 2. We consider two possible trajectories based on this initialization. We know that these trajectories are exhaustive based on lemmas 4.1, B.5, and B.11.

Trajectory 1: there exists some $k \geq 1$, where $\|q_k\|_2^2 = 0$. Since we know that $\|q_k\|_2^2 = 0$ is a fixed point, we know that for any $l > k$, $\|q_l\|_2^2 = 0$. So, $\log\left(1 + \exp\left(-\|q_l\|_2^2 v_l^T y\mathbf{x}\right)\right) = \log\left(1 + \exp\left(-\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right) = \log(2)$.

Trajectory 2: there exists some $k \geq 1$, where $\|q_k\|_2^2 > 0$ and $v_k^T y\mathbf{x} \geq 0$. We know that our loss trajectory will be strictly decreasing for subsequent iterations by considering $\|q_k\|_2^2, v_k^T y\mathbf{x}$ as some initialization in region 1.

Now, for all iterations prior to k, we will show that our loss is strictly decreasing by induction. First, we consider the base case. By corollary 4.1.2, we know that $\|q_1\|_2^2 v_1^T y\mathbf{x} > \|q_0\|_2^2 v_0^T y\mathbf{x}$. This means that $\log\left(1 + \left(-\|q_1\|_2^2 v_1^T y\mathbf{x}\right)\right) < \log\left(1 + \exp\left(\|q_0\|_2^2 v_0^T y\mathbf{x}\right)\right)$.

Next, we consider the inductive case. Assume $\|q_j\|_2^2 > 0, v_j^T y\mathbf{x}$ is in region 2, for $1 \leq j \leq k$. By corollary 4.1.2, we know that $\|q_{j+1}\|_2^2 v_{j+1}^T y\mathbf{x} > \|q_j\|_2^2 v_j^T y\mathbf{x}$, meaning $\log\left(1 + \left(-\|q_{j+1}\|_2^2 v_{j+1}^T y\mathbf{x}\right)\right) < \log\left(1 + \exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)$

Since our loss can only decrease monotonically in this case, it does not meet the EOS condition.

Case 3: $\|q_0\|_2^2 > 0, v_0^T y\mathbf{x}$ in region 3.1. We consider two possible trajectories based on this initialization. We know that these trajectories are exhaustive based on lemmas 4.1, B.6, and B.11.

Trajectory 1: there exists some $k \geq 1$, where $\|q_k\|_2^2, v_k^T y\mathbf{x}$ jumps directly from region 3.1 to 1. We know that our loss trajectory will be strictly decreasing for subsequent iterations by considering $\|q_k\|_2^2, v_k^T y\mathbf{x}$ as some initialization in region 1.

Trajectory 2: there exists some $k \geq 1$, where $\|q_k\|_2^2, v_k^T y\mathbf{x}$ is in region 2. We know that our loss trajectory will be monotonically decreasing for subsequent iterations by considering $\|q_k\|_2^2, v_k^T y\mathbf{x}$ as some initialization in region 2.

Now, for all iterations prior to k, we will show that our loss is monotonically decreasing by induction. For the base case, we know by lemma B.7 that $\|q_1\|_2^2 v_1^T y\mathbf{x} \geq \|q_0\|_2^2 v_0^T y\mathbf{x}$, meaning $\log\left(1 + \left(-\|q_1\|_2^2 v_1^T y\mathbf{x}\right)\right) \leq \log\left(1 + \exp\left(\|q_0\|_2^2 v_0^T y\mathbf{x}\right)\right)$.

Next, we consider the inductive case. Assume $\|q_j\|_2^2 > 0, v_j^T y\mathbf{x}$ is in region 3.1, for $1 \leq j \leq k$. We know by lemma B.7 that $\|q_{j+1}\|_2^2 v_{j+1}^T y\mathbf{x} \geq \|q_j\|_2^2 v_j^T y\mathbf{x}$, meaning $\log\left(1 + \left(-\|q_{j+1}\|_2^2 v_{j+1}^T y\mathbf{x}\right)\right) \leq \log\left(1 + \exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)$.

Since our loss can only decrease monotonically in this case, it does not meet the EOS condition.

Case 4: $\|q_0\|_2^2 > 0, v_0^T y\mathbf{x}$ in region 3.2. We consider three possible trajectories based on this initialization. We know these trajectories are exhaustive based on lemmas 4.1, B.6, B.11, and B.12.

Trajectory 1: there exists some $k \geq 1$, where $\|q_k\|_2^2, v_k^T y\mathbf{x}$ jumps directly from region 3.2 to 1. We know that our loss trajectory will be strictly decreasing for subsequent iterations by considering $\|q_k\|_2^2, v_k^T y\mathbf{x}$ as some initialization in region 1.

Trajectory 2: there exists some $k \geq 1$, where $\|q_k\|_2^2, v_k^T y\mathbf{x}$ jumps directly from region 3.2 to 2. We know that our loss trajectory will be monotonically decreasing for subsequent iterations by considering $\|q_k\|_2^2, v_k^T y\mathbf{x}$ as some initialization in region 2.

Trajectory 3: there exists some $k \geq 1$, where $\|q_k\|_2^2, v_k^T y\mathbf{x}$ is in region 3.1. We know that our loss trajectory will be monotonically decreasing for subsequent iterations by considering $\|q_k\|_2^2, v_k^T y\mathbf{x}$ as some initialization in region 3.1.

Now, for all iterations prior to k, we will show that our loss is strictly increasing by induction. For the base case, we know by lemma B.7 that $\|q_1\|_2^2 v_1^T y\mathbf{x} < \|q_0\|_2^2 v_0^T y\mathbf{x}$, meaning $\log\left(1 + \left(-\|q_1\|_2^2 v_1^T y\mathbf{x}\right)\right) > \log\left(1 + \exp\left(\|q_0\|_2^2 v_0^T y\mathbf{x}\right)\right)$.

Next, we consider the inductive case. Assume $\|q_j\|_2^2 > 0, v_j^T y\mathbf{x}$ is in region 3.2, for $1 \leq j \leq k$. We know by lemma B.7 that $\|q_{j+1}\|_2^2 v_{j+1}^T y\mathbf{x} < \|q_j\|_2^2 v_j^T y\mathbf{x}$, meaning $\log\left(1 + \left(-\|q_{j+1}\|_2^2 v_{j+1}^T y\mathbf{x}\right)\right) > \log\left(1 + \exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)$.

Since our loss has only one local maximum in this case, it does not meet the EOS condition.

Since we have shown that our training loss can have at most one local maximum for every possible trajectory, it therefore does not exhibit EOS behavior according to our definition.

□

Now, we show the supporting lemmas for Theorem 4.2. First, we prove lemma 4.1 that $v_t^T y\mathbf{x}$ is always monotonically increasing. In Lemma B.4, we prove that $\|q_t\|_2^2$ grows monotonically in region 1 (i.e., $v_t^T y\mathbf{x} \geq 0$). In Corollary 4.1.1, we prove that the loss is always decreasing in region 1. In Lemma B.5, we prove that $\|q_t\|_2^2$ shrinks in region 2 (see Figure 6). In Corollary 4.1.2, we prove that, despite $\|q_t\|_2^2$ shrinking (see lemma B.5), the loss is still decreasing in region 2. In Lemma B.6, we prove that for regions 3.1 and 3.2, $\|q_t\|_2^2$ is increasing. In Lemma B.7, we show how the boundary separating region 3.1 and 3.2 (see the red curve in Figure 6) causes the loss to worsen in region 3.2 but not in region 3.1. In Lemma B.8, we define a necessary condition for the system to be in region 3.2, which we use in later lemmas. In Lemma B.9, we prove that the separating boundary for regions 3.1 and region 3.2 is upper bounded in the $v_t^T y\mathbf{x}$ direction by the one of the $\|q_t\|_2^2$ nullclines (see Figure 6). In Lemma B.10, we prove an upper bound in the $v_t^T y\mathbf{x}$ direction for one of the nullclines of $\|q_t\|_2^2$, which we use in Lemma B.12. In Lemma B.11, we show that once the system leaves region 3.2, it is impossible to return back. Lastly, in Lemma B.12, we show that the system can escape region 3.2 in finite time.

First we show that $v_t^T y\mathbf{x}$ is always monotonically increasing, therefore it does not have a finite nullcline, and we don't partition the space according to this variable.

Proof of Lemma 4.1. Assume some $k, l \in \mathbb{N}$, $l > k$ and $\|q_k\|_2^2 > 0$. We consider the iterated map for $v_l^T y\mathbf{x}$ below.

$$
\begin{aligned}
v_l^T y\mathbf{x} &= v_{l-1}^T y\mathbf{x} + \eta \left(1 + \exp\left(\|q_{l-1}\|_2^2 v_{l-1}^T y\mathbf{x}\right)\right)^{-1} \|q_{l-1}\|_2^2 \|y\mathbf{x}\|_2^2 \\
&= v_k^T y\mathbf{x} + \eta \|y\mathbf{x}\|_2^2 \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} \|q_k\|_2^2 \\
&+ \eta \|y\mathbf{x}\|_2^2 \sum_{i=1}^{l-k-1} \left(1 + \exp(\|q_{k+i}\|_2^2 v_{k+i}^T y\mathbf{x})\right)^{-1} \|q_{k+i}\|_2^2 \\
&\geq v_k^T y\mathbf{x} + \eta \|y\mathbf{x}\|_2^2 \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} \|q_k\|_2^2 && \text{since } \forall i \in \{1, ..., l-k-1\}.\ \|q_{k+i}\|_2^2 \geq 0 \\
&> v_k^T y\mathbf{x} && \text{since } \|q_k\|_2^2 > 0
\end{aligned}
$$

□

Next we show that whenever $v_k^T y\mathbf{x} \geq 0$ the norm of q_t is nondecreasing, this is the main property of region 1 (see Figure 6)

Lemma B.4. *For any $k, l \in \mathbb{N}$ such that $l > k$, if $\|q_k\|_2^2 > 0, v_k^T y\boldsymbol{x} \geq 0$, then $\|q_l\|_2^2 \geq \|q_k\|_2^2$.*

Proof. Assume some $k, l \in \mathbb{N}$, $l > k$, $\eta > 0$, $\|q_k\|_2^2 > 0$ and $v_k^T y\mathbf{x} \geq 0$.

We consider the iterated map for $\|q_l\|_2^2$ below.

$$
\begin{aligned}
\|q_l\|_2^2 &= \|q_{l-1}\|_2^2 \left(1 + \eta \left(1 + \exp\left(\|q_{l-1}\|_2^2 v_{l-1}^T y\mathbf{x}\right)\right)^{-1} v_{l-1}^T y\mathbf{x}\right)^2 \\
&= \|q_k\|_2^2 \prod_{i=0}^{l-k-1} \left(1 + \eta \left(1 + \exp\left(\|q_{k+i}\|_2^2 v_{k+i}^T y\mathbf{x}\right)\right)^{-1} v_{k+i}^T y\mathbf{x}\right)^2 \\
&\geq \|q_k\|_2^2 && \text{by lemma 4.1}
\end{aligned}
$$

□

As a simple corollary of the first two Lemmas, it's clear that in region 1 the output of the model is always improving.

Proof of Corollary 4.1.1. Assume some $k, l \in \mathbb{N}$, $l > k$, $\|q_k\|_2^2 > 0$ and $v_k^T y\mathbf{x} \geq 0$. We derive the following.

$$
\begin{aligned}
&v_l^T y\mathbf{x} > v_k^T y\mathbf{x} && \text{by lemma 4.1} \\
&\|q_l\|_2^2 \geq \|q_k\|_2^2 && \text{by lemma B.4} \\
&\|q_l\|_2^2 v_l^T y\mathbf{x} > \|q_k\|_2^2 v_k^T y\mathbf{x}
\end{aligned}
$$

□

Now we consider region 2 of Figure 6. This region is sandwiched between two nullclines of $\|q_t\|_2^2$, and we show that norm of q_t decreases in this region.

Lemma B.5. *For any $k \in \mathbb{N}$, if $\|q_k\|_2^2 > 0, v_k^T y\boldsymbol{x}$ satisfies*

$-\frac{2}{\eta} < \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\boldsymbol{x}\right)\right)^{-1} v_k^T y\boldsymbol{x} < 0$*, then $\|q_{k+1}\|_2^2 < \|q_k\|_2^2$.*

Proof. Assume some $k \in \mathbb{N}$ and $\|q_k\|_2^2 > 0, v_k^T y\mathbf{x}$ that satisfies

$-\frac{2}{\eta} < \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} < 0$. Consider the iterated map for $\|q_{k+1}\|_2^2$ below.

$$\begin{aligned}\|q_{k+1}\|_2^2 &= \|q_k\|_2^2 \left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2 \\ &< \|q_k\|_2^2\end{aligned}$$

We get to the final step by our assumption that $-\frac{2}{\eta} < \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} < 0$. □

Note that even though in region 2, the norm of q_t is decreasing, the loss function is actually still improving as Corollary 4.1.2 states, which we prove below.

Proof of Corollary 4.1.2. Assume some $k \in \mathbb{N}$ and $\|q_k\|_2^2 > 0, v_k^T y\mathbf{x}$ that satisfies $-\frac{2}{\eta} < \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} < 0$. We derive the following.

$$\begin{aligned}&v_{k+1}^T y\mathbf{x} > v_k^T y\mathbf{x} && \text{by lemma 4.1}\\ &\|q_{k+1}\|_2^2 < \|q_k\|_2^2 && \text{by lemma B.5}\\ &\|q_k\|_2^2 v_k^T y\mathbf{x} < \|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\end{aligned}$$

The final step comes from our assumption that $-\frac{2}{\eta} < \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} < 0$. □

Finally, we give lemmas related to region 3.1 and 3.2 in Figure 6. In these regions, the norm of q_t is once again increasing.

Lemma B.6. *For any $k \in \mathbb{N}$, if $\|q_k\|_2^2 > 0, v_k^T y\boldsymbol{x}$ satisfies $\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\boldsymbol{x}\right)\right)^{-1} v_k^T y\boldsymbol{x} < -\frac{2}{\eta}$, then $\|q_{k+1}\|_2^2 > \|q_k\|_2^2$.*

Proof. Assume some $k \in \mathbb{N}$ and $\|q_k\|_2^2 > 0, v_k^T y\mathbf{x}$ that satisfies $\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} < -\frac{2}{\eta}$. We derive the following.

$$\begin{aligned}\|q_{k+1}\|_2^2 &= \|q_k\|_2^2 \left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2 \\ &> \|q_k\|_2^2\end{aligned}$$

The final step is due to our assumption $\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} < -\frac{2}{\eta}$.

□

In region 3, the loss function may move in different directions, so we further partition the region into 3.1 and 3.2 according to whether the loss function is improving. In region 3.2, we show that the loss worsens by the following lemma.

Lemma B.7. *For any $k \in \mathbb{N}$ such that $\|q_k\|_2^2 > 0, v_k^T y\boldsymbol{x}$,*

$$\left(v_k^T y\boldsymbol{x}\right)^2 \left(2+\eta\left(1+exp(\|q_k\|_2^2 v_k^T y\boldsymbol{x})\right)^{-1} v_k^T y\boldsymbol{x}\right) + \|q_k\|_2^2 \|y\boldsymbol{x}\|_2^2 \left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\boldsymbol{x}\right)\right)^{-1} v_k^T y\boldsymbol{x}\right)^2 \geq 0$$

if and only if $\|q_{k+1}\|_2^2 v_{k+1}^T y\boldsymbol{x} \geq \|q_k\|_2^2 v_k^T y\boldsymbol{x}$.

Proof. First, we prove the forward direction of the biconditional. Assume some $k \in \mathbb{N}$ and $\|q_k\|_2^2 > 0, v_k^T y\mathbf{x}$ that satisfies

$$\left(v_k^T y\mathbf{x}\right)^2 \left(2+\eta\left(1+exp(\|q_k\|_2^2 v_k^T y\mathbf{x})\right)^{-1} v_k^T y\mathbf{x}\right) + \|q_k\|_2^2 \|y\mathbf{x}\|_2^2 \left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2 \geq 0.$$

We derive the following.

$$\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x} - \|q_k\|_2^2 v_k^T y\mathbf{x}$$
$$= v_k^T y\mathbf{x}\Bigg(\|q_k\|_2^2 + 2\eta \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} \|q_k\|_2^2 v_k^T y\mathbf{x} + \eta^2 \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-2} \|q_k\|_2^2 \left(v_k^T y\mathbf{x}\right)^2 \Bigg)$$
$$+ \eta \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} \|q_k\|_2^2 \|y\mathbf{x}\|_2^2 \Bigg(\|q_k\|_2^2 + 2\eta \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} \|q_k\|_2^2 v_k^T y\mathbf{x}$$
$$+ \eta^2 \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-2} \|q_k\|_2^2 \left(v_k^T y\mathbf{x}\right)^2 \Bigg) - \|q_k\|_2^2 v_k^T y\mathbf{x}$$

$$= 2\eta \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} \|q_k\|_2^2 \left(v_k^T y\mathbf{x}\right)^2 + \eta^2 \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-2} \|q_k\|_2^2 \left(v_k^T y\mathbf{x}\right)^3$$
$$+ \eta \|q_k\|_2^4 \|y\mathbf{x}\|_2^2 \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} + 2\eta^2 \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-2} \|q_k\|_2^4 v_k^T y\mathbf{x} \|y\mathbf{x}\|_2^2$$
$$+ \eta^3 \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-3} \|q_k\|_2^4 \left(v_k^T y\mathbf{x}\right)^2 \|y\mathbf{x}\|_2^2$$

$$= \eta \|q_k\|_2^2 \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} \Bigg(\left(v_k^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)$$
$$+ \|q_k\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2 \Bigg)$$

$$\geq 0$$

Now, we consider the reverse direction of the biconditional, which we will prove by contrapositive. From our above derivation, we already know that $\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x} - \|q_k\|_2^2 v_k^T y\mathbf{x} < 0$, since we assumed that $\|q_k\|_2^2 > 0$ and

$$\left(v_k^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1+exp(\|q_k\|_2^2 v_k^T y\mathbf{x})\right)^{-1} v_k^T y\mathbf{x}\right) + \|q_k\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2 < 0.$$

□

The boundary separating regions 3.1 and 3.2 is not easy to express explicitly, but we do show an upperbound for the boundary in the following lemmas. In the lemma below, we first show that the separating boundary for regions 3.1 and 3.2 is upper bounded in the $v_t^T y\mathbf{x}$ direction by one of the nullclines of $\|q_t\|$ (see the red and blue curves in Figure 6).

Lemma B.8. *A necessary condition for* $\|q\|_2^2 > 0, v^T y\boldsymbol{x}$ *to satisfy*

$$\left(v^T y\boldsymbol{x}\right)^2 \left(2 + \eta \left(1+\exp(\|q\|_2^2 v_k^T y\boldsymbol{x})\right)^{-1} v^T y\boldsymbol{x}\right) + \|q\|_2^2 \|y\boldsymbol{x}\|_2^2 \left(1 + \eta \left(1+\exp\left(\|q\|_2^2 v^T y\boldsymbol{x}\right)\right)^{-1} v^T y\boldsymbol{x}\right)^2 < 0,$$

is that $\left(1+\exp\left(\|q\|_2^2 v^T y\boldsymbol{x}\right)\right)^{-1} v^T y\boldsymbol{x} < -\frac{2}{\eta}$.

Proof. Assume that some $\|q\|_2^2 > 0$.

We will show that if $\left(1+\exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} v^T y\mathbf{x} \geq -\frac{2}{\eta}$, then

$$\left(v_k^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1+exp(\|q_k\|_2^2 v_k^T y\mathbf{x})\right)^{-1} v_k^T y\mathbf{x}\right) + \|q_k\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta \left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2 \geq 0.$$

We derive the following.

$$\left(v^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1+\exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} v^T y\mathbf{x}\right) \geq 0 \qquad \text{by our assumption}$$
$$\|q\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta \left(1+\exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} v^T y\mathbf{x}\right)^2 \geq 0$$
$$\left(v^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1+\exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} v^T y\mathbf{x}\right)$$
$$+ \|q\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta \left(1+\exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} v^T y\mathbf{x}\right)^2 \geq 0.$$

□

In the following lemma, we show an upper bound in the $v_t^T y\mathbf{x}$ direction one of the nullclines of $\|q_t\|_2^2$, which is also an upper bound for the boundary separating regions 3.1 and 3.2.

Lemma B.9. *For any $\|q\|_2^2 > 0$, the $v_1^T y\boldsymbol{x}$ that satisfies*

$$\left(v_1^T y\boldsymbol{x}\right)^2 \left(2 + \eta\left(1 + \exp\left(\|q\|_2^2 v_1^T y\boldsymbol{x}\right)\right)^{-1} v_1^T y\boldsymbol{x}\right) + \|q\|_2^2 \|y\boldsymbol{x}\|_2^2 \left(1 + \eta\left(1 + \exp\left(\|q\|_2^2 v_1^T y\boldsymbol{x}\right)\right)^{-1} v_1^T y\boldsymbol{x}\right)^2 < 0$$

is less than the $v_2^T y\boldsymbol{x}$ that satisfies $\left(1 + \exp\left(\|q\|_2^2 v_2^T y\boldsymbol{x}\right)\right)^{-1} v_2^T y\boldsymbol{x} = -\frac{2}{\eta}$.

Proof. Consider some $\|q\|_2^2 > 0$, $v_1^T y\mathbf{x}$ that satisfies

$$\left(v_1^T y\mathbf{x}\right)^2 \left(2 + \eta\left(1 + \exp\left(\|q\|_2^2 v_1^T y\mathbf{x}\right)\right)^{-1} v_1^T y\mathbf{x}\right) + \|q\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta\left(1 + \exp\left(\|q\|_2^2 v_1^T y\mathbf{x}\right)\right)^{-1} v_1^T y\mathbf{x}\right)^2 < 0,$$

and $v_2^T y\mathbf{x}$ that satisfies $\left(1 + \exp\left(\|q\|_2^2 v_2^T y\mathbf{x}\right)\right)^{-1} v_2^T y\mathbf{x} = -\frac{2}{\eta}$. Assume for the sake of contradiction that $v_2^T y\mathbf{x} \le v_1^T y\mathbf{x}$. Then,

$$\left(1 + \exp\left(\|q\|_2^2 v_2^T y\mathbf{x}\right)\right)^{-1} v_2^T y\mathbf{x} \le \left(1 + \exp\left(\|q\|_2^2 v_1^T y\mathbf{x}\right)\right)^{-1} v_2^T y\mathbf{x} \qquad \text{by our assumption}$$

$$\left(1 + \exp\left(\|q\|_2^2 v_1^T y\mathbf{x}\right)\right)^{-1} v_1^T y\mathbf{x} \ge \left(1 + \exp\left(\|q\|_2^2 v_2^T y\mathbf{x}\right)\right)^{-1} v_2^T y\mathbf{x} = -\frac{2}{\eta} \quad \text{since } v_2^T y\mathbf{x} \le v_1^T y\mathbf{x}$$

This contradicts the necessary condition we determined in lemma B.8. □

In the following lemma, we show an upper bound in the $v_t^T y\mathbf{x}$ direction for a nullcline of $\|q_t\|_2^2$ (see the bottom blue curve in Figure 6), which is also an upper bound for the boundary separating regions 3.1 and 3.2.

Lemma B.10. *For any $\|q\|_2^2 > 0$, the $v_t^T y\boldsymbol{x}$ that satisfies $\left(1 + \exp\left(\|q\|_2^2 v^T y\boldsymbol{x}\right)\right)^{-1} v^T y\boldsymbol{x} = -\frac{2}{\eta}$ is less than $-\frac{2}{\eta}$.*

Proof. First, we will show that for $0 < \|q_1\|_2^2 < \|q_2\|_2^2$ and $v_1^T y\mathbf{x}, v_2^T y\mathbf{x}$ which satisfy $\left(1 + \exp\left(\|q_1\|_2^2 v_1^T y\mathbf{x}\right)\right)^{-1} v_1^T y\mathbf{x} = \left(1 + \exp\left(\|q\|_2^2 v_1^T y\mathbf{x}\right)\right)^{-1} v_1^T y\mathbf{x} = -\frac{2}{\eta}$, we get that $v_1^T y\mathbf{x} < v_2^T y\mathbf{x}$. Assume for the sake of contradiction that $v_1^T y\mathbf{x} \ge v_2^T y\mathbf{x}$. Then,

$$\|q_1\|_2^2 v_1^T y\mathbf{x} > \|q_2\|_2^2 v_2^T y\mathbf{x} \qquad \text{since } v_1^T y\mathbf{x} \ge v_2^T y\mathbf{x} \text{ and } \|q_1\|_2^2 < \|q_2\|_2^2$$

$$\left(1 + \exp\left(\|q_1\|_2^2 v_1^T y\mathbf{x}\right)\right)^{-1} v_1^T y\mathbf{x} > \left(1 + \exp\left(\|q_2\|_2^2 v_2^T y\mathbf{x}\right)\right)^{-1} v_2^T y\mathbf{x},$$

where the last step comes from our assumption that $v_1^T y\mathbf{x} \ge v_2^T y\mathbf{x}$ and $\left(1 + \exp\left(\|q_1\|_2^2 v_1^T y\mathbf{x}\right)\right)^{-1} v_1^T y\mathbf{x} = -\frac{2}{\eta}$. Now that we have proven the above, we consider $\|q\|_2^2 \to \infty$. Then, we see that $\left(1 + \exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} \to 1$. So, $\left(1 + \exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} v^T y\mathbf{x} = -\frac{2}{\eta}$ becomes $v^T y\mathbf{x} = -\frac{2}{\eta}$. □

We are now ready to show that once the system leaves region 3.2, it cannot go back.

Lemma B.11. *For any $k \in \mathbb{N}$, if $\|q_k\|_2^2 > 0, v_k^T y\boldsymbol{x}$ satisfies*

$$\left(v_k^T y\boldsymbol{x}\right)^2 \left(2 + \eta\left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\boldsymbol{x}\right)\right)^{-1} v_k^T y\boldsymbol{x}\right) + \|q_k\|_2^2 \|y\boldsymbol{x}\|_2^2 \left(1 + \eta\left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\boldsymbol{x}\right)\right)^{-1} v_k^T y\boldsymbol{x}\right)^2 \ge 0,$$

then $\left(\|q_{k+1}\|_2^2 > 0, v_{k+1}^T y\boldsymbol{x}\right)$ satisfies

$$\left(v_{k+1}^T y\boldsymbol{x}\right)^2 \left(2 + \eta\left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\boldsymbol{x}\right)\right)^{-1} v_{k+1}^T y\boldsymbol{x}\right)$$
$$+ \|q_{k+1}\|_2^2 \|y\boldsymbol{x}\|_2^2 \left(1 + \eta\left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\boldsymbol{x}\right)\right)^{-1} v_{k+1}^T y\boldsymbol{x}\right)^2 \ge 0.$$

Proof. Assume some $k \in \mathbb{N}$ and $\|q_k\|_2^2 > 0, v_k^T y\mathbf{x}$ that satisfies

$$\left(v_k^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right) + \|q_k\|_2^2 \|y\mathbf{x}\|_2^2 \left(1 + \eta \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2 \geq 0.$$

We consider 3 cases for $\|q_k\|_2^2 > 0, v_k^T y\mathbf{x}$.

Case 1: $v_k^T y\mathbf{x} \geq 0$. Then, by lemma 4.1, we know that $v_{k+1}^T y\mathbf{x} > 0$. So, $\left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x} > 0 > -\frac{2}{\eta}$. This violates the necessary condition that we derived in lemma B.8.

Case 2: $-\frac{2}{\eta} \leq \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} < 0$.

We derive the following.

$$\begin{aligned}
&\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x} \geq \|q_k\|_2^2 v_k^T y\mathbf{x} && \text{by lemma B.7}\\
&v_k^T y\mathbf{x} \left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} \geq v_k^T y\mathbf{x} \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} && \text{since } v_k^T y\mathbf{x} < 0\\
&v_{k+1}^T y\mathbf{x} \left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} > \left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} && \text{by lemma 4.1}\\
&-\frac{2}{\eta} \leq v_k^T y\mathbf{x} \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} < v_{k+1}^T y\mathbf{x} \left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1}.
\end{aligned}$$

This also violates the necessary condition that we derived in lemma B.8.

Case 3: $\left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x} < -\frac{2}{\eta}$.

If $-\frac{2}{\eta} \leq \left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}$, then we use the violation of the necessary condition derived in lemma B.8 to conclude. Now, we assume $\left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x} < -\frac{2}{\eta}$. We start by deriving a relation between $\left(v_k^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)$ and $\left(v_{k+1}^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}\right)$.

$$\begin{aligned}
&\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x} \geq \|q_k\|_2^2 v_k^T y\mathbf{x} && \text{by lemma B.7}\\
&v_{k+1}^T y\mathbf{x} \left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1}\\
&> v_k^T y\mathbf{x} \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} && \text{since } v_k^T y\mathbf{x} < 0 \text{ and lemma 4.1}
\end{aligned}$$

$$\begin{aligned}
&\left(v_{k+1}^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}\right)\\
&> \left(v_{k+1}^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right) && \text{by lemma 4.1}
\end{aligned}$$

$$\begin{aligned}
&\left(v_{k+1}^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)\\
&> \left(v_k^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right) && \text{since } v_k^T y\mathbf{x}, v_{k+1}^T y\mathbf{x} < 0 \text{ and lemma 4.1}
\end{aligned}$$

$$\begin{aligned}
&\left(v_k^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)\\
&< \left(v_{k+1}^T y\mathbf{x}\right)^2 \left(2 + \eta \left(1 + \exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}\right).
\end{aligned}$$

Now, we derive a relation between $\|q_k\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2$ and $\|q_{k+1}\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}\right)^2$. With our assumption $\left(1+\exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x} < -\frac{2}{\eta}$, we know that

$$
\begin{aligned}
&\|q_{k+1}\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}\right)^2 \\
&= \left(\|q_k\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2\right)\left(1+\eta\left(1+\exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}\right)^2 \\
&> \|q_k\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2.
\end{aligned}
$$

Then we get that

$$
\begin{aligned}
&\left(v_{k+1}^T y\mathbf{x}\right)^2\left(2+\eta\left(1+\exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}\right) \\
&+\|q_{k+1}\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_{k+1}\|_2^2 v_{k+1}^T y\mathbf{x}\right)\right)^{-1} v_{k+1}^T y\mathbf{x}\right)^2 \\
&> \left(v_k^T y\mathbf{x}\right)^2\left(2+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right) \\
&+\|q_k\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\mathbf{x}\right)\right)^{-1} v_k^T y\mathbf{x}\right)^2 \\
&\geq 0.
\end{aligned}
$$

□

Finally, we show that for any starting point in region 3.2, the trajectory will eventually "escape" 3.2 and reach other regions.

Lemma B.12. *For any $k \in \mathbb{N}$, if $\|q_k\|_2^2 > 0, v_k^T y\boldsymbol{x}$ satisfies*

$$
\left(v_k^T y\boldsymbol{x}\right)^2\left(2+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\boldsymbol{x}\right)\right)^{-1} v_k^T y\boldsymbol{x}\right)+\|q_k\|_2^2\|y\boldsymbol{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\boldsymbol{x}\right)\right)^{-1} v_k^T y\boldsymbol{x}\right)^2 < 0,
$$

then there exists some $l \in \mathbb{N}$, where $k < l \leq \lceil * \rceil \frac{-\left(\frac{2}{\eta}+v_k^T y\boldsymbol{x}\right)\left(1+\exp\left(\|q_k\|_2^2 v_k^T y\boldsymbol{x}\right)\right)}{\eta\|q_k\|_2^2\|y\boldsymbol{x}\|_2^2} + k + 1$, such that $\|q_l\|_2^2, v_l^T y\boldsymbol{x}$ satisfies

$$
\left(v_l^T y\boldsymbol{x}\right)^2\left(2+\eta\left(1+\exp\left(\|q_l\|_2^2 v_l^T y\boldsymbol{x}\right)\right)^{-1} v_l^T y\boldsymbol{x}\right)+\|q_l\|_2^2\|y\boldsymbol{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_l\|_2^2 v_l^T y\boldsymbol{x}\right)\right)^{-1} v_l^T y\boldsymbol{x}\right)^2 \geq 0.
$$

Proof. Consider the set $\{v^T y\mathbf{x}\}$ that satisfies

$$
\left(v^T y\mathbf{x}\right)^2\left(2+\eta\left(1+\exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} v^T y\mathbf{x}\right)+\|q\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q\|_2^2 v^T y\mathbf{x}\right)\right)^{-1} v^T y\mathbf{x}\right)^2 < 0
$$

for all $\|q\|_2^2 > 0$. By lemmas 9 and 10, we know that the set is upper bounded by $-\frac{2}{\eta}$.

Assume for the sake of contradiction that there exists some $\|q_j\|_2^2, v_j^T y\mathbf{x}$ that satisfies

$$
\left(v_j^T y\mathbf{x}\right)^2\left(2+\eta\left(1+\exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)^{-1} v_j^T y\mathbf{x}\right)+\|q_j\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)^{-1} v_j^T y\mathbf{x}\right)^2 < 0.
$$

where for all $j < l \leq \lceil * \rceil \frac{-\left(\frac{2}{\eta}+v_j^T y\mathbf{x}\right)\left(1+\exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)}{\eta\|q_j\|_2^2\|y\mathbf{x}\|_2^2} + j + 1$, $\|q_l\|_2^2, v_l^T y\mathbf{x}$ satisfies

$$
\left(v_l^T y\mathbf{x}\right)^2\left(2+\eta\left(1+\exp\left(\|q_l\|_2^2 v_l^T y\mathbf{x}\right)\right)^{-1} v_l^T y\mathbf{x}\right)+\|q_l\|_2^2\|y\mathbf{x}\|_2^2\left(1+\eta\left(1+\exp\left(\|q_l\|_2^2 v_l^T y\mathbf{x}\right)\right)^{-1} v_l^T y\mathbf{x}\right)^2 < 0.
$$

For conciseness, let's set $\xi = \lceil * \rceil \frac{-\left(\frac{2}{\eta} + v_j^T y\mathbf{x}\right)\left(1+\exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)}{\eta \|q_j\|_2^2 \|y\mathbf{x}\|_2^2} + j + 1$. We derive the following.

$$
\begin{aligned}
-\frac{2}{\eta} &\geq v_\xi^T y\mathbf{x} \\
&= v_j^T y\mathbf{x} + \eta \|y\mathbf{x}\|_2^2 \sum_{i=j}^{\xi-1} \left(1 + \exp\left(\|q_i\|_2^2 v_i^T y\mathbf{x}\right)\right)^{-1} \|q_i\|_2^2 \\
&> v_j^T y\mathbf{x} + \eta \|y\mathbf{x}\|_2^2 \|q_j\|_2^2 \sum_{i=j}^{\xi-1} \left(1 + \exp\left(\|q_i\|_2^2 v_i^T y\mathbf{x}\right)\right)^{-1} \qquad \text{by lemmas 6 and 8} \\
&> v_j^T y\mathbf{x} + \eta \|y\mathbf{x}\|_2^2 \|q_j\|_2^2 \left(1 + \exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)^{-1} (\xi - j),
\end{aligned}
$$

where the last step comes from lemma B.7 and our assumption for all $j < l \leq \lceil * \rceil \frac{-\left(\frac{2}{\eta} + v_j^T y\mathbf{x}\right)\left(1+\exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)}{\eta \|q_j\|_2^2 \|y\mathbf{x}\|_2^2} + j + 1$.

Then, we get

$$
\frac{-\left(\frac{2}{\eta} + v_j^T y\mathbf{x}\right)\left(1 + \exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)}{\eta \|y\mathbf{x}\|_2^2 \|q_j\|_2^2} > \xi - j = \lceil * \rceil \frac{-\left(\frac{2}{\eta} + v_j^T y\mathbf{x}\right)\left(1 + \exp\left(\|q_j\|_2^2 v_j^T y\mathbf{x}\right)\right)}{\eta \|q_j\|_2^2 \|y\mathbf{x}\|_2^2} + 1,
$$

which is a contradiction.

□

B.2 Two Datapoint Setting ($N = 2$)

In the following subsection, we provide our fixed analysis, sharpness approximations and analysis, and additional experiments.

Prior to that discussion, we first prove Claim 5.1 below about the strictly increasing behavior of m_t. Recall in Section 5 that we defined m_t as $v_t^T (\mathbf{x}_1 - \mathbf{x}_2)$, the component of v_t in the max-margin direction.

Proof of Claim 5.1. Consider some m_k and $\|q_k\|_2^2 > 0$. Based on the iterated map of m_t, we see the following.

$$
\begin{aligned}
m_l = m_{l-1} + \eta \|q_{l-1}\|_2^2 \left(1 - \mathbf{x}_1^T \mathbf{x}_2\right) \Bigg(&\left(1 + \exp\left(\frac{1}{2}\|q_{l-1}\|_2^2 \left(m_{l-1} + c_{l-1}\right)\right)\right)^{-1} \\
&+ \left(1 + \exp\left(\frac{1}{2}\|q_{l-1}\|_2^2 \left(m_{l-1} - c_{l-1}\right)\right)\right)^{-1} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= m_k + \eta \|q_k\|_2^2 \left(1 - \mathbf{x}_1^T \mathbf{x}_2\right) \Bigg(&\left(1 + \exp\left(\frac{1}{2}\|q_k\|_2^2 \left(m_k + c_k\right)\right)\right)^{-1} \\
&+ \left(1 + \exp\left(\frac{1}{2}\|q_k\|_2^2 \left(m_k - c_k\right)\right)\right)^{-1} \Bigg) \\
+\eta \left(1 - \mathbf{x}_1^T \mathbf{x}_2\right) \sum_{i=k+1}^{l-1} \|q_i\|_2^2 \Bigg(&\left(1 + \exp\left(\frac{1}{2}\|q_i\|_2^2 \left(m_i + c_i\right)\right)\right)^{-1} \\
&+ \left(1 + \exp\left(\frac{1}{2}\|q_i\|_2^2 \left(m_i - c_i\right)\right)^{-1}\right)
\end{aligned}
$$

$$\geq m_k + \eta\|q_k\|_2^2\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\Bigg(\left(1+\exp\left(\frac{1}{2}\|q_k\|_2^2\left(m_k+c_k\right)\right)\right)^{-1} + \left(1+\exp\left(\frac{1}{2}\|q_k\|_2^2\left(m_k-c_k\right)\right)\right)^{-1}\Bigg) \quad \text{since } \|q_k\|_2^2 > 0$$

$$> m_k$$

□

B.2.1 Fixed Point/Direction Analysis

Recall in Section 5, we showed that the dynamics of our model in the two datapoint setting can be captured by the following system $(\|q_t\|_2^2, c_t, m_t)$, where $c_t = v_t^T(\mathbf{x}_1+\mathbf{x}_2)$ (projection of model parameter v in max-margin complement direction) and $m_t = v_t^T(\mathbf{x}_1-\mathbf{x}_2)$ (projection of model parameter v in max-margin direction). We proceed to analyze the fixed points/directions for the two datapoint system below. We make the following claims about the system.

Claim B.13 (Nullclines of $\|q_t\|_2^2$)**.** *The nullclines for $\|q_t\|_2^2$ are.*

$$\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t+c_t\right)\right)\right)^{-1}(m_t+c_t) + \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t-c_t\right)\right)\right)^{-1}(m_t-c_t) = b$$

for $b \in \{0, -\frac{4}{\eta}\}$ and $\|q_t\|_2^2 = 0$.

Claim B.14 (Nullclines of c_t)**.** *The nullclines for c_t are $\|q_t\|_2^2 = 0$ and $c_t = 0$.*

Claim B.15 (Fixed Points/Directions for $(\|q_t\|_2^2, c_t, m_t)$ system)**.** *The fixed points/directions for the system $(\|q_t\|_2^2, c_t, m_t)$ are $\|q_t\|_2^2 = 0$ and $c_t = 0, \|q_t\|_2^2 m_t \to \infty$, where $c_t = 0, \|q_t\|_2^2 m_t \to \infty$ is Lyapunov stable and $\|q_t\|_2^2 = 0$ is stable for $-\frac{4}{\eta} < m_t < 0$.*

Claim B.16 (Stable $c_t = 0$)**.** *If our system satisfies the following, then the $c_t = 0$ plane is stable.*

$$\|q_t\|_2^4\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2 m_t\right)\right)^{-2}\exp\left(\frac{1}{2}\|q_t\|_2^2 m_t\right) < \frac{2}{\eta\left(1+\boldsymbol{x}_1^T\boldsymbol{x}_2\right)}$$

Claim B.17 ($|c_t|$ Growth Boundary)**.** *If our system satisfies the following, then c_t will grow in absolute value.*

$$\|q_t\|_2^2\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t-|c_t|\right)\right)\right)^{-1} - \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t+|c_t|\right)\right)\right)^{-1} > \frac{2|c_t|}{\eta\left(1+\boldsymbol{x}_1^T\boldsymbol{x}_2\right)}$$

We prove the claims below, starting with Claim B.13 about the nullclines of $\|q_t\|_2^2$.

Proof of Claim B.13. To solve for the nullclines of $\|q_t\|_2^2$, we first specify its iterated map.

$$\begin{aligned}
&\|q_{t+1}\|_2^2 = \|q_t\|_2^2 \\
&+ \eta\|q_t\|_2^2\left[\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t+c_t\right)\right)\right)^{-1}(m_t+c_t) + \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t-c_t\right)\right)\right)^{-1}(m_t-c_t)\right] \\
&+ \frac{\eta^2}{4}\|q_t\|_2^2\left[\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t+c_t\right)\right)\right)^{-1}(m_t+c_t) + \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t-c_t\right)\right)\right)^{-1}(m_t-c_t)\right]^2 \\
&= \|q_t\|_2^2\left[1+\frac{\eta}{2}\left(\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t+c_t\right)\right)\right)^{-1}(m_t+c_t) + \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t-c_t\right)\right)\right)^{-1}(m_t-c_t)\right)\right]^2
\end{aligned}$$

With the iterated map above, we find the nullclines by solving $\|q_{t+1}\|_2^2 - \|q_t\|_2^2 = 0$. With some calculation we find get the following.

$$\begin{aligned}
0 = \|q_t\|_2^2&\left[\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t+c_t\right)\right)\right)^{-1}(m_t+c_t) + \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t-c_t\right)\right)\right)^{-1}(m_t-c_t)\right] \\
&\cdot\left[1+\frac{\eta}{4}\left[\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t+c_t\right)\right)\right)^{-1}(m_t+c_t) + \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t-c_t\right)\right)\right)^{-1}(m_t-c_t)\right]\right]
\end{aligned}$$

From the above, we see that the nullclines are $\|q_t\|_2^2 = 0$ and the following:

$$\left[\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t+c_t)\right)\right)^{-1}(m_t+c_t)+\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t-c_t)\right)\right)^{-1}(m_t-c_t)\right]=0$$

$$\left[\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t+c_t)\right)\right)^{-1}(m_t+c_t)+\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t-c_t)\right)\right)^{-1}(m_t-c_t)\right]=-\frac{4}{\eta}.$$

□

Now, we prove Claim B.14 for the nullclines of c_t.

Proof of Claim B.14. To calculate the nullclines of c_t, we first specify its iterated map below.

$$c_{t+1}=c_t+\eta\|q_t\|_2^2\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)\left(\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t+c_t)\right)\right)^{-1}-\left(1+\exp\left(\|q_t\|_2^2\frac{1}{2}\,(m_t-c_t)\right)\right)^{-1}\right)$$

With the iterated map above, we can find the nullclines by solving $c_{t+1}-c_t=0$, which is equivalent to solving

$$\|q_t\|_2^2\left(\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t+c_t)\right)\right)^{-1}-\left(1+\exp\left(\|q_t\|_2^2\frac{1}{2}\,(m_t-c_t)\right)\right)^{-1}\right)=0.$$

We see that solutions to the above are $\|q_t\|_2^2=0$ and $c_t=0$. □

Next, we prove Claim B.15 about the fixed points/directions.

Proof of Claim B.15. Since we already have the nullclines for $\|q_t\|_2^2$ and c_t from Claims B.13 and B.14, we find the fixed points/directions by first calculating the nullclines for m_t. We first specify the iterated map of m_t.

$$m_{t+1}=m_t+\eta\|q_t\|_2^2\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\left(\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t+c_t)\right)\right)^{-1}+\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t-c_t)\right)\right)^{-1}\right)$$

Then, we solve for the nullclines of m_t by solving

$$\|q_t\|_2^2\left(\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t+c_t)\right)\right)^{-1}+\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\,(m_t-c_t)\right)\right)^{-1}\right)=0.$$

From our calculation, we find that the nullclines are $\|q_t\|_2^2=0$ and $\|q_t\|_2^2\,(m_t-c_t)\to\infty$, $\|q_t\|_2^2\,(m_t+c_t)\to\infty$. Considering the above with the nullclines of c_t and $\|q_t\|_2^2$, we find that the fixed points/directions of the system are $\|q_t\|_2^2=0$ and $c_t=0, \|q_t\|_2^2 m_t\to\infty$.

With the fixed points/directions calculated, we proceed to determine their the stability. We do so by solving for the eigenvalues of the Jacobian of the system at each fixed point. First, we form the Jacobian matrix below.

$$J(\|q_{t+1}\|_2^2, c_{t+1}, m_{t+1})=\begin{bmatrix}\frac{\partial\|q_{t+1}\|_2^2}{\partial\|q_t\|_2^2} & \frac{\partial\|q_{t+1}\|_2^2}{\partial c_t} & \frac{\partial\|q_{t+1}\|_2^2}{\partial m_t}\\ \frac{\partial c_{t+1}}{\partial\|q_t\|_2^2} & \frac{\partial c_{t+1}}{\partial c_t} & \frac{\partial c_{t+1}}{\partial m_t}\\ \frac{\partial m_{t+1}}{\partial\|q_t\|_2^2} & \frac{\partial m_{t+1}}{\partial c_t} & \frac{\partial m_{t+1}}{\partial m_t}\end{bmatrix}$$

Next, we evaluate it at the fixed direction $c_t=0, \|q_t\|_2^2 m_t\to\infty$ as shown below.

$$J(\|q_{t+1}\|_2^2, c_{t+1}, m_{t+1})=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Since all the eigenvalues are 1, we know that $c_t = 0, \|q_t\|_2^2 m_t \to \infty$ is Lyapunov stable.

For fixed point $\|q_t\|_2^2 = 0$, the Jacobian reduces to the following.

$$J(\|q_{t+1}\|_2^2, c_{t+1}, m_{t+1}) = \begin{bmatrix} 1 + \eta m_t + \frac{\eta^2}{4} m_t^2 & 0 & 0 \\ 0 & 1 & 0 \\ \eta\left(1 - \mathbf{x}_1^T \mathbf{x}_2\right) & 0 & 1 \end{bmatrix}$$

where the eigenvalues are 1 (with algebraic multiplicity 2) and $1 + \eta m_t + \frac{\eta^2}{4} m_t^2$.

This means that there exists some part of the m_t axis that is stable if the following condition is satisfied:

$$-\frac{8}{\eta^2} < \left(\frac{4}{\eta} + m_t\right) m_t < 0$$

Since $-\frac{8}{\eta^2}$ is not a tight lower bound of the function, we get that the condition is satisfied for $-\frac{4}{\eta} < m_t < 0$.

□

In the following, we prove Claim B.16 about the stable region of $c_t = 0$ plane.

Proof of Claim B.16. To determine the stability of the $c_t = 0$ plane, we study the eigenvalues of the Jacobian of the system at $c_t = 0$. From our calculations, we get that

$$J(\|q_{t+1}\|_2^2, c_{t+1}, m_{t+1}) = \begin{bmatrix} \frac{\partial \|q_{t+1}\|_2^2}{\partial \|q_t\|_2^2}\Big|_{c_t=0} & 0 & \frac{\partial \|q_{t+1}\|_2^2}{\partial m_t}\Big|_{c_t=0} \\ 0 & \frac{\partial c_{t+1}}{\partial c_t}\Big|_{c_t=0} & 0 \\ \frac{\partial m_{t+1}}{\partial \|q_t\|_2^2}\Big|_{c_t=0} & 0 & \frac{\partial m_{t+1}}{\partial m_t}\Big|_{c_t=0} \end{bmatrix}$$

From the Jacobian, the relevant eigenvalue is

$$\lambda = \frac{\partial c_{t+1}}{\partial c_t}\Big|_{c_t=0} = 1 - \eta \|q_t\|_2^4 \left(1 + \mathbf{x}_1^T \mathbf{x}_2\right) \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 m_t\right)\right)^{-2} \exp\left(\frac{1}{2}\|q_t\|_2^2 m_t\right)$$

, since the eigendirection is $\nu = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (i.e., in the direction of c_t).

To determine where the $c_t = 0$ plane is stable, we consider the following condition:

$$-1 < 1 - \eta \|q_t\|_2^4 \left(1 + \mathbf{x}_1^T \mathbf{x}_2\right) \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 m_t\right)\right)^{-2} \exp\left(\frac{1}{2}\|q_t\|_2^2 m_t\right) < 1$$

$$\|q_t\|_2^4 \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 m_t\right)\right)^{-2} \exp\left(\frac{1}{2}\|q_t\|_2^2 m_t\right) < \frac{2}{\eta\left(1 + \mathbf{x}_1^T \mathbf{x}_2\right)}$$

□

Lastly, we prove Claim B.17 about the $|c_t|$ growth boundary.

Proof of Claim B.17. To understand when c_t will grow in absolute value, we look at the nullclines for c_t^2. First, we specify the iterated map for c_t^2 below.

$$\begin{aligned} c_{t+1}^2 = c_t^2 \\ - 2\eta\left(1 + \mathbf{x}_1^T \mathbf{x}_2\right) \|q_t\|_2^2 \sqrt{c_t^2} \left[\left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t - \sqrt{c_t^2}\right)\right)\right)^{-1} - \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t + \sqrt{c_t^2}\right)\right)\right)^{-1}\right] \\ + \eta^2 \left(1 + \mathbf{x}_1^T \mathbf{x}_2\right)^2 \|q_t\|_2^4 \left[\left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t - \sqrt{c_t^2}\right)\right)\right)^{-1} - \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t + \sqrt{c_t}\right)\right)\right)^{-1}\right]^2 \end{aligned}$$

Now, to solve for the nullclines (i.e., $c_{t+1}^2 - c_t^2 = 0$), we solve the following.

$$0 = \|q_t\|_2^2 \left[\left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t - \sqrt{c_t^2}\right)\right)\right)^{-1} - \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t + \sqrt{c_t^2}\right)\right)\right)^{-1} \right]$$
$$\cdot \left[-2\sqrt{c_t^2} + \eta\left(1 + \mathbf{x}_1^T\mathbf{x}_2\right)\|q_t\|_2^2 \left[\left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t - \sqrt{c_t^2}\right)\right)\right)^{-1} - \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t + \sqrt{c_t^2}\right)\right)\right)^{-1} \right] \right]$$

As we are already familiar with the nullclines $\|q_t\|_2^2 = 0$ and $c_t = 0$ from Claim B.14, we see that c_t^2 grows if our system satisfies the following:

$$\|q_t\|_2^2 \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t - \sqrt{c_t^2}\right)\right)\right)^{-1} - \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t + \sqrt{c_t^2}\right)\right)\right)^{-1} > \frac{2\sqrt{c_t^2}}{\eta\left(1 + \mathbf{x}_1^T\mathbf{x}_2\right)}$$

This is equivalent to saying the system should satisfy

$$\|q_t\|_2^2 \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t - |c_t|\right)\right)\right)^{-1} - \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2 \left(m_t + |c_t|\right)\right)\right)^{-1} > \frac{2|c_t|}{\eta\left(1 + \mathbf{x}_1^T\mathbf{x}_2\right)}$$

□

B.2.2 Sharpness Approximations

As mentioned in Section 5, we derived some approximations for the sharpness in our two datapoint setting. In this section, we derive the four sharpness approximations and their error bounds, which we show below.

First, we define the following notation.

$$m_t = v_t^T\left(\mathbf{x}_1 - \mathbf{x}_2\right) \qquad \text{(projection onto max-margin)}$$
$$c_t = v_t^T\left(\mathbf{x}_1 + \mathbf{x}_2\right) \qquad \text{(projection onto max-margin complement)}$$
$$\psi_{1,t} = \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t + c_t\right)\right)\right)^{-2} \exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t + c_t\right)\right)$$
$$\psi_{2,t} = \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t - c_t\right)\right)\right)^{-2} \exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t - c_t\right)\right)$$
$$\phi_{1,t} = \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t + c_t\right)\right)\right)^{-1}$$
$$\phi_{2,t} = \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2\left(m_t - c_t\right)\right)\right)^{-1}$$

Now, we introduce our approximations.

Approximation 1 Our first sharpness approximation is

$$\frac{1}{2}\|q_t\|_2^4 \left(\left(\psi_{1,t} + \psi_{2,t}\right) \pm \sqrt{\left(\psi_{1,t} - \psi_{2,t}\right)^2 + 4\psi_{1,t}\psi_{2,t}\left(\mathbf{x}_1^T\mathbf{x}_2\right)^2} \right)$$

where the error with the true sharpness is upper bounded by

$$\frac{1}{4}\left(\psi_{1,t}\|q_t\|_2^2\left(m_t + c_t\right)^2 - \phi_{1,t}\left(m_t + c_t\right) + \psi_{2,t}\|q_t\|_2^2\left(m_t - c_t\right)^2 - \phi_{2,t}\left(m_t - c_t\right)\right)$$
$$+ \sqrt{\begin{array}{l} \frac{1}{16}\left(\psi_{1,t}\|q_t\|_2^2\left(m_t + c_t\right)^2 - \phi_{1,t}\left(m_t + c_t\right) + \psi_{2,t}\|q_t\|_2^2\left(m_t - c_t\right)^2 - \phi_{2,t}\left(m_t - c_t\right)\right)^2 \\ + 2\|q_t\|_2^2\left(\left(\frac{1}{2}\psi_{1,t}\|q_t\|_2^2\left(m_t + c_t\right) - \phi_{1,t}\right)^2 + \left(-\frac{1}{2}\psi_{2,t}\|q_t\|_2^2\left(m_t - c_t\right) + \phi_{2,t}\right)^2\right) \\ + 4\|q_t\|_2^2\mathbf{x}_1^T\mathbf{x}_2\left(\frac{1}{2}\psi_{1,t}\|q_t\|_2^2\left(m_t + c_t\right) - \phi_{1,t}\right)\left(-\frac{1}{2}\psi_{2,t}\|q_t\|_2^2\left(m_t - c_t\right) + \phi_{2,t}\right) \end{array}}$$

Approximation 2 Our second sharpness approximation is

$$\frac{1}{4}\left(\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right)^2+\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)^2-\phi_{1,t}\left(m_t+c_t\right)-\phi_{2,t}\left(m_t-c_t\right)+\left(\psi_{1,t}+\psi_{2,t}\right)\|q_t\|_2^4\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\right)$$
$$+\frac{1}{2}\sqrt{\begin{array}{l}\frac{1}{4}\Big(\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right)^2+\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)^2-\phi_{1,t}\left(m_t+c_t\right)-\phi_{2,t}\left(m_t-c_t\right)\\+\left(\psi_{1,t}+\psi_{2,t}\right)\|q_t\|_2^4\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\Big)^2-\Big(4\psi_{1,t}\psi_{2,t}\|q_t\|_2^6c_t^2+2\left(\phi_{2,t}\psi_{1,t}-\phi_{1,t}\psi_{2,t}\right)\|q_t\|_2^4c_t\\+3\|q_t\|_2^4\left(\psi_{1,t}\left(m_t+c_t\right)+\psi_{2,t}\left(m_t-c_t\right)\right)\left(\phi_{1,t}+\phi_{2,t}\right)-4\left(\phi_{1,t}+\phi_{2,t}\right)^2\|q_t\|_2^2\Big)\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\end{array}}$$

where the error with the true sharpness is upper bounded by

$$\frac{1}{4}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)$$
$$+\frac{1}{2}\|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2}\sqrt{\begin{array}{l}\frac{1}{4}\left(\psi_{1,t}+\psi_{2,t}\right)^2\|q_t\|_2^6\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)+\left(\psi_{1,t}-\psi_{2,t}\right)^2\|q_t\|_2^6\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\\+4\left(\frac{1}{2}\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right)-\frac{1}{2}\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)-\phi_{1,t}+\phi_{2,t}\right)^2\end{array}}$$

Approximation 3 (Low-Error Approximation) Our third sharpness approximation is

$$\frac{1}{4}\left(\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right)^2+\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)^2-\phi_{1,t}\left(m_t+c_t\right)-\phi_{2,t}\left(m_t-c_t\right)+\left(\psi_{1,t}+\psi_{2,t}\right)\|q_t\|_2^4\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)\right)$$
$$+\frac{1}{2}\sqrt{\begin{array}{l}\frac{1}{4}\Big(\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right)^2+\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)^2-\phi_{1,t}\left(m_t+c_t\right)-\phi_{2,t}\left(m_t-c_t\right)\\+\left(\psi_{1,t}+\psi_{2,t}\right)\|q_t\|_2^4\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)\Big)^2-\Big(4\psi_{1,t}\psi_{2,t}\|q_t\|_2^6m_t^2-2\|q_t\|_2^4\left(\phi_{1,t}\psi_{2,t}+\phi_{2,t}\psi_{1,t}\right)m_t\\+3\|q_t\|_2^4\left(\psi_{1,t}\left(m_t+c_t\right)-\psi_{2,t}\left(m_t-c_t\right)\right)\left(\phi_{1,t}-\phi_{2,t}\right)-4\left(\phi_{1,t}-\phi_{2,t}\right)^2\|q_t\|_2^2\Big)\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)\end{array}}$$

where the error with the true sharpness is upper bounded by

$$\frac{1}{4}\left(\psi_{1,t}+\psi_{2,t}\right)\|q_t\|_2^4\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)$$
$$+\frac{1}{2}\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}\sqrt{\begin{array}{l}\frac{1}{4}\left(\psi_{1,t}+\psi_{2,t}\right)^2\|q_t\|_2^6\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)+\left(\psi_{1,t}-\psi_{2,t}\right)^2\|q_t\|_2^6\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)\\+4\left(\frac{1}{2}\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right)-\phi_{1,t}+\frac{1}{2}\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)-\phi_{2,t}\right)^2\end{array}}$$

Approximation 4 (Interpretable Approximation) Our fourth sharpness approximation is

$$\frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2)$$

where the error with the true sharpness is upper bounded by

$$\sqrt{\begin{array}{l}\frac{1}{4}\left(\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right))^2+\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)^2-\phi_{1,t}\left(m_t+c_t\right)-\phi_{2,t}\left(m_t-c_t\right)\right)^2\\+2\|q_t\|_2^2\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)\left(\frac{1}{2}\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right)-\frac{1}{2}\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)-\phi_{1,t}+\phi_{2,t}\right)^2\\+2\|q_t\|_2^2\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\left(\frac{1}{2}\psi_{1,t}\|q_t\|_2^2\left(m_t+c_t\right)+\frac{1}{2}\psi_{2,t}\|q_t\|_2^2\left(m_t-c_t\right)-\phi_{1,t}-\phi_{2,t}\right)^2\\+\frac{1}{2}\left(\psi_{1,t}-\psi_{2,t}\right)^2\|q_t\|_2^8\left(1-(\mathbf{x}_1^T\mathbf{x}_2)^2\right)+\frac{1}{4}\left(\psi_{1,t}+\psi_{2,t}\right)^2\|q_t\|_2^8\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)^2\end{array}}$$

From our approximations, we claim that approximation 3 is a low-error approximation of the sharpness in our setting. We also denote approximation 4 as our interpretable sharpness approximation based on its simplicity in form.

In the remainder of the section, we perform some preliminary calculations necessary for the approximations, show our derivations of the four sharpness approximations and their error bounds, and provide some empirical results and error bound comparisons to support for our claim about approximation 3.

Preliminary Calculations Before getting into the details of the approximations, we perform some preliminarily calculations on the Loss Hessian. With $\theta = (q_t, u_t, v_t)$, we start with

$$\begin{aligned}&\nabla_{\theta_t} L(q_t, u_t, v_t) - \lambda I_{2h+d} \\ &= \begin{bmatrix} A & B & C \\ B & A & C \\ C^T & C^T & D \end{bmatrix}\end{aligned}$$

where

$$\begin{aligned}A &= \left[\psi_{1,t}\left(v_t^T \mathbf{x}_1\right)^2 + \psi_{2,t}\left(v_t^T \mathbf{x}_2\right)^2\right] q_t q_t^T - \lambda I_h \\ B &= -\left[\phi_{1,t} v_t^T \mathbf{x}_1 - \phi_{2,t} v_t^T \mathbf{x}_2\right] I_h + \left[\psi_{1,t}\left(v_t^T \mathbf{x}_1\right)^2 + \psi_{2,t}\left(v_t^T \mathbf{x}_2\right)^2\right] q_t q_t^T \\ C &= -q_t\left[\phi_{1,t}\mathbf{x}_1^T - \phi_{2,t}\mathbf{x}_2^T\right] + \|q_t\|_2^2 q_t\left[\psi_{1,t} v_t^T \mathbf{x}_1 \mathbf{x}_1^T + \psi_{2,t} v_t^T \mathbf{x}_2 \mathbf{x}_2^T\right] \\ D &= \|q_t\|_2^4\left[\psi_{1,t}\mathbf{x}_1\mathbf{x}_1^T + \psi_{2,t}\mathbf{x}_2\mathbf{x}_2^T\right] - \lambda I_d.\end{aligned}$$

After performing the same similarity transforms as we discussed in the single datapoint setting (see Appendix B.1.2), we look at the following matrix decomposition:

$\begin{bmatrix} 0 & \sqrt{2}C \\ \sqrt{2}C^T & 0 \end{bmatrix} + \begin{bmatrix} A+B & 0 \\ 0 & D \end{bmatrix}$ in which we overload notation a little here by denoting

$$\begin{aligned}A + B &= 2\left[\psi_{1,t}\left(v_t^T \mathbf{x}_1\right)^2 + \psi_{2,t}\left(v_t^T \mathbf{x}_2\right)^2\right] q_t q_t^T - \left[\phi_{1,t} v_t^T \mathbf{x}_1 - \phi_{2,t} v_t^T \mathbf{x}_2\right] I_h \\ D &= \|q_t\|_2^4\left[\psi_{1,t}\mathbf{x}_1\mathbf{x}_1^T + \psi_{2,t}\mathbf{x}_2\mathbf{x}_2^T\right].\end{aligned}$$

In the above decomposition, we treat $\begin{bmatrix} 0 & \sqrt{2}C \\ \sqrt{2}C^T & 0 \end{bmatrix}$ as a perturbation on the matrix $\begin{bmatrix} A+B & 0 \\ 0 & D \end{bmatrix}$.

Based on our original definition for C, we know that $\begin{bmatrix} 0 & \sqrt{2}C \\ \sqrt{2}C^T & 0 \end{bmatrix}$ is rank-2 with eigenvalues $\pm\sqrt{2}\|q\|_2 \sqrt{\begin{array}{l}\left(\psi_{1,t}\|q\|_2^2 v_t^T \mathbf{x}_1 - \phi_{1,t}\right)^2 + \left(\psi_{2,t}\|q\|_2^2 v_t^T \mathbf{x}_2 + \phi_{2,t}\right)^2 \\ + 2\left(\psi_{1,t}\|q\|_2^2 v_t^T \mathbf{x}_1 - \phi_{1,t}\right)\left(\psi_{2,t}\|q\|_2^2 v_t^T \mathbf{x}_2 + \phi_{2,t}\right)\mathbf{x}_1^T \mathbf{x}_2\end{array}}$.

Since the perturbation occurs in a 3-dimensional subspsace spanned by $\begin{bmatrix} q \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \mathbf{x}_1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ \mathbf{x}_2 \end{bmatrix}$, we believe the 2 nonzero eigenvalues from the D block matrix and 1 nonzero eigenvalue from the $A+B$ block matrix have been perturbed.

We now focus on studying this perturbation in the 3-dimensional subspace through the following derivation.

$$U = \begin{bmatrix} q & 0 & 0 \\ 0 & \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$$

$$U^T \begin{bmatrix} A+B & \sqrt{2}C \\ \sqrt{2}C^T & D \end{bmatrix} U = \begin{bmatrix} q_t^T(A+B)q_t & \sqrt{2}q_t^T C\mathbf{x}_1 & \sqrt{2}q_t^T C\mathbf{x}_2 \\ \sqrt{2}q_t^T C\mathbf{x}_1 & \mathbf{x}_1^T D\mathbf{x}_1 & \mathbf{x}_1^T D\mathbf{x}_2 \\ \sqrt{2}q_t^T C\mathbf{x}_2 & \mathbf{x}_1^T D\mathbf{x}_2 & \mathbf{x}_2^T D\mathbf{x}_2 \end{bmatrix} = K$$

$$U^T U = \begin{bmatrix} \|q_t\|_2^2 & 0 & 0 \\ 0 & 1 & \mathbf{x}_1^T\mathbf{x}_2 \\ 0 & \mathbf{x}_1^T\mathbf{x}_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \|q_t\|_2^2 & 0 & 0 \\ 0 & 1+\mathbf{x}_1^T\mathbf{x}_2 & 0 \\ 0 & 0 & 1-\mathbf{x}_1^T\mathbf{x}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= Q\Lambda Q = G$$

$$(U^T U)^{-1} = Q\Lambda^{-1}Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\|q_t\|_2^2} & 0 & 0 \\ 0 & \frac{1}{1+\mathbf{x}_1^T\mathbf{x}_2} & 0 \\ 0 & 0 & \frac{1}{1-\mathbf{x}_1^T\mathbf{x}_2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \text{ (Assuming } \|q_t\|_2^2 \neq 0)$$

This means the sharpness is one of the solutions to the general eigenvalue problem $K\nu = \lambda G\nu$. We can also rewrite this to get that the solutions to the general eigenvalue problem are also the eigenvalues that we get from the matrices $G^{-1}K$, $\Lambda^{-\frac{1}{2}}QKQ\Lambda^{-\frac{1}{2}}$, and $\Lambda^{-1}QKQ$. Since $\Lambda^{-\frac{1}{2}}QKQ\Lambda^{-\frac{1}{2}}$ is real and symmetric, we know that its eigenvalues are real. We focus on the deriving the matrix $\Lambda^{-\frac{1}{2}}QKQ\Lambda^{-\frac{1}{2}}$.

$$\Lambda^{-\frac{1}{2}}QKQ\Lambda^{-\frac{1}{2}}$$

$$= \begin{bmatrix} \frac{1}{\|q_t\|_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}} \end{bmatrix}$$

$$\cdot \begin{bmatrix} q_t^T(A+B)q_t & q_t^T C(\mathbf{x}_1+\mathbf{x}_2) & q_t^T C(\mathbf{x}_1-\mathbf{x}_2) \\ q_t^T C(\mathbf{x}_1+\mathbf{x}_2) & \frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)^T D(\mathbf{x}_1+\mathbf{x}_2) & \frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)^T D(\mathbf{x}_1-\mathbf{x}_2) \\ q_t^T C(\mathbf{x}_1-\mathbf{x}_2) & \frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)^T D(\mathbf{x}_1-\mathbf{x}_2) & \frac{1}{2}(\mathbf{x}_1-\mathbf{x}_2)^T D(\mathbf{x}_1-\mathbf{x}_2) \end{bmatrix}$$

$$\cdot \begin{bmatrix} \frac{1}{\|q_t\|_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\|q_t\|_2^2} q_t^T(A+B)q_t & \frac{1}{\|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2}} q_t^T C(\mathbf{x}_1+\mathbf{x}_2) & \frac{1}{\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}} q_t^T C(\mathbf{x}_1-\mathbf{x}_2) \\ \frac{1}{\|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2}} q_t^T C(\mathbf{x}_1+\mathbf{x}_2) & \frac{1}{2(1+\mathbf{x}_1^T\mathbf{x}_2)}(\mathbf{x}_1+\mathbf{x}_2)^T D(\mathbf{x}_1+\mathbf{x}_2) & \frac{1}{2\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2}}(\mathbf{x}_1+\mathbf{x}_2)^T D(\mathbf{x}_1-\mathbf{x}_2) \\ \frac{1}{\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}} q_t^T C(\mathbf{x}_1-\mathbf{x}_2) & \frac{1}{2\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2}}(\mathbf{x}_1+\mathbf{x}_2)^T D(\mathbf{x}_1-\mathbf{x}_2) & \frac{1}{2(1-\mathbf{x}_1^T\mathbf{x}_2)}(\mathbf{x}_1-\mathbf{x}_2)^T D(\mathbf{x}_1-\mathbf{x}_2) \end{bmatrix}$$

$$
= \begin{bmatrix}
\begin{array}{l} 2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 \\ +2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 \\ -\phi_{1,t}v_t^T\mathbf{x}_1 \\ +\phi_{2,t}v_t^T\mathbf{x}_2 \end{array} & \begin{array}{l} \|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \\ +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ -\phi_{1,t}+\phi_{2,t}\Big) \end{array} & \begin{array}{l} \|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ -\phi_{1,t}-\phi_{2,t}\Big) \end{array} \\
\begin{array}{l} \|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \\ +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ -\phi_{1,t}+\phi_{2,t}\Big) \end{array} & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2) & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} \\
\begin{array}{l} \|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ -\phi_{1,t}-\phi_{2,t}\Big) \end{array} & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2)
\end{bmatrix}
$$

With the above $3x3$ matrix, we use Cauchy Interlacing Theorem (Stewart & Sun, 1990) to produce the approximations of the sharpness. For each approximation, we also calculate the error bound using Weyl's inequality.

Approximation 1 Derivation We consider the first submatrix formed by removing the first column and the first row.

$$
\begin{bmatrix}
\frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2) & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} \\
\frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2)
\end{bmatrix}
$$

We show the eigenvalues for this submatrix below.

$$
\lambda = \frac{1}{2}\|q_t\|_2^4\left((\psi_{1,t}+\psi_{2,t}) \pm \sqrt{(\psi_{1,t}-\psi_{2,t})^2+4\psi_{1,t}\psi_{2,t}\left(\mathbf{x}_1^T\mathbf{x}_2\right)^2}\right)
$$

Now, we use Weyl's inequality to find a bound on this. We write the general $3x3$ matrix using the following decomposition:

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2) & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} \\
0 & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2)
\end{bmatrix}
$$

$$
+\begin{bmatrix}
\begin{array}{l}2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2-\phi_{1,t}v_t^T\mathbf{x}_1\\ +2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2+\phi_{2,t}v_t^T\mathbf{x}_2\end{array} & \begin{array}{l}\|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2}\\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1-\phi_{1,t}\\ +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2+\phi_{2,t}\Big)\end{array} & \begin{array}{l}\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}\\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1-\phi_{1,t}\\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2-\phi_{2,t}\Big)\end{array}\\
\begin{array}{l}\|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2}\\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1-\phi_{1,t}\\ +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2+\phi_{2,t}\Big)\end{array} & 0 & 0\\
\begin{array}{l}\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}\\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1-\phi_{1,t}\\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2-\phi_{2,t}\Big)\end{array} & 0 & 0
\end{bmatrix}
$$

From our calculation, we see that one of the eigenvalues of the second matrix in the decomposition, denoted as our perturbation matrix, is 0. The other two eigenvalues are a symmetric pair that we show below, where the difference between the sharpness and our approximation in this case is upper bounded by the plus direction of the symmetric pair.

$$
\begin{aligned}
\lambda &= \frac{1}{2}\left(2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2-\phi_{1,t}v_t^T\mathbf{x}_1+2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2+\phi_{2,t}v_t^T\mathbf{x}_2\right)\\
&\pm\sqrt{\begin{array}{l}\frac{1}{4}\left(2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2-\phi_{1,t}v_t^T\mathbf{x}_1+2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2+\phi_{2,t}v_t^T\mathbf{x}_2\right)^2\\ +2\|q_t\|_2^2\left(\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1-\phi_{1,t}\right)^2+\left(\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2+\phi_{2,t}\right)^2\right)\\ +4\|q_t\|_2^2\mathbf{x}_1^T\mathbf{x}_2\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1-\phi_{1,t}\right)\left(\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2+\phi_{2,t}\right)\end{array}}
\end{aligned}
$$

Approximation 2 Derivation Next, we consider the second submatrix formed by removing the second column and second row.

$$
\begin{bmatrix}
\begin{array}{l}2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2\\ +2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2\\ -\phi_{1,t}v_t^T\mathbf{x}_1\\ +\phi_{2,t}v_t^T\mathbf{x}_2\end{array} & \begin{array}{l}\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}\\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1\\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2\\ -\phi_{1,t}-\phi_{2,t}\Big)\end{array}\\
\begin{array}{l}\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}\\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1\\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2\\ -\phi_{1,t}-\phi_{2,t}\Big)\end{array} & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2)
\end{bmatrix}
$$

We show the eigenvalues for this submatrix below.

$$\lambda = \frac{1}{2}\left(2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 + 2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 - \phi_{1,t}v_t^T\mathbf{x}_1 + \phi_{2,t}v_t^T\mathbf{x}_2 + \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2)\right)$$
$$\pm \frac{1}{2}\sqrt{\begin{array}{l}\left(2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 + 2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 - \phi_{1,t}v_t^T\mathbf{x}_1 + \phi_{2,t}v_t^T\mathbf{x}_2 + \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2)\right)^2 \\ -\left(4\psi_{1,t}\psi_{2,t}\|q_t\|_2^6\left(v_t^T\mathbf{x}_1 + v_t^T\mathbf{x}_2\right)^2 + 6\|q_t\|_2^4\left(\psi_{1,t}v_t^T\mathbf{x}_1 - \psi_{2,t}v_t^T\mathbf{x}_2\right)(\phi_{1,t}+\phi_{2,t})\right. \\ \left. + 2\left(\phi_{2,t}\psi_{1,t} - \phi_{1,t}\psi_{2,t}\right)\|q_t\|_2^4\left(v_t^T\mathbf{x}_1 + v_t^T\mathbf{x}_2\right) - 4\left(\phi_{1,t}+\phi_{2,t}\right)^2\|q_t\|_2^2\right)\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\end{array}}$$

Next, we solve for the Weyl's inequality bound decomposing the general matrix in the following way.

$$\begin{bmatrix} \begin{array}{l}2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 \\ +2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 \\ -\phi_{1,t}v_t^T\mathbf{x}_1 \\ +\phi_{2,t}v_t^T\mathbf{x}_2\end{array} & 0 & \begin{array}{l}\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \right. \\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ \left. -\phi_{1,t}-\phi_{2,t}\right)\end{array} \\ 0 & 0 & 0 \\ \begin{array}{l}\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \right. \\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ \left. -\phi_{1,t}-\phi_{2,t}\right)\end{array} & 0 & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2) \end{bmatrix}$$

$$+\begin{bmatrix} 0 & \begin{array}{l}\|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \right. \\ +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ \left. -\phi_{1,t}+\phi_{2,t}\right)\end{array} & 0 \\ \begin{array}{l}\|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \right. \\ +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ \left. -\phi_{1,t}+\phi_{2,t}\right)\end{array} & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2) & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} \\ 0 & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} & 0 \end{bmatrix}$$

From our calculations, we find the one of the eigenvalues of our perturbation matrix (the second matrix in the decomposition) is 0. The other two eigenvalues are a symmetric pair that we show below, where the error bound for this approximation is the plus direction of the symmetric pair.

$$\lambda = \frac{1}{4}(\psi_{1,t} + \psi_{2,t})\|q_t\|_2^4(1 + \mathbf{x}_1^T\mathbf{x}_2)$$
$$\pm \frac{1}{2}\|q_t\|_2\sqrt{1 + \mathbf{x}_1^T\mathbf{x}_2}\sqrt{\begin{array}{l}\frac{1}{4}\left(\psi_{1,t} + \psi_{2,t}\right)^2\|q_t\|_2^6(1 + \mathbf{x}_1^T\mathbf{x}_2) + \left(\psi_{1,t} - \psi_{2,t}\right)^2\|q_t\|_2^6\left(1 - \mathbf{x}_1^T\mathbf{x}_2\right) \\ + 4\left(\psi_{1,t}\|q_t\|_2^2 v_t^T\mathbf{x}_1 + \psi_{2,t}\|q_t\|_2^2 v_t^T\mathbf{x}_2 - \phi_{1,t} + \phi_{2,t}\right)^2\end{array}}$$

Approximation 3 Derivation Next, we consider the third submatrix formed by removing the third row and third column.

$$\begin{bmatrix} \begin{array}{l}2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 \\ + 2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 \\ - \phi_{1,t}v_t^T\mathbf{x}_1 \\ + \phi_{2,t}v_t^T\mathbf{x}_2\end{array} & \begin{array}{l}\|q_t\|_2\sqrt{1 + \mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2 v_t^T\mathbf{x}_1 \\ + \psi_{2,t}\|q_t\|_2^2 v_t^T\mathbf{x}_2 \\ - \phi_{1,t} + \phi_{2,t}\Big)\end{array} \\ \begin{array}{l}\|q_t\|_2\sqrt{1 + \mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2 v_t^T\mathbf{x}_1 \\ + \psi_{2,t}\|q_t\|_2^2 v_t^T\mathbf{x}_2 \\ - \phi_{1,t} + \phi_{2,t}\Big)\end{array} & \frac{1}{2}(\psi_{1,t} + \psi_{2,t})\|q_t\|_2^4(1 + \mathbf{x}_1^T\mathbf{x}_2) \end{bmatrix}$$

We show the eigenvalues for this submatrix below.

$$\lambda = \frac{1}{2}\left(2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 + 2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 - \phi_{1,t}v_t^T\mathbf{x}_1 + \phi_{2,t}v_t^T\mathbf{x}_2 + \frac{1}{2}(\psi_{1,t} + \psi_{2,t})\|q_t\|_2^4(1 + \mathbf{x}_1^T\mathbf{x}_2)\right)$$
$$\pm \frac{1}{2}\sqrt{\begin{array}{l}\left(2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 + 2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 - \phi_{1,t}v_t^T\mathbf{x}_1 + \phi_{2,t}v_t^T\mathbf{x}_2 + \frac{1}{2}(\psi_{1,t} + \psi_{2,t})\|q_t\|_2^4(1 + \mathbf{x}_1^T\mathbf{x}_2)\right)^2 \\ - \Big(4\psi_{1,t}\psi_{2,t}\|q_t\|_2^6\left(v_t^T\mathbf{x}_1 - v_t^T\mathbf{x}_2\right)^2 - 2\|q_t\|_2^4\left(\phi_{1,t}\psi_{2,t} + \phi_{2,t}\psi_{1,t}\right)\left(v_t^T\mathbf{x}_1 - v_t^T\mathbf{x}_2\right) \\ + 6\|q_t\|_2^4\left(\psi_{1,t}v_t^T\mathbf{x}_1 + \psi_{2,t}v_t^T\mathbf{x}_2\right)\left(\phi_{1,t} - \phi_{2,t}\right) - 4\left(\phi_{1,t} - \phi_{2,t}\right)^2\|q_t\|_2^2\Big)\left(1 + \mathbf{x}_1^T\mathbf{x}_2\right)\end{array}}$$

Then, we can decompose the general matrix in the following way for the error bound.

$$
\begin{bmatrix}
2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 - \phi_{1,t}v_t^T\mathbf{x}_1 \\
+\,2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 + \phi_{2,t}v_t^T\mathbf{x}_2 & \|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2} \\
& \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 - \phi_{1,t} \\
& +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 + \phi_{2,t}\Big) \\
\|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2} & \\
\cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 - \phi_{1,t} & \tfrac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2) \\
+\,\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 + \phi_{2,t}\Big) & \\
0 & 0
\end{bmatrix}
$$

$$
+\begin{bmatrix}
0 & 0 & \|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \\
& & \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 - \phi_{1,t} \\
& & -\,\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 - \phi_{2,t}\Big) \\
0 & 0 & \tfrac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} \\
\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} & & \\
\cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 - \phi_{1,t} & \tfrac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} & \tfrac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2) \\
-\,\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 - \phi_{2,t}\Big) & &
\end{bmatrix}
$$

For the perturbation matrix (the second matrix in the decomposition), we get that one of the eigenvalues is 0 and the other two form a symmetric pair that we show below, where difference between the sharpness and the approximation from this submatrix is upper bounded by the plus direction of the symmetric pair.

$$
\lambda = \frac{1}{4}\left(\psi_{1,t}+\psi_{2,t}\right)\|q_t\|_2^4\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)
$$
$$
\pm\frac{1}{2}\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2}\sqrt{\begin{array}{l}\frac{1}{4}\left(\psi_{1,t}+\psi_{2,t}\right)^2\|q_t\|_2^6\left(1-\mathbf{x}_1^T\mathbf{x}_2\right) \\ +\left(\psi_{1,t}-\psi_{2,t}\right)^2\|q_t\|_2^6\left(1+\mathbf{x}_1^T\mathbf{x}_2\right) \\ +4\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 - \phi_{1,t} - \psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 - \phi_{2,t}\right)^2\end{array}}
$$

Approximation 4 Derivation In this last approximation, we consider the middle element in the $\Lambda^{-\frac{1}{2}}QKQ\Lambda^{-\frac{1}{2}}$ matrix, $\frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2)$, which is the second $1x1$ principal submatrix. Our motivation for focusing on this principal submatrix is from our analysis of the $\Lambda^{-\frac{1}{2}}QKQ\Lambda^{-\frac{1}{2}}$ matrix under the assumption that $c_t = 0$ (i.e., $v_t^T\mathbf{x}_1 = -v_t^T\mathbf{x}_2$), which causes $\psi_{1,t} = \psi_{2,t}$ and

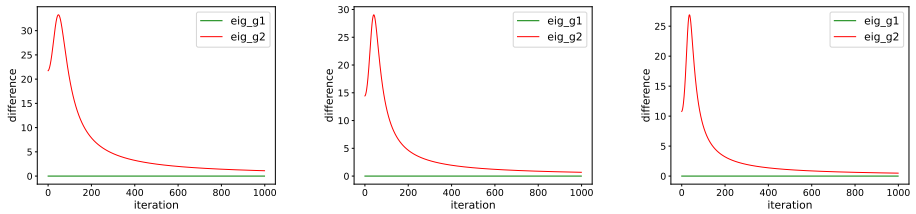
$\phi_{1,t} = \phi_{2,t}$. Then, the $3x3$ matrix reduces to the following.

$$\begin{bmatrix} 4\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 - 2\phi_{1,t}v_t^T\mathbf{x}_1 & 0 & 2\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \cdot \left(\psi_{1,t}\|q_t\|_2^2 v_t^T\mathbf{x}_1 - \phi_{1,t}\right) \\ 0 & \psi_{1,t}\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2) & 0 \\ 2\|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \cdot \left(\psi_{1,t}\|q_t\|_2^2 v_t^T\mathbf{x}_1 - \phi_{1,t}\right) & 0 & \psi_{1,t}\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2) \end{bmatrix}$$

Given the sparsity of the above matrix, we can directly compute the eigenvalues. We know that one eigenvalue is $\psi_{1,t}\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2)$. We show the symmetric pair of eigenvalues below.

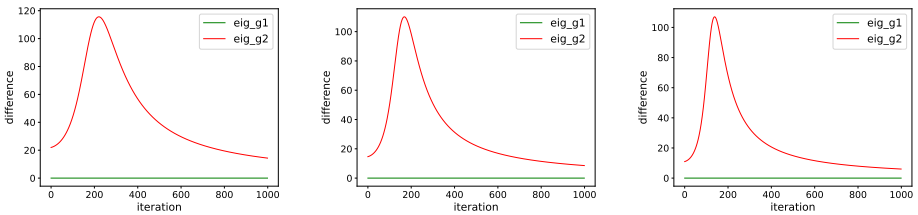
$$2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 - \phi_{1,t}v_t^T\mathbf{x}_1 + \frac{1}{2}\psi_{1,t}\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2)$$
$$\pm \sqrt{\left(2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 - \phi_{1,t}v_t^T\mathbf{x}_1 + \frac{1}{2}\psi_{1,t}\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2)\right)^2 - 6\phi_{1,t}\psi_{1,t}\|q_t\|_2^4 v_t^T\mathbf{x}_1(1-\mathbf{x}_1^T\mathbf{x}_2) - 4\phi_{1,t}^2\|q_t\|_2^2(1-\mathbf{x}_1^T\mathbf{x}_2)}$$

From experimental evaluation, we compare the $\psi_{1,t}\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2)$ eigenvalue with the positive direction of the symmetric pair of eigenvalues above and find that the former is the exact eigenvalue in our setting under the $c_t = 0$ assumption, shown in Figures 7 and 8. For conciseness, we refer to $\psi_{1,t}\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2)$ as eig_g1 and the other eigenvalue as eig_g2 in the figures.



(a) $\eta = 0.10$ (b) $\eta = 0.15$ (c) $\eta = 0.20$

Figure 7: We show the differences between the sharpness and both eigenvalues we calculated for the $c_0 = 0$ case, where $\eta \in \{0.10, 0.15, 0.20\}$, $\mathbf{x}_1^T\mathbf{x}_2 = 0.99$, $\|q_0\|_2^2 = 1.05\sqrt{\frac{8}{0.13 \cdot 1.99}}$, and $m_0 = 0.01$.



(a) $\eta = 0.10$ (b) $\eta = 0.15$ (c) $\eta = 0.20$

Figure 8: We show the differences between the sharpness and both eigenvalues we calculated for the $c_0 = 0$ case, where $\eta \in \{0.10, 0.15, 0.20\}$, $\mathbf{x}_1^T\mathbf{x}_2 = 0.9991$, $\|q_0\|_2^2 = 1.05\sqrt{\frac{8}{0.13 \cdot 1.99}}$, and $m_0 = 0.01$.

Since $\frac{1}{2}(\psi_{1,t} + \psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2)$ reduces to $\psi_{1,t}\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2)$ under the $c_t = 0$ assumption and is easier to interpret than the prior approximations, we give it consideration.

Now, we calculate Weyl's Inequality, starting from the following decomposition.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1+\mathbf{x}_1^T\mathbf{x}_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

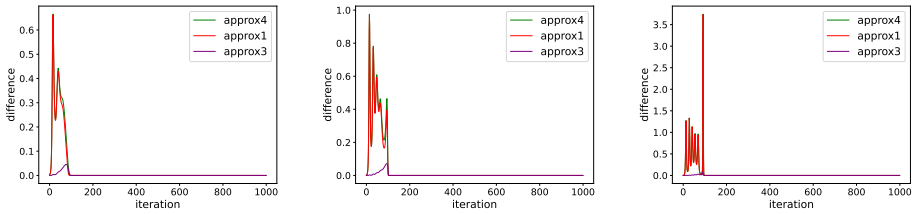
$$+ \begin{bmatrix} \begin{array}{l} 2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2 \\ +2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2 \\ -\phi_{1,t}v_t^T\mathbf{x}_1 \\ +\phi_{2,t}v_t^T\mathbf{x}_2 \end{array} & \begin{array}{l} \|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \\ +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ -\phi_{1,t}+\phi_{2,t}\Big) \end{array} & \begin{array}{l} \|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ -\phi_{1,t}-\phi_{2,t}\Big) \end{array} \\ \begin{array}{l} \|q_t\|_2\sqrt{1+\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \\ +\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ -\phi_{1,t}+\phi_{2,t}\Big) \end{array} & 0 & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} \\ \begin{array}{l} \|q_t\|_2\sqrt{1-\mathbf{x}_1^T\mathbf{x}_2} \\ \cdot\Big(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1 \\ -\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2 \\ -\phi_{1,t}-\phi_{2,t}\Big) \end{array} & \frac{1}{2}(\psi_{1,t}-\psi_{2,t})\|q_t\|_2^4\sqrt{1-(\mathbf{x}_1^T\mathbf{x}_2)^2} & \frac{1}{2}(\psi_{1,t}+\psi_{2,t})\|q_t\|_2^4(1-\mathbf{x}_1^T\mathbf{x}_2) \end{bmatrix}$$

Since the perturbation matrix (second matrix in the decomposition) is essentially as complex as the original $3x3$ matrix, we calculate the Frobenius norm of the perturbation matrix instead as our error bound.

$$\sqrt{\begin{array}{l} \left(2\psi_{1,t}\|q_t\|_2^2(v_t^T\mathbf{x}_1)^2+2\psi_{2,t}\|q_t\|_2^2(v_t^T\mathbf{x}_2)^2-\phi_{1,t}v_t^T\mathbf{x}_1+\phi_{2,t}v_t^T\mathbf{x}_2\right)^2 \\ +2\|q_t\|_2^2\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1+\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2-\phi_{1,t}+\phi_{2,t}\right)^2 \\ +2\|q_t\|_2^2\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)\left(\psi_{1,t}\|q_t\|_2^2v_t^T\mathbf{x}_1-\psi_{2,t}\|q_t\|_2^2v_t^T\mathbf{x}_2-\phi_{1,t}-\phi_{2,t}\right)^2 \\ +\frac{1}{2}\left(\psi_{1,t}-\psi_{2,t}\right)^2\|q_t\|_2^8\left(1-(\mathbf{x}_1^T\mathbf{x}_2)^2\right)+\frac{1}{4}\left(\psi_{1,t}+\psi_{2,t}\right)^2\|q_t\|_2^8\left(1-\mathbf{x}_1^T\mathbf{x}_2\right)^2 \end{array}}$$

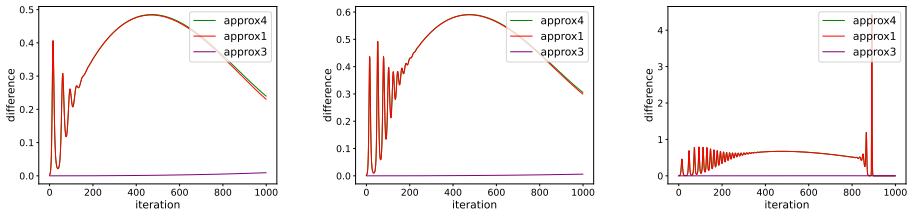
Next, we analyze the error bounds for our different approximations and provide some empirical results to support our claim that Approximation 3 is a low-error approximation for the sharpness in our setting.

Analysis of the Approximations From the approximations we derived, we note that approximation 3 appears to the most accurate in our setting. In comparison to the error bound that we calculated for approximation 2, we note that it is quite similar in form to the error bound for approximation 3. We note that difference in whether $1+\mathbf{x}_1^T\mathbf{x}_2$ or $1-\mathbf{x}_1^T\mathbf{x}_2$ are prominent factors in the error bound. Given that we assume $0.99 \leq \mathbf{x}_1^T\mathbf{x}_2 < 1$, the error bounds suggests that approximation 3 is more accurate than approximation 2 in our setting. As for approximations 1 and 4, we determine that approximation 3 is more accurate than them by experimental evaluation, shown in Figures 9 and 10.



(a) $\eta = 0.10$ (b) $\eta = 0.15$ (c) $\eta = 0.20$

Figure 9: We show the differences between the sharpness and approximations 1, 3, and 4 for $\eta \in \{0.10, 0.15, 0.20\}$, $\mathbf{x}_1^T x2 = 0.99$, $\|q_0\|_2^2 = 1.05\sqrt{\frac{8}{0.13 \cdot 1.99}}$, $c_0 = 0.02$, and $m_0 = 0.01$.



(a) $\eta = 0.10$ (b) $\eta = 0.15$ (c) $\eta = 0.20$

Figure 10: We show the differences between the sharpness and approximations 1, 3, and 4 for $\eta \in \{0.10, 0.15, 0.20\}$, $\mathbf{x}_1^T \mathbf{x}_2 = 0.99$, $\|q_0\|_2^2 = 1.05\sqrt{\frac{8}{0.13 \cdot 1.9991}}$, $c_0 = 0.02$, and $m_0 = 0.01$.

Furthermore, we also note that the differences of approximations 1 and 4 to the true sharpness are quite similar.

From this, we denote approximation 3 as our low-error sharpness approximation and approximation 4 as our interpretable sharpness approximation. We further note that our interpretable approximation supports the idea proposed by Wang et al. (2022) about the correlation between the sharpness and the norm of the output layer of a two-layer linear model. However, we see that the correlation is not direct and more subtle.

Additional Eigenvalues of Loss Hessian In addition to our sharpness approximations, we provide closed-forms for all but the top three largest (in algebraic value) eigenvalues of the Loss Hessian.

We start with

$$\nabla_{\theta_t} L(q_t, u_t, v_t) - \lambda I_{2h+d} = \begin{bmatrix} A & B & C \\ B & A & C \\ C^T & C^T & D \end{bmatrix}$$

where

$$\begin{aligned} A &= \left[\psi_{1,t} \left(v_t^T \mathbf{x}_1\right)^2 + \psi_{2,t} \left(v_t^T \mathbf{x}_2\right)^2\right] q_t q_t^T - \lambda I_h \\ B &= -\left[\phi_{1,t} v_t^T \mathbf{x}_1 - \phi_{2,t} v_t^T \mathbf{x}_2\right] I_h + \left[\psi_{1,t} \left(v_t^T \mathbf{x}_1\right)^2 + \psi_{2,t} \left(v_t^T \mathbf{x}_2\right)^2\right] q_t q_t^T \\ C &= -q_t \left[\phi_{1,t} \mathbf{x}_1^T - \phi_{2,t} \mathbf{x}_2^T\right] + \|q_t\|_2^2 q_t \left[\psi_{1,t} v_t^T \mathbf{x}_1 \mathbf{x}_1^T + \psi_{2,t} v_t^T \mathbf{x}_2 \mathbf{x}_2^T\right] \\ D &= \|q_t\|_2^4 \left[\psi_{1,t} \mathbf{x}_1 \mathbf{x}_1^T + \psi_{2,t} \mathbf{x}_2 \mathbf{x}_2^T\right] - \lambda I_d. \end{aligned}$$

From here, we follow a similar calculation as in the single datapoint case, except we use Woodbury Matrix Inversion instead of the Sherman-Morrison Formula to handle the D matrix. From those calculations, we find a closed-form for all but the top three eigenvalues that are analogous to those

in the single datapoint case. There are h copies of $\phi_{1,t}v_t^T\mathbf{x}_1 - \phi_{2,t}v_t^T\mathbf{x}_2$, $h-1$ copies of $-\phi_{1,t}v_t^T\mathbf{x}_1 + \phi_{2,t}v_t^T\mathbf{x}_2$, and $d-2$ copies of 0.

B.2.3 Explaining Edge of Stability Trajectory

In this section, we revisit the training trajectory that we introduced in Section 5, shown in Figure 11.

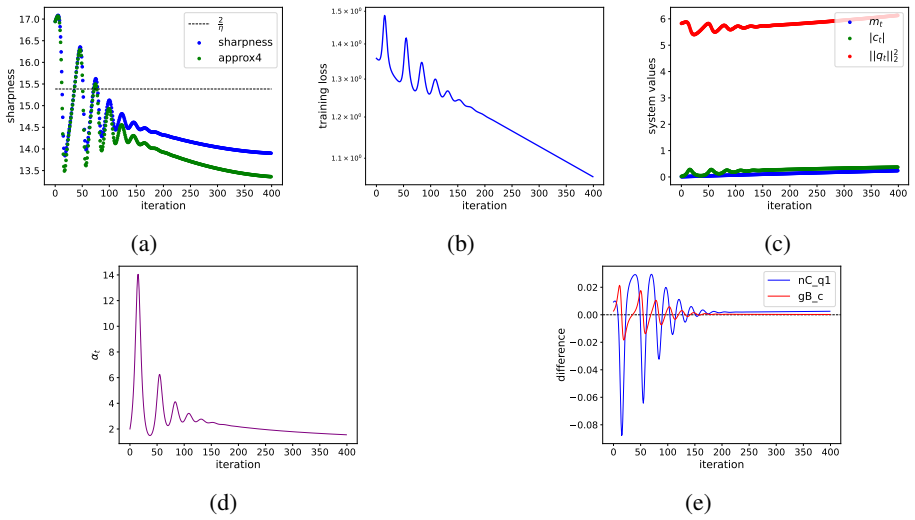


Figure 11: We show the sharpness (a), training loss (b), system values (c), α (i.e., $\frac{|c_t|}{m_t}$) (d), and "location" of the system relative to $\|q_t\|_2^2$ to nullcline 1 ($nC_q1 = 0$) and $|c_t|$ the growth boundary ($gB_c = 0$) (e) for $\eta = 0.13$, $\|q_0\|_2^2 = 1.05\sqrt{\frac{8}{0.13 \cdot 1.9991}}$, $c_0 = 0.02$, and $m_0 = 0.01$. We also show the similarity between our sharpness approximation and the true sharpness.

Recall from Section 5 that we defined $\|q_t\|_2^2$ nullcline 1 where Equation 1 equals 0 the $|c_t|$ growth boundary where Equation 2 equals 0.

$$\left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m_t + c_t\right)\right)\right)^{-1}\left(m_t + c_t\right) + \left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m_t - c_t\right)\right)\right)^{-1}\left(m_t - c_t\right) \quad (1)$$

$$|c_t| - \frac{\eta\|q_t\|_2^2\left(1+\mathbf{x}_1^T\mathbf{x}_2\right)}{2}\left[\left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m_t - |c_t|\right)\right)\right)^{-1} - \left(1 + \exp\left(\tfrac{1}{2}\|q_t\|_2^2\left(m_t + |c_t|\right)\right)\right)^{-1}\right] \quad (2)$$

To understand how trajectory in Figure 11 exhibits Edge of Stability, we explain the first oscillation in loss and sharpness.

To explain the first oscillation in loss and sharpness, we first split the initial trajectory into four phases based on the sign of nC_q1 and gB_c in Figure 11(e), where nC_q1 represents Equation 1 (to capture dynamics about $\|q_t\|_2^2$ nullcline 1) and gB_c represents Equation 2 (to capture dynamics about the $|c_t|$ growth boundary).

In the initial part of the trajectory, we see that the system is inside $\|q_t\|_2^2$ nullcline 1 and the $|c_t|$ growth boundary ($nC_q1 > 0, gB_c > 0$), which we denote as phase 1. Then, the system is outside $\|q_t\|_2^2$ nullcline 1, but still inside the $|c_t|$ growth boundary ($nC_q1 < 0, gB_c > 0$), which we denote as phase 2. In phase 3, we see that the system is outside both $\|q_t\|_2^2$ nullcline 1 and the $|c_t|$ growth boundary ($nC_q1 < 0, gB_c < 0$). Lastly, in phase 4, the system is inside $\|q_t\|_2^2$ nullcline 1, but still outside the $|c_t|$ growth boundary ($nC_q1 > 0, gB_c < 0$).

Now, we provide intuition on how the loss and sharpness behave in each phase.

Loss Trajectory To explain the loss trajectory, we first rewrite our loss using $\alpha_t = \frac{|c_t|}{m_t}$ (see Figure 11(d)) as shown below.

$$L = \log\left(1 + \exp\left(-\frac{1}{2}(1-\alpha_t)\|q_t\|_2^2 m_t\right)\right) + \log\left(1 + \exp\left(-\frac{1}{2}\|q_t\|_2^2 m_t(1+\alpha_t)\right)\right)$$

In the first phase, we observe $\|q_t\|_2^2$ and α_t grow. During the first few iterations of this phase, we observe a decrease in loss. From our rewritten loss above, we see that for small $\alpha_t > 1$, $\|q_t\|_2^2 m_t$ needs to be larger in order for the loss to increase. We observe that $\|q_t\|_2^2 m_t$ is not large enough for the first few steps even as α_t increases. Once α_t and $\|q_t\|_2^2 m_t$ grow large enough, then the loss increases for the remainder of phase 1. In the second phase of the trajectory, we observe α_t increase and $\|q_t\|_2^2$ shrink. The loss begins to decrease, since $\|q_t\|_2^2 m_t$ is shrinking due to $\|q_t\|_2^2$. After this, in the third phase of the system's trajectory, we observe $\|q_t\|_2^2$ and α_t shrink, and this causes the loss to continue to decrease. Once the system enters the fourth phase, $\|q_t\|_2^2$ begins to grow and α_t continues to shrink. We continue to see decreases in loss due to the shrinkage of α_t.

Following this last phase, we observe that $\|q_t\|_2^2$ continues to grow and α_t continues to shrink until the system is inside the $|c_t|$ growth boundary ($gB_c > 0$), where we enter phase 1 again. Once this happens, the cycle repeats.

Sharpness Trajectory To explain the sharpness trajectory, we use the interpretable sharpness approximation we derived in Appendix B.2.2, shown below, to explain the first oscillation in sharpness.

$$\frac{1}{2}(\psi_{1,t} + \psi_{2,t})\|q_t\|_2^4(1 + \mathbf{x}_1^T\mathbf{x}_2),$$

where

$$\psi_{1,t} = \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2(m_t + c_t)\right)\right)^{-2}\exp\left(\frac{1}{2}\|q_t\|_2^2(m_t + c_t)\right)$$
$$\psi_{2,t} = \left(1 + \exp\left(\frac{1}{2}\|q_t\|_2^2(m_t - c_t)\right)\right)^{-2}\exp\left(\frac{1}{2}\|q_t\|_2^2(m_t - c_t)\right).$$

By rewriting the sharpness in terms of $\alpha_t = \frac{|c_t|}{m_t}$, we note an initial increase in the sharpness as the norm of $\|q_t\|_2^2$ and α_t grow in phase 1. However, the sigmoid second derivative terms in the approximation start to dominate the growth of $\|q_t\|_2^2$ due to the fast growth of α_t, which causes the sharpness to begin to shrink for the remainder of phase 1. In phase 2, the sharpness continues to shrink α_t still grows quite fast, causing the sigmoid second derivative terms to continue to dominate even as $\|q_t\|_2^2$ shrinks. Initially in phase 3, we find that the sharpness continues to decrease as α_t initially shrinks slowly. However, as α_t shrinks faster, we begin to observe an increase in sharpness. We can think of this as the sigmoid second derivative terms allowing more of $\|q_t\|_2^2$ to be "present" in the sharpness, where the "presence" $\|q_t\|_2^2$ increases over time from some small initial amount. In phase 4, the shrinkage in α_t and growth in $\|q_t\|_2^2$ continues to cause the sharpness to increase, as $\|q_t\|_2^2$ dominates the sigmoid second derivative terms. As seen from Figure 11(e), the system can enter phase 1, and the cycle repeats.

We note that the dampening in the oscillations for both loss and sharpness is likely the result of the growth of m_t, which constrains the magnitude of α_t (see Figure 11(d)).

B.3 Difference Between Squared Loss and Logistic Loss

Based on our analyses in Sections 4 and 5

we highlight some of the differences between our two layer rank-1 linear model to minimize logistic loss and the two-layer linear model studied by Kalra et al. (2025) to minimize mean squared loss.

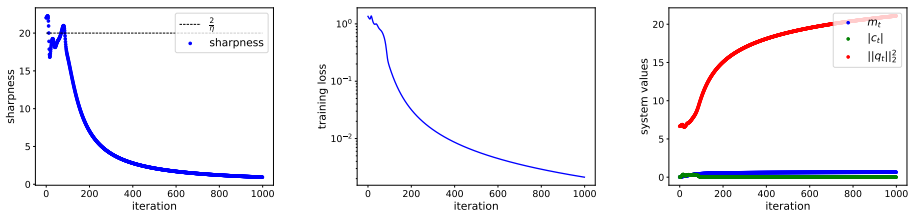
Training Dataset Size Kalra et al. (2025) highlight that Edge of Stability can occur in a two-layer linear model even with a single training datapoint, as long as the corresponding label $y \neq 0$. However, in our work, we find that Edge of Stability does not occur with only a single training datapoint. The simplest setting we found for Edge of Stability in our model was with two training datapoints.

Existence of critical learning rate η_c Furthermore, Kalra et al. (2025) state that Edge of Stability occurs in their setting once the learning rate η crosses some critical learning rate η_c. In our work, we claim that the occurrence of the Edge of Stability relies on more than just crossing some η_c. We find that it is necessary for $\mathbf{x}_t^T\mathbf{x}_2$ to be above some critical threshold, $\left(\mathbf{x}_t^T\mathbf{x}_2\right)_c$. While we know that for any choice of η and $\mathbf{x}_t^T\mathbf{x}_2$, there exists a region of the phase space where the system moves away from the $c_t = 0$ plane, our model is not guaranteed to exhibit Edge of Stability behavior for $\mathbf{x}_t^T\mathbf{x}_2$ that is not large enough. Intuitively, we know that a large $\mathbf{x}_t^T\mathbf{x}_2$ corresponds to a smaller optimal margin of separation, since the datapoints are "harder" to distinguish. The small margin would then constrain the growth of m_t, requiring more iterations to escape the region where the $c_t = 0$ is unstable (assuming initialization in this region). This allows the system to movement across the $\|q_t\|_2^2$ nullclines for sufficient iterations until m_t becomes large enough. So, the faster growth rate of m_t with smaller $\mathbf{x}_1^T\mathbf{x}_2$ may cause the system to "escape" the c_t growth region quite early during training, resulting in no sufficient oscillations in $|c_t|$ or $\|q_t\|_2^2$ to produce to oscillations in the loss or sharpness.

Edge of Stability Source In the regression setting, Kalra et al. (2025) claim that Edge of Stability occurs when a fixed point, defined as the line where the residual of their model is 0 for appropriate values of the Loss Hessian trace (zero-loss line), becomes unstable. The instability of the zero-loss line causes oscillations in the residual that cause the parameters to move towards the Edge of Stability manifold. In our classification setting, we find an analogous condition, where Edge of Stability can occur when our 3-dimensional system enters the region where the $c_t = 0$ plane is unstable and remains there for a sufficient number of training steps. However, we find that oscillations in $|c_t|$ that produce movement of the system in and out of $\|q_t\|_2^2$ nullcline 1 causes Edge of Stability behavior. This is because oscillations of the system about $\|q_t\|_2^2$ nullcline 1 causes $\|q_t\|_2^2$ to oscillate, and we find that $\|q_t\|_2^2$ is a significant component in the sharpness based on our interpretable approximation in Appendix B.2.2.

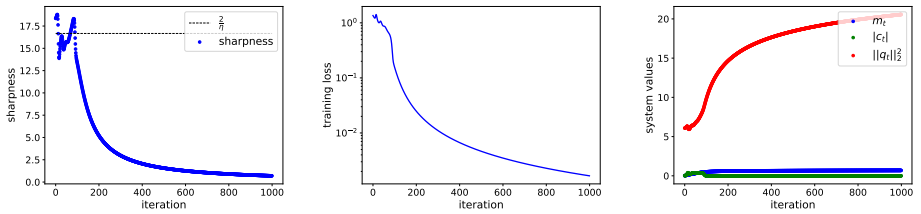
B.4 Additional Experiments For Edge of Stability

In this section, we provide additional experiments to show Edge of Stability behavior in our two datapoint setting. Our experiments consider $\mathbf{x}_1^T\mathbf{x}_2 \in \{0.99, 0.9991\}$, $\eta \in \{0.10, 0.12, 0.15, 0.17, 0.20\}$, $\|q_0\|_2^2 = 1.05\sqrt{\frac{8}{\eta(1+\mathbf{x}_1^T x2)}}$, $c_0 = 0.02$, and $m = 0.01$, where each model is trained for 1000 iterations with hidden width 1024. In each of the figures below, we show the sharpness, loss, and system trajectories.



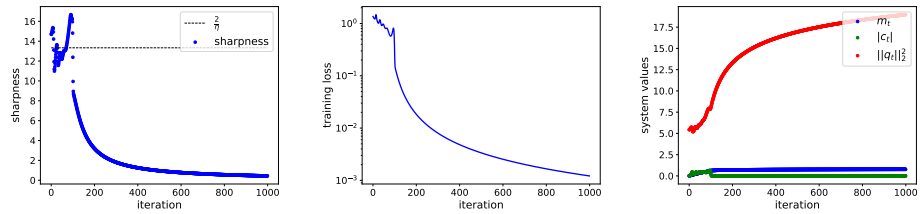
(a) (b) (c)

Figure 12: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.99$ and $\eta = 0.10$.



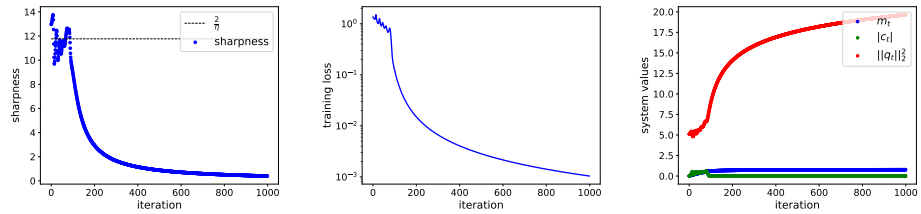
(a) (b) (c)

Figure 13: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.99$ and $\eta = 0.12$.



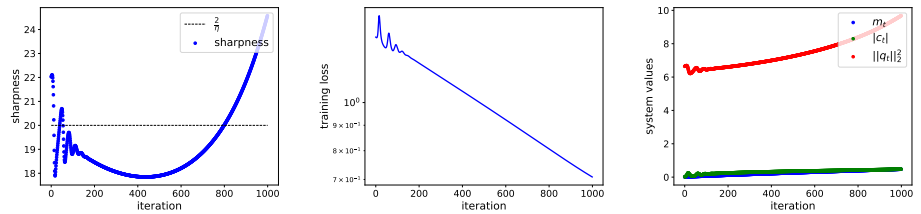
(a) (b) (c)

Figure 14: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.99$ and $\eta = 0.15$.



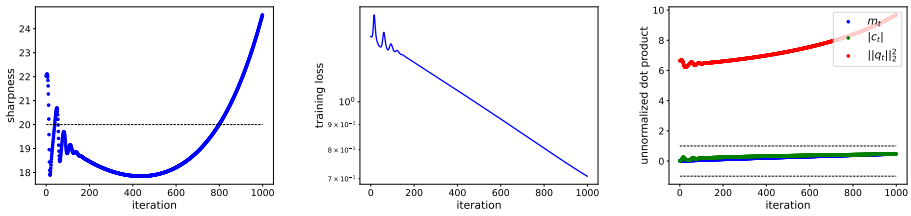
(a) (b) (c)

Figure 15: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.99$ and $\eta = 0.17$.



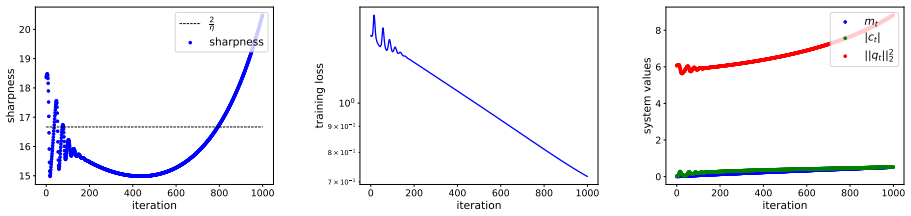
(a) (b) (c)

Figure 16: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.99$ and $\eta = 0.20$.



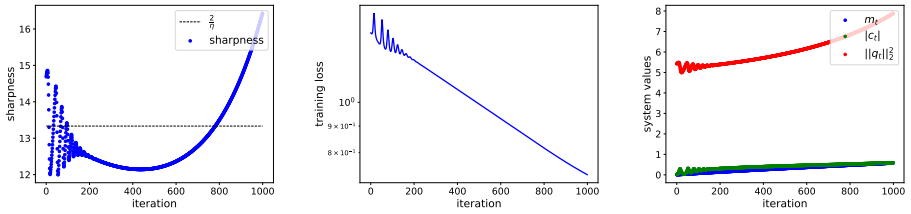
(a) (b) (c)

Figure 17: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.9991$ and $\eta = 0.10$.



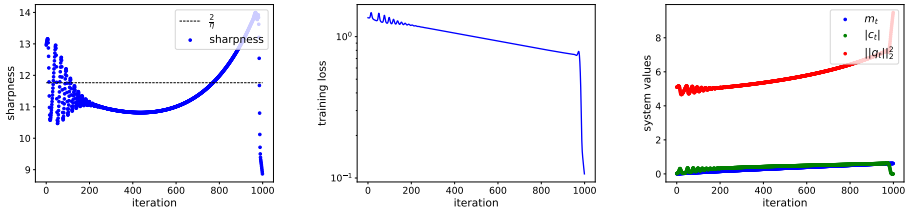
(a) (b) (c)

Figure 18: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.9991$ and $\eta = 0.12$.



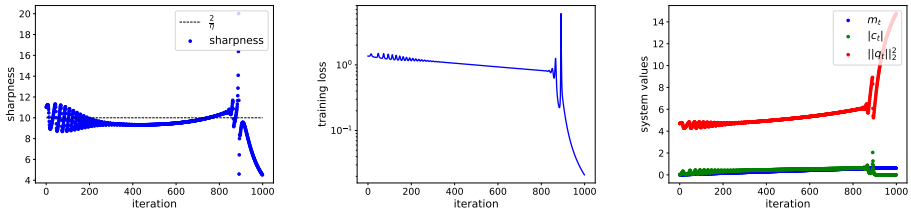
(a) (b) (c)

Figure 19: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.9991$ and $\eta = 0.15$.



(a) (b) (c)

Figure 20: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T\mathbf{x}_2 = 0.9991$ and $\eta = 0.17$.



(a) (b) (c)

Figure 21: We show the sharpness (a), training loss (b), and system trajectory (c) for $\mathbf{x}_1^T \mathbf{x}_2 = 0.9991$ and $\eta = 0.20$.

B.5 Asymptotic Setting $\left(\mathbf{x}_1^T \mathbf{x}_2 \to 1\right)$

In the following subsection, we provide the proofs of Theorem 5.2, B.18, and B.19 and asymptotic simulation results.

Recall from Section 5, the dynamical system in the asymptotic setting $\left(\mathbf{x}_1^T \mathbf{x}_2 \to 1\right)$ reduces down to a 2D system $\|q_t\|_2^2, c_t$, where we focus on a reparameterized version of it, $\|q_t\|_2^2, c_t^2$, shown below.

$$\|q_{t+1}\|_2^2 = \|q_t\|_2^2 \left(1 + \frac{\eta}{2}\left[\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m+\sqrt{c_t^2}\right)\right)\right)^{-1}\left(m+\sqrt{c_t^2}\right) + \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m-\sqrt{c_t^2}\right)\right)\right)^{-1}\left(m-\sqrt{c_t^2}\right)\right]\right)^2 \tag{5}$$

$$c_{t+1}^2 = \left(\sqrt{c_t^2} - 2\eta\|q_t\|_2^2\left[\left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m-\sqrt{c_t^2}\right)\right)\right)^{-1} - \left(1+\exp\left(\frac{1}{2}\|q_t\|_2^2\left(m+\sqrt{c_t^2}\right)\right)\right)^{-1}\right]\right)^2. \tag{6}$$

For the asymptotic dynamical system, the fixed point of interest, $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$, is defined as a point of intersection of $\|q_t\|_2^2$ nullcline 1 (where Equation 7 equals 0) and the c_t^2 growth boundary (where Equation 8 equals 0).

$$\left(1+\exp\left(\frac{1}{2}q\left(\sqrt{c^2}+m\right)\right)\right)^{-1}\left(\sqrt{c^2}+m\right) - \left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}-m\right) \tag{7}$$

$$\sqrt{c^2} - nq\left(\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1} - \left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\right) \tag{8}$$

Note that in the following subsections, we overload notation by referring to $\|q\|_2^2$ as q and c^2 as c.

Further note that to prove theorem 5.3, we split the theorem into two smaller theorems shown below about the eigenvalues of the system Jacobian being complex-valued and crossing the unit circle.

Theorem B.18. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$, the eigenvalues of the Jacobian of the asymptotic dynamical system defined by Equations 5 and 6 evaluated at fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$ are complex-valued.*

Theorem B.19. *For $m \in [0.15, 0.25]$, the eigenvalues of the Jacobian of the asymptotic dynamical system defined by Equations 5 and 6 evaluated at fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$ cross the unit-circle at a unique $\eta_{c,m} \in [0.2, 0.3]$.*

We prove the above theorems in Sections B.5.2 and B.5.3.

B.5.1 Existence of Fixed Point $(\tilde{q}, \tilde{c})$

In this section, we prove Theorem 5.2, where we will show that the fixed point $(\tilde{q}, \tilde{c})$ exists and is contained within the containment rectangle, defined where $\tilde{c} \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025], \tilde{q} \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$. We first introduce a preliminary theorem, the Poincare-Miranda Theorem, followed by the proof of Theorem 5.2 and the supporting lemmas.

Theorem B.20. *Consider n continuous, real-valued function of n variables, $f_1, .., f_n : [-1, 1]^n \to \mathbb{R}$. Assume that for each variable x_i, the function f_i is nonpositive when $x_i = -1$ and nonnegative when $x_i = 1$. Then, there is a point in the n-dimensional cube $[-1, 1]^n$ in which all functions are simultaneously 0.*

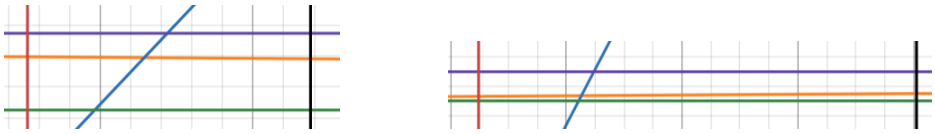
For the proof of Theorem 5.2, our proof mainly relies on applying Poincare Miranda Theorem to the following two continuous function.

$$\sqrt{c}\left(\sqrt{c} - m\right)\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\right) - 2m\eta q \tag{9}$$

$$\sqrt{c} + \sqrt{c}\exp\left(qm\right) + \left(\sqrt{c} - \eta q\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right) + \left(\sqrt{c} + \eta q\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right) \tag{10}$$

We consider these functions as alternate "equivalent" functions to the equations defining $\|q_t\|_2^2$ nullcline 1 and the c_t^2 growth boundary in the sense that if $(\tilde{q}, \tilde{c})$, for $\tilde{q} > 0$, is a root of these alternate functions, then $(\tilde{q}, \tilde{c})$ is an intersection point of $\|q_t\|_2^2$ nullcline 1 and the c_t^2 growth boundary.

To prove that fixed point $(\tilde{q}, \tilde{c})$ exists in the containment rectangle, we show that Equation 9 is negative for $c = 2m\eta + 0.25m + 0.02$ and positive for $c = 2m\eta + 0.25m + 0.025$ and Equation 10 is negative $q = \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15$ and positive for $q = \frac{0.9m^2}{\eta} + \frac{0.52}{\eta} + 1.944$. For visual intuition, please refer to Figure 22.



(a) (b)

Figure 22: We show that the fixed point $(\tilde{q}, \tilde{c})$ equivalently defined as the root to Equation 9 (orange line) and Equation 10 (blue line) is contained in the rectangle where the top purple line is $c = 2m\eta + 0.25m + 0.025$, the bottom green line is $2m\eta + 0.25m + 0.02$, the left red line is $q = \frac{0.9m^2}{\eta} + \frac{0.52}{\eta} + 1.944$, and the right black line is $q = \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15$. In (a), we consider the case where $\eta = 0.2, m = 0.25$, and, in (b), we consider $\eta = 0.3, m = 0.15$.

Proof of Theorem 5.2. To prove the Theorem, we first show that Equation 9 is negative for $c = 2m\eta + 0.25m + 0.02$ in lemma B.22 and positive for $c = 2m\eta + 0.25m + 0.025$ in lemma B.21. Next, we show that Equation 10 is negative for $q = \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15$ in lemma B.24 and positive for $q = \frac{0.9m^2}{\eta} + \frac{0.52}{\eta} + 1.944$ in lemma B.23. Then, by Poincare-Miranda Theorem, we know that there exists an intersection point, $(\tilde{q}, \tilde{c})$, for Equations 9 and 10 contained within the rectangle defined by $\tilde{c} \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025], \tilde{q} \in \left[\frac{0.9m^2}{\eta} + \frac{0.52}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$.

By lemma B.25, we know that $(\tilde{q}, \tilde{c})$ is a fixed point for the dynamical system.

This completes the proof of the theorem

□

Next, we introduce the supporting lemmas for Theorem 5.2.

Supporting Lemmas for Theorem 5.2 Below, we provide the supporting lemmas for Theorem 5.2. Lemmas B.21, B.22, B.23, and B.24 are related to showing that the containment rectangle defined by $c \in [], q \in []$ contains a root for Equations 9 and 10. Lemma B.25 shows that a root for Equations 9 and 10 is also an intersection point of q_t nullcline 1 and the c_t growth boundary. Refer to Section B.5.1 for the proofs of the below lemmas.

Lemma B.21. *For* $\eta \in [0.2, 0.3]$, $m \in [0.15, 0.25]$, *and* $q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$, *Equation 9 evaluated at* $c = 2m\eta + 0.25m + 0.025$ *is positive.*

Lemma B.22. *For* $\eta \in [0.2, 0.3]$, $m \in [0.15, 0.25]$, *and* $q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$, *Equation 9 evaluated at* $c = 2m\eta + 0.25m + 0.02$ *is negative.*

Lemma B.23. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], c \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025]$, *Equation 10 evaluated at* $q = \frac{0.9m^2}{\eta} + \frac{0.52}{\eta} + 1.944$ *is positive.*

Lemma B.24. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], c \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025]$, *Equation 10 evaluated at* $q = \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15$ *is negative.*

Lemma B.25. *If there exists $(\tilde{q}, \tilde{c})$, such that $\tilde{q} > 0$, that is a root for Equations 9 and 10, then $(\tilde{q}, \tilde{c})$ is an intersection point of $\|q_t\|_2^2$ nullcline 1 and the c_t^2 growth boundary. Meaning, $(\tilde{q}, \tilde{c})$ satisfies the conditions*

$$\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(m+\sqrt{\tilde{c}}\right)+\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(m-\sqrt{\tilde{c}}\right)=0$$

$$\sqrt{\tilde{c}}-\eta\tilde{q}\left[\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\right]=0.$$

Supporting Lemmas for Lemma B.21 Below, we provide the supporting lemmas for lemma B.21. Refer to Section B.5.1 for the proofs of the below lemmas.

Lemma B.26. *For $m \in [0.21, 0.25]$ and $\eta \in [0.2, 0.3]$,*

$$\frac{0.9m^2}{\eta}+\frac{0.52}{\eta}+1.944$$
$$-\frac{2}{m+\sqrt{2m\eta+0.25m+0.025}}\left(\ln(4m\eta)-\frac{1}{2}\ln(2m\eta+0.25m+0.025)-\ln(2m\eta+0.25m+0.025-m^2)\right)$$

is positive.

Lemma B.27. *For $m \in [0.21, 0.25]$,*

$$\left(0.9m^2+1.1032\right)\left(m+\sqrt{0.85m+0.025}\right)-0.6\left(\ln(1.2m)-\frac{1}{2}\ln(0.85m+0.025)-\ln(0.85m+0.025-m^2)\right) \tag{11}$$

is positive.

Lemma B.28. *For $m \in [0.21, 0.25]$ and $\eta \in [0.2, 0.3]$,*

$$\left(0.9m^2+0.52+1.944\eta\right)\left(m+\sqrt{2m\eta+0.25m+0.025}\right)$$
$$-2\eta\left(\ln(4m\eta)-\frac{1}{2}\ln(2m\eta+0.25m+0.025)-\ln(2m\eta+0.25m+0.025-m^2)\right)$$

is decreasing in η.

Lemma B.29. *For $m \in [0.21, 0.25]$ and $\eta \in [0.2, 0.3]$,*

$$\left(0.9m^2+0.52+1.944\eta\right)\left(m+\sqrt{2m\eta+0.25m+0.025}\right)$$
$$-2\eta\left(\ln(4m\eta)-\frac{1}{2}\ln(2m\eta+0.25m+0.025)-\ln(2m\eta+0.25m+0.025-m^2)\right)$$

is convex in η.

Lemma B.30. *For $m \in [0.15, 0.21], \eta \in [0.2, 0.3]$,*

$$-\ln(4\eta m)+\frac{1}{2}\ln(2m\eta+0.25m+0.025)+\ln(2m\eta+0.25m+0.025-m^2) \tag{12}$$

can be lower bounded by $\ln(0.2805)$.

Lemma B.31. *For $m \in [0.15, 0.21], \eta \in [0.2, 0.3]$,*

$$-\ln(4\eta m)+\frac{1}{2}\ln(2m\eta+0.25m+0.025)+\ln(2m\eta+0.25m+0.025-m^2) \tag{12}$$

is concave in m.

Lemma B.32. *For $m \in [0.21, 0.25], \eta \in [0.2, 0.3]$,*

$$-1.888m\eta-0.79m+0.025-1.8m^3-m\sqrt{2m\eta+0.25m+0.025}$$
$$+\sqrt{2m\eta+0.25m+0.025}\left(\sqrt{2m\eta+0.25m+0.025}-m\right)$$
$$\cdot\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{2m\eta+0.25m+0.025}\right)\right) \tag{13}$$

is convex in η.

Lemma B.33. *For $m \in [0.21, 0.25], \eta \in [0.2, 0.3]$,*

$$-0.45m^4\eta^2-0.4975m^3\eta+0.013m\eta+0.13m^2\eta+0.0065m+0.0200625m^2+0.225m^4\eta+0.03375m^4-0.9m^5\eta \\ -0.225m^5+0.000325-0.11875m^3+2.916m^2\eta^3-0.26m^2\eta^2 \quad (14)$$

is positive.

Lemma B.34. *For $m \in [0.21, 0.25], \eta \in [0.2, 0.3]$,*

$$-0.45m^4\eta^2-0.4975m^3\eta+0.013m\eta+0.13m^2\eta+0.0065m+0.0200625m^2+0.225m^4\eta+0.03375m^4-0.9m^5\eta \\ -0.225m^5+0.000325-0.11875m^3+2.916m^2\eta^3-0.26m^2\eta^2 \quad (14)$$

grows in η.

Lemma B.35. *For $m \in [0.21, 0.25]$,*

$$-1.888m-\frac{m^2}{\sqrt{0.73m+0.025}} \\ +\left(2m+\frac{-m^2}{\sqrt{0.73m+0.025}}+\frac{-0.45}{0.24}m^4+\frac{-0.49328}{0.24}m^2+\frac{0.45}{0.24^2}m^4\sqrt{0.73m+0.025}+\frac{0.24875}{0.24^2}m^2\sqrt{0.73m+0.025}\right. \\ \left.+\frac{-0.45\,(0.49)}{0.24^2}m^3\sqrt{0.73m+0.025}+\frac{-0.0714128}{0.24^2}m\sqrt{0.73m+0.025}+\frac{-0.26\,(0.025)}{0.24^2}\sqrt{0.73m+0.025}\right) \\ \cdot\exp\left(\frac{0.45}{0.24}m^3+\frac{0.49328}{0.24}m+\frac{0.45}{0.24}m^2\sqrt{0.73m+0.025}+\frac{0.49328}{0.24}\sqrt{0.73m+0.025}\right) \quad (15)$$

grows in m and is positive.

Lemma B.36. *The global minimizer in q for the following equation*

$$\sqrt{c}\left(\sqrt{c}-m\right)\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\right)-2m\eta q \quad (9)$$

is $q_{min}=\frac{2}{m+\sqrt{c}}\left[\ln(4\eta m)-\frac{1}{2}\ln(c)-\ln(c-m^2)\right]$

Lemma B.37. *For $m \in [0.15, 0.21], \eta \in [0.2, 0.3]$,*

$$\frac{-m^2\,(n+0.125)}{\sqrt{2mn+0.25m+0.025}}+(0.375+3n-2m)\sqrt{2nm+0.25m+0.025}+4n(1+\ln(0.2805)) \quad (16)$$

grows in η.

Lemma B.38. *For $m \in [0.21, 0.25]$,*

$$16.18652m^6+1.77291m^4-6.70944m^3+2.37458m^2-0.94068m+1.24492-\frac{1.28321m^6}{\sqrt{0.73m+0.025}}-\frac{5.58668m^4}{\sqrt{0.73m+0.025}} \\ -\frac{0.02852m^3}{\sqrt{0.73m+0.025}}-\frac{2.02096m^2}{\sqrt{0.73m+0.025}}-\frac{0.95214m}{\sqrt{0.73m+0.025}}-\frac{0.04119}{\sqrt{0.73m+0.025}}-\frac{0.79298m^2}{0.73m+0.025} \\ +43.94531m^6\sqrt{0.73m+0.025}+33.20816m^4\sqrt{0.73m+0.025}+8.70052m^3\sqrt{0.73m+0.025} \\ +4.25697m^2\sqrt{0.73m+0.025}+6.08893m\sqrt{0.73m+0.025} \quad (17)$$

is positive and grows in m.

Lemma B.39. *For $m \in [0.21, 0.25]$,*

$$-1.24312m+0.025-1.8m^3-m\sqrt{0.73m+0.025} \\ +\sqrt{0.73m+0.025}\left(\sqrt{0.73m+0.025}-m\right)\exp\left(\left(\frac{0.45}{0.24}m^2+\frac{0.49328}{0.24}\right)\left(m+\sqrt{0.73m+0.025}\right)\right) \quad (18)$$

is convex in m.

Lemma B.40. *For $m \in [0.21, 0.25]$,*

$$\frac{-(0.365)^2 m - (0.365)\,0.025}{(0.73\,(0.21) + 0.025)^{\frac{3}{2}}} - 31.64063m^5\sqrt{0.73\,(0.25) + 0.025} + 21.81445m^4\sqrt{0.73m + 0.025}$$
$$-22.41938m^3\sqrt{0.73\,(0.25) + 0.025} - 8.43382m^2\sqrt{0.73\,(0.25) + 0.025} + 5.74222m\sqrt{0.73x + 0.025}$$
$$-2.47502\sqrt{0.73\,(0.25) + 0.025} - 15.39844m^5 + 11.44555m^4 - 4.98488m^3 + 6.70438m^2 + 0.14066m$$
$$+\frac{0.46836\,(0.25)^5 + 0.03208\,(0.25)^4 - 3.07942\,(0.25)^3 + 1.81890\,(0.25)^2 - 0.90338\,(0.25) + 0.49501}{\sqrt{0.73\,(0.25) + 0.025}}$$
$$+\frac{-0.24980x^3 - 0.27383x}{(0.73\,(0.21) + 0.025)} + 2.16879 \quad (19)$$

decreases in m and is positive.

Lemma B.41. *For $m \in [0.21, 0.25]$,*

$$0.46836m^5 + 0.03208m^4 - 3.07942m^3 + 1.81890m^2 - 0.90338m + 0.49501 \quad (20)$$

decreases in m and is positive.

Lemma B.42. *For $m \in [0.21, 0.25], \eta \in [0.2, 0.3]$,*

$$-1.888m\eta - 0.79m + 0.025 - 1.8m^3 - m\sqrt{2m\eta + 0.25m + 0.025}$$
$$+\sqrt{2m\eta + 0.25m + 0.025}\left(\sqrt{2m\eta + 0.25m + 0.025} - m\right)$$
$$\cdot \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right) \quad (13)$$

is positive.

Supporting Lemmas for Lemma B.22 Below, we provide the supporting lemmas for lemma B.22. Refer to Section B.5.1 for the proofs of the below lemmas.

Lemma B.43. *For $\eta \in [0.2, 0.3], m \in [0.15, 0.25]$,*

$$-1.8m^3 - 0.79m - 1.888m\eta + 0.02 - m\sqrt{2m\eta + 0.25m + 0.02}$$
$$+\sqrt{2m\eta + 0.25m + 0.02}\left(\sqrt{2m\eta + 0.25m + 0.02} - m\right)$$
$$\cdot \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \quad (21)$$

is convex in η.

Lemma B.44. *For $m \in [0.15, 0.25]$,*

$$-1.8m^3 - 1.1676m + 0.02 - m\sqrt{0.65m + 0.02}$$
$$+\sqrt{0.65m + 0.02}\left(\sqrt{0.65m + 0.02} - m\right)\exp\left(\left(2.25m^2 + 2.272\right)\left(m + \sqrt{0.65m + 0.02}\right)\right) \quad (22)$$

is negative.

Lemma B.45. *For $m \in [0.15, 0.25]$*

$$-1.8m^3 - 1.3564m + 0.02 - m\sqrt{0.85m + 0.02}$$
$$+\sqrt{0.85m + 0.02}\left(\sqrt{0.85m + 0.02} - m\right)\exp\left(\left(\frac{0.45m^2}{0.3} + \frac{0.5516}{0.3}\right)\left(m + \sqrt{0.85m + 0.02}\right)\right) \quad (23)$$

is negative.

Lemma B.46. *For $\eta \in [0.2, 0.3], m \in [0.15, 0.25]$,*

$$-1.88m^3 - 0.71m - 2.3m\eta + 0.02 - m\sqrt{2m\eta + 0.25m + 0.02} + \sqrt{2m\eta + 0.25m + 0.02}\left(\sqrt{2m\eta + 0.25m + 0.02} - m\right) \cdot \exp\left(\left(\frac{0.47}{\eta}m^2 + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{2m\eta + 0.25 + 0.02}\right)\right) \quad (24)$$

is convex in η.

Lemma B.47. *For $m \in [0.15, 0.25]$,*

$$-1.88m^3 - 1.17m + 0.02 - m\sqrt{0.65m + 0.02} + \sqrt{0.65m + 0.02}\left(\sqrt{0.65m + 0.02} - m\right)\exp\left(\left(2.35m^2 + 2.275\right)\left(m + \sqrt{0.65m + 0.02}\right)\right) \quad (25)$$

is negative.

Lemma B.48. *For $m \in [0.15, 0.25]$*

$$-1.88m^3 - 1.4m + 0.02 - m\sqrt{0.85m + 0.02} + \sqrt{0.85m + 0.02}\left(\sqrt{0.85m + 0.02} - m\right)\exp\left(\left(\frac{0.47}{0.3}m^2 + \frac{0.5625}{0.3}\right)\left(m + \sqrt{0.85m + 0.02}\right)\right) \quad (26)$$

is negative.

Lemma B.49. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$-0.9m^5\eta - 0.45m^4\eta^2 - 0.121m^2 + 0.225m^4\eta - 0.502m^3\eta - 0.26m^2\eta^2 + 0.1508m^2\eta + 0.003825m^4 + 0.02506m^2 - 0.225m^5 + 0.0026m + 0.000208 + 2.916m^2\eta^3 - 0.0104m\eta \quad (27)$$

is positive.

Lemma B.50. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$*

$$\left(-\frac{0.45\,(0.2)^2}{0.1175} - \frac{0.9\,(0.2)}{\sqrt{0.2325}} - \frac{0.225}{\sqrt{0.2325}}\right)m^5 + \left(-\frac{0.45\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.225\,(0.2)}{\sqrt{0.2325}} + \frac{0.03825}{\sqrt{0.2325}}\right)m^4 + \left(\frac{(0.2)^3}{(0.2325)^{\frac{3}{2}}} - \frac{0.26\,(0.2)^2}{0.1175} - \frac{0.973\,(0.2)^3}{0.1175} - \frac{0.121}{\sqrt{0.2325}} - \frac{0.502\,(0.2)}{\sqrt{0.2325}}\right)m^3 + \left(-\frac{0.26\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.1508\,(0.2)}{\sqrt{0.2325}} + \frac{0.02506}{\sqrt{0.2325}} + \frac{2.916\,(0.2)^3}{\sqrt{0.2325}}\right)m^2 + \left(\frac{0.0026}{\sqrt{0.2325}} - \frac{0.0104\,(0.2)}{\sqrt{0.2325}}\right)m + \frac{0.000208}{\sqrt{0.2325}} \quad (28)$$

is concave in m.

Lemma B.51. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\frac{m^3\eta^3}{(0.2325)^{\frac{3}{2}}} - \frac{0.45}{0.1175}m^5\eta^2 - \frac{0.26}{0.1175}m^3\eta^2 - \frac{0.973}{0.1175}m^3\eta^3 - \frac{0.9}{\sqrt{0.2325}}m^5\eta - \frac{0.45}{\sqrt{0.2325}}m^4\eta^2 - \frac{0.121}{\sqrt{0.2325}}m^3 + \frac{0.225}{\sqrt{0.2325}}m^4\eta - \frac{0.502}{\sqrt{0.2325}}m^3\eta - \frac{0.26}{\sqrt{0.2325}}m^2\eta^2 + \frac{0.1508}{\sqrt{0.2325}}m^2\eta + \frac{0.03825}{\sqrt{0.2325}}m^4 + \frac{0.02506}{\sqrt{0.2325}}m^2 - \frac{0.225}{\sqrt{0.2325}}m^5 + \frac{0.0026}{\sqrt{0.2325}}m + \frac{0.000208}{\sqrt{0.2325}} + \frac{2.916}{\sqrt{0.2325}}m^2\eta^3 - \frac{0.0104}{\sqrt{0.2325}}m\eta \quad (29)$$

increases in η.

Lemma B.52. *For* $m \in [0.15, 0.25]$,

$$-5.4m^2 - 1.1676 - \frac{0.975m + 0.02}{\sqrt{0.1825}} + \left(0.81083m^2 + 0.86826m + 0.68602 + \frac{-1.60665m^2}{\sqrt{0.65m + 0.02}} + \frac{-0.54329m}{\sqrt{0.65m + 0.02}}\right) \cdot \exp\left(0.96121(0.25)^2 + 2.41263m + 2.272\sqrt{0.65m + 0.02}\right). \quad (30)$$

is concave in m.

Lemma B.53. *For* $m \in [0.15, 0.25]$

$$-5.64m^2 - 1.17 - \frac{0.975m + 0.02}{\sqrt{0.65\,(0.25) + 0.02}} + \left(1.30511m^2 + 0.87501m + 0.69844 - \frac{1.30510m^3}{\sqrt{0.65m + 0.02}} - \frac{1.53406x^2}{\sqrt{0.65m + 0.02}} - \frac{0.54284m}{\sqrt{0.65m + 0.02}} - \frac{0.00521}{\sqrt{0.65\,(0.25) + 0.02}}\right) \cdot \exp\left(1.00393(0.25)^2 + 2.42188m + 2.275\sqrt{0.65m + 0.02}\right). \quad (31)$$

is concave in m.

Lemma B.54. *For* $m \in [0.15, 0.25]$,

$$1.30511m^2 + 0.87501m + 0.69844 - \frac{1.30510m^3}{\sqrt{0.65m + 0.02}} - \frac{1.53406x^2}{\sqrt{0.65m + 0.02}} - \frac{0.54284m}{\sqrt{0.65m + 0.02}} - \frac{0.00521}{\sqrt{0.65\,(0.25) + 0.02}} \quad (32)$$

is positive.

Lemma B.55. *For* $m \in [0.15, 0.25]$,

$$-11.28m + \frac{-(0.02)\,(0.85) - (1.275)\,(0.425)\,m}{(0.85m + 0.02)^{\frac{3}{2}}} + \left(-\frac{1.64928m^5}{(0.85m + 0.02)^{\frac{3}{2}}} - \frac{4.38490m^4}{(0.85m + 0.02)^{\frac{3}{2}}} - \frac{5.64255m^3}{(0.85m + 0.02)^{\frac{3}{2}}} - \frac{2.77694m^2}{(0.85m + 0.02)^{\frac{3}{2}}} + \frac{0.24584m}{(0.85m + 0.02)^{\frac{3}{2}}} + \frac{0.00299}{(0.85m + 0.02)^{\frac{3}{2}}} + \frac{1.64926m^4}{0.85m + 0.02} + \frac{2.33813m^3}{0.85m + 0.02} + \frac{3.56982m^2}{0.85m + 0.02} + \frac{1.87183m}{0.85m + 0.02} + \frac{0.05006}{0.85m + 0.02}\right) \cdot \exp\left(\frac{0.22663}{0.3}(0.15)^2 + \frac{0.591875}{0.3}m + \frac{0.5625}{0.3}\sqrt{0.85(0.15) + 0.02}\right) \quad (33)$$

decreases in m.

Supporting Lemmas for Lemma B.23 Below, we provide the supporting lemmas for lemma B.23. Refer to Section B.5.1 for the proofs of the below lemmas.

Lemma B.56. *For* $m \in [0.15, 0.25]$,

$$1 + \exp\left(3m^3 + \frac{1.1032}{0.3}m\right) + \left(1 + \frac{-0.9m^2 - 1.1032}{\sqrt{0.85m + 0.02}}\right)\exp\left(\left(1.5m^2 + \frac{0.5516}{0.3}\right)\left(m + \sqrt{0.85m + 0.02}\right)\right) + \left(1 + \frac{0.9m^2 + 1.1032}{\sqrt{0.85m + 0.02}}\right)\exp\left(\left(1.5m^2 + \frac{0.5516}{0.3}\right)\left(m - \sqrt{0.85m + 0.02}\right)\right) \quad (34)$$

is concave in m.

Lemma B.57. *For* $m \in [0.15, 0.25]$,

$$b_{2,1} + b_{2,2} \quad (35)$$

is negative, where

$$b_{2,1} = \frac{\begin{array}{r}-13.16756m^8 - 8.39216m^7 - 6.65690m^6 - 27.87899m^5 - 3.32475m^4 \\ -8.80771m^3 - 1.54734m^2 + 0.54641m - 0.58446\end{array}}{(0.85m + 0.02)^{\frac{5}{2}}}$$

and

$$b_{2,2} = \frac{\begin{array}{r}-21.94593m^7 + 13.70115m^6 - 31.93304m^5 - 0.12630m^4 \\ -8.99522m^3 - 0.73222m^2 - 2.22881m + 1.04799\end{array}}{(0.85m + 0.02)^{2}}.$$

Lemma B.58. *For* $m \in [0.15, 0.25]$,

$$b_{1,1} + b_{1,2} \tag{36}$$

is positive, where

$$b_{1,1} = \frac{\begin{array}{r}13.16757m^8 + 2.17416m^7 + 21.85326m^6 + 9.86506m^5 + 16.45667m^4 \\ +3.59784m^3 + 3.47969m^2 - 0.49787m + 0.58447\end{array}}{(0.85m + 0.02)^{\frac{5}{2}}}.$$

and

$$b_{1,2} = \frac{\begin{array}{r}-21.94593m^7 + 13.70115m^6 - 31.93304m^5 - 0.12630m^4 \\ -8.99522m^3 - 0.73222m^2 - 2.22881m + 1.04799\end{array}}{(0.85m + 0.02)^{2}}.$$

Lemma B.59. *For* $m \in [0.15, 0.25]$,

$$\begin{array}{r}13.78622m^8 - 52.72855m^7 + 106.12827m^6 - 61.36707m^5 + 53.25291m^4 - 2.36911m^3 \\ +10.33671m^2 - 4.67231m + 2.11237\end{array} \tag{37}$$

is positive.

Lemma B.60. *For* $m \in [0.15, 0.25]$,

$$\begin{aligned}1 + \exp\left(4.5m^3 + 4.544m\right) + \left(1 + \frac{-0.9m^2 - 0.9088}{\sqrt{0.65m + 0.02}}\right) \exp\left(\left(2.25m^2 + 2.272\right)\left(m + \sqrt{0.65m + 0.02}\right)\right) \\ + \left(1 + \frac{0.9m^2 + 0.9088}{\sqrt{0.65m + 0.02}}\right) \exp\left(\left(2.25m^2 + 2.272\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)\end{aligned} \tag{38}$$

is concave in m.

Lemma B.61. *For* $m \in [0.15, 0.25]$,

$$\begin{array}{r}19.05333m^8 - 67.87193m^7 + 134.36985m^6 - 64.26385m^5 + 60.36050m^4 - 1.33340m^3 + 10.08152m^2 \\ -3.07063m + 1.18559\end{array} \tag{39}$$

is positive.

Lemma B.62. *For* $m \in [0.15, 0.25]$,

$$a_{2,1} + a_{2,2} \tag{40}$$

is positive, where

$$a_{2,1} = \frac{\begin{array}{r}17.32515m^8 + 8.88654m^7 + 8.94527m^6 + 23.90324m^5 + 4.16069m^4 \\ +6.38985m^3 + 1.26793m^2 - 0.33042m + 0.27707\end{array}}{(0.65m + 0.02)^{\frac{5}{2}}}$$

and

$$a_{2,2} = \frac{\begin{array}{r}-28.87523m^7 + 17.65092m^6 - 34.85621m^5 + 1.71461m^4 \\ -8.24483m^3 - 0.21430m^2 - 1.65229m + 0.60463\end{array}}{(0.65m + 0.02)^{2}}.$$

Lemma B.63. *For $m \in [0.15, 0.25]$,*

$$a_{1,1} + a_{1,2} \tag{41}$$

is negative, where

$$a_{1,1} = \frac{\begin{array}{r}-17.32514m^8 - 8.88653m^7 - 8.94526m^6 - 24.15990m^5 - 4.17647m^4 \\ -6.47647m^3 - 1.29224m^2 + 0.32859m - 0.27709\end{array}}{(0.65m + 0.02)^{\frac{5}{2}}}$$

and

$$a_{1,2} = \frac{\begin{array}{r}-28.87523m^7 + 17.65092m^6 - 34.85621m^5 + 1.50072m^4 \\ -8.25668m^3 - 0.25766m^2 - 1.65362m + 0.60463\end{array}}{(0.65m + 0.02)^2}.$$

Lemma B.64. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\begin{aligned}&\left(\sqrt{2m\eta + 0.25m + 0.02} + 0.9m^2 + 0.52 + 1.944\eta\right) \exp\left(-\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\sqrt{2m\eta + 0.25m + 0.02}\right) \\ &+ \left(\sqrt{2m\eta + 0.25m + 0.02} - 0.9m^2 - 0.52 - 1.944\eta\right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\sqrt{2m\eta + 0.25m + 0.02}\right) \\ &+ \sqrt{2m\eta + 0.25m + 0.02} \exp\left(-\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right) m\right) \\ &+ \sqrt{2m\eta + 0.25m + 0.02} \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right) m\right)\end{aligned} \tag{42}$$

is concave in η.

Lemma B.65. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\begin{aligned}&\sqrt{2m\eta + 0.25m + 0.02} \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right) m\right) \\ &+ \left(\sqrt{2m\eta + 0.25m + 0.02} - 0.9m^2 - 0.52 - 1.944\eta\right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\sqrt{2m\eta + 0.25m + 0.02}\right)\end{aligned} \tag{43}$$

is concave in η.

Lemma B.66. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\left(\sqrt{2m\eta + 0.25m + 0.02} + 0.9m^2 + 0.52 + 1.944\eta\right) \exp\left(-\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\sqrt{2m\eta + 0.25m + 0.02}\right) \tag{44}$$

is concave in η.

Lemma B.67. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\begin{aligned}
&\frac{1}{2\sqrt{0.1175}} + \frac{1}{2\sqrt{0.1175}} \exp\left(\frac{0.9m^3}{\eta} + \frac{0.52m}{\eta} + 1.944m\right) \\
&+ \left(-\frac{1}{\eta\sqrt{0.1175}} \left(0.45m^2 + 0.26 + 0.972\eta\right)^2 + \frac{1}{2\sqrt{0.1175}} + \frac{1}{2\eta} \left(0.45m^2 + 0.26 + 0.972\eta\right)\right) \\
&\cdot \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{0.1175}\right)\right) \\
&+ \left(-\frac{1}{\eta\sqrt{0.1175}} \left(0.45m^2 + 0.26 + 0.972\eta\right)^2 + \frac{1}{2\sqrt{0.1175}} - \frac{1}{2\eta} \left(0.45m^2 + 0.26 + 0.972\eta\right)\right) \\
&\cdot \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{0.1175}\right)\right)
\end{aligned} \tag{45}$$

is positive.

Lemma B.68. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$*

$$\begin{aligned}
&- \frac{1.8\eta}{\sqrt{0.1175}} \left(0.45m^3 + 0.26m + 0.972\eta m\right) - 0.45\eta m - \frac{0.45\left(0.15\right)^2 + 0.26 + 0.972\eta}{\sqrt{0.1175}} \left(0.45m^2 + 0.26 + 0.972\eta\right)^2 \\
&+ \frac{0.225\left(0.15\right)^2\eta + 0.13\eta + 0.486\eta^2}{\sqrt{0.1175}} - \left(0.225\left(0.15\right)^2 + 0.13 + 0.486\eta\right)\left(0.45m^2 + 0.26 + 0.972\eta\right) \\
&+ \frac{\left(0.45\right)\left(0.15\right)^2\left(0.38\right)\eta + \left(0.26\right)\left(0.38\right)\eta + \left(0.972\right)\left(0.38\right)\eta^2}{\sqrt{0.1175}} \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{0.1175}\right)\right)
\end{aligned} \tag{46}$$

is negative.

Lemma B.69. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$*

$$\begin{aligned}
&- \frac{1.8}{\sqrt{0.1175}} \left(0.45\eta m^3 + 0.26\eta m + 0.972\eta^2 m\right) + 0.45\eta m - \frac{0.288125 + 0.972\eta}{\sqrt{0.1175}} \left(0.45m^2 + 0.26 + 0.972\eta\right)^2 \\
&+ \frac{0.1440625\eta + 0.486n^2}{\sqrt{0.1175}} + \left(0.1440625 + 0.486\eta\right)\left(0.45m^2 + 0.26 + 0.972\eta\right) \\
&+ \frac{\left(0.270125\right)\left(0.62\right)\eta + \left(0.972\right)\left(0.62\right)\eta^2}{\sqrt{0.1175}} \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{0.1175}\right)\right)
\end{aligned} \tag{47}$$

is negative.

Lemma B.70. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3], c \in [0.1175, 0.2375]$,*

$$\begin{aligned}
&\sqrt{c} + \sqrt{c} \exp\left(\frac{0.9m^3}{\eta} + \frac{0.52m}{\eta} + 1.944m\right) \\
&\quad + \left(\sqrt{c} - 0.9m^2 - 0.52 - 1.944\eta\right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{c}\right)\right) \\
&\quad + \left(\sqrt{c} + 0.9m^2 + 0.52 + 1.944\eta\right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{c}\right)\right)
\end{aligned} \tag{48}$$

is convex in c.

Lemma B.71. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$-0.235\left(0.45m^2+0.26\right)\left(0.45m^2+0.26+0.972\eta\right)-\eta\sqrt{0.1175}\left(0.45m^2+0.26\right)+2\eta\left(0.45m^2+0.26+0.972\eta\right)^2$$
$$-4\eta\left(0.45m^2+0.26\right)\left(0.45m^2+0.26+0.972\eta\right)-2\sqrt{0.1175}\left(0.45m^2+0.26+0.972\eta\right)^3$$
$$+6\sqrt{0.1175}\left(0.45m^2+0.26\right)\left(0.45m^2+0.26+0.972\eta\right)^2 \tag{49}$$

Lemma B.72. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$-0.235\left(0.45m^2+0.26\right)\left(0.45m^2+0.26+0.972\eta\right)+\eta\sqrt{0.1175}\left(0.45m^2+0.26\right)+2\eta\left(0.45m^2+0.26+0.972\eta\right)^2$$
$$-4\eta\left(0.45m^2+0.26\right)\left(0.45m^2+0.26+0.972\eta\right)+2\sqrt{0.1175}\left(0.45m^2+0.26+0.972\eta\right)^3$$
$$-6\sqrt{0.1175}\left(0.45m^2+0.26\right)\left(0.45m^2+0.26+0.972\eta\right)^2 \tag{50}$$

Lemma B.73. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], c \in [0.1175, 0.2375]$,

$$\frac{c}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^2-1-\frac{\sqrt{c}}{\eta}\left(0.45m^2+0.26+0.972\eta\right)$$
$$+\frac{2}{n}\left(0.45m^2+0.26+0.972\eta\right)^2+\frac{2\sqrt{c}}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^3 \tag{51}$$

is positive and grows in c.

Lemma B.74. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], c \in [0.1175, 0.2375]$,

$$\frac{c}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^2-1+\frac{\sqrt{c}}{\eta}\left(0.45m^2+0.26+0.972\eta\right)$$
$$+\frac{2}{\eta}\left(0.45m^2+0.26+0.972\eta\right)^2-\frac{2\sqrt{c}}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^3 \tag{52}$$

is positive and grows in c.

Supporting Lemmas for Lemma B.24 Below, we provide the supporting lemmas for lemma B.24. Refer to Section B.5.1 for the proofs of the below lemmas.

Lemma B.75. *For* $x \in [0.15, 0.25]$,

$$\frac{-0.011x}{0.672x+0.025} \tag{53}$$

is convex in x *and can be upper bounded by* $-0.01132x - 0.0114$.

Lemma B.76. *For* $m \in [0.15, 0.25]$,

$$\left(\frac{0.07804m^4+0.00291m^3+0.03985m^2+0.66249m+0.025}{0.672m+0.025}\right)\exp\left(\frac{0.94}{0.211}m^3+\frac{0.93365}{0.211}m\right)$$
$$+\left(\mu_{1,1}+\mu_{1,2}\right)\exp\left(\left(\frac{0.47}{0.211}m^2+\frac{0.466825}{0.211}\right)\left(m+\sqrt{0.672m+0.025}\right)\right)$$
$$+\left(\mu_{2,1}+\mu_{2,2}\right)\exp\left(\left(\frac{0.47}{0.211}m^2+\frac{0.466825}{0.211}\right)\left(m-\sqrt{0.672m+0.025}\right)\right) \tag{54}$$

is concave in m, where

$$\mu_{1,1} = \frac{0.05354m^3 - 0.93709m^2 + 0.01552m - 0.90851}{\sqrt{0.672m + 0.025}}$$

$$\mu_{1,2} = \frac{-0.05032m^5 - 0.00272m^4 - 0.06456m^3 - 0.00410m^2 + 0.64652m + 0.02362}{0.672m + 0.025}$$

$$\mu_{2,1} = \frac{-0.05353m^3 + 0.93710m^2 - 0.01551m + 0.90852}{\sqrt{0.672m + 0.025}}$$

$$\mu_{2,2} = \frac{-0.05032m^5 - 0.00272m^4 - 0.06456m^3 - 0.00410m^2 + 0.64652m + 0.02362}{0.672m + 0.025}.$$

Lemma B.77. *For $x \in [0.15, 0.25]$,*

$$\exp\left(-\left(\frac{0.47}{0.211}x^2 + \frac{0.466825}{0.211}\right)\left(x + \sqrt{0.672x + 0.025}\right)\right) \tag{55}$$

is convex in x and can be upper bounded by $-1.21526x + 0.502$.

Lemma B.78. *For $x \in [0.15, 0.25]$,*

$$\left(\frac{0.47}{0.211}x^2 + \frac{0.466825}{0.211}\right)\left(\sqrt{0.672x + 0.025} - x\right) \tag{56}$$

is concave in x and can be minimized at $x = 0.25$.

Lemma B.79. *For $x \in [0.15, 0.25]$,*

$$\sqrt{0.672x + 0.025} \tag{57}$$

can be lower bounded by $10\left(\sqrt{0.193} - \sqrt{0.1258}\right)x + 0.227$.

Lemma B.80. *For $m \in [0.15, 0.25]$,*

$$\delta_{2,1}\sqrt{0.672m + 0.025} + \delta_{2,2} \tag{58}$$

is positive, where

$$\begin{aligned}\delta_{2,1} =& 1.69123m^{10} - 0.87479m^9 + 21.38686m^8 + 10.73075m^7 + 9.84447m^6 + 27.05522m^5 \\ &+ 4.63531m^4 + 6.87129m^3 + 1.41681m^2 - 0.32436m + 0.29412\end{aligned}$$

and

$$\begin{aligned}\delta_{2,2} =& -1.01473m^{11} - 0.60389m^{10} - 0.84169m^9 - 23.62301m^8 + 10.53287m^7 - 25.57167m^6 \\ &- 1.13405m^5 - 6.00502m^4 - 0.60382m^3 - 1.15742m^2 + 0.37493m + 0.01514.\end{aligned}$$

Lemma B.81. *For $m \in [0.15, 0.25]$,*

$$\delta_{1,1}\sqrt{0.672m + 0.025} + \delta_{1,2} \tag{59}$$

is negative, where

$$\begin{aligned}\delta_{1,1} =& -1.69122m^{10} + 0.87500m^9 - 21.38654m^8 - 10.73028m^7 - 9.84375m^6 - 27.05472m^5 \\ &- 4.63493m^4 - 6.87112m^3 - 1.41675m^2 + 0.32437m - 0.29411\end{aligned}$$

and

$$\begin{aligned}\delta_{1,2} =& -1.01473m^{11} - 0.60389m^{10} - 0.84147m^9 - 23.62276m^8 + 10.53324m^7 - 25.57118m^6 \\ &- 1.13379m^5 - 6.00481m^4 - 0.60377m^3 - 1.15740m^2 + 0.37493m + 0.01514.\end{aligned}$$

Lemma B.82. *For* $m \in [0.15, 0.25]$,

$$\begin{aligned}&\Big(\left(-4.86669m^3 - 0.26392m^2 - 1.41011m - 2.28476\right)\sqrt{0.672m + 0.025} + 4.57470m^5 + 0.24809m^4 \\ &+ 5.86930m^3 + 0.37310m^2 + 2.31656m + 0.12583\Big)\exp\left(\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\sqrt{0.672m + 0.025}\right) \\ &+ \left(-7.09418m^4 - 0.26392m^3 - 3.62256m^2 + 0.86524m\right)\exp\left(\frac{0.47}{0.211}m^3 + \frac{0.466825}{0.211}m\right) \\ &+\Big(\left(4.86670m^3 + 0.26392m^2 + 1.41012m + 2.28477\right)\sqrt{0.672m + 0.025} + 4.57470m^5 + 0.24809m^4 + 5.86930m^3 \\ &\quad + 0.37310m^2 + 2.31656m + 0.12583\Big)\exp\left(-\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\sqrt{0.672m + 0.025}\right) \\ &+ \left(7.09419m^4 + 0.26393m^3 + 3.62257m^2 + 1.13477m\right)\exp\left(\frac{-0.47}{0.211}m^3 + \frac{-0.466825}{0.211}m\right)\end{aligned} \tag{60}$$

is negative.

Lemma B.83. *For* $m \in [0.15, 0.25]$,

$$\begin{aligned}&\Big(\left(-4.86669m^3 - 0.26392m^2 - 1.41011m - 2.28476\right)\sqrt{0.672m + 0.025} + 4.57470m^5 + 0.24809m^4 + 5.86930m^3 \\ &\quad + 0.37310m^2 + 2.31656m + 0.12583\Big)\exp\left(\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\sqrt{0.672m + 0.025}\right) \\ &+ \left(-7.09418m^4 - 0.26392m^3 - 3.62256m^2 + 0.86524m\right)\exp\left(\frac{0.47}{0.211}m^3 + \frac{0.466825}{0.211}m\right)\end{aligned} \tag{61}$$

is concave in m.

Lemma B.84. *For* $x \in [0.15, 0.25]$,

$$\exp\left(\left(\frac{0.47}{0.211}x^2 + \frac{0.466825}{0.211}\right)\left(x - \sqrt{0.672x + 0.025}\right)\right) \tag{62}$$

is convex in x.

Lemma B.85. *For* $m \in [0.15, 0.25]$,

$$\begin{aligned}&\Big(\left(4.86670m^3 + 0.26392m^2 + 1.41012m + 2.28477\right)\sqrt{0.672m + 0.025} + 4.57470m^5 + 0.24809m^4 \\ &+ 5.86930m^3 + 0.37310m^2 + 2.31656m + 0.12583\Big)\exp\left(-\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\sqrt{0.672m + 0.025}\right)\end{aligned} \tag{63}$$

is concave in m.

Lemma B.86. *For* $m \in [0.15, 0.25]$,

$$\left(7.09419m^4 + 0.26393m^3 + 3.62257m^2 + 1.13477m\right)\exp\left(\frac{-0.47}{0.211}m^3 + \frac{-0.466825}{0.211}m\right) \tag{64}$$

is concave in m.

Lemma B.87. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\left(\sqrt{2m\eta + 0.25m + 0.025} + 0.94m^2 + 0.48 + 2.15\eta\right) \exp\left(-\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\sqrt{2m\eta + 0.25m + 0.025}\right)$$

$$+ \left(\sqrt{2m\eta + 0.25m + 0.025} - 0.94m^2 - 0.48 - 2.15\eta\right) \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\sqrt{2m\eta + 0.25m + 0.025}\right)$$

$$+ \sqrt{2m\eta + 0.25m + 0.025} \exp\left(\frac{-0.47m^3}{\eta} + \frac{-0.24m}{\eta} - 1.075m\right)$$

$$+ \sqrt{2m\eta + 0.25m + 0.025} \exp\left(\frac{0.47m^3}{\eta} + \frac{0.24m}{\eta} + 1.075m\right) \tag{65}$$

is concave in η.

Lemma B.88. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\sqrt{2m\eta + 0.25m + 0.025} \exp\left(\frac{0.47m^3}{\eta} + \frac{0.24m}{\eta} + 1.075m\right)$$

$$+ \left(\sqrt{2m\eta + 0.25m + 0.025} - 0.94m^2 - 0.48 - 2.15\eta\right) \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\sqrt{2m\eta + 0.25m + 0.025}\right) \tag{66}$$

is concave in η.

Lemma B.89. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\left(\sqrt{2m\eta + 0.25m + 0.025} + 0.94m^2 + 0.48 + 2.15\eta\right) \exp\left(-\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\sqrt{2m\eta + 0.25m + 0.025}\right) \tag{67}$$

is concave in η.

Lemma B.90. *For* $\eta \in [0.2, 0.3], m \in [0.15, 0.25]$,

$$\begin{aligned}
&- 3.225m^2\eta - 0.2975m^2 - 0.05375m + 0.47m^4 \\
&+ \Bigg(\frac{1}{\eta}\left(0.2209m^6 - 0.027025m^4 - 0.02526m^3 - 0.0714m^2 - 0.0129m\right) \\
&\quad + \frac{1}{\eta^2}\left(0.11045m^6 + 0.01105m^5 + 0.1128m^4 + 0.01128m^3 + 0.0288m^2 + 0.00288m\right) \\
&\quad + \frac{1}{\eta^3}\left(0.01381m^6 + 0.00277m^5 + 0.01424m^4 + 0.00282m^3 + 0.00375m^2 + 0.00072m + 0.00004\right) \\
&\quad - 1.0105m^4 - 0.516m^2 + 1.15563m^2\eta\Bigg)\sqrt{2m\eta + 0.25m + 0.025}
\end{aligned} \tag{68}$$

decreases in η.

Lemma B.91. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\sqrt{2m\eta + 0.25m + 0.025} \exp\left(\frac{-0.47m^3}{\eta} + \frac{-0.24m}{\eta} - 1.075m\right) \tag{69}$$

is concave in η.

Lemma B.92. *For* $m \in [0.2, 0.25], \eta \in [0.2, 0.3]$

$$\begin{aligned}
&\frac{1}{2\sqrt{0.15}} + \frac{1}{2\sqrt{0.15}} \exp\left(\frac{0.94m^3}{\eta} + \frac{0.48m}{\eta} + 2.15m\right) \\
&+\left(\frac{1}{2\sqrt{0.15}} + \frac{1}{2\eta}\left(0.47m^2 + 0.24 + 1.075\eta\right)\right. \\
&\quad \left. - \frac{1}{\eta\sqrt{0.15}}\left(0.47m^2 + 0.24 + 1.075\eta\right)^2\right) \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.15}\right)\right) \\
&+\left(\frac{1}{2\sqrt{0.15}} - \frac{1}{2\eta}\left(0.47m^2 + 0.24 + 1.075\eta\right)\right. \\
&\quad \left. - \frac{1}{\eta\sqrt{0.15}}\left(0.47m^2 + 0.24 + 1.075\eta\right)^2\right) \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)
\end{aligned} \tag{70}$$

is positive.

Lemma B.93. *For* $m \in [0.2, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&-(1.075)^2\eta^2 + \left(\frac{1}{2} - (1.075)(0.94)m^2 - (1.075)(0.48) - 0.5375\sqrt{0.15}\right)\eta \\
&+\left(-0.235\sqrt{0.15}m^2 - 0.12\sqrt{0.15} - (0.47)^2 m^4 - (0.48)(0.47)m^2 - (0.24)^2\right)
\end{aligned} \tag{71}$$

is negative.

Lemma B.94. *For* $m \in [0.15, 0.2], \eta \in [0.2, 0.3]$.

$$\begin{aligned}
&\frac{1}{2\sqrt{0.1175}} + \frac{1}{2\sqrt{0.1175}} \exp\left(\frac{0.94m^3}{\eta} + \frac{0.48m}{\eta} + 2.15m\right) \\
&+\left(\frac{1}{2\sqrt{0.1175}} + \frac{1}{2}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)\right. \\
&\quad \left. - \frac{\eta}{\sqrt{0.1175}}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)^2\right) \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.1175}\right)\right) \\
&+\left(\frac{1}{2\sqrt{0.1175}} - \frac{1}{2}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)\right. \\
&\quad \left. - \frac{\eta}{\sqrt{0.1175}}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)^2\right) \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.1175}\right)\right)
\end{aligned} \tag{72}$$

is positive.

Lemma B.95. *For* $m \in [0.15, 0.2], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&-\frac{(1.075)^2}{\sqrt{0.1175}}\eta^2 + \left(\frac{1}{2\sqrt{0.1175}} - 0.5375 - \frac{(1.075)(0.94)}{\sqrt{0.1175}}m^2 - \frac{(1.075)(0.48)}{\sqrt{0.1175}}\right)\eta \\
&+\left(-0.235m^2 - 0.12 - \frac{(0.47)^2}{\sqrt{0.1175}}m^4 - \frac{(0.48)(0.47)}{\sqrt{0.1175}}m^2 - \frac{(0.24)^2}{\sqrt{0.1175}}\right)
\end{aligned} \tag{73}$$

is negative.

Lemma B.96. *For* $m \in [0.15, 0.2], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&-\frac{(1.075)^2}{\sqrt{0.1175}}\eta^2 + \left(\frac{1}{2\sqrt{0.1175}} + 0.5375 - \frac{(2.15)(0.47)}{\sqrt{0.1175}}m^2 - \frac{(1.075)(0.48)}{\sqrt{0.1175}}\right)\eta \\
&+\left(0.235m^2 + 0.12 - \frac{(0.47)^2}{\sqrt{0.1175}}m^4 - \frac{(0.24)(0.94)}{\sqrt{0.1175}}m^2 - \frac{(0.24)^2}{\sqrt{0.1175}}\right)
\end{aligned} \tag{74}$$

is negative.

Lemma B.97. *For* $m \in [0.15, 0.2], \eta \in [0.2, 0.3]$,

$$\frac{1}{2\sqrt{0.1175}} + \frac{1}{2\sqrt{0.1175}} \exp\left(\frac{0.94m^3}{0.3} + \frac{0.48m}{0.3} + 2.15m\right)$$

$$+\left(\frac{1}{2\sqrt{0.1175}} + \frac{1}{2}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)\right.$$
$$\left. - \frac{\eta}{\sqrt{0.1175}}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)^2\right) \exp\left(\left(\frac{0.47m^2}{0.2} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.1175}\right)\right)$$

$$+\left(\frac{1}{2\sqrt{0.1175}} - \frac{1}{2}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)\right.$$
$$\left. - \frac{\eta}{\sqrt{0.1175}}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)^2\right) \exp\left(\left(\frac{0.47m^2}{0.3} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.1175}\right)\right) \tag{75}$$

is concave in η.

Lemma B.98. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], c \in [0.1175, 0.2375]$,

$$\sqrt{c} + \sqrt{c}\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{c}\right)\right) + \sqrt{c}\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{c}\right)\right)$$

$$+ \sqrt{c}\exp\left(\frac{0.94m^3}{\eta} + \frac{0.48m}{\eta} + 2.15m\right) - \left(0.94m^2 + 0.48 + 2.15\eta\right)\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{c}\right)\right)$$

$$+ \left(0.94m^2 + 0.48 + 2.15\eta\right)\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{c}\right)\right) \tag{76}$$

is convex in c.

Lemma B.99. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], c \in [0.1175, 0.2375]$,

$$-1 + \left(\frac{\sqrt{c}}{\eta}\right)\left(0.47m^2 + 0.24 + 1.075\eta\right) + \left(\frac{c}{\eta^2}\right)\left(0.47m^2 + 0.24 + 1.075\eta\right)^2$$
$$+ \left(\frac{2}{\eta}\right)\left(0.47m^2 + 0.24 + 1.075\eta\right)^2 - \left(\frac{2\sqrt{c}}{\eta^2}\right)\left(0.47m^2 + 0.24 + 1.075\eta\right)^3. \tag{77}$$

is positive and grows in c.

Lemma B.100. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], c \in [0.1175, 0.2375]$,

$$-1 - \left(\frac{\sqrt{c}}{\eta}\right)\left(0.47m^2 + 0.24 + 1.075\eta\right) + \left(\frac{c}{\eta^2}\right)\left(0.47m^2 + 0.24 + 1.075\eta\right)^2$$
$$+ \left(\frac{2}{\eta}\right)\left(0.47m^2 + 0.24 + 1.075\eta\right)^2 + \left(\frac{2\sqrt{c}}{\eta^2}\right)\left(0.47m^2 + 0.24 + 1.075\eta\right)^3 \tag{78}$$

is positive and grows in c.

Proofs for Lemmas in Section B.5.1

Proof of Lemma B.21. To prove the theorem, first note that the Equation 9 is convex in q. So, we split up our analysis into two cases, since $q_l > q_{min}$ for $m \in [0.21, 0.25]$, according to lemma B.26. First, for $m \in [0.15, 0.21]$, we will show that q_{min} that minimizes Equation 9 is positive, which we prove in lemma B.101. Second, we will show that the lower bound for q, $q_l = \frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944$ is positive for $m \in [0.21, 0.25]$, which we prove in lemma B.42. This concludes the proof of the theorem. □

Proof of Lemma B.22. We begin the proof by first noting that

$$\sqrt{c}\left(\sqrt{c}-m\right)\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\right)-2m\eta q \tag{79}$$

is convex in q. So, by Jensen's inequality, it is sufficient to evaluate Equation 79 at $q=\frac{0.9m^2}{\eta}+\frac{0.5}{\eta}+1.944$ and $q=\frac{0.94m^2}{\eta}+\frac{0.48}{\eta}+2.15$ and show both are negative.

First, we start with $\frac{0.9m^2}{\eta}+\frac{0.5}{\eta}+1.944$. By lemma B.43, we know that Equation 79 evaluated at $q=\frac{0.9m^2}{\eta}+\frac{0.5}{\eta}+1.944$ is convex in η. So, by Jensen's inequality, it is sufficient to show that for $\eta=0.2,0.3$ that Equation 79 evaluated at $q=\frac{0.9m^2}{\eta}+\frac{0.5}{\eta}+1.944$ is negative. We prove this in lemmas B.44 and B.45 respectively.

For $q=\frac{0.94m^2}{\eta}+\frac{0.48}{\eta}+2.15$, we follow the same approach as above. By convexity in η for Equation 79 evaluated at $q=\frac{0.94m^2}{\eta}+\frac{0.48}{\eta}+2.15$, we further verify in lemmas B.47 and B.48 that at $\eta=0.2,0.3$, the equation is negative.

□

Proof of Lemma B.24. To prove the theorem, we proceed as follows. First, we will show that Equation 10 evaluated at $q=\frac{0.94m^2}{\eta}+\frac{0.48}{\eta}+2.15$ is convex in c for $c\in[0.1175,0.2375]$ (see lemma B.98) and the first partial derivative with respect to c is positive at $c=0.1175$ for $m\in[0.15,0.20]$ and positive at $c=0.15$ for $m\in[0.20,0.25]$ (see lemmas B.94 and B.92).

After upper bounding Equation 10 (evaluated at $q=\frac{0.94m^2}{\eta}+\frac{0.48}{\eta}+2.15$) by $c=2m\eta+0.25m+0.025$, we scale by $\exp\left(-\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)m\right)$ and show that the resulting expression is concave in η in lemma B.87.

Then, we linearize Equation 10 (evaluated at $q=\frac{0.94m^2}{\eta}+\frac{0.48}{\eta}+2.15, c=2m\eta+0.25m+0.025$) at $\eta=0.211$ and evaluate it at $\eta=0.2$ (see lemma B.82 to validate first partial η of Equation 10 is negative at $\eta=0.211$).

Consider the linearization below (simplified by rounding up after five digits of precision).

$$\begin{aligned}
&\left(1+\frac{-0.011m}{0.672m+0.025}-0.11612m^3-0.05929m\right)\\
&+\left(\frac{0.07804m^4+0.00291m^3+0.03985m^2+0.66249m+0.025}{0.672m+0.025}\right)\exp\left(\frac{0.94}{0.211}m^3+\frac{0.93365}{0.211}m\right)\\
&+(\nu_{1,1}+\nu_{1,2})\exp\left(\left(\frac{0.47}{0.211}m^2+\frac{0.466825}{0.211}\right)\left(m+\sqrt{0.672m+0.025}\right)\right)\\
&+(\nu_{2,1}+\nu_{2,2})\exp\left(\left(\frac{0.47}{0.211}m^2+\frac{0.466825}{0.211}\right)\left(m-\sqrt{0.672m+0.025}\right)\right),
\end{aligned} \tag{80}$$

where

$$\nu_{1,1} = \frac{0.05354m^3 - 0.93709m^2 + 0.01552m - 0.90851}{\sqrt{0.672m + 0.025}}$$

$$\nu_{1,2} = \frac{-0.05032m^5 - 0.00272m^4 - 0.06456m^3 - 0.00410m^2 + 0.64652m + 0.02362}{0.672m + 0.025}$$

$$\nu_{2,1} = \frac{-0.05353m^3 + 0.93710m^2 - 0.01551m + 0.90852}{\sqrt{0.672m + 0.025}}$$

$$\nu_{2,2} = \frac{-0.05032m^5 - 0.00272m^4 - 0.06456m^3 - 0.00410m^2 + 0.64652m + 0.02362}{0.672m + 0.025}.$$

Note that we upper bound the above by replacing $\frac{-0.011m}{0.672m+0.025}$ by linear local upper bound $-0.01132m - 0.0114$ (see lemma B.75.

Since

$$1 + (-0.01132m - 0.0114) - 0.11612m^3 - 0.05929m \tag{81}$$

is concave in m, we show that the remaining expression for Equation 80 is concave in m in lemma B.76.

So, we create another linearization at $m = 0.176$, which we evaluate at $m = 0.15$ (negative), since the first partial m of our upper bound for Equation 80 is negative at $m = 0.176$.

This completes the proof.

□

Proof of Lemma B.23. To prove the theorem, we proceed as follows. First, we will show that Equation 10 evaluated at $q = \frac{0.9m^2}{\eta} + \frac{0.52}{\eta} + 1.944$ is convex in c for $c \in [0.1175, 0.2375]$ (see lemma B.70) and the first partial derivative with respect to c is positive at $c = 0.1175$ (see lemma B.67).

After lower bounding Equation 10 (evaluated at $q = \frac{0.9m^2}{\eta} + \frac{0.52}{\eta} + 1.944$) by $c = 2m\eta + 0.25m + 0.02$, we scale by $\exp\left(-\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)m\right)$ and show that the resulting expression is concave in η in lemma B.64.

Then, for each $\eta = 0.2, 0.3$, we show that the resulting equation for each case is concave in m in lemmas B.60 and B.56, so we evaluate $m = 0.15, 0.25$, which are positive.

□

Proof of Lemma B.25. We consider a point $(\tilde{q} > 0, \tilde{c})$ that is an intersection point for our alternate equivalent conditions for $\|q_t\|_2^2$ nullcline 1 and the c_t^2 growth boundary. Meaning the below conditions are satisfied:

$$\sqrt{\tilde{c}} + \sqrt{\tilde{c}}\exp\left(\tilde{q}m\right) + \sqrt{\tilde{c}}\exp\left(\frac{1}{2}\tilde{q}\left(m + \sqrt{\tilde{c}}\right)\right) + \sqrt{\tilde{c}}\exp\left(\frac{1}{2}\tilde{q}\left(m - \sqrt{\tilde{c}}\right)\right) - \eta\tilde{q}\left[\exp\left(\frac{1}{2}\tilde{q}\left(m + \sqrt{\tilde{c}}\right)\right) - \exp\left(\frac{1}{2}\tilde{q}\left(m - \sqrt{\tilde{c}}\right)\right)\right] = 0$$

$$\sqrt{\tilde{c}}\left(\sqrt{\tilde{c}} - m\right)\left(1 + \exp\left(\frac{1}{2}\tilde{q}\left(m + \sqrt{\tilde{c}}\right)\right)\right) - 2m\eta\tilde{q} = 0.$$

To prove the lemma, we will first show that the condition

$$\sqrt{\tilde{c}} + \sqrt{\tilde{c}}\exp\left(\tilde{q}m\right) + \sqrt{\tilde{c}}\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) + \sqrt{\tilde{c}}\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right) - \eta\tilde{q}\left[\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) - \exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right] = 0$$

gives us that

$$\sqrt{\tilde{c}} - \eta\tilde{q}\left[\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1} - \left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\right] = 0$$

.

We proceed as follows:

$$\begin{aligned}
&\sqrt{\tilde{c}} + \sqrt{\tilde{c}}\exp\left(\tilde{q}m\right) + \sqrt{\tilde{c}}\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) + \sqrt{\tilde{c}}\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right) \\
&- \eta\tilde{q}\left[\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) - \exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right] \\
&= \sqrt{\tilde{c}}\left[1+\exp\left(\tilde{q}m\right) + \exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) + \exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right] \\
&- \eta\tilde{q}\left[\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) - \exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right] \\
&= \sqrt{\tilde{c}}\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right]\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right] \\
&- \eta\tilde{q}\left[\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) - \exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right]
\end{aligned}$$

Since $\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right]^{-1}\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right]^{-1} > 0,$

and

$$\sqrt{\tilde{c}}\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right]\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right] - \eta\tilde{q}\left[\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) - \exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right] = 0$$

then

$$\begin{aligned}
&\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right]^{-1}\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right]^{-1} \\
&\cdot\left(\sqrt{\tilde{c}}\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right]\left[1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right]\right. \\
&\left.- \eta\tilde{q}\left[\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) - \exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right]\right) \\
&= \sqrt{\tilde{c}} - \eta\tilde{q}\left[\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1} - \left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\right] \\
&= 0
\end{aligned}$$

Next, we will show that how the condition

$$\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(m+\sqrt{\tilde{c}}\right)+\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(m-\sqrt{\tilde{c}}\right)=0$$

is satisfied.

Since $\tilde{q}>0$, then

$$\frac{\sqrt{\tilde{c}}}{\eta\tilde{q}}\left(\sqrt{\tilde{c}}-m\right)\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)-2m=0.$$

Since $\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}>0$, then

$$\frac{\sqrt{\tilde{c}}}{\eta\tilde{q}}\left(\sqrt{\tilde{c}}-m\right)-2m\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}=0$$

.

Since we proved

$$\sqrt{\tilde{c}}-\eta\tilde{q}\left[\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\right]=0$$

for $(\tilde{q},\tilde{c})$, then

$$\begin{aligned}&\left[\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\right]\left(\sqrt{\tilde{c}}-m\right)\\&-2m\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\\&=\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(\sqrt{\tilde{c}}-m\right)-\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(\sqrt{\tilde{c}}+m\right)\\&=0\end{aligned}$$

Since

$$\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(\sqrt{\tilde{c}}-m\right)-\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(\sqrt{\tilde{c}}+m\right)=0$$

then

$$\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(\sqrt{\tilde{c}}+m\right)+\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1}\left(m-\sqrt{\tilde{c}}\right)=0$$

.

This completes the proof. □

Proofs for Lemmas in Section B.5.1

Proof of Lemma B.26. To prove the lemma, we begin by scaling

$$\begin{aligned}&\frac{0.9m^2}{\eta}+\frac{0.52}{\eta}+1.944\\&-\frac{2}{m+\sqrt{2m\eta+0.25m+0.025}}\left(\ln(4m\eta)-\frac{1}{2}\ln(2m\eta+0.25m+0.025)-\ln(2m\eta+0.25m+0.025-m^2)\right)\end{aligned}$$

by $\eta\left(m+\sqrt{2m\eta+0.25m+0.025}\right)$.

By lemma B.29, we know that

$$\left(0.9m^2+0.52+1.944\eta\right)\left(m+\sqrt{2m\eta+0.25m+0.025}\right) - 2\eta\left(\ln(4m\eta)-\frac{1}{2}\ln(2m\eta+0.25m+0.025)-\ln(2m\eta+0.25m+0.025-m^2)\right) \quad (82)$$

is convex in η for the parameter range.

Since we also know that Equation 82 is decreasing for $\eta \in [0.2, 0.3]$ by lemma B.28, we evaluate Equation 82 at $\eta = 0.3$, and show that it is positive for $0.21 \leq m \leq 0.25$. We prove this in lemma B.27.

This completes the proof. □

Proof of Lemma B.27. To proceed with the proof, we show that the derivative of Equation 11 is positive for $m \in [0.21, 0.25]$. Consider the derivative below.

$$2.7m^2+1.1032+1.8m\sqrt{0.85m+0.025}+\frac{(0.85)(0.9m^2+1.1032)}{2\sqrt{0.85m+0.025}}-\frac{0.6}{m} + \frac{(0.3)(0.85)}{0.85m+0.025}+\frac{0.6(0.85-2m)}{0.85m+0.025-m^2}$$

After scaling the above by m, we can lower bound it with the following:

$$2.7(0.21)^3+1.1032(0.21)+1.8(0.21)^2\sqrt{0.85(0.21)+0.025}+\frac{(0.85)(0.21)(0.9(0.21)^2+1.1032)}{2\sqrt{0.85(0.25)+0.025}}-0.6 + \frac{(0.3)(0.85)(0.21)}{0.85(0.25)+0.025}+\frac{0.6(0.85-2(0.25))(0.21)}{0.85(0.25)+0.025-(0.21)^2},$$

which is positive.

For, we lower bound Equation 11 by evaluating at $m = 0.21$, which is positive.

This completes the proof. □

Proof of LemmaB.28. To prove the lemma, we note that

$$\left(0.9m^2+0.52+1.944\eta\right)\left(m+\sqrt{2m\eta+0.25m+0.025}\right) - 2\eta\left(\ln(4m\eta)-\frac{1}{2}\ln(2m\eta+0.25m+0.025)-\ln(2m\eta+0.25m+0.025-m^2)\right)$$

is convex in η according to lemma B.29. So, it is sufficient to show that the partial derivative of the above equation with respect to η evaluated at $n = 0.3$ is negative for $m \in [0.21, 0.25]$.

We calculate the partial derivative with respect to η evaluated at $\eta = 0.3$ below.

$$1.944\left(m+\sqrt{0.85m+0.025}\right)+\frac{m\left(0.9m^2+1.1032\right)}{\sqrt{0.85m+0.025}}-2\ln(1.2m)+\ln(0.85m+0.025) + 2\ln(0.85m+0.025-m^2)-2+\frac{0.6m}{0.85m+0.025}+\frac{1.2m}{0.85m+0.025-m^2}$$

We upper bound the above by upper bounding the term $\frac{m\left(0.9m^2+1.1032\right)}{\sqrt{0.85m+0.025}}$ by $\frac{m\left(0.9m^2+1.1032\right)}{\sqrt{0.85(0.21)+0.025}}$.

Next, we'll show the upper bound grows in m (so we can upper bound it by evaluating at $m = 0.25$). The partial derivative of the upper bound with respect to m is shown below.

$$1.944+\frac{0.8262}{\sqrt{0.85m+0.025}}+\frac{2.7m^2+1.1032}{\sqrt{0.2035}}-\frac{2}{m}+\frac{0.85}{0.85m+0.025}+\frac{2\left(0.85-2m\right)}{0.85m+0.025-m^2}+\frac{0.015}{\left(0.85m+0.025\right)^2} + \frac{0.03+1.2m^2}{\left(0.85m+0.025-m^2\right)^2}$$

After scaling by m, we derive the following lower bound for the (scaled) partial derivative.

$$1.944m+\frac{0.8262m}{\sqrt{0.85\left(0.25\right)+0.025}}+\frac{2.7m^3+1.1032m}{\sqrt{0.2035}}-2+\frac{0.85m}{0.85\left(0.25\right)+0.025}+\frac{2\left(0.85-2m\right)m}{0.85\left(0.25\right)+0.025-\left(0.21\right)^2} + \frac{0.015m}{\left(0.85\left(0.25\right)+0.025\right)^2}+\frac{0.03m+1.2m^3}{\left(0.85\left(0.25\right)+0.025-0.21^2\right)^2}$$

By taking another partial derivative with respect to m, we get the following

$$1.944+\frac{0.8262}{\sqrt{0.2375}}+\frac{8.1m^2+1.1032}{\sqrt{0.2035}}+\frac{0.85}{0.2375}+\frac{2\left(0.85-4m\right)}{0.1934}+\frac{0.015}{\left(0.2375\right)^2}+\frac{0.03+3.6m^2}{\left(0.1934\right)^2}$$

and can lower bound by

$$1.944+\frac{0.8262}{\sqrt{0.2375}}+\frac{8.1(0.21)^2+1.1032}{\sqrt{0.2035}}+\frac{0.85}{0.2375}+\frac{2\left(0.85-4(0.25)\right)}{0.1934}+\frac{0.015}{\left(0.2375\right)^2}+\frac{0.03+3.6(0.21)^2}{\left(0.1934\right)^2},$$

which is positive.

So, we can further lower bound

$$1.944m+\frac{0.8262m}{\sqrt{0.85\left(0.25\right)+0.025}}+\frac{2.7m^3+1.1032m}{\sqrt{0.2035}}-2+\frac{0.85m}{0.85\left(0.25\right)+0.025}+\frac{2\left(0.85-2m\right)m}{0.85\left(0.25\right)+0.025-\left(0.21\right)^2} + \frac{0.015m}{\left(0.85\left(0.25\right)+0.025\right)^2}+\frac{0.03m+1.2m^3}{\left(0.85\left(0.25\right)+0.025-0.21^2\right)^2}$$

by evaluating at $m = 0.21$, which is positive.

So, we can further upper bound the following term

$$1.944\left(m+\sqrt{0.85m+0.025}\right)+\frac{m\left(0.9m^2+1.1032\right)}{\sqrt{0.85(0.21)+0.025}}-2\ln(1.2m)+\ln(0.85m+0.025) + 2\ln(0.85m+0.025-m^2)-2+\frac{0.6m}{0.85m+0.025}+\frac{1.2m}{0.85m+0.025-m^2}$$

by evaluating it at $m = 0.25$, which is negative.

This completes the proof.

□

Proof of Lemma B.29. To prove convexity, we start by calculating the second partial derivative of

$$\left(0.9m^2+0.52+1.944\eta\right)\left(m+\sqrt{2m\eta+0.25m+0.025}\right) - 2\eta\left(\ln(4m\eta)-\frac{1}{2}\ln(2m\eta+0.25m+0.025)-\ln(2m\eta+0.25m+0.025-m^2)\right)$$

with respect to η and showing that it is positive.

The second derivative with respect with η is

$$\frac{3.888m}{\sqrt{2m\eta+0.25m+0.025}}-\frac{m^2\left(0.9m^2+0.52+1.944\eta\right)}{\left(2m\eta+0.25m+0.025\right)^{3/2}}-\frac{2}{\eta}+\frac{4m}{2m\eta+0.25m+0.025}+\frac{8m}{2m\eta+0.25m+0.025-m^2}$$
$$-\frac{4m^2\eta}{\left(2m\eta+0.25m+0.025\right)^2}-\frac{8m^2\eta}{\left(2m\eta+0.25m+0.025-m^2\right)^2}.$$

After scaling the above by η, we lower bound it by the following:

$$\frac{3.888m\eta}{\sqrt{2(0.25)\eta+0.25(0.25)+0.025}}-\frac{(0.25)^2(0.3)\left(0.9(0.25)^2+0.52+1.944(0.3)\right)}{\left(2m\eta+0.25m+0.025\right)^{3/2}}-2+\frac{4m^2\eta^2+m^2\eta+0.1m\eta}{\left(2m\eta+0.25m+0.025\right)^2}$$
$$+\frac{8m^2\eta^2+2m^2\eta+0.2m\eta-8m^3\eta}{\left(2(0.25)\eta+0.25(0.25)+0.025-(0.21)^2\right)^2}. \quad (83)$$

With this lower bound, we will show that its partial derivative with respect to m is positive, so we can lower bound Equation 83 by evaluating it at $m=0.21$.

Taking the partial derivative with respect to m, we get the following:

$$\frac{3.888\eta}{\sqrt{2(0.25)\eta+0.25(0.25)+0.025}}+\frac{3(0.25)^2(0.3)\left(0.9(0.25)^2+0.52+1.944(0.3)\right)(2\eta+0.25)}{2\left(2m\eta+0.25m+0.025\right)^{5/2}}$$
$$+\frac{0.025m\eta+0.0025\eta}{\left(2m\eta+0.25m+0.025\right)^3}+\frac{16m\eta^2+4m\eta+0.2\eta-24m^2\eta}{\left(2(0.25)\eta+0.25(0.25)+0.025-(0.21)^2\right)^2}. \quad (84)$$

To show that the Equation 84 is positive, we note that

$$\begin{aligned}&16m\eta+4m+0.2-24m^2\\&\geq 6(0.21)(0.2)+4(0.21)+0.2-24(0.25)^2\\&>0.\end{aligned}$$

So, we can further lower bound Equation 84, by evaluating at $m=0.21$, shown below.

$$\frac{3.888(0.21)\eta}{\sqrt{2(0.25)\eta+0.25(0.25)+0.025}}-\frac{(0.25)^2(0.3)\left(0.9(0.25)^2+0.52+1.944(0.3)\right)}{\left(2(0.21)\eta+0.25(0.21)+0.025\right)^{3/2}}-2$$
$$+\frac{4(0.21)^2\eta^2+(0.21)^2\eta+0.1(0.21)\eta}{\left(2(0.21)\eta+0.25(0.21)+0.025\right)^2}+\frac{8(0.21)^2\eta^2+2(0.21)^2\eta+0.2(0.21)\eta-8(0.21)^3\eta}{\left(2(0.25)\eta+0.25(0.25)+0.025-(0.21)^2\right)^2}. \quad (85)$$

Next, we will show that the partial derivative of Equation 85 with respect to η is positive, we can lower bound it by evaluating at $\eta=0.2$.

Taking the partial derivative with respect to η, we get the following:

$$\frac{0.20412\eta+0.071442}{\left(0.5\eta+0.0875\right)^{\frac{3}{2}}}+\frac{3\left(0.21\right)\left(0.25\right)^2\left(0.3\right)\left(0.9\left(0.25\right)^2+0.52+1.944\left(0.3\right)\right)}{\left(0.42\eta+0.0775\right)^{\frac{5}{2}}}$$
$$+\frac{0.024025\left(0.21\right)}{\left(0.42\eta+0.0775\right)^3}+\frac{\left(0.21\right)\left(0.012224\eta+0.01159648\right)}{\left(0.5\eta+0.0434\right)^3},$$

which is positive.

So, we can lower bound Equation 85 by evaluating it at $\eta=0.2$, which is positive.

So,

$$\left(0.9m^2+0.52+1.944\eta\right)\left(m+\sqrt{2m\eta+0.25m+0.025}\right)$$
$$-2\eta\left(\ln(4m\eta)-\frac{1}{2}\ln(2m\eta+0.25m+0.025)-\ln(2m\eta+0.25m+0.025-m^2)\right)$$

is convex in η for the specified parameter range.

□

Proof of Lemma B.36. To proceed with the proof, we begin by noting that Equation 9 is convex in q. So, we can compute the global minimizer by evaluating where the partial derivative with respect to q is zero. Note the first partial derivative of Equation 9 with respect to q below.

$$\frac{1}{2}\sqrt{c}\left(\sqrt{c}-m\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)-2\eta m$$

By solving where the above is zero, we get that $q_{min}=\frac{2}{m+\sqrt{c}}\left[\ln(4\eta m)-\frac{1}{2}\ln(c)-\ln(c-m^2)\right]$.

This completes the proof. □

Lemma B.101. *For $m\in[0.15,0.21],\eta\in[0.2,0.3],c=2m\eta+0.25m+0.025$,*

$$\sqrt{c}\left(\sqrt{c}-m\right)\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\right)-2m\eta q \tag{??}$$

evaluated at the global minimizer in q, $q_m in$, is positive.

Proof of Lemma B.101. To begin the proof, we note that Equation 9 is convex in q, and we have the global minimizer from lemma B.36. So, we will show that the following is positive for $m\in[0.15,0.21],\eta\in[0.2,0.3]$.

$$\sqrt{2m\eta+0.25m+0.025}\left(\sqrt{2m\eta+0.25m+0.025}-m\right)$$
$$+\frac{4\eta m}{m+\sqrt{2m\eta+0.25m+0.025}}\left(1-\ln(4\eta m)+\frac{1}{2}\ln(2m\eta+0.25m+0.025)+\ln(2m\eta+0.25m+0.025-m^2)\right)$$

For convenience, we rescale the above by $m+\sqrt{2m\eta+0.25m+0.025}$ and show it is positive.

To do so, we lower bound the rescaled equation by the following:

$$\sqrt{2m\eta+0.25m+0.025}(2m\eta+0.25m+0.025-m^2)+4\eta m(1+\ln(0.2805))$$

We show that this lower bound is valid in lemma B.30. Before going further, we will first establish that the partial derivative of our lower bound with respect to m is locally concave in m. Consider the partial derivative with respect to m below.

$$\frac{-m^2\left(\eta+0.125\right)}{\sqrt{2m\eta+0.25m+0.025}}+\left(0.375+3\eta-2m\right)\sqrt{2\eta m+0.25m+0.025}+4\eta(1+\ln(0.2805))$$

First, we establish concavity in m. We calculate the second partial derivative with respect to m of the above expression below.

$$\frac{-\left(\eta+0.125\right)\left(6\eta m+0.75m+0.75\eta+0.196875+3\eta^2\right)}{\left(2\eta m+0.25m+0.025\right)^{3/2}}+\frac{-3m^2\left(\eta+0.125\right)^3}{\left(2m\eta+0.25m+0.025\right)^{5/2}}$$

Since both terms are negative, it is concave in m.

Leveraging the fact that the expression

$$\frac{-m^2\left(\eta+0.125\right)}{\sqrt{2m\eta+0.25m+0.025}}+\left(0.375+3\eta-2m\right)\sqrt{2\eta m+0.25m+0.025}+4\eta(1+\ln(0.2805))$$

grows with η from lemma B.37, we split our analysis into two cases.

Case 1: $\eta\in[0.27,0.3]$. We will show that

$$\frac{-m^2\left(\eta+0.125\right)}{\sqrt{2m\eta+0.25m+0.025}}+\left(0.375+3\eta-2m\right)\sqrt{2nm+0.25m+0.025}+4\eta(1+\ln(0.2805))$$

is locally positive.

By lemma B.37 and concavity in m, we evaluate

$$\frac{-m^2\left(\eta+0.125\right)}{\sqrt{2m\eta+0.25m+0.025}}+\left(0.375+3\eta-2m\right)\sqrt{2\eta m+0.25m+0.025}+4\eta(1+\ln(0.2805))$$

at $m=0.15, 0.21$ and $\eta=0.27$, both of which are positive.

So, we can lower bond

$$\sqrt{2m\eta+0.25m+0.025}(2m\eta+0.25m+0.025-m^2)+4\eta m(1+\ln(0.2805))$$

by evaluating at $m=0.15$, yielding the following.

$$\sqrt{0.3\eta+0.0625}(0.3\eta+0.04)+0.6\eta(1+\ln(0.2805))$$

Next, we will show that the above is convex in η for $\eta>0$, so we can lower bound it further by linearizing at $\eta=0.27$. We show convexity in η by showing the second derivative is locally positive. The second derivative is shown below:

$$\frac{\left(0.15\right)^2\left(0.9\eta+0.21\right)}{\left(0.3\eta+0.0625\right)^{3/2}}.$$

Since the numerator is positive, we have convexity. Note that the derivative of

$$\sqrt{0.3\eta+0.0625}(0.3\eta+0.04)+0.6\eta(1+\ln(0.2805))$$

at $\eta=0.27$ is negative, so we evaluate the $\eta=0.27$ linearization at $\eta=0.3$, which is positive.

This concludes case 1.

Case 2: $\eta\in[0.2,0.27]$ We will show that

$$\frac{-m^2\left(\eta+0.125\right)}{\sqrt{2m\eta+0.25m+0.025}}+\left(0.375+3\eta-2m\right)\sqrt{2\eta m+0.25m+0.025}+4\eta(1+\ln(0.2805))$$

is locally decreasing in m. Since we showed the above is concave, it is sufficient to prove that the partial derivative of the above with respect to m at $m=0.15$ is locally negative.

At $m=0.15$, the partial derivative is as follows:

$$\frac{\left(3\eta-0.225\right)\left(\eta+0.125\right)}{\sqrt{0.3\eta+0.0625}}+\frac{\left(0.15\right)^2\left(\eta+0.125\right)^2}{\left(0.3\eta+0.0625\right)^{\frac{3}{2}}}-2\sqrt{0.3\eta+0.0625}$$

To prove the above is negative, we scale it by $\sqrt{0.3\eta+0.0625}$ and lower bound the scaled partial derivative by the following

$$\left(3\left(0.27\right)-0.45\right)\left(0.27\right)-0.153125+\frac{\left(0.15\right)^2\left(0.27+0.125\right)^2}{\left(0.3\left(0.2\right)+0.0625\right)},$$

which is negative.

This means that

$$\sqrt{2m\eta+0.25m+0.025}(2m\eta+0.25m+0.025-m^2)+4\eta m(1+\ln(0.2805))$$

is locally concave in this case. So, we will show that the above evaluated at $m=0.15, 0.21$ is positive using Jensen's inequality.

We'll consider $m=0.15$ first, which yields the following.

$$\sqrt{0.3\eta+0.0625}(0.3\eta+0.04)+0.6\eta(1+\ln(0.2805))$$

Earlier in the lemma, we already proved that the above expression is convex, noting that the second derivative is

$$\frac{\left(0.15\right)^2\left(0.9\eta+0.21\right)}{\left(0.3\eta+0.0625\right)^{3/2}},$$

which has a positive numerator for $\eta > 0$.

Since the derivative of

$$\sqrt{0.3\eta + 0.0625}(0.3\eta + 0.04) + 0.6\eta(1 + \ln(0.2805))$$

is negative at $\eta = 0.27$, then we know that

$$\sqrt{0.3\eta + 0.0625}(0.3\eta + 0.04) + 0.6\eta(1 + \ln(0.2805))$$

is decreasing for $\eta \in [0.2, 0.27]$. So, we can evaluate

$$\sqrt{0.3\eta + 0.0625}(0.3\eta + 0.04) + 0.6\eta(1 + \ln(0.2805))$$

at $\eta = 0.27$, which is positive.

Next, we consider $m = 0.21$, which yields the following expression

$$\sqrt{0.42\eta + 0.0775}\,(0.42\eta + 0.0334) + 0.84\eta(1 + \ln(0.2805))$$

Note that the above expression is also convex in η, as seen by the second derivative below, which is positive for $\eta > 0$.

$$\frac{(0.21)^2\,(1.26\eta + 0.2766)}{(0.42\eta + 0.0775)^{3/2}}$$

By evaluating the derivative of

$$\sqrt{0.42\eta + 0.0775}\,(0.42\eta + 0.0334) + 0.84\eta(1 + \ln(0.2805))$$

at $\eta = 0.2$, we see that it is positive, so we can lower bound

$$\sqrt{0.42\eta + 0.0775}\,(0.42\eta + 0.0334) + 0.84\eta(1 + \ln(0.2805))$$

by evaluating at $\eta = 0.2$, which is positive.

This concludes the case.

This completes the proof.

□

Proof of Lemma B.37. Consider the partial derivative of Equation 16 below. We will show that it is locally positive.

$$\frac{\begin{array}{c}-5m^3n - 0.625m^3 + 0.20625m^2 + 18\eta^2m^2 + 4.5\eta m^2 + 0.375m\eta \\ + 0.046875m + 0.001875 + 4\,(1 + \ln\,(0.2805))\,(2m\eta + 0.25m + 0.025)^{3/2}\end{array}}{(2\eta m + 0.25m + 0.025)^{3/2}}$$

To proceed, we will focus on the numerator of the above expression. We will first show that the partial derivative of the numerator with respect to η is positive.

Consider the partial derivative with respect to η below.

$$\left(-5m^2 + 36\eta m + 4.5m + 0.375 + 12\,(1 + \ln\,(0.2805))\,\sqrt{2\eta m + 0.25m + 0.025}\right) m$$

To show that it is positive, we will lower bound the inner expression $-5m^2 + 36\eta m + 4.5m + 0.375 + 12\,(1 + \ln\,(0.2805))\,\sqrt{2\eta m + 0.25m + 0.025}$ by $(-5\,(0.21) + 36\,(0.2) + 4.5)\,(0.15) + 0.375 + 12\,(1 + \ln\,(0.2805))\,\sqrt{2\,(0.21)\,(0.3) + 0.25\,(0.21) + 0.025}$, which is positive.

So, we lower bound the expression

$$\begin{array}{l}-5m^3n - 0.625m^3 + 0.20625m^2 + 18\eta^2m^2 + 4.5\eta m^2 + 0.375m\eta \\ + 0.046875m + 0.001875 + 4\,(1 + \ln\,(0.2805))\,(2m\eta + 0.25m + 0.025)^{3/2}\end{array}$$

by evaluating at $\eta = 0.2$.

We show this lower bound below:

$$-1.625m^3 + 1.82625m^2 + 0.121875m + 0.001875 + 4\left(1 + \ln\left(0.2805\right)\right)\left(0.65m + 0.025\right)^{3/2}$$

Next, we show that this lower bound grows in m. So, we will show that the derivative of the lower bound is locally positive. The derivative of the lower bound is shown below.

$$-4.875m^2 + 3.6525m + 0.121875 + 3.9\left(1 + \ln\left(0.2805\right)\right)\sqrt{0.65m + 0.025}$$

We can lower bound the above by

$$\left(-4.875\left(0.21\right) + 3.6525\right)\left(0.15\right) + \left(0.121875 + 3.9\left(1 + \ln\left(0.2805\right)\right)\sqrt{0.65\left(0.21\right) + 0.025}\right),$$

which is positive.

So, Equation 16 can be lower bounded by

$$-1.625(0.15)^3 + 1.82625(0.15)^2 + 0.121875(0.15) + 0.001875 + 4\left(1 + \ln\left(0.2805\right)\right)\left(0.65(0.15) + 0.025\right)^{3/2},$$

which is positive.

This completes the proof. □

Proof of Lemma B.30. By lemma B.31, we know that Equation 12 is locally concave in m.

So, we can lower bound Equation 12, by checking which of $m = 0.15, 0.21$ is smaller. First, we check $m = 0.15$. Then, Equation 12 reduces to the following.

$$-ln(0.6\eta) + \frac{1}{2}\ln(0.3\eta + 0.0625) + \ln(0.3\eta + 0.04)$$

We will show that the derivative is locally negative. Consider the expression for the derivative below.

$$\frac{0.045\eta^2 - 0.006\eta - 0.0025}{\eta\left(0.3\eta + 0.0625\right)\left(0.3\eta + 0.04\right)}$$

We will show that the numerator is negative, by upper bounding it by the following, $\left(0.045\left(0.3\right) - 0.006\right)\left(0.3\right) - 0.0025$, which, is negative.

So, we can lower bound

$$\frac{0.045\eta^2 - 0.006\eta - 0.0025}{\left(0.3\eta + 0.0625\right)\left(0.3\eta + 0.04\right)}$$

by evaluating at $\eta = 0.3$, which is greater than -1.26572.

Next, we consider Equation 12 at $m = 0.21$, which reduces to the following.

$$-\ln\left(0.84\eta\right) + \frac{1}{2}\ln\left(0.42\eta + 0.0775\right) + \ln\left(0.42\eta + 0.0334\right)$$

We will first show that the above is locally convex. Consider the second derivative below.

$$\frac{-\left(0.1764\right)\left(0.0882\right)\eta^4 + \left(0.028056\right)\left(0.0882\right)\eta^3 + \left(\left(0.0334\right)^2\left(0.0882\right) + \left(0.028056\right)\left(0.0651\right)\right)\eta^2 + \left(\left(0.0334\right)^2\left(0.0651\right) + \left(0.028056\right)\left(0.0775\right)^2\right)\eta + \left(0.0334\right)^2\left(0.0775\right)^2}{x^2\left(0.42\eta + 0.0775\right)^2\left(0.42\eta + 0.0334\right)^2}$$

To proceed, we'll show that the numerator of the above expression is locally positive.

Consider the following lower bound on the numerator,

$$-\left(0.1764\right)\left(0.0882\right)\left(0.3\right)^4 + \left(0.028056\right)\left(0.0882\right)\left(0.2\right)^3 + \left(\left(0.0334\right)^2\left(0.0882\right) + \left(0.028056\right)\left(0.0651\right)\right)\left(0.2\right)^2$$
$$+ \left(\left(0.0334\right)^2\left(0.0651\right) + \left(0.028056\right)\left(0.0775\right)^2\right)\left(0.2\right) + \left(0.0334\right)^2\left(0.0775\right)^2,$$

which is positive.

So, by locally convexity, we can calculate the local minima by setting the first derivative to zero. From this, we get the following expression for the local minimum:

$$\eta_{min} = \frac{(0.0334)(0.21) + \sqrt{(0.0334)^2(0.21)^2 + 8(0.21)^2(0.0334)(0.0775)}}{4(0.21)^2}.$$

Evaluating

$$-\ln(0.84\eta) + \frac{1}{2}\ln(0.42\eta + 0.0775) + \ln(0.42\eta + 0.0334)$$

at η_{min}, we get that it is greater than -1.27092, which is smaller than what we obtained for the $m = 0.15$ case.

Since $\ln(0.2805)$ is less than -1.27092, it is suitable lower bound.

This completes the proof.

□

Proof of Lemma B.31. To proceed, we will show that the second partial derivative of Equation 12 is locally negative.

The second partial derivative with respect to m is shown below.

$$\frac{1}{m^2} - \frac{2(\eta + 0.125)^2}{(2m\eta + 0.25m + 0.025)^2} - \frac{2}{2m\eta + 0.(2x + 0.25 - 2m)25m + 0.025 - m^2} - \frac{(2\eta + 0.25 - 2m)^2}{(2m\eta + 0.25m + 0.025 - m^2)^2}$$

We first scale the above by m^2 and show that the partial derivative with respect to m of the rescaled second derivative is locally negative. We show the partial derivative below.

$$-\frac{4m(\eta + 0.125)^2}{(2m\eta + 0.25m + 0.025)^2} + \frac{8m^2(\eta + 0.125)^3}{(2m\eta + 0.25m + 0.025)^3} - \frac{4m}{2m\eta + 0.25m + 0.025 - m^2}$$

$$+\frac{2m^2(2\eta + 0.25 - 2m)}{(2m\eta + 0.25m + 0.025 - m^2)^2} - \frac{2\left(2m\eta + 0.25m - 2m^2\right)(2\eta + 0.25 - 4m)}{(2m\eta + 0.25m + 0.025 - m^2)^2} + \frac{2m^2(2\eta + 0.25 - 2m)^3}{(2m\eta + 0.25m + 0.025 - m^2)^3}$$

By factoring out $2m$, we rewrite the above as follows.

$$\frac{-0.05(\eta + 0.125)^2}{(2m\eta + 0.25m + 0.025)^3} + \frac{0.15m\eta - 2m^3\eta - 0.25m^3 - 0.15m^2 + 0.01875m - 0.0028125 - 0.1\eta^2 - 0.025\eta}{(2m\eta + 0.25m + 0.025 - m^2)^3}$$

To show the above is negative, we will show that the expression $0.15m\eta - 2m^3\eta - 0.25m^3 - 0.15m^2 + 0.01875m - 0.0028125 - 0.1\eta^2 - 0.025\eta$ is negative.

First, note that the expression is concave in η, by the negative quadratic and linear η coefficients.

To upper bound the expression in η, we will show that the partial derivative of the expression with respect to η is negative. Consider the partial derivative below.

$$-0.2\eta - 0.025 + 0.15m - 2m^3$$

We upper bound the above by $-0.2(0.2) - 0.025 + 0.15(0.25) - 2(0.15)^3$, which is negative.

So, we can upper bound the expression

$$0.15m\eta - 2m^3\eta - 0.25m^3 - 0.15m^2 + 0.01875m - 0.0028125 - 0.1\eta^2 - 0.025\eta$$

by evaluating at $\eta = 0.2$, which simplifies to the following.

$$-0.65m^3 - 0.15m^2 + 0.04875m - 0.0118125$$

Note that the above is concave in m for $m \geq 0$. We will further upper bound by showing that the derivative at $m = 0.15$ is negative. The derivative of the above is shown below

$$-1.95m^2 - 0.3m + 0.04875,$$

which is negative at $m = 0.15$.

So, we upper bound our expression $-0.65m^3 - 0.15m^2 + 0.04875m - 0.0118125$ by evaluating at $m = 0.15$, which is negative.

Since the partial derivative with respect to m of the rescaled second partial derivative is negative, we can upper bound it at $m = 0.15$, shown below.

$$1 - \frac{0.045\left(\eta + 0.125\right)^2}{\left(0.3\eta + 0.0625\right)^2} - \frac{0.045}{0.3\eta + 0.04} - \frac{\left(0.15\right)^2\left(2\eta - 0.05\right)^2}{\left(0.3\eta + 0.04\right)^2}$$

To further upper bound this, we'll that the the partial derivative with respect to η is negative. Consider the partial derivative below.

$$\frac{-0.00225\eta - \left(0.0625\right)\left(0.01125\right) + \left(0.027\right)\left(0.125\right)^2}{\left(0.3\eta + 0.0625\right)^3}$$

$$+ \frac{-0.0045\eta + 0.04\left(0.015\right) + \left(0.6\right)\left(0.04\right)\left(0.005\right) + \left(0.6\right)\left(0.05\right)^2\left(0.15\right)^2}{\left(0.3\eta + 0.04\right)^3}$$

Since $\frac{-0.00225x - (0.0625)(0.01125) + (0.027)(0.125)^2}{(0.3x + 0.0625)^3}$ is negative, and the numerator of $\frac{-0.0045x + 0.04(0.015) + (0.6)(0.04)(0.005) + (0.6)(0.05)^2(0.15)^2}{(0.3x + 0.04)^3}$ is decreasing in η, we can upper bound the numerator by $-0.0045(0.2) + 0.04\left(0.015\right) + \left(0.6\right)\left(0.04\right)\left(0.005\right) + \left(0.6\right)\left(0.05\right)^2\left(0.15\right)^2$, which is negative.

So, we can upper bound the term

$$1 - \frac{0.045\left(\eta + 0.125\right)^2}{\left(0.3\eta + 0.0625\right)^2} - \frac{0.045}{0.3\eta + 0.04} - \frac{\left(0.15\right)^2\left(2\eta - 0.05\right)^2}{\left(0.3\eta + 0.04\right)^2}$$

by evaluating it at $\eta = 0.2$, which is negative.

This completes the proof.

□

Proof of Lemma B.43. We proceed by showing the second partial derivative of Equation 13 with respect to η is locally positive.

Consider the second partial derivative below.

$$\frac{m^3}{(2m\eta + 0.25m + 0.025)^{3/2}}$$

$$+ \frac{m^3}{(2m\eta + 0.25m + 0.025)^{3/2}} \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right)$$

$$+ 2m\alpha_1\left(2 - \frac{m}{\sqrt{2m\eta + 0.25m + 0.025}}\right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{2}m\eta + 0.25m + 0.025\right)\right)$$

$$+ \alpha_2\sqrt{2m\eta + 0.25m + 0.025}\left(\sqrt{2m\eta + 0.25m + 0.025} - m\right)$$
$$\cdot \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{2}m\eta + 0.25m + 0.025\right)\right)$$

$$+ \alpha_1^2\sqrt{2m\eta + 0.25m + 0.025}\left(\sqrt{2m\eta + 0.25m + 0.025} - m\right)$$
$$\cdot \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{2}m\eta + 0.25m + 0.025\right)\right),$$

where

$$\alpha_1 = -\left(\frac{0.45m^2}{\eta^2} + \frac{0.26}{\eta^2}\right)\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right) + \frac{m\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)}{\sqrt{2m\eta + 0.25m + 0.025}}$$

and

$$\alpha_2 = 2\left(\frac{0.45m^2}{\eta^3} + \frac{0.26}{\eta^3}\right)\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right) - \frac{2m\left(\frac{0.45m^2}{\eta^2} + \frac{0.26}{\eta^2}\right)}{\sqrt{2m\eta + 0.25m + 0.025}}$$
$$- \frac{m^2\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)}{(2m\eta + 0.25m + 0.025)^{3/2}}.$$

We will show that

$$+ \frac{m^3}{(2m\eta + 0.25m + 0.025)^{3/2}}$$

$$+ 2m\alpha_1\left(2 - \frac{m}{\sqrt{2m\eta + 0.25m + 0.025}}\right)$$

$$+ \sqrt{2m\eta + 0.25m + 0.025}\left(\sqrt{2m\eta + 0.25m + 0.025} - m\right)\alpha_2$$

is locally positive.

Note that after scaling by η^3, we can simplify the above expression to the following.

$$\frac{m^3\eta^3}{(2m\eta+0.25m+0.025)^{3/2}}$$

$$-\frac{m^3\eta^2\left(0.45m^2+0.26+0.972\eta\right)}{2m\eta+0.25m+0.025}$$

$$+\frac{\begin{array}{r}-0.45m^4\eta^2-0.4975m^3\eta+0.013m\eta+0.13m^2\eta+0.0065m+0.0200625m^2+0.225m^4\eta+0.03375m^4\\-0.9m^5\eta-0.225m^5+0.000325-0.11875m^3+2.916m^2\eta^3-0.26m^2\eta^2\end{array}}{\sqrt{2m\eta+0.25m+0.025}}$$

By lemma B.33, we know that

$$-0.45m^4\eta^2-0.4975m^3\eta+0.013m\eta+0.13m^2\eta+0.0065m+0.0200625m^2+0.225m^4\eta+0.03375m^4-0.9m^5\eta$$
$$-0.225m^5+0.000325-0.11875m^3+2.916m^2\eta^3-0.26m^2\eta^2$$

is positive.

Using lemma B.34, we can lower bound

$$\frac{m^3\eta^3}{(2m\eta+0.25m+0.025)^{3/2}}$$

$$-\frac{m^3\eta^2\left(0.45m^2+0.26+0.972\eta\right)}{2m\eta+0.25m+0.025}$$

$$+\frac{\begin{array}{r}-0.45m^4\eta^2-0.4975m^3\eta+0.013m\eta+0.13m^2\eta+0.0065m+0.0200625m^2+0.225m^4\eta+0.03375m^4\\-0.9m^5\eta-0.225m^5+0.000325-0.11875m^3+2.916m^2\eta^3-0.26m^2\eta^2\end{array}}{\sqrt{2m\eta+0.25m+0.025}}$$

by

$$\frac{m^3\left(0.2\right)^3}{\left(2\left(0.25\right)\left(0.2\right)+0.25\left(0.25\right)+0.025\right)^{\frac{3}{2}}}$$

$$-\frac{m^3\left(0.3\right)^2\left(0.45m^2\ +\ 0.26\ +\ 0.972\left(0.3\right)\right)}{2\left(0.21\right)\left(0.2\right)+0.25\left(0.21\right)\ +\ 0.025}$$

$$+\frac{\begin{array}{r}-0.45m^4\left(0.2\right)^2-\ 0.4975m^3\left(0.2\right)+\ 0.013m\left(0.2\right)+0.13m^2\left(0.2\right)+0.0065m+0.0200625m^2+0.225m^4\left(0.2\right)\\+0.03375m^4-0.9m^5\left(0.2\right)-0.225m^5+0.000325-0.11875m^3+2.916m^2\left(0.2\right)^3-0.26m^2\left(0.2\right)^2\end{array}}{\sqrt{2\left(0.25\right)\left(0.3\right)+0.25\left(0.25\right)\ +\ 0.025}}.$$

To show the above is positive, we will show that the derivative is negative. Consider the derivative below.

$$\left(-\frac{\left(0.3\right)^2\left(0.45\right)}{0.1615}-\frac{0.18}{\sqrt{0.2375}}\right)\cdot 5m^4-\frac{0.16425}{\sqrt{0.2375}}\cdot 4m^3+\left(\frac{\left(0.2\right)^3}{\left(0.1875\right)^{\frac{3}{2}}}-\frac{\left(0.3\right)^2\left(0.5516\right)}{0.1615}-\frac{0.21825}{\sqrt{0.2375}}\right)\cdot 3m^2$$
$$+\frac{0.0589905}{\sqrt{0.2375}}\cdot 2m+\frac{0.0091}{\sqrt{0.2375}}$$

We can upper bound it by

$$\left(-\frac{(0.3)^2(0.45)}{0.1615}-\frac{0.18}{\sqrt{0.2375}}\right)\cdot 5(0.21)^4-\frac{0.16425}{\sqrt{0.2375}}\cdot 4(0.21)^3$$
$$+\left(\frac{(0.2)^3}{(0.1875)^{\frac{3}{2}}}-\frac{(0.3)^2(0.5516)}{0.1615}-\frac{0.21825}{\sqrt{0.2375}}\right)\cdot 3(0.21)^2$$
$$+\frac{0.0589905}{\sqrt{0.2375}}\cdot 2(0.25)+\frac{0.0091}{\sqrt{0.2375}},$$

which is negative.

So we evaluate

$$\frac{m^3(0.2)^3}{(2(0.25)(0.2)+0.25(0.25)+0.025)^{\frac{3}{2}}}$$

$$-\frac{m^3(0.3)^2\left(0.45m^2+0.26+0.972(0.3)\right)}{2(0.21)(0.2)+0.25(0.21)+0.025}$$

$$+\frac{\begin{array}{r}-0.45m^4(0.2)^2-0.4975m^3(0.2)+0.013m(0.2)+0.13m^2(0.2)+0.0065m+0.0200625m^2+0.225m^4(0.2)\\+0.03375m^4-0.9m^5(0.2)-0.225m^5+0.000325-0.11875m^3+2.916m^2(0.2)^3-0.26m^2(0.2)^2\end{array}}{\sqrt{2(0.25)(0.3)+0.25(0.25)+0.025}}.$$

at $m=0.25$, which is positive.

This completes the proof.

□

Proof of Lemma B.33. To proceed, we begin by lower bounding Equation 14 by evaluating at $\eta = 0.2$, since the equation increases in η according to lemma B.34.

We can further lower bound the $\eta = 0.2$ lower bound with the following

$$-0.405(0.25)^5+0.06075(0.21)^4-0.21825(0.25)^3+0.0589905(0.21)^2+0.0091(0.21)+0.000325,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.34. We prove the lemma showing that the partial derivative of Equation 14 with respect to η is positive. Consider the partial derivative below.

$$\left(-0.9m^4+(-0.9\eta+0.225)m^3-0.4975m^2+\left(8.748\eta^2-0.52\eta+0.13\right)m+0.013\right)m$$

We can lower bound the partial derivative the following

$$-0.9(0.25)^4+(-0.9(0.3)+0.225)(0.25)^3-0.4975(0.25)^2+\left(8.748(0.2)^2-0.52(0.3)+0.13\right)(0.21)+0.013,$$

which is positive.

This completes the proof. □

Proof of Lemma B.35. We begin the proof by showing that the derivative of Equation 15 is positive. Consider the derivative below, simplified by rounding down terms after five digits of precision.

$$
\begin{aligned}
&-1.888+\frac{-1.095m^2-0.05m}{(0.73m+0.025)^{3/2}}\\
&+\Bigg(\frac{0.365m^2}{(0.73m+0.025)^{3/2}}+16.18652m^6-12.36695m^5+4.86465m^4-6.70944m^3-1.48700m^2-0.94068m\\
&+1.91534+\frac{-1.28321m^6-5.58668m^4-0.02852m^3-2.02096m^2-0.95214m-0.04119}{\sqrt{0.73m+0.025}}\\
&+\frac{-0.68438m^4-0.75020m^2}{0.73m+0.025}+43.94531m^6\sqrt{0.73m+0.025}-28.56446m^5\sqrt{0.73m+0.025}\\
&+40.34928m^4\sqrt{0.73m+0.025}+8.70052m^3\sqrt{0.73m+0.025}+4.25697m^2\sqrt{0.73m+0.025}\\
&+6.08893m\sqrt{0.73m+0.025}-1.47175\sqrt{0.73m+0.025}\Bigg)\\
&\cdot\exp\left(\frac{0.45}{0.24}m^3+\frac{0.49328}{0.24}m+\frac{0.45}{0.24}m^2\sqrt{0.73m+0.025}+\frac{0.49328}{0.24}\sqrt{0.73m+0.025}\right)
\end{aligned}
$$

We can lower bound the derivative with the following:

$$
\begin{aligned}
&-1.888+\frac{-1.095\left(0.25\right)^2-0.05\left(0.25\right)}{(0.73m+0.025)^{\frac{3}{2}}}\\
&+\Bigg(\frac{0.365m^2}{(0.73\left(0.25\right)+0.025)^{\frac{3}{2}}}+16.18652m^6-12.36695\left(0.25\right)m^4+4.86465m^4-6.70944m^3-1.48700m^2\\
&-0.94068m+1.91534+\frac{-1.28321m^6-5.58668m^4-0.02852m^3-2.02096m^2-0.95214m-0.04119}{\sqrt{0.73m+0.025}}\\
&+\frac{-0.68438\left(0.25\right)^2m^2-0.75020x^2}{0.73m+0.025}+43.94531m^6\sqrt{0.73m+0.025}-28.56446\left(0.25\right)m^4\sqrt{0.73m+0.025}\\
&+40.34928m^4\sqrt{0.73m+0.025}+8.70052m^3\sqrt{0.73m+0.025}+4.25697m^2\sqrt{0.73m+0.025}\\
&+6.08893m\sqrt{0.73m+0.025}-1.47175\sqrt{0.73\left(0.25\right)+0.025}\Bigg)\\
&\cdot\exp\left(\frac{0.45}{0.24}m^3+\frac{0.49328}{0.24}m+\frac{0.45}{0.24}m^2\sqrt{0.73m+0.025}+\frac{0.49328}{0.24}\sqrt{0.73m+0.025}\right)
\end{aligned}
$$

After rounding down several terms, we can further lower bound the above by the following using lemma B.38

$$
\begin{aligned}
&-1.888+\frac{-0.08094}{(0.73m+0.025)^{\frac{3}{2}}}+\Bigg(16.18652m^6+1.77291m^4-6.70944m^3+2.37458m^2-0.94068m+1.24492\\
&-\frac{1.28321m^6}{\sqrt{0.73m+0.025}}-\frac{5.58668m^4}{\sqrt{0.73m+0.025}}-\frac{0.02852m^3}{\sqrt{0.73m+0.025}}-\frac{2.02096m^2}{\sqrt{0.73m+0.025}}-\frac{0.95214m}{\sqrt{0.73m+0.025}}\\
&-\frac{0.04119}{\sqrt{0.73m+0.025}}-\frac{0.79298m^2}{0.73m+0.025}+43.94531m^6\sqrt{0.73m+0.025}+33.20816m^4\sqrt{0.73m+0.025}\\
&+8.70052m^3\sqrt{0.73m+0.025}+4.25697m^2\sqrt{0.73m+0.025}+6.08893m\sqrt{0.73m+0.025}\Bigg)\\
&\cdot\exp\left(\frac{0.45}{0.24}\left(0.21\right)^3+\frac{0.49328}{0.24}\left(0.21\right)+\frac{0.45}{0.24}\left(0.21\right)^2\sqrt{0.73\left(0.21\right)+0.025}+\frac{0.49328}{0.24}\sqrt{0.73\left(0.21\right)+0.025}\right).
\end{aligned}
$$

Since $\frac{-0.08094}{(0.73m+0.025)^{\frac{3}{2}}}$ and the exponential coefficient grow in m, according to lemma B.38, we can lower bound the above by evaluating at $m=0.21$, which is positive.

Since the lower bound for the derivative is positive, we can lower bound Equation 15 by evaluating it at $m = 0.21$, which is positive.

This completes the proof.

□

Proof of Lemma B.38. To proceed, we will first show that Equation 17 grows in m.

We will show that the derivative is positive. Consider the derivative below, simplified by rounding down after five digits of precision.

$$97.11912m^5 + 7.09164m^3 - 20.12832m^2 + 4.74916m - 0.94068$$

$$+ \frac{0.46837m^6 + 2.03913m^4 + 0.01040m^3 + 0.73765m^2 + 0.34753m + 0.01503}{(0.73m + 0.025)^{\frac{3}{2}}} + \frac{-0.57888m^2 - 0.03965m}{(0.73m + 0.025)^2}$$

$$+ \frac{208.52049m^6 - 1.10746m^5 + 109.08880m^4 + 3.20392m^3 + 8.33594m^2 + 2.83830m - 0.79992}{\sqrt{0.73m + 0.025}}$$

We lower bound the above by the following

$$\frac{0.46837}{(0.73\,(0.25) + 0.025)^{\frac{3}{2}}} m^6 + \left(97.11912 + \frac{(208.52049\,(0.21) - 1.10746)}{\sqrt{0.73\,(0.25) + 0.025}}\right) m^5$$

$$+ \left(\frac{109.08880}{\sqrt{0.73\,(0.25) + 0.025}} + \frac{2.03913}{(0.73\,(0.25) + 0.025)^{\frac{3}{2}}}\right) m^4$$

$$+ \left(7.09164 + \frac{3.20392}{\sqrt{0.73\,(0.25) + 0.025}} + \frac{0.01040}{(0.73\,(0.25) + 0.025)^{\frac{3}{2}}}\right) m^3$$

$$+ \left(-20.12832 - \frac{0.57888}{(0.73\,(0.21) + 0.025)^2} + \frac{0.73765}{(0.73\,(0.25) + 0.025)^{\frac{3}{2}}}\right) m^2$$

$$+ \left(4.74916 - \frac{0.03965}{(0.73\,(0.21) + 0.025)^2} + \frac{0.34753}{(0.73\,(0.25) + 0.025)^{\frac{3}{2}}}\right) m$$

$$+ \left(-0.94068 + \frac{\left(8.33594\,(0.21)^2 + 2.83830\,(0.21) - 0.79992\right)}{\sqrt{0.73\,(0.25) + 0.025}} + \frac{0.01503}{(0.73\,(0.25) + 0.025)^{\frac{3}{2}}}\right),$$

which we further simplify by rounding down after five digits of precision to get the following.

$$4.95520x^6 + 190.81793x^5 + 261.05438x^4 + 14.23518x^3 - 30.53320x^2 + 7.17870x - 0.42222$$

We can lower bound the above by the following

$$\left(4.95520\,(0.21)^4 + 190.81793\,(0.21)^3 + 261.05438\,(0.21)^2 + 14.23518\,(0.21) - 30.53320\right)(0.25)^2 + 7.17870\,(0.21) - 0.42222,$$

which is positive.

Using this, we can lower bound Equation 17 by evaluating at $m = 0.21$, which is positive.

This completes the proof.

□

Proof of Lemma B.39. We proceed by showing that the second derivative of Equation 18 is locally positive.

Consider the second derivative below, simplified by rounding down several terms after five digits of precision.

$$
\begin{aligned}
&-10.8m + \frac{-3\left(0.365\right)^2 m - 0.025\left(0.73\right)}{\left(0.73m + 0.025\right)^{\frac{3}{2}}} \\
&+ \left(\frac{-\left(0.365\right)^2 m - \left(0.365\right)0.025}{\left(0.73m + 0.025\right)^{\frac{3}{2}}} - 31.64063m^5\sqrt{0.73m + 0.025} + 21.81445m^4\sqrt{0.73m + 0.025} \right. \\
&- 22.41938m^3\sqrt{0.73m + 0.025} - 8.43382m^2\sqrt{0.73m + 0.025} + 5.74222m\sqrt{0.73m + 0.025} \\
&- 2.47502\sqrt{0.73m + 0.025} - 15.39844m^5 + 11.44555m^4 - 4.98488m^3 + 6.70438m^2 + 0.14066m \\
&+ \frac{0.46836m^5 + 0.03208m^4 - 3.07942m^3 + 1.81890m^2 - 0.90338m + 0.49501}{\sqrt{0.73m + 0.025}} \\
&\left. + \frac{-0.24980m^3 - 0.27383m}{\left(0.73m + 0.025\right)} + 2.16879 \right) \\
&\cdot \exp\left(\frac{0.45}{0.24}m^3 + \frac{0.49328}{0.24}m + \frac{0.45}{0.24}m^2\sqrt{0.73m + 0.025} + \frac{0.49328}{0.24}\sqrt{0.73m + 0.025} \right)
\end{aligned}
$$

Using lemma B.41 and B.40, we lower bound the above with the following.

$$
\begin{aligned}
&-10.8\left(0.25\right) + \frac{-3\left(0.365\right)^2\left(0.25\right) - 0.025\left(0.73\right)}{\left(0.73\left(0.21\right) + 0.025\right)^{\frac{3}{2}}} \\
&+ \left(\frac{-\left(0.365\right)^2 m - \left(0.365\right)0.025}{\left(0.73\left(0.21\right) + 0.025\right)^{\frac{3}{2}}} - 31.64063m^5\sqrt{0.73\left(0.25\right) + 0.025} + 21.81445m^4\sqrt{0.73m + 0.025} \right. \\
&- 22.41938m^3\sqrt{0.73\left(0.25\right) + 0.025} - 8.43382m^2\sqrt{0.73\left(0.25\right) + 0.025} + 5.74222m\sqrt{0.73x + 0.025} \\
&- 2.47502\sqrt{0.73\left(0.25\right) + 0.025} - 15.39844m^5 + 11.44555m^4 - 4.98488m^3 + 6.70438m^2 + 0.14066m \\
&+ \frac{0.46836\left(0.25\right)^5 + 0.03208\left(0.25\right)^4 - 3.07942\left(0.25\right)^3 + 1.81890\left(0.25\right)^2 - 0.90338\left(0.25\right) + 0.49501}{\sqrt{0.73\left(0.25\right) + 0.025}} \\
&\left. + \frac{-0.24980x^3 - 0.27383x}{\left(0.73\left(0.21\right) + 0.025\right)} + 2.16879 \right) \\
&\cdot \exp\left(\frac{0.45}{0.24}(0.21)^3 + \frac{0.49328}{0.24}(0.21) + \frac{0.45}{0.24}(0.21)^2\sqrt{0.73(0.21) + 0.025} + \frac{0.49328}{0.24}\sqrt{0.73(0.21) + 0.025} \right)
\end{aligned}
$$

Since the above decreases in m by lemma B.40, we evaluate it at $m = 0.25$, which is positive.

This completes the proof.

□

Proof of Lemma B.40. We start the proof by show that the derivative of Equation 19 is negative.

Consider the derivative below.

$$87.2578m^3\sqrt{0.73m+0.025}+5.74222\sqrt{0.73m+0.025}+\frac{21.81445\,(0.365)\,m^4+5.74222\,(0.365)\,m}{\sqrt{0.73m+0.025}}$$
$$+\left(-76.9922-158.20315\sqrt{0.2075}\right)m^4+45.78224m^3+\left(-14.95464+\frac{-0.7494}{0.1783}-67.25814\sqrt{0.2075}\right)m^2$$
$$+\left(13.40877-16.86764\sqrt{0.2075}\right)m+\left(0.14066+\frac{-0.27383}{0.1783}+\frac{-(0.365)^2}{(0.1783)^{\frac{3}{2}}}\right)$$

We upper bound the derivative by the following.

$$5.74222\sqrt{0.73m+0.025}$$

$$+\left(45.78224+87.2578\sqrt{0.73\,(0.25)+0.025}\right.$$
$$\left.+\left(-76.9922-158.20315\sqrt{0.2075}+\frac{21.81445\,(0.365)}{\sqrt{0.73\,(0.21)+0.025}}\right)(0.21)\right)m^3$$

$$+\left(-14.95464+\frac{-0.7494}{0.1783}-67.25814\sqrt{0.2075}\right)m^2+\left(13.40877-16.86764\sqrt{0.2075}+\frac{5.74222\,(0.365)}{\sqrt{0.73\,(0.21)+0.025}}\right)m$$

$$+\left(0.14066+\frac{-0.27383}{0.1783}+\frac{-(0.365)^2}{(0.1783)^{\frac{3}{2}}}\right)$$

After simplifying the above by rounding up after five digits of precision, we calculate the derivative of the above and want to show that it is positive. Consider the derivative below.

$$\frac{5.74222\,(0.365)}{\sqrt{0.73m+0.025}}+174.56379m^2-99.5904m+10.68880$$

In order to show that the above is positive, we will first show that the derivative of the above is negative. Consider the derivative below.

$$\frac{-5.74222\,(0.365)^2}{(0.73m+0.025)^{\frac{3}{2}}}+349.12758m-99.5904,$$

We can upper bound the above by

$$\frac{-5.74222\,(0.365)^2}{(0.73\,(0.25)+0.025)^{\frac{3}{2}}}+349.12758\,(0.25)-99.5904,$$

which is negative.

So, we can lower bound

$$\frac{5.74222\,(0.365)}{\sqrt{0.73m+0.025}}+174.56379m^2-99.5904m+10.68880$$

by evaluating at $m=0.25$, which is positive.

Then, we can upper bound

$$5.74222\sqrt{0.73m+0.025}$$

$$+\left(45.78224+87.2578\sqrt{0.73\left(0.25\right)+0.025}\right.$$

$$+\left(-76.9922-158.20315\sqrt{0.2075}+\frac{21.81445\left(0.365\right)}{\sqrt{0.73\left(0.21\right)+0.025}}\right)\left(0.21\right)\Big)m^3$$

$$+\left(-14.95464+\frac{-0.7494}{0.1783}-67.25814\sqrt{0.2075}\right)m^2+\left(13.40877-16.86764\sqrt{0.2075}+\frac{5.74222\left(0.365\right)}{\sqrt{0.73\left(0.21\right)+0.025}}\right)m$$

$$+\left(0.14066+\frac{-0.27383}{0.1783}+\frac{-\left(0.365\right)^2}{\left(0.1783\right)^{\frac{3}{2}}}\right)$$

by evaluating at $m=0.25$, which is negative. This shows that Equation 19 decreases in m.

We show Equation 19 is positive, by evaluating at $m=0.25$, which is positive.

This completes the proof.

□

Proof of Lemma B.41. We start by showing that the derivative of Equation 20 is negative.

Consider the derivative below.

$$2.3418m^4+0.12832m^3-9.23826m^2+3.6378m-0.90338$$

We can upper bound the above by the following

$$2.3418\left(0.25\right)^4+0.12832\left(0.25\right)^3-9.23826\left(0.21\right)^2+3.6378\left(0.25\right)-0.90338,$$

which is negative.

So, we can lower bound Equation 20 by evaluating it at $m=0.25$, which is positive.

This completes the proof. □

Proof of Lemma B.42. To prove the lemma, we use convexity of Equation 13 in η from lemma B.43 and prove Equation 13 is positive by linearizing at $\eta=0.24$.

Since the partial derivative of Equation 13 with respect to η evaluating at $\eta=0.24$ is positive, according to lemma B.35, we evaluating the $\eta=0.24$ linearization at $\eta=0.2$.

We lower bound the linearization by lower bounding Equation 13 evaluated at $\eta=0.24$ and upper bounding the partial derivative of Equation 13 with respect to η evaluated at $\eta=0.24$.

From lemma B.35, we can upper bound the partial derivative of Equation 13 with respect to η evaluated at $\eta=0.24$, by evaluating at $m=0.25$, which is at most 0.05776.

We can lower bound Equation 13 evaluated at $\eta=0.24$ (also Equation 18) using local convexity in m established in lemma B.39 by linearizing Equation 18 at $m=0.23$.

Since the derivative of Equation 18, shown below, at $m = 0.23$ is positive, we evaluate the linearization at $m = 0.21$.

$$
\begin{aligned}
&- 1.24312 - 5.4m^2 - \sqrt{0.73m + 0.025} - \frac{0.365m}{\sqrt{0.73m + 0.025}} \\
&+ \left(\frac{0.187532}{0.24} - \frac{0.365m}{\sqrt{0.73m + 0.025}} + \frac{0.45\,(0.365)}{0.24} m^3 + \frac{(0.45)\,(0.025)}{0.24} m^2 + \frac{0.49328\,(0.365)}{0.24} m \right. \\
&- \frac{1.35}{0.24} m^3 \sqrt{0.73m + 0.025} + \frac{5\,(0.45)\,(0.365)}{0.24} m^2 \sqrt{0.73m + 0.025} + \frac{-0.47078}{0.24} m \sqrt{0.73m + 0.025} \\
&\left. + \frac{-0.0599528}{0.24} \sqrt{0.73m + 0.025} \right) \\
&\cdot \exp\left(\frac{0.45}{0.24} m^3 + \frac{0.49328}{0.24} x + \frac{0.45}{0.24} m^2 \sqrt{0.73m + 0.025} + \frac{0.49328}{0.24} \sqrt{0.73m + 0.025} \right)
\end{aligned}
$$

The $m = 0.23$ linearization evaluated at $m = 0.21$ is at least 0.00429.

So, our lower bound for the $\eta = 0.24$ linearization evaluated $\eta = 0.2$ is positive.

This completes the proof.

□

Proofs for Lemmas in Section B.5.1

Proof of Lemma B.43. We proceed with the proof by calculating the second derivative of Equation21 with respect to η and showing that it is positive.

We provide an expression for the second derivative below.

$$
\begin{aligned}
&\frac{m^3}{(2m\eta + 0.25m + 0.02)^{3/2}} \\
&+ \left(\frac{m^3}{(2m\eta + 0.25m + 0.02)^{3/2}} \right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972 \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) \right) \\
&+ 4m\beta_1 \left(2 - \frac{m}{\sqrt{2m\eta + 0.25m + 0.02}} \right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972 \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) \right) \\
&+ \beta_2 \sqrt{2m\eta + 0.25m + 0.02} \left(\sqrt{2m\eta + 0.25m + 0.02} - m \right) \\
&\cdot \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972 \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) \right) \\
&+ \beta_1^2 \sqrt{2m\eta + 0.25m + 0.02} \left(\sqrt{2m\eta + 0.25m + 0.02} - m \right) \\
&\cdot \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972 \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) \right),
\end{aligned}
$$

where

$$
\beta_1 = - \left(\frac{0.45m^2}{\eta^2} + \frac{0.26}{\eta^2} \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) + \frac{m \left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972 \right)}{\sqrt{2m\eta + 0.25m + 0.02}}
$$

and

$$\beta_2 = 2\left(\frac{0.45m^2}{\eta^3} + \frac{0.26}{\eta^3}\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right) - \frac{m\left(\frac{0.45m^2}{\eta^2} + \frac{0.26}{\eta^2}\right)}{\sqrt{2m\eta + 0.25m + 0.02}} - \frac{m^2\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)}{(2m\eta + 0.25m + 0.02)^{3/2}}.$$

We proceed with the rest of the proof by showing the below is positive.

$$+ \frac{m^3}{(2m\eta + 0.25m + 0.02)^{3/2}}$$

$$+ 4m\beta_1\left(2 - \frac{m}{\sqrt{2m\eta + 0.25m + 0.02}}\right)$$

$$+ \beta_2\sqrt{2m\eta + 0.25m + 0.02}\left(\sqrt{2m\eta + 0.25m + 0.02} - m\right)$$

By scaling the above by η^3, we simplify the expression to the following.

$$\frac{m^3\eta^3}{(2m\eta + 0.25m + 0.02)^{3/2}} - \frac{m^3\eta^2\left(0.45m^2 + 0.26 + 0.972\right)}{2m\eta + 0.25m + 0.02}$$

$$+ \frac{\begin{array}{c}-0.9m^5\eta - 0.45m^4\eta^2 - 0.121m^2 + 0.225m^4\eta - 0.502m^3\eta - 0.26m^2\eta^2 + 0.1508m^2\eta \\ + 0.003825m^4 + 0.02506m^2 - 0.225m^5 + 0.0026m + 0.000208 + 2.916m^2\eta^3 - 0.0104m\eta\end{array}}{\sqrt{2m\eta + 0.25m + 0.02}}$$

By lemma B.49, we can lower bound the above by

$$\frac{m^3\eta^3}{(2(0.25)(0.3)) + 0.25(0.25) + 0.02)^{3/2}} - \frac{m^3\eta^2\left(0.45m^2 + 0.26 + 0.972\right)}{2(0.15)(0.2)) + 0.25(0.15) + 0.02}$$

$$+ \frac{\begin{array}{c}-0.9m^5\eta - 0.45m^4\eta^2 - 0.121m^2 + 0.225m^4\eta - 0.502m^3\eta - 0.26m^2\eta^2 + 0.1508m^2\eta \\ + 0.003825m^4 + 0.02506m^2 - 0.225m^5 + 0.0026m + 0.000208 + 2.916m^2\eta^3 - 0.0104m\eta\end{array}}{\sqrt{2(0.25)(0.3) + 0.25(0.25) + 0.02}}$$

, which simplifies to

$$\frac{m^3\eta^3}{(0.2325)^{\frac{3}{2}}} - \frac{0.45}{0.1175}m^5\eta^2 - \frac{0.26}{0.1175}m^3\eta^2 - \frac{0.973}{0.1175}m^3\eta^3 - \frac{0.9}{\sqrt{0.2325}}m^5\eta - \frac{0.45}{\sqrt{0.2325}}m^4\eta^2 - \frac{0.121}{\sqrt{0.2325}}m^3$$
$$+ \frac{0.225}{\sqrt{0.2325}}m^4\eta - \frac{0.502}{\sqrt{0.2325}}m^3\eta - \frac{0.26}{\sqrt{0.2325}}m^2\eta^2 + \frac{0.1508}{\sqrt{0.2325}}m^2\eta + \frac{0.03825}{\sqrt{0.2325}}m^4 + \frac{0.02506}{\sqrt{0.2325}}m^2 - \frac{0.225}{\sqrt{0.2325}}m^5$$
$$+ \frac{0.0026}{\sqrt{0.2325}}m + \frac{0.000208}{\sqrt{0.2325}} + \frac{2.916}{\sqrt{0.2325}}m^2\eta^3 - \frac{0.0104}{\sqrt{0.2325}}m\eta$$

By lemma B.51, we can further lower bound by $\eta = 0.2$, shown below.

$$\left(-\frac{0.45\,(0.2)^2}{0.1175} - \frac{0.9\,(0.2)}{\sqrt{0.2325}} - \frac{0.225}{\sqrt{0.2325}}\right)m^5 + \left(-\frac{0.45\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.225\,(0.2)}{\sqrt{0.2325}} + \frac{0.03825}{\sqrt{0.2325}}\right)m^4$$

$$+ \left(\frac{(0.2)^3}{(0.2325)^{\frac{3}{2}}} - \frac{0.26\,(0.2)^2}{0.1175} - \frac{0.973\,(0.2)^3}{0.1175} - \frac{0.121}{\sqrt{0.2325}} - \frac{0.502\,(0.2)}{\sqrt{0.2325}}\right)m^3$$

$$+ \left(-\frac{0.26\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.1508\,(0.2)}{\sqrt{0.2325}} + \frac{0.02506}{\sqrt{0.2325}} + \frac{2.916\,(0.2)^3}{\sqrt{0.2325}}\right)m^2 + \left(\frac{0.0026}{\sqrt{0.2325}} - \frac{0.0104\,(0.2)}{\sqrt{0.2325}}\right)m + \frac{0.000208}{\sqrt{0.2325}}$$

Since the above is concave in m, by lemma B.50, by Jensen's inequality, we evaluate the lower bound at $m = 0.15, 0.25$, which are both positive.

This completes the proof.

□

Proof of Lemma B.50. To proceed with the proof, we will show that second partial derivative of Equation 28 with respect to m, shown below, is negative.

$$\left(\left(-\frac{0.45\,(0.2)^2}{0.1175} - \frac{0.9\,(0.2)}{\sqrt{0.2325}} - \frac{0.225}{\sqrt{0.2325}}\right)(20)\,m + \left(-\frac{0.45\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.225\,(0.2)}{\sqrt{0.2325}} + \frac{0.03825}{\sqrt{0.2325}}\right)(12)\right) m^2$$

$$+ \left(\frac{(0.2)^3}{(0.2325)^{\frac{3}{2}}} - \frac{0.26\,(0.2)^2}{0.1175} - \frac{0.973\,(0.2)^3}{0.1175} - \frac{0.121}{\sqrt{0.2325}} - \frac{0.502\,(0.2)}{\sqrt{0.2325}}\right)(4)\,m$$

$$+ \left(-\frac{0.26\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.1508\,(0.2)}{\sqrt{0.2325}} + \frac{0.02506}{\sqrt{0.2325}} + \frac{2.916\,(0.2)^3}{\sqrt{0.2325}}\right)(2)$$

Note that $\left(-\frac{0.45(0.2)^2}{0.1175} - \frac{0.9(0.2)}{\sqrt{0.2325}} - \frac{0.225}{\sqrt{0.2325}}\right)(20)\,m + \left(-\frac{0.45(0.2)^2}{\sqrt{0.2325}} + \frac{0.225(0.2)}{\sqrt{0.2325}} + \frac{0.03825}{\sqrt{0.2325}}\right)(12)$ can be lower upper bounded by $\left(-\frac{0.45(0.2)^2}{0.1175} - \frac{0.9(0.2)}{\sqrt{0.2325}} - \frac{0.225}{\sqrt{0.2325}}\right)(20)\,(0.15) + \left(-\frac{0.45(0.2)^2}{\sqrt{0.2325}} + \frac{0.225(0.2)}{\sqrt{0.2325}} + \frac{0.03825}{\sqrt{0.2325}}\right)(12)$ which is negative.

So, we can upper bound the second partial derivative by

$$\left(\left(-\frac{0.45\,(0.2)^2}{0.1175} - \frac{0.9\,(0.2)}{\sqrt{0.2325}} - \frac{0.225}{\sqrt{0.2325}}\right)(20)\,(0.15) + \left(-\frac{0.45\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.225\,(0.2)}{\sqrt{0.2325}} + \frac{0.03825}{\sqrt{0.2325}}\right)(12)\right)(0.15)^2$$

$$+ \left(\frac{(0.2)^3}{(0.2325)^{\frac{3}{2}}} - \frac{0.26\,(0.2)^2}{0.1175} - \frac{0.973\,(0.2)^3}{0.1175} - \frac{0.121}{\sqrt{0.2325}} - \frac{0.502\,(0.2)}{\sqrt{0.2325}}\right)(4)\,(0.15)$$

$$+ \left(-\frac{0.26\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.1508\,(0.2)}{\sqrt{0.2325}} + \frac{0.02506}{\sqrt{0.2325}} + \frac{2.916\,(0.2)^3}{\sqrt{0.2325}}\right)(2)$$

which is negative. So, our lower bound

$$\left(-\frac{0.45\,(0.2)^2}{0.1175} - \frac{0.9\,(0.2)}{\sqrt{0.2325}} - \frac{0.225}{\sqrt{0.2325}}\right) m^5 + \left(-\frac{0.45\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.225\,(0.2)}{\sqrt{0.2325}} + \frac{0.03825}{\sqrt{0.2325}}\right) m^4$$

$$+ \left(\frac{(0.2)^3}{(0.2325)^{\frac{3}{2}}} - \frac{0.26\,(0.2)^2}{0.1175} - \frac{0.973\,(0.2)^3}{0.1175} - \frac{0.121}{\sqrt{0.2325}} - \frac{0.502\,(0.2)}{\sqrt{0.2325}}\right) m^3$$

$$+ \left(-\frac{0.26\,(0.2)^2}{\sqrt{0.2325}} + \frac{0.1508\,(0.2)}{\sqrt{0.2325}} + \frac{0.02506}{\sqrt{0.2325}} + \frac{2.916\,(0.2)^3}{\sqrt{0.2325}}\right) m^2 + \left(\frac{0.0026}{\sqrt{0.2325}} - \frac{0.0104\,(0.2)}{\sqrt{0.2325}}\right) m + \frac{0.000208}{\sqrt{0.2325}}$$

is concave in m.

This completes the proof.

□

Proof of Lemma B.51. To prove the lemma, we'll show that the partial derivative of Equation 29 with respect to η is locally positive. Consider the partial derivative with respect to η, shown below.

$$\frac{3m^3\eta^2}{(0.2325)^{\frac{3}{2}}} - \frac{0.45\,(2)}{0.1175}m^5\eta - \frac{0.26\,(2)}{0.1175}m^3\eta - \frac{0.973\,(3)}{0.1175}m^3\eta^2 - \frac{0.9}{\sqrt{0.2325}}m^5 - \frac{0.45\,(2)}{\sqrt{0.2325}}m^4\eta + \frac{0.225}{\sqrt{0.2325}}m^4$$
$$- \frac{0.502}{\sqrt{0.2325}}m^3 - \frac{0.26\,(2)}{\sqrt{0.2325}}m^2\eta + \frac{0.1508}{\sqrt{0.2325}}m^2 + \frac{2.916\,(3)}{\sqrt{0.2325}}m^2\eta^2 - \frac{0.0104}{\sqrt{0.2325}}m$$

To show that the above partial derivative is positive, we take another partial derivative with respect to η, shown below.

$$\left(\frac{6m}{(0.2325)^{\frac{3}{2}}} - \frac{0.973\,(6)}{0.1175}m + \frac{2.916\,(6)}{\sqrt{0.2325}}\right)m^2\eta - \frac{0.45\,(2)}{0.1175}m^5 - \frac{0.26\,(2)}{0.1175}m^3 - \frac{0.45\,(2)}{\sqrt{0.2325}}m^4 - \frac{0.26\,(2)}{\sqrt{0.2325}}m^2$$

Note that we can lower bound the coefficient linear in η by the following $\left(\frac{6(0.15)}{(0.2325)^{\frac{3}{2}}} - \frac{0.973(6)}{0.1175}\,(0.25) + \frac{2.916(6)}{\sqrt{0.2325}}\right)$, which is positive. So, we can lower bound the second partial derivative by $\eta = 0.2$, shown below.

$$\left(-\frac{0.45\,(2)}{0.1175}m^3 - \frac{0.45\,(2)}{\sqrt{0.2325}}m^2 + \left(\frac{6\,(0.2)}{(0.2325)^{\frac{3}{2}}} - \frac{0.973\,(6)\,(0.2)}{0.1175} - \frac{0.26\,(2)}{0.1175}\right)m + \left(\frac{2.916\,(6)\,(0.2)}{\sqrt{0.2325}} - \frac{0.26\,(2)}{\sqrt{0.2325}}\right)\right)m^2$$

Focusing on the inner expression

$$-\frac{0.45\,(2)}{0.1175}m^3 - \frac{0.45\,(2)}{\sqrt{0.2325}}m^2 + \left(\frac{6\,(0.2)}{(0.2325)^{\frac{3}{2}}} - \frac{0.973\,(6)\,(0.2)}{0.1175} - \frac{0.26\,(2)}{0.1175}\right)m + \left(\frac{2.916\,(6)\,(0.2)}{\sqrt{0.2325}} - \frac{0.26\,(2)}{\sqrt{0.2325}}\right),$$

note that the coefficients for terms of degree at least 1 in m are negative, we can lower bound the inner expression by evaluating it at $m = 0.25$, which is positive.

Since the second partial derivative in η is positive, we can lower bound the first partial derivative in η by $\eta = 0.2$, shown below.

$$\left(\left(-\frac{0.9}{\sqrt{0.2325}} - \frac{0.45\,(2)\,(0.2)}{0.1175}\right)m^4\right.$$
$$+ \left(-\frac{0.45\,(2)\,(0.2)}{\sqrt{0.2325}} + \frac{0.225}{\sqrt{0.2325}}\right)m^3$$
$$+ \left(\frac{3\,(0.2)^2}{(0.2325)^{\frac{3}{2}}} - \frac{0.26\,(2)\,(0.2)}{0.1175} - \frac{0.973\,(3)\,(0.2)^2}{0.1175} - \frac{0.502}{\sqrt{0.2325}}\right)m^2$$
$$\left.+ \left(-\frac{0.26\,(2)\,(0.2)}{\sqrt{0.2325}} + \frac{0.1508}{\sqrt{0.2325}} + \frac{2.916\,(3)\,(0.2)^2}{\sqrt{0.2325}}\right)m - \frac{0.0104}{\sqrt{0.2325}}\right)m$$

Focusing on the inner expression,

$$\left(-\frac{0.9}{\sqrt{0.2325}} - \frac{0.45\,(2)\,(0.2)}{0.1175}\right)m^4$$
$$+ \left(-\frac{0.45\,(2)\,(0.2)}{\sqrt{0.2325}} + \frac{0.225}{\sqrt{0.2325}}\right)m^3$$
$$+ \left(\frac{3\,(0.2)^2}{(0.2325)^{\frac{3}{2}}} - \frac{0.26\,(2)\,(0.2)}{0.1175} - \frac{0.973\,(3)\,(0.2)^2}{0.1175} - \frac{0.502}{\sqrt{0.2325}}\right)m^2$$
$$+ \left(-\frac{0.26\,(2)\,(0.2)}{\sqrt{0.2325}} + \frac{0.1508}{\sqrt{0.2325}} + \frac{2.916\,(3)\,(0.2)^2}{\sqrt{0.2325}}\right)m - \frac{0.0104}{\sqrt{0.2325}}$$

we'll show that it is concave in m.

Consider the second partial derivative of the inner expression with respect to m below:

$$\left(\left(-\frac{0.9}{\sqrt{0.2325}}-\frac{0.45\left(2\right)\left(0.2\right)}{0.1175}\right)\left(12\right)m+\left(-\frac{0.45\left(2\right)\left(0.2\right)}{\sqrt{0.2325}}+\frac{0.225}{\sqrt{0.2325}}\right)\left(6\right)\right)m$$

$$+\left(\frac{3\left(0.2\right)^{2}}{\left(0.2325\right)^{\frac{3}{2}}}-\frac{0.26\left(2\right)\left(0.2\right)}{0.1175}-\frac{0.973\left(3\right)\left(0.2\right)^{2}}{0.1175}-\frac{0.502}{\sqrt{0.2325}}\right)\left(2\right)$$

Note that we can upper bound the term $\left(-\frac{0.9}{\sqrt{0.2325}}-\frac{0.45(2)(0.2)}{0.1175}\right)(12)\,m+\left(-\frac{0.45(2)(0.2)}{\sqrt{0.2325}}+\frac{0.225}{\sqrt{0.2325}}\right)(6)$ by $\left(-\frac{0.9}{\sqrt{0.2325}}-\frac{0.45(2)(0.2)}{0.1175}\right)(12)\,(0.15)+\left(-\frac{0.45(2)(0.2)}{\sqrt{0.2325}}+\frac{0.225}{\sqrt{0.2325}}\right)(6)$, which is negative.

So, we can upper bound the second partial derivative by

$$\left(\left(-\frac{0.9}{\sqrt{0.2325}}-\frac{0.45\left(2\right)\left(0.2\right)}{0.1175}\right)\left(12\right)\left(0.15\right)+\left(-\frac{0.45\left(2\right)\left(0.2\right)}{\sqrt{0.2325}}+\frac{0.225}{\sqrt{0.2325}}\right)\left(6\right)\right)\left(0.15\right)$$

$$+\left(\frac{3\left(0.2\right)^{2}}{\left(0.2325\right)^{\frac{3}{2}}}-\frac{0.26\left(2\right)\left(0.2\right)}{0.1175}-\frac{0.973\left(3\right)\left(0.2\right)^{2}}{0.1175}-\frac{0.502}{\sqrt{0.2325}}\right)\left(2\right)$$

, which is negative and proves concavity of the inner expression.

So, by Jensen's inequality, we can evaluate the inner expression

$$\begin{aligned}&\left(-\frac{0.9}{\sqrt{0.2325}}-\frac{0.45\left(2\right)\left(0.2\right)}{0.1175}\right)m^{4}\\&+\left(-\frac{0.45\left(2\right)\left(0.2\right)}{\sqrt{0.2325}}+\frac{0.225}{\sqrt{0.2325}}\right)m^{3}\\&+\left(\frac{3\left(0.2\right)^{2}}{\left(0.2325\right)^{\frac{3}{2}}}-\frac{0.26\left(2\right)\left(0.2\right)}{0.1175}-\frac{0.973\left(3\right)\left(0.2\right)^{2}}{0.1175}-\frac{0.502}{\sqrt{0.2325}}\right)m^{2}\\&+\left(-\frac{0.26\left(2\right)\left(0.2\right)}{\sqrt{0.2325}}+\frac{0.1508}{\sqrt{0.2325}}+\frac{2.916\left(3\right)\left(0.2\right)^{2}}{\sqrt{0.2325}}\right)m-\frac{0.0104}{\sqrt{0.2325}}\end{aligned}$$

at $m=0.15,0.25$, both of which are positive.

This completes the proof.

□

Proof of Lemma B.49. To prove the lemma, we consider the partial derivative of Equation 27 with respect to η, shown below, and prove that it is positive for $\eta\in[0.2,0.3], m\in[0.15,0.25]$.

$$-0.9m^{5}-0.9m^{4}\eta+0.225m^{4}-0.502m^{3}-0.52m^{2}\eta+0.1508m^{2}+8.748m^{2}\eta^{2}-0.0104m$$

We take another partial derivative with respect to η and show it is positive for $\eta\in[0.2,0.3], m\in[0.15,0.25]$ by lower bounding it. Consider the second partial derivative of Equation 27 with respect to η:

$$\left(0.9m^{2}-0.52+17.496\eta\right)m^{2}$$

.

Note that the inner expression $0.9m^{2}-0.52+17.496\eta$ can be lower bounded by $0.9(0.15)^{2}-0.52+17.496(0.2)$ which is positive.

So, our first partial derivative with respect to η can be lower bounded at $\eta = 0.2$, shown below.

$$\left(-0.9m^4 + 0.045m^3 - 0.502m^2 + 0.39672m - 0.0104\right) m$$

Next, we'll show that the inner expression $-0.9m^4 + 0.045m^3 - 0.502m^2 + 0.39672m - 0.0104$ is concave in m for $m \in [0.15, 0.25]$.

The second partial derivative of the inner expression with respect to m is $-10.8m^2 + 0.27m - 1.004$, and we can upper bound it by $-10.8(0.15)^2 + 0.27(0.25) - 1.004$, which is negative.

So, by concavity, we evaluate

$$-0.9m^4 + 0.045m^3 - 0.502m^2 + 0.39672m - 0.0104$$

at $m = 0.15, 0.25$, which are both positive.

So, our first partial derivative of Equation 27 with respect to η is positive.

So, we lower bound Equation 27 by $\eta = 0.2$, shown below:

$$-0.405m^5 + 0.06525m^4 - 0.2214m^3 + 0.068148m^2 + 0.00052m + 0.000208$$

Now, we will prove concavity in m (for $m \in [0.15, 0.25]$) for our lower bound. The second partial derivative of the lower bound with respect to m is shown below.

$$-8.1m^3 + 0.783m^2 - 1.3284m + 0.136296$$

Since $-8.1(0.15) + 0.783$ is negative, then the second partial derivative is decreasing in m, so we evaluate it at $m = 0.15$, which is negative.

So, our lower bound is locally concave in m, and we evaluate it at $m = 0.15, 0.25$, both of which are positive.

This completes the proof. □

Proof of Lemma B.44. We proceed with the proof by first locally upper bounding Equation 22 with the following

$$\begin{aligned}
&-1.8m^3 - 1.1676m + 0.02 - m\sqrt{0.65m + 0.02} \\
&+ \sqrt{0.65m + 0.02}\left(\sqrt{0.65m + 0.02} - m\right) \\
&\cdot \exp\left(2.25\left(0.25\right)^2 m + 2.25m^2\sqrt{0.65\left(0.25\right) + 0.02} + 2.272m + 2.272\sqrt{0.65m + 0.02}\right),
\end{aligned}$$

which simplifies down to

$$\begin{aligned}
&-1.8m^3 - 1.1676m + 0.02 - m\sqrt{0.65m + 0.02} \\
&+ \sqrt{0.65m + 0.02}\left(\sqrt{0.65m + 0.02} - m\right)\exp\left(0.96121m^2 + 2.41263m + 2.272\sqrt{0.65m + 0.02}\right)
\end{aligned}$$

by rounding up after five digits of precision.

We will show that the above is locally decreasing. Consider the derivative of the upper bound.

$$\begin{aligned}
&-5.4m^2 - 1.1676 - \frac{0.975m + 0.02}{\sqrt{0.65m + 0.02}} \\
&+ \left(\begin{aligned}
&\left(1.92242\right)\left(0.65\right)m^2 + \left(\left(2.41263\right)\left(0.65\right) - \left(2.272\right)\left(0.325\right) + \left(1.92242\right)\left(0.02\right)\right)m + \left(0.65 + \left(2.41263\right)\left(0.02\right)\right) \\
&- \left(1.92242\right)\left(0.65\right)m^3 + \left(-\left(1.92242\right)\left(0.02\right) - \left(2.41263\right)\left(0.65\right)\right)m^2 \\
&+ \frac{+\left(-\left(2.41263\right)\left(0.02\right) + \left(2.272\right)\left(0.325\right)\left(0.65\right) - 0.975\right)m + \left(\left(2.272\right)\left(0.325\right)\left(0.02\right) - 0.02\right)}{\sqrt{0.65m + 0.02}}
\end{aligned}\right) \\
&\cdot \exp\left(0.96121m^2 + 2.41263m + 2.272\sqrt{0.65m + 0.02}\right)
\end{aligned}$$

We will show that the derivative is negative. We simplify the above expression by rounding up several terms after five digits of precision, shown below.

$$
\begin{aligned}
&-5.4m^2 - 1.1676 - \frac{0.975m + 0.02}{\sqrt{0.65\,(0.25) + 0.02}} \\
&+ \left(1.24958m^2 + 0.86826m + 0.69826 + \frac{((2.272)\,(0.325)\,(0.02) - 0.02)}{\sqrt{0.65\,(0.25) + 0.02}} + \frac{-1.24957\,(0.15)\,m^2}{\sqrt{0.65\,(0.25) + 0.02}} + \frac{-1.60665m^2}{\sqrt{0.65m + 0.02}} \right. \\
&\left. + \frac{-0.54329m}{\sqrt{0.65m + 0.02}}\right) \\
&\cdot \exp\left(0.96121m^2 + 2.41263m + 2.272\sqrt{0.65m + 0.02}\right)
\end{aligned}
$$

can be simplified to

$$
\begin{aligned}
&-5.4m^2 - 1.1676 - \frac{0.975m + 0.02}{\sqrt{0.1825}} \\
&+ \left(0.81083m^2 + 0.86826m + 0.68602 + \frac{-1.60665m^2}{\sqrt{0.65m + 0.02}} + \frac{-0.54329m}{\sqrt{0.65m + 0.02}}\right) \\
&\cdot \exp\left(0.96121m^2 + 2.41263m + 2.272\sqrt{0.65m + 0.02}\right).
\end{aligned}
$$

Before simplifying the above further, we must first show that the exponential coefficient is locally positive. We do so with the following lower bound on the coefficient $0.81083(0.15)^2 + 0.86826(0.15) + 0.68602 + \frac{-1.60665(0.25)^2}{\sqrt{0.65(0.15)+0.02}} + \frac{-0.54329(0.25)}{\sqrt{0.65(0.15)+0.02}}$, which is positive.

So we can further upper bound our derivative expression with the following

$$
\begin{aligned}
&-5.4m^2 - 1.1676 - \frac{0.975m + 0.02}{\sqrt{0.1825}} \\
&+ \left(0.81083m^2 + 0.86826m + 0.68602 + \frac{-1.60665m^2}{\sqrt{0.65m + 0.02}} + \frac{-0.54329m}{\sqrt{0.65m + 0.02}}\right) \\
&\cdot \exp\left(0.96121(0.25)^2 + 2.41263m + 2.272\sqrt{0.65m + 0.02}\right).
\end{aligned}
$$

By lemma B.52, the above upper bound is locally concave, so we prove it is negative by upper bounding it by linearization at $m = 0.19$.

Note the derivative of the term below.

$$
\begin{aligned}
&-10.8m - \frac{0.975}{\sqrt{0.1825}} \\
&+ \left((2.41263)\,(0.81083)\,m^2 + (1.62166 + (2.41263)\,(0.86826))m + 0.86826 + (2.41263)\,(0.68602)\right. \\
&+ \frac{-(3)\,(1.60665)\,(0.325)\,m^2 - (2)\,(1.60665)\,(0.02)\,m - 0.54329\,(0.325)\,m - 0.54329\,(0.02)}{(0.65m + 0.02)^{3/2}} \\
&+ \frac{(2.272)\,(0.325)\,(0.81083) - (1.60665)\,(2.41263)}{\sqrt{0.65m + 0.02}}m^2 + \frac{(2.272)\,(0.325)\,(0.86826) - (0.54329)\,(2.41263)}{\sqrt{0.65m + 0.02}}m \\
&\left. + \frac{(2.272)\,(0.325)\,(0.68602)}{\sqrt{0.65m + 0.02}} + \frac{-(1.60665)\,(2.272)\,(0.325)\,m^2}{0.65x + 0.02} + \frac{-(0.54329)\,(2.272)\,(0.325)\,m}{0.65x + 0.02}\right) \\
&\cdot \exp\left(2.41263m + 0.96121\,(0.25)^2 + 2.272\sqrt{0.65m + 0.02}\right)
\end{aligned}
$$

Evaluated at $m = 0.19$, the derivative is negative. So, we consider the $m = 0.19$ linearization at $m = 0.15$, which is negative.

This means that

$$-1.8m^3 - 1.1676m + 0.02 - m\sqrt{0.65m + 0.02}$$
$$+ \sqrt{0.65m + 0.02}\left(\sqrt{0.65m + 0.02} - m\right)\exp\left(0.96121m^2 + 2.41263m + 2.272\sqrt{0.65m + 0.02}\right)$$

is locally decreasing. So, we evaluate the above at $m = 0.15$, which is negative. This completes the proof.

□

Proof of Lemma B.52. We proceed by showing that the second derivative of Equation 30 is negative. Consider the second derivative below, simplified by rounding several terms up after five digits of precision.

$$\begin{aligned}
&-10.8 \\
&+\Bigg(4.71967m^2 + 12.87889m + 9.80442 + \frac{-6.46299m^2 - 5.42642m + 2.41576}{\sqrt{0.65m + 0.02}} \\
&+\frac{-3.59014m^2 - 3.27044m - 0.43168}{(0.65x + 0.02)^{3/2}} + \frac{1.52733m^2 + 0.23482m + 0.01060}{(0.65m + 0.02)^{5/2}} \\
&+\frac{-5.28235m^2 - 3.83502m - 0.02712}{0.65m + 0.02} + \frac{-0.38556m^2 + 0.08293m - 0.00802}{(0.65m + 0.02)^2}\Bigg) \\
&\cdot \exp\left(2.41263m + 0.96121\,(0.25)^2 + 2.272\sqrt{0.65m + 0.02}\right)
\end{aligned}$$

We further simplify the second derivative below, rounding up several terms after five digits of precision.

$$\begin{aligned}
&-10.8 \\
&+\Bigg(4.71967m^2 + 12.87889m + 9.80442 + \frac{-2.73061m^4 - 4.79429m^3 + 0.20673m^2 - 0.05054m + 0.00294}{(0.65m + 0.02)^{5/2}} \\
&+\frac{-3.43352m^3 - 2.98397m^2 - 0.01139m - 0.00856}{(0.65m + 0.02)^2}\Bigg) \\
&\cdot \exp\left(2.41263m + 0.96121\,(0.25)^2 + 2.272\sqrt{0.65m + 0.02}\right)
\end{aligned}$$

We upper bound the above with the following

$$\begin{aligned}
&-10.8 \\
&+\Bigg(4.71967m^2 + 12.87889m + 9.80442 \\
&+\frac{-2.73061\,(0.15)^4 - 4.79429m^3 + 0.20673m^2 - 0.05054m + 0.00294}{(0.65m + 0.02)^{5/2}} \\
&+\frac{-3.43352\,(0.15)^3 - 2.98397m^2 - 0.01139m - 0.00856}{(0.65m + 0.02)^2}\Bigg) \\
&\cdot \exp\left(2.41263m + 0.96121\,(0.25)^2 + 2.272\sqrt{0.65m + 0.02}\right)
\end{aligned}$$

which we simplify below by rounding up several terms after five digits of precision.

$$
\begin{aligned}
&-10.8 \\
&+\Bigg(4.71967m^2+12.87889m+9.80442 \\
&+\frac{-4.79429m^3+0.20673m^2-0.05054m+0.00156}{(0.65m+0.02)^{5/2}} \\
&+\frac{-2.98397m^2-0.01139m-0.02014}{(0.65m+0.02)^{2}}\Bigg) \\
&\cdot\exp\left(2.41263m+0.96121\,(0.25)^2+2.272\sqrt{0.65m+0.02}\right)
\end{aligned}
$$

To further simplify the above expression, we will first show that the exponential coefficient is locally positive. To do, we will show that the derivative of the exponential coefficient is positive. Consider the derivative of the coefficient below.

$$
\begin{aligned}
&9.43934m+12.87889+\frac{-14.38287m^2+0.41346m-0.05054}{(0.65m+0.02)^{5/2}} \\
&+\frac{-5\,(0.325)\left(-4.79429m^3+0.20673m^2-0.05054m+0.00156\right)}{(0.65m+0.02)^{7/2}} \\
&+\frac{-5.96794m-0.01139}{(0.65m+0.02)^{2}}+\frac{-2\,(0.65)\left(-2.98397m^2-0.01139m-0.02014\right)}{(0.65m+0.02)^{3}}
\end{aligned}
$$

We simplify and lower bound the derivative with the following:

$$
\begin{aligned}
&9.43934\,(0.15)+12.87889-\frac{1.55815m^3}{(0.65m+0.02)^{7/2}}-\frac{0.35485m^2}{(0.65m+0.02)^{7/2}}+\frac{0.05754m}{(0.65m+0.02)^{7/2}} \\
&-\frac{0.00355}{(0.65m+0.02)^{7/2}}-\frac{0.11196m}{(0.65m+0.02)^{3}}+\frac{0.02595}{(0.65m+0.02)^{3}}
\end{aligned}
$$

To proceed further, we will show that the derivative of the above lower bound is locally negative. Consider the derivative below:

$$
\begin{aligned}
&\frac{\begin{aligned}
&-(0.65)\,(4.67445)\,m^3+(7)\,(0.325)\,(1.55815)\,m^3-(0.02)\,(4.67445)\,m^2-(0.65)\,(0.7097)\,m^2 \\
&+(7)\,(0.325)\,(0.35485)\,m^2-(0.02)\,(0.7097)\,m+(0.65)\,(0.05754)\,m-(7)\,(0.325)\,(0.05754)\,m \\
&+(0.02)\,(0.05754)+(7)\,(0.325)\,(0.00355)+(0.65)\,(0.22392)\,m\sqrt{0.65m+0.02} \\
&-(0.11196)\,(0.02)\,\sqrt{0.65m+0.02}-(0.65)\,(0.07785)\,\sqrt{0.65m+0.02}
\end{aligned}}{(0.65m+0.02)^{9/2}}
\end{aligned}
$$

We will show that the numerator is locally negative. We simplify the numerator, shown below.

$$
0.50640m^3+0.25249m^2-0.10769m+0.00923+0.14555m\sqrt{0.65m+0.02}-0.05285\sqrt{0.65m+0.02}
$$

Note the derivative of the above below.

$$
1.5192m^2+0.50498m-0.10769+\frac{(3)\,(0.14555)\,(0.325)\,m+0.14555\,(0.02)-0.05285\,(0.325)}{\sqrt{0.65m+0.02}}
$$

Since $1.5192m^2+0.50498m-0.10769$ can be lower bounded by $1.5192(0.15)^2+0.50498(0.15)-0.10769$ (which is positive), and $(3)\,(0.14555)\,(0.325)\,m+0.14555\,(0.02)-0.05285\,(0.325)$ can be lower bounded by $(3)\,(0.14555)\,(0.325)\,(0.15)+0.14555\,(0.02)-0.05285\,(0.325)$ (which is positive), the derivative is locally positive.

So, we can upper bound the numerator by $0.50640(0.25)^3+0.25249(0.25)^2-0.10769(0.25)+0.00923+0.14555(0.25)\sqrt{0.65(0.25)+0.02}-0.05285\sqrt{0.65(0.25)+0.02}$, which is negative.

Then, the term

$$9.43934\,(0.15) + 12.87889 - \frac{1.55815m^3}{(0.65m+0.02)^{7/2}} - \frac{0.35485m^2}{(0.65m+0.02)^{7/2}} + \frac{0.05754m}{(0.65m+0.02)^{7/2}} - \frac{0.00355}{(0.65m+0.02)^{7/2}} - \frac{0.11196m}{(0.65m+0.02)^3} + \frac{0.02595}{(0.65m+0.02)^3}$$

can be lower bounded by evaluating at $m = 0.25$, which is positive.

It follows then that the term

$$\begin{aligned} &-10.8 \\ &+\left(4.71967m^2 + 12.87889m + 9.80442 \right. \\ &+ \frac{-4.79429m^3 + 0.20673m^2 - 0.05054m + 0.00156}{(0.65m+0.02)^{5/2}} \\ &\left. + \frac{-2.98397m^2 - 0.01139m - 0.02014}{(0.65m+0.02)^2}\right) \\ &\cdot \exp\left(2.41263m + 0.96121\,(0.25)^2 + 2.272\sqrt{0.65m+0.02}\right) \end{aligned}$$

can be further upper bounded by the following

$$\begin{aligned} &-10.8 \\ &+\left(4.71967m^2 + 12.87889m + 9.80442 \right. \\ &+ \frac{-4.79429m^3 + 0.20673m^2 - 0.05054m + 0.00156}{(0.65m+0.02)^{5/2}} \\ &\left. + \frac{-2.98397m^2 - 0.01139m - 0.02014}{(0.65m+0.02)^2}\right) \\ &\cdot \exp\left(2.41263(0.25) + 0.96121\,(0.25)^2 + 2.272\sqrt{0.65(0.25)+0.02}\right). \end{aligned}$$

Since the derivative of the above is positive, then we can upper bound it by evaluating at $m = 0.25$, which is negative.

This completes the proof. □

Proof of Lemma B.45. We proceed with the proof by upper bounding Equation 23 with the following

$$\begin{aligned} &-1.8m^3 - 1.3564m + 0.02 - m\sqrt{0.85m+0.02} \\ &+\sqrt{0.85m+0.02}\left(\sqrt{0.85m+0.02} - x\right) \\ &\exp\left(\frac{0.45}{0.3}\,(0.25)\,x^2 + \frac{0.5516}{0.3}x + \frac{0.45}{0.3}x^2\sqrt{0.85\,(0.25)+0.02} + \frac{0.5516}{0.3}\sqrt{0.85x+0.02}\right), \end{aligned}$$

which we simplify to the following by rounding up after five digits of precision

$$\begin{aligned} &-1.8m^3 - 1.3564m + 0.02 - m\sqrt{0.85m+0.02} \\ &+\sqrt{0.85m+0.02}\left(\sqrt{0.85m+0.02} - m\right)\exp\left(1.0983m^2 + \frac{0.5516}{0.3}m + \frac{0.5516}{0.3}\sqrt{0.85m+0.02}\right). \end{aligned}$$

We will show that the above upper bound is convex for $m \in [0.15, 0.25]$.

We calculate the second derivative of the upper bound below, rounding several terms down after five digits of precision.

$$
\begin{aligned}
&-10.8m + \frac{-0.541875m - 0.017}{(0.85m + 0.02)^{3/2}} \\
&\\
&+ \Bigg(4.10129m^3 + 5.24599m^2 + 7.19967m + 2.45584 \\
&+ \frac{-3.48611m^5 - 6.00015m^4 - 6.58697m^3 - 2.69105m^2 + 0.23158m + 0.00259}{(0.85m + 0.02)^{3/2}} \\
&+ \frac{-1.45903m^3 - 1.25561m^2 - 0.50603m - 0.00342}{(0.85m + 0.02)}\Bigg) \\
&\cdot \exp\left(1.0983m^2 + \frac{0.5516}{0.3}m + \frac{0.5516}{0.3}\sqrt{0.85m + 0.02}\right)
\end{aligned}
$$

We further lower bound the above by the following

$$
\begin{aligned}
&-10.8m + \frac{-0.541875m - 0.017}{(0.85m + 0.02)^{3/2}} \\
&\\
&\Bigg(4.10129\,(0.15)\,m^2 + 5.24599m^2 + 7.19967m + 2.45584 \\
&+ \frac{-6.00015\,(0.25)^3\,m - 6.58697\,(0.25)\,m^2 - 2.69105m^2 + 0.23158m}{(0.85m + 0.02)^{3/2}} + \frac{0.00259 - 3.48611\,(0.25)^5}{(0.85\,(0.15) + 0.02)^{3/2}} \\
&- \frac{1.45903\,(0.25)^2\,m}{(0.85m + 0.02)} - \frac{1.25561\,(0.25)\,m}{(0.85m + 0.02)} - \frac{0.50603m}{(0.85m + 0.02)} - \frac{0.00342}{(0.85\,(0.15) + 0.02)}\Bigg) \\
&\cdot \exp\left(1.0983m^2 + \frac{0.5516}{0.3}m + \frac{0.5516}{0.3}\sqrt{0.85m + 0.02}\right),
\end{aligned}
$$

which we further simplify to the following, rounding several terms down after five digits of precision

$$
\begin{aligned}
&-10.8m + \frac{-0.541875m - 0.017}{(0.85m + 0.02)^{3/2}} \\
&\\
&\left(5.86118m^2 + 7.19967m + 2.41827 + \frac{-4.33780m^2 + 0.13782m}{(0.85m + 0.02)^{3/2}} + \frac{-0.91113m}{(0.85m + 0.02)}\right) \\
&\cdot \exp\left(1.0983m^2 + \frac{0.5516}{0.3}m + \frac{0.5516}{0.3}\sqrt{0.85m + 0.02}\right).
\end{aligned}
$$

To further lower bound the above, we will first show that the exponential coefficient is positive.

We do so by first showing that the exponential coefficient is convex for $m \in [0.15, 0.25]$.

We calculate the second derivative of coefficient, simplifying the terms by rounding down after five digits of precision below.

$$
11.72236 + \frac{0.78350m^2 + 0.22216m - 0.01051}{(0.85m + 0.02)^{\frac{7}{2}}} + \frac{0.03097}{(0.85m + 0.02)^3}
$$

Observe that $0.78350m^2 + 0.22216m - 0.01051$ is positive by the following lower bound $0.78350\,(0.15)^2 + 0.22216\,(0.15) - 0.01051$, which is positive.

So, the exponential coefficient is convex locally. We prove the coefficient is positive by linearizing at $m = 0.16$. Note the derivative of the coefficient below

$$11.72236m + 7.19967$$

$$+ \frac{-(0.425)(4.33780)m^2 - (0.425)(0.13782)m - (0.02)(8.6756)m + (0.02)(0.13782)}{(0.85m + 0.02)^{5/2}}$$

$$+ \frac{-(0.02)(0.91113)}{(0.85m + 0.02)^2},$$

which is negative at $m = 0.16$. So, we evaluate the $m = 0.16$ linearization at $m = 0.25$, which is positive.

Since the exponential coefficient is positive, we can further lower bound our previous lower bound for the second derivative by the following:

$$-10.8m + \frac{-0.541875m - 0.017}{(0.85m + 0.02)^{3/2}}$$

$$\left(5.86118m^2 + 7.19967m + 2.41827 + \frac{-4.33780m^2 + 0.13782m}{(0.85m + 0.02)^{3/2}} + \frac{-0.91113m}{(0.85m + 0.02)}\right)$$
$$\cdot \exp\left(1.0983m^2 + \frac{0.5516}{0.3}(0.15) + \frac{0.5516}{0.3}\sqrt{0.85(0.15) + 0.02}\right).$$

We will show that the above is locally convex. Consider the second derivative of the above lower bound below, simplified by rounding down after five digits of precision.

$$\frac{-0.29363m + 0.02764}{(0.85m + 0.02)^{7/2}}$$

$$+ \left(11.72236 + \frac{0.78350m^2 + 0.22216m - 0.01051}{(0.85m + 0.02)^{7/2}} + \frac{0.03097}{(0.85m + 0.02)^3}\right)$$
$$\cdot \exp\left(1.0983m^2 + \frac{0.5516}{0.3}(0.15) + \frac{0.5516}{0.3}\sqrt{0.85(0.15) + 0.02}\right)$$

We lower bound the above by the following

$$\frac{-0.29363(0.25) + 0.02764}{(0.85(0.15) + 0.02)^{7/2}}$$

$$+ \left(11.72236 + \frac{0.78350(0.15)^2 + 0.22216(0.15) - 0.01051}{(0.85(0.25) + 0.02)^{7/2}} + \frac{0.03097}{(0.85(0.25) + 0.02)^3}\right)$$
$$\cdot \exp\left(1.0983m^2 + \frac{0.5516}{0.3}(0.15) + \frac{0.5516}{0.3}\sqrt{0.85(0.15) + 0.02}\right),$$

which is positive.

So, the expression

$$-10.8m + \frac{-0.541875m - 0.017}{(0.85m + 0.02)^{3/2}}$$

$$\left(5.86118m^2 + 7.19967m + 2.41827 + \frac{-4.33780m^2 + 0.13782m}{(0.85m + 0.02)^{3/2}} + \frac{-0.91113m}{(0.85m + 0.02)}\right)$$
$$\cdot \exp\left(1.0983m^2 + \frac{0.5516}{0.3}(0.15) + \frac{0.5516}{0.3}\sqrt{0.85(0.15) + 0.02}\right).$$

is locally convex.

We will show that the linearization at $m = 0.24$ is locally positive. We first note the derivative of the above expression below

$$-10.8 + \frac{-0.541875}{(0.85m + 0.02)^{3/2}} + \frac{-3\,(0.425)\,(-0.541875m - 0.017)}{(0.85m + 0.02)^{5/2}}$$

$$+\left(5.86118\,(2)\,m + 7.19967 + \frac{-4.33780\,(2)\,m + 0.13782}{(0.85m + 0.02)^{3/2}} + \frac{-3\,(0.425)\left(-4.33780m^2 + 0.13782m\right)}{(0.85m + 0.02)^{5/2}}\right.$$
$$\left. + \frac{-0.91113}{(0.85m + 0.02)} + \frac{(0.85)\,0.91113m}{(0.85m + 0.02)^2}\right)$$
$$\cdot \exp\left(1.0983m^2 + \frac{0.5516}{0.3}(0.15) + \frac{0.5516}{0.3}\sqrt{0.85(0.15) + 0.02}\right),$$

which is positive at $m = 0.24$.

So, we evaluate the $m = 0.24$ linearization $m = 0.15$, which is positive.

So, our expression

$$-1.8m^3 - 1.3564m + 0.02 - m\sqrt{0.85m + 0.02}$$
$$+\sqrt{0.85m + 0.02}\left(\sqrt{0.85m + 0.02} - m\right)\exp\left(1.0983m^2 + \frac{0.5516}{0.3}m + \frac{0.5516}{0.3}\sqrt{0.85m + 0.02}\right).$$

is locally convex.

By evaluating at $m = 0.15, 0.25$, we see that both are negative. This completes the proof.

□

Proof of Lemma B.46. We proceed with the proof by showing that the second derivative of Equation 24 with respect to η is positive for $\eta \in [0.2, 0.3], m \in [0.15, 0.25]$.

We show the second partial derivative with respect to η below.

$$\frac{m^3}{(2m\eta + 0.25m + 0.02)^{3/2}}$$

$$+ \left(\frac{m^3}{(2m\eta + 0.25m + 0.02)^{3/2}} \right) \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075 \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) \right)$$

$$+ 2\sigma_1 m \left(2 - \frac{m}{\sqrt{2m\eta + 0.25m + 0.02}} \right) \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075 \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) \right)$$

$$+ \sigma_2 \sqrt{2m\eta + 0.25m + 0.02} \left(\sqrt{2m\eta + 0.25m + 0.02} - m \right)$$
$$\cdot \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075 \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) \right)$$

$$+ \sigma_1^2 \sqrt{2m\eta + 0.25m + 0.02} \left(\sqrt{2m\eta + 0.25m + 0.02} - m \right)$$
$$\cdot \exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075 \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) \right),$$

where

$$\sigma_1 = -\left(\frac{0.47}{\eta^2} + \frac{0.24}{\eta^2} \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) + \frac{m\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075 \right)}{\sqrt{2m\eta + 0.25m + 0.02}}$$

and

$$\sigma_2 = 2\left(\frac{0.47m^2}{\eta^3} + \frac{0.24}{\eta^3} \right) \left(m + \sqrt{2m\eta + 0.25m + 0.02} \right) - \frac{2m\left(\frac{0.47}{\eta^2} + \frac{0.24}{\eta^2} \right)}{\sqrt{2m\eta + 0.25m + 0.02}} - \frac{m^2\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075 \right)}{(2m\eta + 0.25m + 0.02)^{3/2}}.$$

From the expression above, we'll show the below is positive.

$$+ \left(\frac{m^3}{(2m\eta + 0.25m + 0.02)^{3/2}} \right)$$

$$+ 2m\sigma_1 \left(2 - \frac{m}{\sqrt{2m\eta + 0.25m + 0.02}} \right)$$

$$+ \sigma_2 \sqrt{2m\eta + 0.25m + 0.02} \left(\sqrt{2m\eta + 0.25m + 0.02} - m \right)$$

After scaling by η^3, we simplify the above expression to the following:

$$\frac{m^3\eta^3}{(2m\eta + 0.25m + 0.02)^{3/2}} - \frac{m^3\eta^2\left(0.47m^2 + 0.24 + 1.075\eta\right)}{2m\eta + 0.25m + 0.02}$$

$$\frac{\begin{aligned} &-0.235m^5 + 0.020776m^2 + 0.0096m\eta + 0.0048m + 0.000192 + 3.225m^2\eta^3 - 0.4612m^3\eta - 0.47m^4\eta^2 - 0.94m^5\eta \\ &\qquad - 0.24m^2\eta^2 + 0.03995m^4 + 0.12m^2\eta + 0.235m^4\eta - 0.1106m^3 \end{aligned}}{\sqrt{2m\eta + 0.25m + 0.02}}$$

Next, we will present a lower bound for the above expression. To do so, we must first prove that
$-0.235m^5 + 0.020776m^2 + 0.0096m\eta + 0.0048m + 0.000192 + 3.225m^2\eta^3 - 0.4612m^3\eta - 0.47m^4\eta^2 - 0.94m^5\eta$
$- 0.24m^2\eta^2 + 0.03995m^4 + 0.12m^2\eta + 0.235m^4\eta - 0.1106m^3$

is positive for our parameter range.

To prove this, we start by showing that the first partial derivative of the expression with respect to η is positive. We show the first partial derivative below.

$$0.0096m + 9.675m^2\eta^2 - 0.4612m^3 - 0.94m^4\eta - 0.94m^5 - 0.48m^2\eta$$

We take another partial derivative with respect to η to obtain the following expression $\left(9.675\left(2\right)\eta - 0.94m^2 - 0.48\right)m^2$, where the inner expression $\left(9.675\left(2\right)\eta - 0.94m^2 - 0.48\right)$ can be lower bounded by $\left(9.675\left(2\right)\left(0.2\right) - 0.48 - 0.94\left(0.25\right)^2\right)$, which is positive.

So, we can lower bound the first partial derivative with respect to η by evaluating at $\eta = 0.2$ to get the following expression

$$\left(0.0096 + 0.291m - 0.4612m^2 - 0.188m^3 - 0.94m^4\right)m$$

The inner expression $0.0096 + 0.291m - 0.4612m^2 - 0.188m^3 - 0.94m^4$ can be lower bonded by $0.0096 + 0.291\left(0.15\right) - 0.4612\left(0.25\right)^2 - 0.188\left(0.25\right)^3 - 0.94\left(0.25\right)^4$, which is positive.

So, we can lower bound our original expression

$$-0.235m^5 + 0.020776m^2 + 0.0096m\eta + 0.0048m + 0.000192 + 3.225m^2\eta^3 - 0.4612m^3\eta - 0.47m^4\eta^2 - 0.94m^5\eta$$
$$-0.24m^2\eta^2 + 0.03995m^4 + 0.12m^2\eta + 0.235m^4\eta - 0.1106m^3$$

by evaluating at $\eta = 0.2$, shown below.

$$-0.235m^5 - 0.11985m^4 - 0.20284m^3 + 0.164176m^2 + 0.00672m + 0.000192$$

We will show that the first partial derivative of our lower bound with respect to m is positive.

Consider the first partial derivative with respect to m below.

$$-0.235\left(5\right)m^4 - 0.11985\left(4\right)m^3 - 0.20284\left(3\right)m^2 + 0.164176\left(2\right)m + 0.00672$$

Since the terms of degree at least 2 have negative coefficients, we know that the first partial derivative is concave in m. So, we evaluate it at $m = 0.15, 0.25$, both of which are positive.

So, we can lower bound our expression,

$$-0.235m^5 - 0.11985m^4 - 0.20284m^3 + 0.164176m^2 + 0.00672m + 0.000192$$

by evaluating at $m = 0.15$, which is positive.

So now, we lower bound the expression

$$+\left(\frac{m^3}{\left(2m\eta + 0.25m + 0.02\right)^{3/2}}\right)$$

$$+2m\left(2 - \frac{m}{\sqrt{2m\eta + 0.25m + 0.02}}\right)$$
$$\cdot\left[-\left(\frac{0.47}{\eta^2} + \frac{0.24}{\eta^2}\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right) + \frac{m\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)}{\sqrt{2m\eta + 0.25m + 0.02}}\right]$$

$$+\sqrt{2m\eta + 0.25m + 0.02}\left(\sqrt{2m\eta + 0.25m + 0.02} - m\right)$$
$$\cdot\left[2\left(\frac{0.47m^2}{\eta^3} + \frac{0.24}{\eta^3}\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right) - \frac{2m\left(\frac{0.47}{\eta^2} + \frac{0.24}{\eta^2}\right)}{\sqrt{2m\eta + 0.25m + 0.02}} - \frac{m^2\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)}{\left(2m\eta + 0.25m + 0.02\right)^{3/2}}\right]$$

by

$$\frac{m^3\left(0.2\right)^3}{\left(2(0.25)\left(0.3\right)+0.25(0.25)+0.02\right)^{\frac{3}{2}}}+\frac{-m^3\left(0.3\right)^2\left(0.47m^2+0.24+1.075\left(0.3\right)\right)}{2(0.15)\left(0.2\right)+0.25(0.15)+0.02}$$

$$+\frac{\begin{array}{c}-0.235m^5+0.020776m^2+0.0096\left(0.2\right)m+0.0048m+0.000192+3.225\left(0.2\right)^2m^2-0.4612\left(0.2\right)m^3\\-0.47\left(0.2\right)^2m^4-0.94\left(0.2\right)m^4-0.24\left(0.2\right)^2m^2+0.03995m^4+0.12\left(0.2\right)m^2+0.235\left(0.2\right)m^4-0.1106m^3\end{array}}{\sqrt{2(0.25)\left(0.3\right)+0.25(0.25)+0.02}}.$$

We simplify the above lower bound to the following

$$\left(-\frac{0.0423}{0.1175}-\frac{0.235}{\sqrt{0.2325}}\right)m^5-\frac{0.11985}{\sqrt{0.2325}}m^4+\left(\frac{\left(0.2\right)^3}{\left(0.2325\right)^{\frac{3}{2}}}-\frac{0.050625}{0.1175}-\frac{0.20284}{\sqrt{0.2325}}\right)m^3$$
$$+\frac{0.164176}{\sqrt{0.2325}}m^2+\frac{0.00672}{\sqrt{0.2325}}m+\frac{0.000192}{\sqrt{0.2325}}.$$

Now we'll show that the first partial derivative of our lower bound with respect to m is positive. Consider the first partial derivative with respect to m below.

$$\left(-\frac{0.0423}{0.1175}-\frac{0.235}{\sqrt{0.2325}}\right)\left(5\right)m^4-\frac{0.11985\left(4\right)}{\sqrt{0.2325}}m^3+\left(\frac{\left(0.2\right)^3}{\left(0.2325\right)^{\frac{3}{2}}}-\frac{0.050625}{0.1175}-\frac{0.20284}{\sqrt{0.2325}}\right)\left(3\right)m^2$$
$$+\frac{0.164176\left(2\right)}{\sqrt{0.2325}}m+\frac{0.00672}{\sqrt{0.2325}}$$

Since the terms of degree at least 2 have negative coefficients, we know the first partial derivative is concave in m. So, we evaluate the first partial derivative at $m=0.15,0.25$, which are both positive.

So, we evaluate our lower bound

$$\left(-\frac{0.0423}{0.1175}-\frac{0.235}{\sqrt{0.2325}}\right)m^5-\frac{0.11985}{\sqrt{0.2325}}m^4+\left(\frac{\left(0.2\right)^3}{\left(0.2325\right)^{\frac{3}{2}}}-\frac{0.050625}{0.1175}-\frac{0.20284}{\sqrt{0.2325}}\right)m^3$$
$$+\frac{0.164176}{\sqrt{0.2325}}m^2+\frac{0.00672}{\sqrt{0.2325}}m+\frac{0.000192}{\sqrt{0.2325}}.$$

at $m=0.15$, which is positive.

This completes the proof.

□

Proof of Lemma B.47. We proceed with the proof by first deriving the following upper bound for Equation 25 (withing our parameter range).

$$-1.88m^3-1.17m+0.02-m\sqrt{0.65m+0.02}$$

$$+\sqrt{0.65m+0.02}\left(\sqrt{0.65m+0.02}-m\right)$$
$$\cdot\exp\left(2.35\left(0.25\right)^2m+2.275m+2.35m^2\sqrt{0.65\left(0.25\right)+0.02}+2.275\sqrt{0.65m+0.02}\right)$$

For convenience, we simplify the upper bound by rounding up coefficients after five digits of precision. The simplified expression is shown below.

$$-1.88m^3-1.17m+0.02-m\sqrt{0.65m+0.02}$$
$$+\sqrt{0.65m+0.02}\left(\sqrt{0.65m+0.02}-m\right)\exp\left(1.00393m^2+2.42188m+2.275\sqrt{0.65m+0.02}\right)$$

We will show that our upper bound is decreasing (within our parameter range). To do so, we will show that the derivative of our upper bound is negative. We show the derivative of our upper bound below.

$$
\begin{aligned}
&-5.64m^2 - 1.17 - \frac{0.975m + 0.02}{\sqrt{0.65m + 0.02}} \\
&\\
&+ \Bigg((0.65)\,(2.00786)\,m^2 + ((0.02)\,(2.00786) + (0.65)\,(2.42188) - (2.275)\,(0.325))\,m + (0.65 + (0.02)\,(2.42188)) \\
&- \frac{0.975m + 0.02}{\sqrt{0.65m + 0.02}} - 2.00786m^2\sqrt{0.65m + 0.02} - 2.42188m\sqrt{0.65m + 0.02} + (2.275)\,(0.325)\,\sqrt{0.65m + 0.02} \Bigg) \\
&\cdot \exp\left(1.00393m^2 + 2.42188m + 2.275\sqrt{0.65m + 0.02}\right)
\end{aligned}
$$

We can upper bound the derivative above by the following expression, which involves rounding up come coefficients after five digits of precision.

$$
\begin{aligned}
&-5.64m^2 - 1.17 - \frac{0.975m + 0.02}{\sqrt{0.65\,(0.25) + 0.02}} \\
&\\
&+ \Bigg(1.30511m^2 + 0.87501m + 0.69844 - \frac{1.30510m^3}{\sqrt{0.65m + 0.02}} - \frac{1.53406x^2}{\sqrt{0.65m + 0.02}} - \frac{0.54284m}{\sqrt{0.65m + 0.02}} \\
&- \frac{0.00521}{\sqrt{0.65\,(0.25) + 0.02}} \Bigg) \\
&\cdot \exp\left(1.00393m^2 + 2.42188m + 2.275\sqrt{0.65m + 0.02}\right)
\end{aligned}
$$

We can further upper bound this expression by showing that the exponential coefficient is positive for $m \in [0.15, 0.25]$.

Since the exponential coefficient is locally positive by lemma B.54, we can further upper bound the above by the following.

$$
\begin{aligned}
&-5.64m^2 - 1.17 - \frac{0.975m + 0.02}{\sqrt{0.65\,(0.25) + 0.02}} \\
&\\
&+ \Bigg(1.30511m^2 + 0.87501m + 0.69844 - \frac{1.30510m^3}{\sqrt{0.65m + 0.02}} - \frac{1.53406x^2}{\sqrt{0.65m + 0.02}} - \frac{0.54284m}{\sqrt{0.65m + 0.02}} \\
&- \frac{0.00521}{\sqrt{0.65\,(0.25) + 0.02}} \Bigg) \\
&\cdot \exp\left(1.00393(0.25)^2 + 2.42188m + 2.275\sqrt{0.65m + 0.02}\right).
\end{aligned}
$$

Since the above is concave in m by lemma B.53, we show it is negative by upper bounding it by linearizing at $m = 0.18$. We first calculate the derivative of our upper bound below.

$$
\begin{aligned}
&-11.28m - \frac{0.975}{\sqrt{0.1825}} \\
&\\
&+ \gamma_1 \exp\left(1.00393\,(0.25)^2 + 2.42188m + 2.275\sqrt{0.65m + 0.02}\right),
\end{aligned}
$$

where

$$\begin{aligned}\gamma_1 =& (2.42188)(1.30511)m^2 + 1.30511(2)m + (2.42188)(0.87501)m + 0.87501 + (2.42188)(0.69844)\\ &- \frac{(0.00521)(2.42188)}{\sqrt{0.1825}} - \frac{(2.42188)(1.30510)m^3}{\sqrt{0.65m+0.02}} - \frac{1.30510(3)m^2}{\sqrt{0.65m+0.02}} + \frac{(2.275)(0.325)(1.30511)m^2}{\sqrt{0.65m+0.02}}\\ &- \frac{(2.42188)(1.53406)m^2}{\sqrt{0.65m+0.02}} - \frac{1.53406(2)m}{\sqrt{0.65m+0.02}} - \frac{(2.42188)(0.54284)m}{\sqrt{0.65m+0.02}} + \frac{(2.275)(0.325)(0.87501)m}{\sqrt{0.65x+0.02}}\\ &- \frac{0.54284}{\sqrt{0.65x+0.02}} + \frac{(2.275)(0.325)(0.69844)}{\sqrt{0.65m+0.02}} - \frac{(2.275)(0.325)(0.00521)}{\sqrt{0.1825}\sqrt{0.65m+0.02}} + \frac{1.30510(0.325)m^3}{(0.65m+0.02)^{3/2}}\\ &+ \frac{1.53406(0.325)m^2}{(0.65m+0.02)^{3/2}} + \frac{0.54284(0.325)m}{(0.65m+0.02)^{3/2}} - \frac{2.275(0.325)(1.30510)m^3}{0.65m+0.02} - \frac{(2.275)(0.325)(1.53406)m^2}{0.65m+0.02}\\ &- \frac{(2.275)(0.325)(0.54284)m}{0.65m+0.02}.\end{aligned}$$

Evaluating the above $m = 0.18$ is negative, so we check the linearization at $m = 0.15$, which is negative.

This tells us that our upper bound for Equation 25 (shown below) is decreasing for $m \in [0.15, 0.25]$.

$$\begin{aligned}&-1.88m^3 - 1.17m + 0.02 - m\sqrt{0.65m+0.02}\\ &+ \sqrt{0.65m+0.02}\left(\sqrt{0.65m+0.02} - m\right)\exp\left(1.00393m^2 + 2.42188m + 2.275\sqrt{0.65m+0.02}\right)\end{aligned}$$

Evaluating the above at $m = 0.15$ is negative, which completes the proof.

□

Proof of Lemma B.53. We show Equation 31 is concave by showing the second derivative is negative. Consider the second derivative below (rounded up terms after five digits of precision):

$$\begin{aligned}&-11.28\\ &+\left(7.65513m^2 + 17.77565m + 10.87373 + \frac{-7.65506m^3 - 23.28874m^2 - 18.88228m - 1.94590}{\sqrt{0.65m+0.02}}\right.\\ &+ \frac{1.34106m^3 + 3.80765m^2 + 2.34181m + 0.18795}{(0.65m+0.02)^{3/2}} + \frac{-0.41355m^3 - 0.48610m^2 - 0.17201m}{(0.65m+0.02)^{5/2}}\\ &\left.+ \frac{-4.67402m^3 - 10.57029m^2 - 6.00273m - 0.42757}{0.65m+0.02} + \frac{0.94084m^3 + 1.10589m^2 + 0.39133m}{(0.65m+0.02)^2}\right)\\ &\cdot \exp\left(1.00393(0.25)^2 + 2.42188m + 2.275\sqrt{0.65m+0.02}\right)\end{aligned}$$

To show that this is negative, we will show that the exponential coefficient is negative. To do so, we rewrite the coefficient below (rounding up some coefficients after five digits of precision).

$$\frac{\begin{aligned}&-3.23425m^5 - 9.16683m^4 - 6.49808m^3 - 0.21016m^2 - 0.06114m + 0.00299 + 0.19619m^4\sqrt{0.65m+0.02}\\ &+ 1.68593m^3\sqrt{0.65m+0.02} + 2.05209m^2\sqrt{0.65m+0.02} + 0.28319m\sqrt{0.65m+0.02} - 0.00420\sqrt{0.65m+0.02}\end{aligned}}{(0.65m+0.02)^{5/2}}$$

We will show that the numerator of the above expression is negative. Consider the following upper bound for the numerator:

$$\begin{aligned}&-3.23425m^5 - 9.16683m^4 - 6.49808m^3 - 0.21016m^2 - 0.06114m + 0.00299 + 0.19619m^4\sqrt{0.65(0.25)+0.02}\\ &+ 1.68593m^3\sqrt{0.65(0.25)+0.02} + 2.05209m^2\sqrt{0.65(0.25)+0.02} + 0.28319m\sqrt{0.65m+0.02}\\ &- 0.00420\sqrt{0.65m+0.02}\end{aligned}$$

We will show that the derivative of the upper bound is negative. Consider the derivative below, which we upper bound.

$$-3.23425\,(5)\,m^4+(4)\left(-9.16683+0.19619\sqrt{0.1825}\right)m^3+(3)\left(-6.49808+1.68593\sqrt{0.1825}\right)m^2$$
$$+(2)\left(-0.21016+2.05209\sqrt{0.1825}\right)m-0.06114+0.28319\sqrt{0.65m+0.02}+\frac{0.28319\,(0.325)\,m-0.00420\,(0.325)}{\sqrt{0.65m+0.02}}$$

$$\leq -3.23425\,(5)\,m^4+(4)\left(-9.16683+0.19619\sqrt{0.1825}\right)m^3$$
$$+\left((3)\left(-6.49808+1.68593\sqrt{0.1825}\right)(0.15)+(2)\left(-0.21016+2.05209\sqrt{0.1825}\right)\right)m$$
$$-0.06114+0.28319\sqrt{0.65\,(0.25)+0.02}+\frac{0.28319\,(0.325)\,m-0.00420\,(0.325)}{\sqrt{0.65\,(0.15)+0.02}}$$

Since the terms of degree at least one are negative, we can upper bound the above by $m=0.15$, which is negative.

So, we can upper bound

$$-3.23425m^5-9.16683m^4-6.49808m^3-0.21016m^2-0.06114m+0.00299+0.19619m^4\sqrt{0.65\,(0.25)+0.02}$$
$$+1.68593m^3\sqrt{0.65\,(0.25)+0.02}+2.05209m^2\sqrt{0.65\,(0.25)+0.02}+0.28319m\sqrt{0.65m+0.02}$$
$$-0.00420\sqrt{0.65m+0.02}$$

by evaluating at $m=0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.54. To show that Equation 32 is positive, we will first show that the derivative of the exponential coefficient is negative. Consider the derivative below.

$$2.61022m+0.87501$$
$$+\frac{-(0.325)\,(6.5255)\,m^3-((0.02)\,(3.9153)+(4.60218)\,(0.325))\,m^2}{(0.65m+0.02)^{3/2}}$$
$$\frac{-((0.02)\,(3.06812)+(0.54284)\,(0.325))\,m-(0.02)\,(0.54284)}{(0.65m+0.02)^{3/2}}$$

We can upper bound the above derivative by the following (rounding up coefficients after five digits of precision):

$$2.61022m+0.87501+\frac{-1.57401\,(0.15)\,m}{(0.65m+0.02)^{3/2}}+\frac{-0.23778m}{(0.65\,(0.25)+0.02)^{3/2}}+\frac{-0.01085-2.12078\,(0.15)^3}{(0.65\,(0.25)+0.02)^{3/2}}$$

$$\leq -0.43964m+0.64404+\frac{-1.57401\,(0.15)\,m}{(0.65m+0.02)^{3/2}}$$

We take another derivative of this upper bound and show that it is positive below.

$$\left(-0.43964+\frac{(0.325)\,(1.57401)\,(0.15)\,m-(1.57401)\,(0.15)\,(0.02)}{(0.65m+0.02)^{\frac{5}{2}}}\right)$$

$$\geq\left(-0.43964+\frac{(0.325)\,(1.57401)\,(0.15)\,(0.15)-(1.57401)\,(0.15)\,(0.02)}{(0.65\,(0.25)+0.02)^{\frac{5}{2}}}\right)$$

$$>0$$

So, we can upper bound the term

$$-0.43964m + 0.64404 + \frac{-1.57401\,(0.15)\,m}{(0.65m + 0.02)^{3/2}}$$

by evaluating it at $m = 0.25$, which is negative.

So, we can lower bound the term

$$-5.64m^2 - 1.17 - \frac{0.975m + 0.02}{\sqrt{0.65\,(0.25) + 0.02}}$$

$$+ \left(1.30511m^2 + 0.87501m + 0.69844 - \frac{1.30510m^3}{\sqrt{0.65m + 0.02}} - \frac{1.53406x^2}{\sqrt{0.65m + 0.02}} - \frac{0.54284m}{\sqrt{0.65m + 0.02}} - \frac{0.00521}{\sqrt{0.65\,(0.25) + 0.02}}\right)$$
$$\cdot \exp\left(1.00393m^2 + 2.42188m + 2.275\sqrt{0.65m + 0.02}\right)$$

by evaluating it at $m = 0.25$, which is positive.

□

Proof of Lemma B.48. We begin by upper bounding Equation 26 (within our parameter range) by the following.

$$-1.88m^3 - 1.4m + 0.02 - m\sqrt{0.85m + 0.02}$$

$$+ \sqrt{0.85m + 0.02}\left(\sqrt{0.85m + 0.02} - m\right)$$
$$\cdot \exp\left(\frac{0.47}{0.3}\,(0.25)^2\,m + \frac{0.5625}{0.3}m + \frac{0.47}{0.3}m^2\sqrt{0.85\,(0.25) + 0.02} + \frac{0.5625}{0.3}\sqrt{0.85x + 0.02}\right)$$

We simplify the above expression below by rounding up the coefficients after five digits of precision.

$$-1.88m^3 - 1.4m + 0.02 - m\sqrt{0.85m + 0.02}$$
$$+ \sqrt{0.85m + 0.02}\left(\sqrt{0.85m + 0.02} - m\right)\exp\left(\frac{0.22663}{0.3}m^2 + \frac{0.591875}{0.3}m + \frac{0.5625}{0.3}\sqrt{0.85m + 0.02}\right)$$

We will show that the above upper bound is convex for $m \in [0.15, 0.25]$ by showing the second derivative is positive.

For simplicity, we show the second derivative of our upper bound below after rounding down coefficients after 5 digits of precision.

$$-11.28m + \frac{-(0.02)\,(0.85) - (1.275)\,(0.425)\,m}{(0.85m + 0.02)^{\frac{3}{2}}}$$

$$+ \left(-\frac{1.64928m^5}{(0.85m + 0.02)^{\frac{3}{2}}} - \frac{4.38490m^4}{(0.85m + 0.02)^{\frac{3}{2}}} - \frac{5.64255m^3}{(0.85m + 0.02)^{\frac{3}{2}}} - \frac{2.77694m^2}{(0.85m + 0.02)^{\frac{3}{2}}} + \frac{0.24584m}{(0.85m + 0.02)^{\frac{3}{2}}} + \frac{0.00299}{(0.85m + 0.02)^{\frac{3}{2}}} + \frac{1.64926m^4}{0.85m + 0.02} + \frac{2.33813m^3}{0.85m + 0.02} + \frac{3.56982m^2}{0.85m + 0.02} + \frac{1.87183m}{0.85m + 0.02} + \frac{0.05006}{0.85m + 0.02}\right)$$
$$\cdot \exp\left(\frac{0.22663}{0.3}m^2 + \frac{0.591875}{0.3}m + \frac{0.5625}{0.3}\sqrt{0.85m + 0.02}\right)$$

In order to further lower bound the above expression, we will first show that the expression

$$-\frac{1.64928m^5}{(0.85m+0.02)^{\frac{3}{2}}}-\frac{4.38490m^4}{(0.85m+0.02)^{\frac{3}{2}}}-\frac{5.64255m^3}{(0.85m+0.02)^{\frac{3}{2}}}-\frac{2.77694m^2}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{0.24584m}{(0.85m+0.02)^{\frac{3}{2}}}$$
$$+\frac{0.00299}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{1.64926m^4}{0.85m+0.02}+\frac{2.33813m^3}{0.85m+0.02}+\frac{3.56982m^2}{0.85m+0.02}+\frac{1.87183m}{0.85m+0.02}+\frac{0.05006}{0.85m+0.02}$$

is positive for $m \in [0.15, 0.25]$.

First, we write the expression in the following manner:

$$\frac{\begin{array}{l}-1.64928m^5-4.38490m^4-5.64255m^3-2.77694m^2+0.24584m+0.00299+1.64926m^4\sqrt{0.85m+0.02}\\ +2.33813m^3\sqrt{0.85m+0.02}+3.56982m^2\sqrt{0.85m+0.02}+1.87183m\sqrt{0.85m+0.02}+0.05006\sqrt{0.85m+0.02}\end{array}}{(0.85m+0.02)^{\frac{3}{2}}}$$

We will show that the numerator of the above expression is positive. To do so, we create the following linear lower bound for the numerator.

$$\left(0.24584+1.87183\sqrt{0.85\,(0.15)+0.02}+\left(-2.77694+3.56982\sqrt{0.85\,(0.15)+0.02}\right.\right.$$
$$\left.\left.+\left(-5.64255+2.33813\sqrt{0.85\,(0.15)+0.02}\right)(0.25)\right)(0.25)\right)m$$

$$+\left(0.05006\sqrt{0.85\,(0.15)+0.02}-1.64928\,(0.25)^5+0.00299+\left(-4.38490+1.64926\sqrt{0.85\,(0.15)+0.02}\right)(0.25)^4\right)$$

Note that the coefficient for the linear term is positive, so we can lower bound the expression, by evaluating at $m = 0.15$, which is positive.

So, we can lower bound our simplified second derivative expression by the following.

$$-11.28m+\frac{-(0.02)\,(0.85)-(1.275)\,(0.425)\,m}{(0.85m+0.02)^{\frac{3}{2}}}$$

$$+\left(-\frac{1.64928m^5}{(0.85m+0.02)^{\frac{3}{2}}}-\frac{4.38490m^4}{(0.85m+0.02)^{\frac{3}{2}}}-\frac{5.64255m^3}{(0.85m+0.02)^{\frac{3}{2}}}-\frac{2.77694m^2}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{0.24584m}{(0.85m+0.02)^{\frac{3}{2}}}\right.$$
$$\left.+\frac{0.00299}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{1.64926m^4}{0.85m+0.02}+\frac{2.33813m^3}{0.85m+0.02}+\frac{3.56982m^2}{0.85m+0.02}+\frac{1.87183m}{0.85m+0.02}+\frac{0.05006}{0.85m+0.02}\right)$$
$$\cdot\exp\left(\frac{0.22663}{0.3}(0.15)^2+\frac{0.591875}{0.3}m+\frac{0.5625}{0.3}\sqrt{0.85(0.15)+0.02}\right)$$

Since the derivative of the above is negative be lemma B.55, we can lower bound the above by evaluating it at $m = 0.25$, which is positive.

So, our upper bound for Equation 26 is convex. By evaluating it at $m = 0.15, 0.25$, we see that both terms are negative. This completes the proof.

□

Proof of Lemma B.55. We proceed by showing that the derivative of Equation 33 is negative. Consider the derivative below.

$$
\begin{aligned}
&-11.28+\frac{(0.425)^2(1.275)m+(1.275)(0.425)(0.02)}{(0.85m+0.02)^{\frac{5}{2}}}\\
&+t_1\exp\left(\frac{0.22663}{0.3}(0.15)^2+\frac{0.591875}{0.3}m+\frac{0.5625}{0.3}\sqrt{0.85(0.15)+0.02}\right)\\
&+t_2\left(\frac{0.591875}{0.3}\right)\exp\left(\frac{0.22663}{0.3}(0.15)^2+\frac{0.591875}{0.3}m+\frac{0.5625}{0.3}\sqrt{0.85(0.15)+0.02}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
t_1=&\frac{-1.64928(5)m^4}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{3(0.425)(1.64928)m^5}{(0.85m+0.02)^{\frac{5}{2}}}+\frac{-4.38490(4)m^3}{(0.8mx+0.02)^{\frac{3}{2}}}\\
&+\frac{3(0.425)(4.38490)m^4}{(0.85m+0.02)^{\frac{5}{2}}}+\frac{-5.64255(3)m^2}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{3(0.425)(5.64255)m^3}{(0.85m+0.02)^{\frac{5}{2}}}+\frac{-2.77694(2)m}{(0.85m+0.02)^{\frac{3}{2}}}\\
&+\frac{3(0.425)(2.77694)m^2}{(0.85m+0.02)^{\frac{5}{2}}}+\frac{0.24584}{(0.85x+0.02)^{\frac{3}{2}}}+\frac{-3(0.425)(0.24584)m}{(0.85m+0.02)^{\frac{5}{2}}}+\frac{-3(0.425)(0.00299)}{(0.85m+0.02)^{\frac{5}{2}}}\\
&+\frac{1.64926(4)m^3}{0.85m+0.02}+\frac{-(0.85)(1.64926)m^4}{(0.85m+0.02)^2}+\frac{2.33813(3)m^2}{0.85m+0.02}+\frac{-(0.85)(2.33813)m^3}{(0.85m+0.02)^2}+\frac{3.56982(2)m}{0.85m+0.02}\\
&+\frac{-(0.85)(3.56982)m^2}{(0.85m+0.02)^2}+\frac{1.87183}{0.85m+0.02}+\frac{-(0.85)(1.87183)m}{(0.85m+0.02)^2}+\frac{-(0.85)(0.05006)}{(0.85m+0.02)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
t_2=&\frac{-1.64928m^5}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{-4.38490m^4}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{-5.64255m^3}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{-2.77694m^2}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{0.24584m}{(0.85m+0.02)^{\frac{3}{2}}}\\
&+\frac{0.00299}{(0.85m+0.02)^{\frac{3}{2}}}+\frac{1.64926m^4}{0.85m+0.02}+\frac{2.33813m^3}{0.85m+0.02}+\frac{3.56982m^2}{0.85m+0.02}+\frac{1.87183m}{0.85m+0.02}+\frac{0.05006}{0.85m+0.02}.
\end{aligned}
$$

To prove the derivative is negative, we will show the coefficients for the expression are negative. First, consider

$$
-11.28+\frac{(0.425)^2(1.275)m+(1.275)(0.425)(0.02)}{(0.85m+0.02)^{\frac{5}{2}}}.
$$

We can upper bound it by

$$
-11.28+\frac{(0.425)^2(1.275)(0.25)+(1.275)(0.425)(0.02)}{(0.85(0.15)+0.02)^{\frac{5}{2}}},
$$

which is negative.

Next, consider the coefficient of the exponential term. We present a simplified version of this coefficient below by rounding the terms up after five digits of precision.

$$
\begin{aligned}
&\frac{\begin{aligned}&-2.76580m^6-12.32506m^5-19.11829m^4-12.42455m^3-1.21605m^2-0.20084m+0.00123\\&+2.76579m^5\sqrt{0.85m+0.02}+8.19169m^4\sqrt{0.85m+0.02}+10.18554m^3\sqrt{0.85m+0.02}\\&+6.45452m^2\sqrt{0.85m+0.02}+0.30061m\sqrt{0.85m+0.02}-0.00313\sqrt{0.85m+0.02}\end{aligned}}{(0.85m+0.02)^{\frac{5}{2}}}
\end{aligned}
$$

We will show that the numerator is negative. To do so, we upper bound the numerator by the following expression:

$$\left(-19.11829 + 8.19169\sqrt{0.85\,(0.25) + 0.02}\right) m^4$$

$$+ \left(-1.21605 + 6.45452\sqrt{0.85\,(0.25) + 0.02} + \left(-12.42455 + 10.18554\sqrt{0.85\,(0.25) + 0.02}\right)(0.15)\right.$$

$$\left. + \left(-12.32506 + 2.76579\sqrt{0.85\,(0.25) + 0.02}\right)(0.15)^3\right) m^2$$

$$- 0.20084m + 0.30061m\sqrt{0.85m + 0.02} + \left(-0.00313\sqrt{0.85\,(0.15) + 0.02} + 0.00123 - 2.76580\,(0.15)^6\right).$$

We will show that the first derivative of the upper bound is negative. Consider the derivative below:

$$\left(-19.11829 + 8.19169\sqrt{0.85\,(0.25) + 0.02}\right)(4)\, m^3$$

$$+ \left(-1.21605 + 6.45452\sqrt{0.85\,(0.25) + 0.02} + \left(-12.42455 + 10.18554\sqrt{0.85\,(0.25) + 0.02}\right)(0.15)\right.$$

$$\left. + \left(-12.32506 + 2.76579\sqrt{0.85\,(0.25) + 0.02}\right)(0.15)^3\right)(2)\, m$$

$$- 0.20084 + 0.30061\sqrt{0.85m + 0.02} + \frac{0.30061\,(0.425)\, m}{\sqrt{0.85m + 0.02}}.$$

We can upper bound the derivative by the following:

$$\left(-19.11829 + 8.19169\sqrt{0.85\,(0.25) + 0.02}\right)(4)\, m^3$$

$$+ \left(-1.21605 + 6.45452\sqrt{0.85\,(0.25) + 0.02} + \left(-12.42455 + 10.18554\sqrt{0.85\,(0.25) + 0.02}\right)(0.15)\right.$$

$$\left. + \left(-12.32506 + 2.76579\sqrt{0.85\,(0.25) + 0.02}\right)(0.15)^3\right)(2)\, m$$

$$+ \frac{(3)\,(0.30061)\,(0.425)}{\sqrt{0.85\,(0.15) + 0.02}} m - 0.20084 + \frac{(0.02)\,(0.30061)}{\sqrt{0.85\,(0.15) + 0.02}}.$$

Since the cubic term is negative, the above is concave. So, we check the derivative of the above at $m = 0.15$, which is negative.

So, we can further upper bound the above by evaluating at $m = 0.15$, which is negative.

Then, we can further upper bound our numerator upper bound by evaluating at $m = 0.15$, which is negative.

Since both coefficients are negative, this means that the derivative of Equation 33 is negative.

This completes the proof. □

Proofs for Lemmas in Section B.5.1

Proof of Lemma B.56. To prove the lemma, we show that the second partial m of Equation 34 is negative. Consider the expression (simplified by rounding up after five digits of precision with

$\exp\left(3m^3 + \frac{1.1032}{0.3}m\right)$ factored out).

$$81m^4 + 66.192m^2 + 18m + 13.52279 + (b_{1,1} + b_{1,2}) \exp\left(-\left(1.5m^2 + \frac{0.5516}{0.3}\right)\left(m + \sqrt{0.85m + 0.02}\right)\right) + (b_{2,1} + b_{2,2}) \exp\left(\left(\left(1.5m^2 + \frac{0.5516}{0.3}\right)\left(\sqrt{0.85m + 0.02} - m\right)\right)\right), \tag{86}$$

where

$$b_{1,1} = \frac{\begin{array}{c}13.16757m^8 + 2.17416m^7 + 21.85326m^6 + 9.86506m^5 \\ + 16.45667m^4 + 3.59784m^3 + 3.47969m^2 - 0.49787m + 0.58447\end{array}}{(0.85m + 0.02)^{\frac{5}{2}}},$$

$$b_{1,2} = \frac{\begin{array}{c}-21.94593m^7 + 13.70115m^6 - 31.93304m^5 - 0.12630m^4 - 8.99522m^3 \\ - 0.73222m^2 - 2.22881m + 1.04799\end{array}}{(0.85m + 0.02)^2},$$

$$b_{2,1} = \frac{\begin{array}{c}-13.16756m^8 - 8.39216m^7 - 6.65690m^6 - 27.87899m^5 \\ - 3.32475m^4 - 8.80771m^3 - 1.54734m^2 + 0.54641m - 0.58446\end{array}}{(0.85m + 0.02)^{\frac{5}{2}}},$$

and

$$b_{2,2} = \frac{\begin{array}{c}-21.94593m^7 + 13.70115m^6 - 31.93304m^5 - 0.12630m^4 \\ - 8.99522m^3 - 0.73222m^2 - 2.22881m + 1.04799\end{array}}{(0.85m + 0.02)^2}.$$

To proceed, we first show that $b_{1,1} + b_{1,2}$ is positive and $b_{2,1} + b_{2,2}$ is negative in lemmas B.58 and B.57.

To upper bound Equation 86, we replace $\sqrt{0.85m + 0.02} - m$ in $\exp\left(\left(1.5m^2 + \frac{0.5516}{0.3}\right)\left(\sqrt{0.85m + 0.02} - m\right)\right)$ with a local linear lower bound $\left(10\sqrt{0.2325} - 10\sqrt{0.1475} - 1\right)m + 0.236$. The linear lower bound can be verified by evaluating the difference between the square root and linear expression at $m = 0.15, 0.25$, since the difference is concave in m.

Furthermore, note that $\left(1.5m^2 + \frac{0.5516}{0.3}\right)\left(\left(10\sqrt{0.2325} - 10\sqrt{0.1475} - 1\right)m + 0.236\right)$ is increasing in m. To see this, note that the first partial m can be lower bounded by

$$\left(10\sqrt{0.2325} - 10\sqrt{0.1475} - 1\right)(4.5)(0.25)^2 + 0.236\,(3)\,(0.15) + \frac{0.5516\left(10\sqrt{0.2325} - 10\sqrt{0.1475} - 1\right)}{0.3},$$

which is positive.

So, we can further upper bound Equation 86, by replacing $\exp\left(\left(1.5m^2 + \frac{0.5516}{0.3}\right)\left(\left(10\sqrt{0.2325} - 10\sqrt{0.1475} - 1\right)m + 0.236\right)\right)$ with $\exp\left(\left(\left(1.5\,(0.15)^2 + \frac{0.5516}{0.3}\right)\left(\left(10\sqrt{0.2325} - 10\sqrt{0.1475} - 1\right)(0.15) + 0.236\right)\right)\right)$.

We also replace $\exp\left(-\left(1.5m^2 + \frac{0.5516}{0.3}\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ with $\exp\left(-\left(1.5\,(0.15)^2 + \frac{0.5516}{0.3}\right)\left(m + \sqrt{0.85\,(0.15) + 0.02}\right)\right)$.

Since $\exp\left(-\left(1.5\,(0.15)^2 + \frac{0.5516}{0.3}\right)\left(m + \sqrt{0.85\,(0.15) + 0.02}\right)\right)$ is convex in m, we can replace it with a local linear upper bound $-0.62819m + 0.463$. To see that the upper bound is valid locally, one can evaluate the difference between the linear bound and the exponential term at $0.15, 0.25$, since the difference is concave in m.

With the above, we can scale our upper bound for Equation 86 by $(0.85m + 0.02)^{\frac{5}{2}}$, yielding the following expression (simplified by rounding up after five digits of precision).

$$\begin{aligned}&-8.27173m^9 - 15.64560m^8 - 25.70797m^7 - 6.38042m^6 - 48.91229m^5 + 0.21436m^4\\&-14.14977m^3 - 0.47060m^2 + 0.24788m - 0.63382\\&+\Big(13.78622m^8 - 52.72855m^7 + 106.12827m^6 - 61.36707m^5 + 53.25291m^4\\&\quad - 2.36911m^3 + 10.33671m^2 - 4.67231m + 2.11237\Big)\sqrt{0.85m + 0.02}\end{aligned} \tag{87}$$

After showing that the square root coefficient for Equation 87 is positive in lemma B.59, we can linearize $\sqrt{0.85m + 0.02}$ at $m = 0.2$. We then show that the resulting expression is concave in m and linearize it. To see concavity in m, note the second partial m below (simplified by rounding up after five digits of precision).

$$\begin{aligned}372.24864m^7 - 3569.21712m^6 + 2732.83248m^5 - 1219.479m^4 - 235.447m^3 + 128.79036m^2\\-27.85176m - 5.07236\end{aligned} \tag{88}$$

We upper bound the above by

$$\begin{aligned}372.24864\,(0.25)\,m^6 - 3569.21712m^6 + 2732.83248\,(0.25)\,m^4 - 1219.479m^4 - 235.447m^3\\+\,(128.79036\,(0.25) - 27.85176)\,(0.25) - 5.07236,\end{aligned} \tag{89}$$

which simplifies (by rounding up after five digits of precision) to $-3476.15496m^6 - 536.27088m^4 - 235.447m^3 - 3.98590$.

We then linearize our upper bound of Equation 87 at $m = 0.18$ (whose first partial m is negative) and evaluate the linearization at $m = 0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.57. To prove the lemma, we first scale Equation 35 by $(0.85m + 0.02)^{\frac{5}{2}}$ and show that the resulting expression decreases in m. Consider the partial m below (simplified by rounding up after five digits of precision), which we split into two parts. We will show that both are negative. Note that we scaled the second part by $\sqrt{0.85m + 0.02}$.

$$\begin{aligned}-105.34048m^7 - 58.74512m^6 - 39.9414m^5 - 139.39495m^4 - 13.299m^3 - 26.42313m^2\\-3.09468m + 0.54641\end{aligned} \tag{90}$$

$$\begin{aligned}-139.90530m^7 + 72.62643m^6 - 147.64282m^5 - 3.67640m^4 - 26.77088m^3 - 2.09568m^2\\-2.87102m + 0.40082\end{aligned} \tag{91}$$

Note that Equation 90 can be upper bounded by fixing $m = 0.15$, which is negative.

For Equation 91, we upper bound it by

$$\begin{aligned}-139.90530\,(0.15)^7 + 72.62643\,(0.25)^6 - 147.64282\,(0.15)^5 - 3.67640\,(0.15)^4 - 26.77088\,(0.15)^3\\-2.09568\,(0.15)^2 - 2.87102\,(0.15) + 0.40082,\end{aligned}$$

which is negative.

So, we upper bound Equation 35 (scaled by $(0.85m + 0.02)^{\frac{5}{2}}$) in m by $m = 0.15$, which is negative. This completes the proof.

□

Proof of Lemma B.58. To prove the lemma, we show that numerators for $b_{1,1}$ and $b_{2,2}$ in Equation 36 are positive.

We begin with the numerator for $b_{1,1}$, shown below.

$$13.16757m^8 + 2.17416m^7 + 21.85326m^6 + 9.86506m^5 + 16.45667m^4 + 3.59784m^3 + 3.47969m^2 - 0.49787m + 0.58447 \tag{92}$$

We can lower bound the above by

$$13.16757\,(0.15)^8 + 2.17416\,(0.15)^7 + 21.85326\,(0.15)^6 + 9.86506\,(0.15)^5 + 16.45667\,(0.15)^4 + 3.59784\,(0.15)^3 + 3.47969\,(0.15)^2 - 0.49787\,(0.25) + 0.58447,$$

which is positive.

For $b_{1,2}$, note the numerator below.

$$-21.94593m^7 + 13.70115m^6 - 31.93304m^5 - 0.12630m^4 - 8.99522m^3 - 0.73222m^2 - 2.22881m + 1.04799 \tag{93}$$

We can lower bound the above by

$$-21.94593\,(0.25)^7 + 13.70115\,(0.15)^6 - 31.93304\,(0.25)^5 - 0.12630\,(0.25)^4 - 8.99522\,(0.25)^3 - 0.73222\,(0.25)^2 - 2.22881\,(0.25) + 1.04799,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.59. To prove the lemma, we first show that Equation 37 is convex in m and show the linearization is positive.

Consider the second partial m of Equation 37 below.

$$772.02832m^6 - 2214.5991m^5 + 3183.8481m^4 - 1227.3414m^3 + 639.03492m^2 - 14.21466m + 20.67342 \tag{94}$$

We lower bound the above by

$$772.02832m^6 - 2214.5991\,(0.25)\,m^4 + 3183.8481m^4 - 1227.3414\,(0.25)\,m^2 + 639.03492m^2 - 14.21466\,(0.25) + 20.67342, \tag{95}$$

which simplifies (by rounding down after five digits of precision) to $772.02832m^6 + 2630.19832m^4 + 332.19957m^2 + 17.11975$.

We then linearize Equation 37 at $m = 0.18$ (where first partial m is positive) and evaluate the linearization at $m = 0.15$, which is positive.

This completes the proof.

□

Proof of Lemma B.60. To prove the lemma, we start with the second partial m of Equation 38 below, with $\exp\left(4.5m^3 + 4.544m\right)$ factored out and simplified(by rounding up after five digits of precision).

$$182.25m^4 + 122.688m^2 + 27m + 20.64794 + (a_{1,1} + a_{1,2}) \exp\left(-\left(2.25m^2 + 2.272\right)\left(m - \sqrt{0.65m + 0.02}\right)\right) + (a_{2,1} + a_{2,2}) \exp\left(-\left(2.25m^2 + 2.272\right)\left(m + \sqrt{0.65m + 0.02}\right)\right), \tag{96}$$

where

$$a_{1,1} = \frac{\begin{array}{r}-17.32514m^8 - 8.88653m^7 - 8.94526m^6 - 24.15990m^5 - 4.17647m^4 \\ -6.47647m^3 - 1.29224m^2 + 0.32859m - 0.27709\end{array}}{(0.65m + 0.02)^{\frac{5}{2}}},$$

$$a_{1,2} = \frac{\begin{array}{r}-28.87523m^7 + 17.65092m^6 - 34.85621m^5 + 1.50072m^4 \\ -8.25668m^3 - 0.25766m^2 - 1.65362m + 0.60463\end{array}}{(0.65m + 0.02)^{2}},$$

$$a_{2,1} = \frac{\begin{array}{r}17.32515m^8 + 8.88654m^7 + 8.94527m^6 + 23.90324m^5 + 4.16069m^4 \\ +6.38985m^3 + 1.26793m^2 - 0.33042m + 0.27707\end{array}}{(0.65m + 0.02)^{\frac{5}{2}}},$$

and

$$a_{2,2} = \frac{\begin{array}{r}-28.87523m^7 + 17.65092m^6 - 34.85621m^5 + 1.71461m^4 \\ -8.24483m^3 - 0.21430m^2 - 1.65229m + 0.60463\end{array}}{(0.65m + 0.02)^{2}}.$$

To proceed, we first show that $a_{1,1} + a_{1,2}$ is negative and $a_{2,1} + a_{2,2}$ is positive in lemmas B.63 and B.62.

Then, we use $\left(10\sqrt{0.1825} - 10\sqrt{0.1175} - 1\right) m + 0.215$ as a local linear lower bound for $\sqrt{0.65m + 0.02} - m$ in $\exp\left(-\left(2.25m^2 + 2.272\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ from Equation 96. To see that the local linear lower bound it valid, note that the difference between the $\sqrt{0.65m + 0.02} - m$ and $\left(10\sqrt{0.1825} - 10\sqrt{0.1175} - 1\right) m + 0.215$ is concave in m, so we evaluate the difference at $m = 0.15, 0.25$, which are positive.

Further note that

$$\left(2.25m^2 + 2.272\right)\left(\left(10\sqrt{0.1825} - 10\sqrt{0.1175} - 1\right) m + 0.215\right) \tag{97}$$

is decreasing in m. To see this, note that the first partial m can be upper bounded by

$$\begin{array}{r}\left(\left(10\sqrt{0.1825} - 10\sqrt{0.1175} - 1\right)(4.5) + 2.25\left(10\sqrt{0.1825} - 10\sqrt{0.1175} - 1\right)\right)(0.15)^2 \\ +0.215\,(4.5)\,(0.25) + 2.272\left(10\sqrt{0.1825} - 10\sqrt{0.1175} - 1\right),\end{array}$$

which is negative.

So, we can further upper bound Equation 96 by replacing $\exp\left(\left(2.25m^2 + 2.272\right)\left(\left(10\sqrt{0.1825} - 10\sqrt{0.1175} - 1\right) m + 0.215\right)\right)$ with $\exp\left(\left(2.25\,(0.25)^2 + 2.272\right)\left(\left(10\sqrt{0.1825} - 10\sqrt{0.1175} - 1\right)(0.25) + 0.215\right)\right)$.

Note that we can further upper bound Equation 96 by replacing $\exp\left(-\left(2.25m^2 + 2.272\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)$ with $\exp\left(-\left(2.25\,(0.15)^2 + 2.272\right)\left(m + \sqrt{0.65\,(0.15) + 0.02}\right)\right)$.

Since $\exp\left(-\left(2.25\left(0.15\right)^2+2.272\right)\left(m+\sqrt{0.65\left(0.15\right)+0.02}\right)\right)$ is convex in m, we can replace it with a linear upper bound $10\left(\exp\left(-1.376805\right)-\exp\left(-1.144543\right)\right)x+0.418$. To see that this upper bound is valid, note that the difference between the linear function and the exponential function is concave in m, so we evaluate at $m=0.15, 0.25$, both of which are positive.

With the above, we scale our upper bound for Equation 96 by $(0.65m+0.02)^{\frac{5}{2}}$, which simplifies (by rounding up after five digits of precision) to

$$\begin{aligned}&-11.43200m^9-25.11495m^8-15.77697m^7-25.71225m^6-29.69846m^5-8.86370m^4\\&\qquad-8.06930m^3-1.22802m^2+0.18153m-0.30789\\&+\Big(19.05333m^8-67.87193m^7+134.36985m^6-64.26385m^5+60.36050m^4\\&\qquad-1.33340m^3+10.08152m^2-3.07063m+1.18559\Big)\sqrt{0.65m+0.02}\end{aligned}\tag{98}$$

To proceed, we then show that the coefficient for the square root term is positive in lemma B.61, so we can linearize $\sqrt{0.65m+0.02}$ at $m=0.2$. We then show that the resulting expression is concave in m, linearize it at $m=0.185$, and evaluate the linearization at $m=0.15$.

To see concavity of the our upper bound, note the second partial m below (simplified by rounding up after five digits of precision).

$$328.07592m^7-4361.73024m^6+3447.51288m^5-1504.473m^4+136.9814m^3+39.1764m^2+0.58782m-3.18428\tag{99}$$

We then upper bound the above by

$$\begin{aligned}&328.07592\left(0.25\right)m^6-4361.73024m^6+3447.51288\left(0.25\right)m^4-1504.473m^4+136.9814m^3\\&\qquad+39.1764\left(0.25\right)^2+0.58782\left(0.25\right)-3.18428,\end{aligned}\tag{100}$$

which simplifies (after rounding up after five digits of precision) to

$$-4279.71126m^6-642.59478m^4+136.9814m^3-0.5888.\tag{101}$$

The above can then be upper bounded by

$$\left(-4279.71126\left(0.15\right)^3-642.59478\left(0.15\right)+136.9814\right)\left(0.25\right)^3-0.5888,$$

which is negative.

For the $m=0.2$ linearization of our upper bound for Equation 96, note that the first partial m at $m=0.2$ is negative, so we evaluate the linearization at $m=0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.61. To prove the lemma, we first show that Equation 39 is locally increasing in m.

Consider the first partial m below.

$$\begin{aligned}&152.42664m^7-475.10351m^6+806.2191m^5-321.31925m^4+241.442m^3-4.0002m^2\\&\qquad+20.16304m-3.07063\end{aligned}\tag{102}$$

To show that the above is positive, we lower bound it by

$$\begin{aligned}&152.42664m^7-475.10351\left(0.25\right)m^5+806.2191m^5-321.31925\left(0.25\right)m^3+241.442m^3\\&\qquad-4.0002\left(0.25\right)m+20.16304m-3.07063,\end{aligned}\tag{103}$$

which simplifies (by rounding down after five digits of precision) to

$$152.42664m^7 + 687.44322m^5 + 161.11218m^3 + 19.16299m - 3.07063. \tag{104}$$

We then lower bound the above in m by $m = 0.15$, which is positive.

So, we evaluate Equation 39 at $m = 0.15$, which is positive.

This completes the proof. □

Proof of Lemma B.62. To prove the lemma, we show that the numerators for $a_{2,1}$ and $a_{2,2}$ in Equation 40 are positive.

We start with the numerator for $a_{2,1}$, shown below.

$$\begin{aligned} 17.32515m^8 + 8.88654m^7 + 8.94527m^6 + 23.90324m^5 + 4.16069m^4 + 6.38985m^3 \\ +1.26793m^2 - 0.33042m + 0.27707 \end{aligned} \tag{105}$$

Note that we can lower bound the above by

$$\begin{aligned} 17.32515\,(0.15)^8 + 8.88654\,(0.15)^7 + 8.94527\,(0.15)^6 + 23.90324\,(0.15)^5 + 4.16069\,(0.15)^4 \\ +6.38985\,(0.15)^3 + 1.26793\,(0.15)^2 - 0.33042\,(0.25) + 0.27707, \end{aligned}$$

which is positive.

Next, we consider the numerator for $a_{2,2}$, shown below.

$$\begin{aligned} -28.87523m^7 + 17.65092m^6 - 34.85621m^5 + 1.71461m^4 - 8.24483m^3 - 0.21430m^2 \\ -1.65229m + 0.60463 \end{aligned} \tag{106}$$

Note that we can lower bound the above by

$$\begin{aligned} -28.87523\,(0.25)^7 + 17.65092\,(0.15)^6 - 34.85621\,(0.25)^5 + 1.71461\,(0.15)^4 - 8.24483\,(0.25)^3 \\ -0.21430\,(0.25)^2 - 1.65229\,(0.25) + 0.60463, \end{aligned}$$

which is positive.

This completes the proof.

□

Proof of Lemma B.63. To prove the lemma, we first scale Equation 41 by $(0.65m + 0.02)^{\frac{5}{2}}$. We then show that the resulting expression is decreasing in m. We split the partial m into two parts (shown below) and show that both (simplified by rounding up after five digits of precision) are negative. Note that we scale the second part by $\sqrt{0.65m + 0.02}$

$$\begin{aligned} -138.60112m^7 - 62.20571m^6 - 53.67156m^5 - 120.7995m^4 - 16.70588m^3 - 19.42941m^2 \\ -2.58448m + 0.32859 \end{aligned} \tag{107}$$

$$\begin{aligned} -140.76674m^7 + 70.53261m^6 - 122.49284m^5 + 0.90399m^4 - 18.66388m^3 - 0.91409m^2 \\ -1.62258m + 0.16344 \end{aligned} \tag{108}$$

We first show that Equation 107 is negative. Since the expression is decreasing in m, we upper bound by $m = 0.15$, which is negative.

For Equation 108, we first upper bound it by

$$\begin{aligned} -140.76674m^7 + 70.53261\,(0.25)\,m^5 - 122.49284m^5 + 0.90399\,(0.25)\,m^3 - 18.66388m^3 \\ -0.91409m^2 - 1.62258m + 0.16344, \end{aligned} \tag{109}$$

which simplifies (by rounding up after five digits of precision) to

$$-140.76674m^7 - 104.85968m^5 - 18.43788m^3 - 0.91409m^2 - 1.62258m + 0.16344. \quad (110)$$

The above is decreasing in m, so we upper bound by $m = 0.15$, which is negative.

So, we evaluate Equation 41 (scaled by $(0.65m + 0.02)^{\frac{5}{2}}$) at $m = 0.15$, which is negative.

This completes the proof.

□

Proof. To prove the lemma, we divide Equation 42 into the following parts and prove each is concave in η.

Part 1

$$\sqrt{2m\eta + 0.25m + 0.02}\exp\left(-\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)m\right) \quad (111)$$

Part 2

$$\left(\sqrt{2m\eta + 0.25m + 0.02} + 0.9m^2 + 0.52 + 1.944\eta\right)\exp\left(-\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\sqrt{2m\eta + 0.25m + 0.02}\right) \quad (112)$$

Part 3

$$\sqrt{2m\eta + 0.25m + 0.02}\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)m\right)$$
$$+\left(\sqrt{2m\eta + 0.25m + 0.02} - 0.9m^2 - 0.52 - 1.944\eta\right)\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\sqrt{2m\eta + 0.25m + 0.02}\right) \quad (113)$$

To see that Equation 111 is concave in η, consider the second partial η below.

$$-\frac{m^2}{(2m\eta + 0.25m + 0.02)^{\frac{3}{2}}} + \frac{\frac{0.9}{\eta^2}m^4 + \frac{0.52}{\eta^2}m^2 - \left(\frac{0.9}{\eta^3}m^3 + \frac{0.52}{\eta^3}m\right)(2m\eta + 0.25m + 0.02)}{\sqrt{2m\eta + 0.25m + 0.02}} \quad (114)$$

To show the above is negative, we note that the numerator for the $\frac{1}{\sqrt{2m\eta+0.25m+0.02}}$ term simplifies to

$$-\frac{0.9}{\eta^2}m^4 - \frac{0.52}{\eta^2}m^2 - \frac{0.9\,(0.25)}{\eta^3}m^4 - \frac{0.9\,(0.02)}{\eta^3}m^3 - \frac{0.52\,(0.25)}{\eta^3}m^2 - \frac{0.52\,(0.02)}{\eta^3}m. \quad (115)$$

To see that the remaining parts are concave in η, refer to lemmas B.66 and B.65.

This completes the proof.

□

Proof. To prove that Equation 43, we divide up the second partial η and show that each part is negative.

Part 1

$$\begin{aligned}
&\frac{-m^2}{(2m\eta+0.25m+0.02)^{\frac{3}{2}}}\exp\left(\frac{0.45}{\eta}m^3+\frac{0.26}{\eta}m+0.972m\right)\\
&-\frac{2m}{\sqrt{2m\eta+0.25m+0.02}}\left(\frac{0.45}{\eta^2}m^3+\frac{0.26}{\eta^2}m\right)\exp\left(\frac{0.45}{\eta}m^3+\frac{0.26}{\eta}m+0.972m\right)\\
&+\sqrt{2m\eta+0.25m+0.02}\left(\frac{0.45}{\eta^2}m^3+\frac{0.26}{\eta^2}m\right)^2\exp\left(\frac{0.45}{\eta}m^3+\frac{0.26}{\eta}m+0.972m\right)
\end{aligned} \tag{116}$$

Part 2

$$\begin{aligned}
&2\kappa_1\left(\frac{m}{\sqrt{2m\eta+0.25m+0.02}}-1.944\right)\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\sqrt{2m\eta+0.25m+0.02}\right)\\
&+\kappa_2\left(\sqrt{2m\eta+0.25m+0.02}-0.9m^2-0.52-1.944\eta\right)\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\sqrt{2m\eta+0.25m+0.02}\right)\\
&+2\sqrt{2m\eta+0.25m+0.02}\left(\frac{0.45}{\eta^3}m^3+\frac{0.26}{\eta^3}m\right)\exp\left(\frac{0.45}{\eta}m^3+\frac{0.26}{\eta}m+0.972m\right)\\
&+\frac{-m^2}{(2m\eta+0.25m+0.02)^{\frac{3}{2}}}\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\sqrt{2m\eta+0.25m+0.02}\right)
\end{aligned} \tag{117}$$

Part 3

$$\kappa_1^2\left(\sqrt{2m\eta+0.25m+0.02}-0.9m^2-0.52-1.944\eta\right)\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\sqrt{2m\eta+0.25m+0.02}\right) \tag{118}$$

where

$$\kappa_1=-\left(\frac{0.45}{\eta^2}m^2+\frac{0.26}{\eta^2}\right)\sqrt{2m\eta+0.25m+0.02}+\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)\frac{m}{\sqrt{2m\eta+0.25m+0.02}}$$

$$\begin{aligned}
\kappa_2=&2\left(\frac{0.45}{\eta^3}m^2+\frac{0.26}{\eta^3}\right)\sqrt{2m\eta+0.25m+0.02}-2\left(\frac{0.45}{\eta^2}m^2+\frac{0.26}{\eta^2}\right)\frac{m}{\sqrt{2m\eta+0.25m+0.02}}\\
&-\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)\frac{m^2}{(2m\eta+0.25m+0.02)^{\frac{3}{2}}}
\end{aligned}$$

We start by showing that Equation 116 is negative. Note that we first factor out the exponential term and scale by $(2m\eta + 0.25m + 0.02)^{\frac{3}{2}}$, resulting in the following expression.

$$\left(\frac{(0.25)^2(0.45)^2}{\eta^4} + \frac{0.2025}{\eta^3} + \frac{4(0.45)^2}{\eta^2}\right)m^8 + \left(\frac{0.002025}{\eta^4} + \frac{0.0162}{\eta^3}\right)m^7$$
$$+ \left(\frac{0.014706}{\eta^4} + \frac{0.234}{\eta^3} + \frac{4(0.52)(0.45)}{\eta^2}\right)m^6 + \left(\frac{0.00234}{\eta^4} + \frac{0.01872}{\eta^3} + \frac{-0.9(0.25)}{\eta^2} + \frac{-1.8}{\eta}\right)m^5$$
$$+ \left(\frac{0.0043186}{\eta^4} + \frac{0.0676}{\eta^3} + \frac{0.2524}{\eta^2}\right)m^4 + \left(\frac{(0.02)(0.26)^2(0.5)}{\eta^4} + \frac{0.08(0.26)^2}{\eta^3} + \frac{-0.52(0.25)}{\eta^2} + \frac{-1.04}{\eta}\right)m^3$$
$$+ \left(\frac{(0.02)^2(0.26)^2}{\eta^4} + \frac{-0.52(0.02)}{\eta^2} - 1\right)m^2 \tag{119}$$

We then upper bound the above by

$$\left(\frac{(0.25)^2(0.45)^2}{\eta^4} + \frac{0.2025}{\eta^3} + \frac{4(0.45)^2}{\eta^2}\right)m^8 + \left(\frac{0.002025}{\eta^4} + \frac{0.0162}{\eta^3}\right)m^7$$
$$+ \left(\frac{0.014706}{\eta^4} + \frac{0.234}{\eta^3} + \frac{4(0.52)(0.45)}{\eta^2}\right)m^6 + \left(\frac{0.00234}{\eta^4} + \frac{0.01872}{\eta^3} + \frac{-0.9(0.25)}{(0.3)^2} + \frac{-1.8}{0.3}\right)m^5$$
$$+ \left(\frac{0.0043186}{\eta^4} + \frac{0.0676}{\eta^3} + \frac{0.2524}{\eta^2}\right)m^4 + \left(\frac{(0.02)(0.26)^2(0.5)}{\eta^4} + \frac{0.08(0.26)^2}{\eta^3} + \frac{-0.52(0.25)}{(0.3)^2} + \frac{-1.04}{0.3}\right)m^3$$
$$+ \left(\frac{(0.02)^2(0.26)^2}{\eta^4} + \frac{-0.52(0.02)}{(0.3)^2} - 1\right)m^2 \tag{120}$$

and show that the upper bound is decreasing in η.

Note the first partial η below, where all the terms are negative.

$$\left(\frac{-4(0.25)^2(0.45)^2}{\eta^5} + \frac{-3\cdot 0.2025}{\eta^4} + \frac{-2\cdot 4(0.45)^2}{\eta^3}\right)m^8 + \left(\frac{-4\cdot 0.002025}{\eta^5} + \frac{-3\cdot 0.0162}{\eta^4}\right)m^7$$
$$+ \left(\frac{-4\cdot 0.014706}{\eta^5} + \frac{-3\cdot 0.234}{\eta^4} + \frac{-2\cdot 4(0.52)(0.45)}{\eta^3}\right)m^6 + \left(\frac{-4\cdot 0.00234}{\eta^5} + \frac{-3\cdot 0.01872}{\eta^4}\right)m^5$$
$$+ \left(\frac{-4\cdot 0.0043186}{\eta^5} + \frac{-3\cdot 0.0676}{\eta^4} + \frac{-2\cdot 0.2524}{\eta^3}\right)m^4 + \left(\frac{-4(0.02)(0.26)^2(0.5)}{\eta^5} + \frac{-3\cdot 0.08(0.26)^2}{\eta^4}\right)m^3$$
$$+ \left(\frac{-4(0.02)^2(0.26)^2}{\eta^5}\right)m^2 \tag{121}$$

So, we can upper bound Equation 120 by fixing $\eta = 0.2$.

The resulting expression (with m^2 factored out) simplifies (by rounding up after five digits of precision) to

$$53.47266m^6 + 3.29063m^5 + 61.84125m^4 - 4.6975m^3 + 17.45913m^2 - 3.81261m - 1.09865 \tag{122}$$

We can then upper bound the above by

$$53.47266(0.25)^6 + 3.29063(0.25)^5 + 61.84125(0.25)^4 - 4.6975(0.15)^3 + 17.45913(0.25)^2$$
$$-3.81261(0.15) - 1.09865,$$

which is negative.

For Equation 117, we first scale the expression by $\exp\left(\frac{0.45}{\eta}m^3 + \frac{0.26}{\eta}m + 0.972m\right)$ and factor out $\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{2mn + 0.25m + 0.02}\right)\right)$, resulting in the following.

$$\frac{\lambda_1}{2m\eta + 0.25m + 0.02} + \frac{\lambda_{2,1} + \lambda_{2,2} + \lambda_{2,3}\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right)}{(2m\eta + 0.25m + 0.02)^{\frac{3}{2}}} \tag{123}$$

where

$$\begin{aligned}
\lambda_1 =& \frac{0.45}{\eta}m^4 + \frac{0.26}{\eta}m^2 + 1.944m^2 - 0.972m^2 \\
&+ \left(\frac{0.45\,(0.5)}{\eta^3}m^3 + \frac{0.26\,(0.5)}{\eta^3}m + \frac{0.45\,(0.04)}{\eta^3}m^2 + \frac{0.26\,(0.04)}{\eta^3}\right)(2m\eta + 0.25m + 0.02),
\end{aligned}$$

$$\begin{aligned}
\lambda_{2,1} =& -2.916\,(1.944)\,m^2\eta - 0.226\,(1.944)\,m^2 - 0.03888\,(1.944)\,m + 0.45\,(1.944)\,m^4 + \frac{2\,(0.45)^2}{\eta}m^6 \\
&+ \frac{2\,(0.52)\,(0.45)}{\eta}m^4 + \frac{2\,(0.26)^2}{\eta}m^2 + 0.9\,(0.972)\,m^4 + 0.52\,(0.972)\,m^2 - m^2,
\end{aligned}$$

$$\begin{aligned}
\lambda_{2,2} =& \frac{-4\,(0.45)^2}{\eta^2}m^5\,(2m\eta + 0.25m + 0.02) + \frac{-\,(0.45)^2}{\eta^3}m^5\,(2m\eta + 0.25m + 0.02) \\
&+ \frac{-\,(0.45)^2\,(0.08)}{\eta^3}m^4\,(2m\eta + 0.25m + 0.02) + \frac{-4\,(0.52)\,(0.45)}{\eta^2}m^3\,(2m\eta + 0.25m + 0.02) \\
&+ \frac{-\,(0.52)\,(0.45)}{\eta^3}m^3\,(2m\eta + 0.25m + 0.02) + \frac{-\,(0.52)\,(0.45)\,(0.08)}{\eta^3}m^2\,(2m\eta + 0.25m + 0.02) \\
&+ \frac{-4\,(0.26)^2}{\eta^2}m\,(2m\eta + 0.25m + 0.02) + \frac{-\,(0.26)^2}{\eta^3}m\,(2m\eta + 0.25m + 0.02) \\
&+ \frac{-\,(0.26)^2\,(0.08)}{\eta^3}\,(2m\eta + 0.25m + 0.02),
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{2,3} =& \frac{3.6}{\eta}m^5 + \frac{1.8\,(0.25)}{\eta^2}m^5 + \frac{1.8\,(0.02)}{\eta^2}m^4 + \frac{0.9\,(0.5)}{\eta^2}m^5 + \frac{0.9\,(0.25)^2}{\eta^3}m^5 + \frac{0.9\,(0.25)\,(0.02)}{\eta^3}m^4 \\
&+ \frac{0.9\,(0.04)}{\eta^2}m^4 + \frac{0.9\,(0.02)\,(0.25)}{\eta^3}m^4 + \frac{0.9\,(0.02)^2}{\eta^3}m^3 + \frac{2.08}{\eta}m^3 + \frac{1.04\,(0.25)}{\eta^2}m^3 + \frac{1.04\,(0.02)}{\eta^2}m^2 \\
&+ \frac{0.52\,(0.5)}{\eta^2}m^3 + \frac{0.52\,(0.25)^2}{\eta^3}m^3 + \frac{0.52\,(0.25)\,(0.02)}{\eta^3}m^2 \\
&+ \frac{0.52\,(0.04)}{\eta^2}m^2 + \frac{0.52\,(0.02)\,(0.25)}{\eta^3}m^2 + \frac{0.52\,(0.02)^2}{\eta^3}m
\end{aligned}$$

To proceed, we first replace $\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right)$ with $\exp\left(\left(\frac{0.45(0.15)^2}{0.3} + \frac{0.26}{0.3} + 0.972\right)\left(0.25 - \sqrt{2\,(0.15)\,(0.2) + 0.25\,(0.15) + 0.02}\right)\right)$ and scale the resulting expression by $\eta^3\,(2m\eta + 0.25m + 0.02)^{\frac{3}{2}}$.

We then show that the resulting expression is decreasing in η. To see this, note the first partial η below (simplified by rounding up after five digits of precision with m factored out and scaled by $\sqrt{2mn + 0.25m + 0.02}$).

$$6.804m^2\eta^3 + 2.25m^4\eta^2 + 2.029m^2\eta^2 + 0.05832m\eta^2 + 1.575m^4\eta + 0.126m^3\eta + 0.91m^2\eta + 0.0728m\eta + 0.16875m^4 + 0.027m^3 + 0.09858m^2 + 0.0156m + 0.000624$$

$$+\Big(-22.67481m\eta^3 + 5.2488m^3\eta^2 - 2.80171m\eta^2 - 0.22674\eta^2 - 2.43m^5\eta + 6.05182m^4\eta - 2.808m^3\eta + 3.49661m^2\eta - 0.8112m\eta - 0.6075m^5 + 0.707877m^4 - 0.64148m^3 + 0.38092m^2 - 0.16783m - 0.016224\Big)\sqrt{2m\eta + 0.25m + 0.02} \tag{124}$$

Since

$$\begin{aligned} &-22.67481m\eta^3 + \left(5.2488m^3 - 2.80171m - 0.22674\right)\eta^2 \\ &+ \left(-2.43m^4 + 6.05182m^3 - 2.808m^2 + 3.49661m - 0.8112\right)m\eta \\ &+ \left(-0.6075m^5 + 0.707877m^4 - 0.64148m^3 + 0.38092m^2 - 0.16783m - 0.016224\right) \end{aligned} \tag{125}$$

can be upper bounded by

$$\begin{aligned} &-22.67481\,(0.15)\,(0.2)^3 + \Big(5.2488\,(0.25)^3 - 2.80171\,(0.15) - 0.22674\Big)\,(0.2)^2 \\ &+ \Big(-2.43\,(0.15)^4 + 6.05182\,(0.25)^3 - 2.808\,(0.15)^2 + 3.49661\,(0.25) - 0.8112\Big)\,(0.25)(0.3) \\ &+ \Big(-0.6075\,(0.15)^5 + 0.707877\,(0.25)^4 - 0.64148\,(0.15)^3 + 0.38092\,(0.25)^2 - 0.16783\,(0.15) - 0.016224\Big), \end{aligned}$$

which is negative, we can replace $\sqrt{2m\eta + 0.25m + 0.02}$ with $\sqrt{2\,(0.15)\,(0.2) + 0.25\,(0.15) + 0.02}$ in Equation 124, resulting in the following (simplified by rounding up after five digits of precision).

$$\begin{aligned} &(6.804m - 7.77247)\,m\eta^3 + \left(2.25m^4 + 1.79919m^3 + 2.029m^2 - 0.90205m - 0.07772\right)\eta^2 \\ &+ \left(-0.83295m^4 + 3.64945m^3 - 0.83652m^2 + 2.10857m - 0.20526\right)m\eta \\ &+ \left(-0.20823m^5 + 0.41140m^4 - 0.19288m^3 + 0.22916m^2 - 0.04192m - 0.00493\right) \end{aligned} \tag{126}$$

To proceed, we iteratively upper bound the above in η. First, we upper bound

$$(6.804m - 7.77247)\,m\eta + \left(2.25m^4 + 1.79919m^3 + 2.029m^2 - 0.90205m - 0.07772\right) \tag{127}$$

by $\eta = 0.2$, which simplifies (by rounding up after five digits of precision) to $2.25m^4 + 1.79919m^3 + 3.3898m^2 - 2.45654m - 0.07772$. Since

$$\Big(2.25\,(0.25)^4 + 1.79919\,(0.25)^3 + 3.3898\,(0.25)^2 - 2.45654\,(0.15) - 0.07772\Big)$$

is negative, we can upper bound

$$\begin{aligned} &\left(2.25m^4 + 1.79919m^3 + 3.3898m^2 - 2.45654m - 0.07772\right)\eta \\ &+ \left(-0.83295m^4 + 3.64945m^3 - 0.83652m^2 + 2.10857m - 0.20526\right)m \end{aligned} \tag{128}$$

by $\eta = 0.2$, which simplifies (by rounding up after five digits of precision) to

$$-0.83295m^5 + 4.09945m^4 - 0.47668m^3 + 2.78653m^2 - 0.69656m - 0.01554. \tag{129}$$

Since

$$(((((-0.83295\,(0.15) + 4.09945)\,(0.25) - 0.47668)\,(0.25) + 2.78653)\,(0.25) - 0.69656)\,(0.25) - 0.01554)$$

is negative, we can upper bound

$$\begin{aligned}&\left(-0.83295m^5+4.09945m^4-0.47668m^3+2.78653m^2-0.69656m-0.01554\right)\eta\\&+\left(-0.20823m^5+0.41140m^4-0.19288m^3+0.22916m^2-0.04192m-0.00493\right)\end{aligned}\tag{130}$$

by $\eta = 0.2$, which simplifies (by rounding up after five digits of precision) to

$$-0.37482m^5+1.23129m^4-0.28821m^3+0.78647m^2-0.18123m-0.00803.\tag{131}$$

The above can then be upper bounded by

$$((((-0.37482\,(0.15)+1.23129)\,(0.25)-0.28821)\,(0.25)+0.78647)\,(0.25)-0.18123)\,(0.25)-0.00803,$$

which is negative.

Since partial η is negative, we can further upper bound our earlier bound for Equation 123 by $\eta = 0.2$.

We then further upper bound the expression by the following

$$\begin{aligned}&\Big(0.972m^2\,(0.2)^3+0.45m^4\,(0.2)^2+0.26m^2\,(0.2)^2+0.9\,(0.5)\,m^4\,(0.2)\\&\quad+0.45\,(0.5)\,(0.25)\,m^4+0.26m^2\,(0.2)+0.26\,(0.5)\,(0.02)\,m+0.009m^3+0.03286m^2+0.26\,(0.08)\,m\,(0.2)\\&\quad+0.26\,(0.04)\,(0.25)\,m+0.45\,(0.08)\,m^3\,(0.2)+0.26\,(0.04)\,(0.02)\Big)\sqrt{2\,(0.25)\,(0.2)+0.25\,(0.25)+0.02}\\&\\&+\Big(-2.916\,(1.944)\,m^2\,(0.2)^4-0.03888\,(1.944)\,m\,(0.2)^3+1.7496m^4\,(0.2)^3-0.933904m^2\,(0.2)^3\\&\quad-(0.45)^2\,(0.25)\,m^6-3\,(0.45)^2\,m^6\,(0.2)-6\,(0.45)^2\,m^6\,(0.2)^2+0.03918m^5+0.707877m^5\,(0.2)\\&\quad+3.6\,(0.84053)\,m^5\,(0.2)^2-0.05125m^4-0.64148m^4\,(0.2)-6\,(0.52)\,(0.45)\,m^4\,(0.2)^2+0.01826m^3\\&\quad+0.38092m^3\,(0.2)+2.08\,(0.84053)\,m^3\,(0.2)^2-0.0128944m^2-0.16783m^2\,(0.2)\\&\quad-6\,(0.26)^2\,m^2\,(0.2)^2-0.002522m-0.016224m\,(0.2)-(0.26)^2\,(0.08)\,(0.02)\Big),\end{aligned}\tag{132}$$

which simplifies (by rounding up after five digits of precision) to

$$-0.22072m^6+0.30180m^5-0.15153m^4+0.17130m^3-0.03520m^2-0.00237m-0.00001.\tag{133}$$

Last, we show the above is convex in m. Note that we lower bound the second partial m by

$$((-6.6216\,(0.25)+6.036)\,(0.15)-1.81836)\,x^2+1.0278x-0.0704,\tag{134}$$

which is concave. Evaluating the above at $m = 0.15, 0.25$ is positive, so we can then evaluate Equation 133 at $m = 0.15, 0.25$, which are negative.

For Equation 118, note that

$$\sqrt{2\,(0.25)\,(0.3)+0.25\,(0.25)+0.02}-0.9\,(0.15)^2-0.52$$

is negative.

This completes the proof.

□

Proof. To show that Equation 44 is concave in η, we split the second partial η into the following parts (with the exponential term factored out). Note that we factor out $2\left(0.45m^2+0.26\right)$ from the first part and divide out 1.944 for the third part.

Part 1

$$\frac{\frac{-0.9(0.25)}{\eta^3}m^3+\frac{-0.9(0.02)}{\eta^3}m^2+\frac{-0.52(0.25)}{\eta^3}m+\frac{-0.52(0.02)}{\eta^3}+\frac{-0.9}{\eta^2}m^3+\frac{-0.52}{\eta^2}m}{\sqrt{2m\eta+0.25m+0.02}}$$
$$+\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)\cdot\frac{m^2}{(2m\eta+0.25m+0.02)^{\frac{3}{2}}}+\left(\frac{0.45}{\eta^2}m^2+\frac{0.26}{\eta^2}\right)^2(2m\eta+0.25m+0.02)$$
$$-2\left(\frac{0.45}{\eta^2}m^2+\frac{0.26}{\eta^2}\right)\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)m+\frac{\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)^2m^2}{2m\eta+0.25m+0.02} \tag{135}$$

Part 2

$$\frac{-0.9\,(0.25)}{\eta^3}m^3+\frac{-0.9\,(0.02)}{\eta^3}m^2+\frac{-0.52\,(0.25)}{\eta^3}m+\frac{-0.52\,(0.02)}{\eta^3}+\frac{-0.9}{\eta^2}m^3+\frac{-0.52}{\eta^2}m$$
$$+\frac{-2\left(\frac{0.45}{\eta^2}m^2+\frac{0.26}{\eta^2}\right)\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)m\,(2m\eta+0.25m+0.02)+\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)^2m^2}{\sqrt{2m\eta+0.25m+0.02}}$$
$$+2m\left(\frac{0.45}{\eta^2}m^2+\frac{0.26}{\eta^2}\right)+\frac{-\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)m^2}{2m\eta+0.25m+0.02}+\left(\frac{0.45}{\eta^2}m^2+\frac{0.26}{\eta^2}\right)^2(2m\eta+0.25m+0.02)^{\frac{3}{2}} \tag{136}$$

Part 3

$$\frac{\frac{0.9}{\eta}m^3+\frac{0.52}{\eta}m}{\sqrt{2m\eta+0.25m+0.02}}+\frac{\left(0.45m^2+0.26+0.972\eta\right)m^2}{(2m\eta+0.25m+0.02)^{\frac{3}{2}}}+\left(\frac{0.45}{\eta^2}m^2+\frac{0.26}{\eta^2}\right)^2\left(2m\eta^2+0.25m\eta+0.02\eta\right)$$
$$-2\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}\right)\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)m+\frac{\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)^2m^2\eta}{2m\eta+0.25m+0.02}$$
$$-2\left(\frac{0.45}{\eta}m^2+\frac{0.26}{\eta}+0.972\right)\frac{m}{\sqrt{2m\eta+0.25m+0.02}} \tag{137}$$

From Equation 135, we factor out $2\left(0.45m^2+0.26\right)$, scale the remaining expression by $(2m\eta+0.25m+0.02)^{\frac{3}{2}}$, and show that the resulting expression below.

$$\left(\frac{-0.78}{\eta}m^2+\frac{-1.35}{\eta}m^4+\frac{-0.675}{\eta^2}m^4+\frac{-0.9\left(0.25\right)^2}{\eta^3}m^4+\frac{-0.009}{\eta^3}m^3+\frac{-0.054}{\eta^2}m^3+\frac{-0.03286}{\eta^3}m^2+\frac{-0.39}{\eta^2}m^2 + \frac{-0.0052}{\eta^3}m+\frac{-0.0312}{\eta^2}m+\frac{-0.52\left(0.02\right)^2}{\eta^3}+0.972m^2\right)$$

$$+\left(-\frac{\left(2\right)\left(0.972\right)0.45}{\eta}m^4-\frac{\left(2\right)\left(0.972\right)0.26}{\eta}m^2+\frac{5\left(0.45\right)^2}{\eta^2}m^6+\frac{0.9513}{\eta^2}m^4-\frac{\left(0.04\right)\left(0.972\right)0.45}{\eta^2}m^3 +\frac{0.21164}{\eta^2}m^2-\frac{\left(0.04\right)\left(0.972\right)0.26}{\eta^2}m-\frac{3\left(0.45\right)^2}{\eta^3}m^6+\frac{0.08\left(0.45\right)^2}{\eta^3}m^5-\frac{3\left(0.52\right)\left(0.45\right)}{\eta^3}m^4 +\frac{0.08\left(0.52\right)\left(0.45\right)}{\eta^3}m^3-\frac{3\left(0.26\right)^2}{\eta^3}m^2+\frac{0.08\left(0.26\right)^2}{\eta^3}m-\frac{0.25\left(1.75\right)\left(0.45\right)^2}{\eta^4}m^6-\frac{0.03\left(0.45\right)^2}{\eta^4}m^5 +\frac{-0.102294}{\eta^4}m^4-\frac{0.03\left(0.52\right)\left(0.45\right)}{\eta^4}m^3+\frac{-0.0294814}{\eta^4}m^2-\frac{0.03\left(0.26\right)^2}{\eta^4}m +\frac{\left(0.02\right)^2\left(0.26\right)^2}{\eta^4}+\left(0.972\right)^2m^2\right)\sqrt{2m\eta+0.25m+0.02} \tag{138}$$

We then upper bound the the above by

$$\left(\frac{-0.78}{\eta}m^2+\frac{-1.35}{\eta}m^4+\frac{-0.675}{\eta^2}m^4+\frac{-0.9\left(0.25\right)^2}{\eta^3}m^4+\frac{-0.009}{\eta^3}m^3+\frac{-0.054}{\eta^2}m^3+\frac{-0.03286}{\eta^3}m^2+\frac{-0.39}{\eta^2}m^2 +\frac{-0.0052}{\eta^3}m+\frac{-0.0312}{\eta^2}m+\frac{-0.52\left(0.02\right)^2}{\eta^3}+0.972m^2\right)$$

$$+\left(\frac{-\left(2\right)\left(0.972\right)0.45}{\eta}m^4+\frac{-\left(2\right)\left(0.972\right)0.26}{\eta}m^2+\frac{-\left(0.04\right)\left(0.972\right)0.45}{\eta^2}m^3+\frac{-\left(0.04\right)\left(0.972\right)0.26}{\eta^2}m +\frac{-3\left(0.52\right)\left(0.45\right)}{\eta^3}m^4+\left(0.972\right)^2m^2+\left(\frac{\frac{\left(0.02\right)^2\left(0.26\right)^2}{\left(0.2\right)}-3\left(0.26\right)^2\left(0.15\right)^2}{\eta^3}\right) +\left(\frac{0.08\left(0.52\right)\left(0.45\right)+\frac{-0.03(0.52)(0.45)}{(0.3)}}{\eta^3}\right)m^3+\left(\frac{0.08\left(0.45\right)^2+\frac{-0.03(0.45)^2}{(0.3)}}{\eta^3}\right)m^5+\left(\frac{0.21164+\frac{-0.0294814}{(0.3)^2}}{\eta^2}\right)m^2 +\left(\frac{0.9513+\frac{-0.102294}{(0.3)^2}}{\eta^2}\right)m^4+\left(\frac{0.08\left(0.26\right)^2+\frac{-0.03(0.26)^2}{(0.3)}}{\eta^3}\right)m +\left(\frac{5\left(0.45\right)^2+\frac{-0.25(1.75)(0.45)^2}{(0.3)^2}+\frac{-3(0.45)^2}{(0.3)}}{\eta^2}\right)m^6\right)\sqrt{2m\eta+0.25m+0.02}, \tag{139}$$

which simplifies (by rounding up after five digits of precision) to

$$\left(\frac{-0.78}{\eta}m^2+\frac{-1.35}{\eta}m^4+\frac{-0.675}{\eta^2}m^4+\frac{-0.9\left(0.25\right)^2}{\eta^3}m^4+\frac{-0.009}{\eta^3}m^3+\frac{-0.054}{\eta^2}m^3+\frac{-0.03286}{\eta^3}m^2+\frac{-0.39}{\eta^2}m^2\right.$$
$$\left.+\frac{-0.0052}{\eta^3}m+\frac{-0.0312}{\eta^2}m+\frac{-0.52\left(0.02\right)^2}{\eta^3}+0.972m^2\right)$$

$$+\left(\frac{-\left(2\right)\left(0.972\right)0.45}{\eta}m^4+\frac{-\left(2\right)\left(0.972\right)0.26}{\eta}m^2+\frac{-\left(0.04\right)\left(0.972\right)0.45}{\eta^2}m^3+\frac{-\left(0.04\right)\left(0.972\right)0.26}{\eta^2}m\right.$$
$$+\frac{-3\left(0.52\right)\left(0.45\right)}{\eta^3}m^4+\frac{-0.00442}{\eta^3}+\frac{-0.00468}{\eta^3}m^3+\frac{-0.00405}{\eta^3}m^5+\frac{-0.1853}{\eta^2}m^4+\frac{-0.00135}{\eta^3}m$$
$$\left.+\frac{-1.99687}{\eta^2}m^6+\frac{-0.11593}{\eta^2}m^2+\left(0.972\right)^2m^2\right)\sqrt{2m\eta+0.25m+0.02} \tag{140}$$

We then upper bound $\frac{-0.11593}{\eta^2}m^2+\left(0.972\right)^2m^2$ by $\frac{-0.11593}{\left(0.3\right)^2}m^2+\left(0.972\right)^2m^2$, which simplifies (by rounding up after five digits of precision) to $-0.34332m^2$.

We the above we can then further upper bound Equation 140 by replacing $\sqrt{2m\eta+0.25m+0.02}$ with $\sqrt{2\left(0.15\right)\left(0.2\right)+0.25\left(0.15\right)+0.02}$.

To proceed, note that all the terms for the first partial η for the resulting expression (shown below) are positive, so we can further upper bound by fixing $\eta=0.3$

$$\left(\frac{0.78}{\eta^2}m^2+\frac{1.35}{\eta^2}m^4+\frac{2\cdot0.675}{\eta^3}m^4+\frac{3\cdot0.9\left(0.25\right)^2}{\eta^4}m^4+\frac{3\cdot0.009}{\eta^4}m^3+\frac{2\cdot0.054}{\eta^3}m^3+\frac{3\cdot0.03286}{\eta^4}m^2+\frac{2\cdot0.39}{\eta^3}m^2\right.$$
$$\left.+\frac{3\cdot0.0052}{\eta^4}m+\frac{2\cdot0.0312}{\eta^3}m+\frac{3\cdot0.52\left(0.02\right)^2}{\eta^4}\right)$$

$$+\left(\frac{\left(2\right)\left(0.972\right)0.45\left(0.34278\right)}{\eta^2}m^4+\frac{\left(2\right)\left(0.972\right)0.26\left(0.34278\right)}{\eta^2}m^2+\frac{2\cdot\left(0.04\right)\left(0.972\right)0.45\left(0.34278\right)}{\eta^3}m^3\right.$$
$$+\frac{2\cdot\left(0.04\right)\left(0.972\right)0.26\left(0.34278\right)}{\eta^3}m+\frac{3\cdot3\left(0.52\right)\left(0.45\right)\left(0.34278\right)}{\eta^4}m^4+\frac{3\cdot0.00442\left(0.34278\right)}{\eta^4}$$
$$+\frac{3\cdot0.00468\left(0.34278\right)}{\eta^4}m^3+\frac{3\cdot0.00405\left(0.34278\right)}{\eta^4}m^5+\frac{2\cdot0.1853\left(0.34278\right)}{\eta^3}m^4$$
$$\left.+\frac{3\cdot0.00135\left(0.34278\right)}{\eta^4}m+\frac{2\cdot1.99687\left(0.34278\right)}{\eta^3}m^6\right), \tag{141}$$

After fixing $\eta=0.3$, we get the following upper bound (rounded up after five digits of precision).

$$-7.60541m^6-0.05141m^5-24.70090m^4-1.05938m^3-7.87356m^2-0.59489m-0.06381 \tag{142}$$

For Equation 136, we first scale by $2m\eta + 0.25m + 0.02$, resulting in the following.

$$\left(\frac{0.45}{\eta}m^3 + \frac{0.45\,(0.25)}{\eta^2}m^3 + \frac{0.45\,(0.02)}{\eta^2}m^2 + \frac{0.26}{\eta}m + \frac{0.26\,(0.25)}{\eta^2}m + \frac{0.26\,(0.02)}{\eta^2} - 0.972m\right)^2 \sqrt{2m\eta + 0.25m + 0.02}$$

$$+\left(\frac{-0.9\,(0.5)}{\eta^2}m^4 + \frac{-0.9\,(0.25)^2}{\eta^3}m^4 + \frac{-0.9\,(0.25)\,(0.02)}{\eta^3}m^3 + \frac{-0.9\,(0.04)}{\eta^2}m^3 + \frac{-0.9\,(0.02)\,(0.25)}{\eta^3}m^3 + \frac{-0.9\,(0.02)^2}{\eta^3}m^2 + \frac{-0.52\,(0.5)}{\eta^2}m^2 + \frac{-0.52\,(0.25)^2}{\eta^3}m^2 + \frac{-0.52\,(0.25)\,(0.02)}{\eta^3}m + \frac{-0.52\,(0.04)}{\eta^2}m + \frac{-0.52\,(0.02)\,(0.25)}{\eta^3}m + \frac{-0.52\,(0.02)^2}{\eta^3} - \frac{0.45}{\eta}m^4 - \frac{0.26}{\eta}m^2 - 0.972m^2\right) \tag{143}$$

We first upper bound the above by replacing $\sqrt{2m\eta + 0.25m + 0.02}$ by $\sqrt{2\,(0.25)\,(0.3) + 0.25\,(0.25) + 0.02}$ and scale the subsequent expression by η^4.

We then show that the resulting expressions decreases in η, so we can upper bound by $\eta = 0.2$. To see this, consider the first partial η below (simplified by rounding up after five digits of precision).

$$\begin{aligned}
&- 2.06573m^2\eta^3 - 2.61545m^4\eta^2 - 1.51115m^2\eta^2 \\
&+ \left(0.19529m^5 - 0.88524m^3 - 0.08887m^2 - 0.57666m - 0.05134\right) m\eta \\
&+ \left(0.04883m^6 + 0.00391m^5 + 0.00017m^4 - 0.00448m^3 - 0.01656m^2 - 0.00389m - 0.00020\right)
\end{aligned} \tag{144}$$

We then upper bound the above by

$$\begin{aligned}
&- 2.06573\,(0.15)^2\,(0.2)^3 - 2.61545\,(0.15)^4\,(0.2)^2 - 1.51115\,(0.15)^2\,(0.2)^2 \\
&+ \left(0.19529\,(0.25)^5 - 0.88524\,(0.15)^3 - 0.08887\,(0.15)^2 - 0.57666\,(0.15) - 0.05134\right)(0.15)\,(0.2) \\
&+ \Big(0.04883\,(0.25)^6 + 0.00391\,(0.25)^5 + 0.00017\,(0.25)^4 - 0.00448\,(0.15)^3 - 0.01656\,(0.15)^2 \\
&\qquad - 0.00389\,(0.15) - 0.00020\Big),
\end{aligned}$$

which is negative.

The resulting expression from upper bounding by $\eta = 0.2$ is shown below.

$$\begin{aligned}
&\left(0.2025m^3 + 0.009m^2 + 0.07812m + 0.0052\right)^2 (0.48219) \\
&+ \left(-0.03285m^4 - 0.00324m^3 - 0.0206072m^2 - 0.001872m - 0.52\,(0.02)^2\,(0.2)\right)
\end{aligned} \tag{145}$$

We show that the above is decreasing in m by upper bounding the first partial m by

$$0.11864\,(0.25)^5 + 0.00879\,(0.25)^4 - 0.07022\,(0.15)^3 - 0.00463\,(0.15)^2 - 0.03523\,(0.15) - 0.00148,$$

which is negative.

So, we can upper bound Equation 145 in m by $m = 0.15$, which is negative.

For Equation 137, we scale by $(2m\eta + 0.25m + 0.02)^{\frac{3}{2}}$, resulting in the following expression.

$$
\begin{aligned}
&-2.916m^2\eta - 0.226m^2 - 0.03888m + 0.45m^4 \\
&+\Bigg(\left(0.45m^2+0.26\right)^2\frac{m^2}{\eta} + 0.5\left(0.45m^2+0.26\right)^2\frac{m^2}{\eta^2} + 0.04\left(0.45m^2+0.26\right)^2\frac{m}{\eta^2} \\
&\quad + (0.25)^2\left(0.45m^2+0.26\right)^2\frac{m^2}{\eta^3} + (0.5)(0.02)\left(0.45m^2+0.26\right)^2\frac{m}{\eta^3} \\
&\quad + (0.02)^2\left(0.45m^2+0.26\right)^2\frac{1}{\eta^3} - 2(0.972)\left(0.45m^2+0.26\right)m^2 \\
&\quad - 0.04(0.972)\left(0.45m^2+0.26\right)\frac{m}{\eta} - 0.5(0.972)\left(0.45m^2+0.26\right)\frac{m^2}{\eta} + (0.972)^2 m^2\eta\Bigg)\sqrt{2m\eta + 0.25m + 0.02}
\end{aligned}
\tag{146}
$$

To upper bound the above, we show that it is decreasing in η. To do so, we first scale the partial η by $\sqrt{2m\eta + 0.25m + 0.02}$ and replace $\sqrt{2m\eta + 0.25m + 0.02}$ with $\sqrt{2(0.15)(0.2) + 0.25(0.15) + 0.02}$, since the corresponding coefficient ($-2.916m^2$) is negative.

Note that the terms of upper bound for the partial η (shown below) are all negative.

$$
\begin{aligned}
&-2\left(0.45m^2+0.26\right)^2\frac{m^3}{\eta^2} - (0.12)(0.25)\left(0.45m^2+0.26\right)^2\frac{m^2}{\eta^3} - (0.06)(0.02)\left(0.45m^2+0.26\right)^2\frac{m}{\eta^3} \\
&-2(0.5)(0.06)\left(0.45m^2+0.26\right)^2\frac{m^2}{\eta^3} - (6)(0.02)^2\left(0.45m^2+0.26\right)^2\frac{m}{\eta^3} - (3)(0.25)(0.02)^2\left(0.45m^2+0.26\right)^2\frac{m}{\eta^4} \\
&-(3)(0.02)^3\left(0.45m^2+0.26\right)^2\frac{1}{\eta^4} - 2(0.972)(0.45)m^5 - 0.83538m^2 - 0.01837\left(0.45m^2+0.26\right)\frac{m^2}{\eta^2} \\
&-0.00102\left(0.45m^2+0.26\right)\frac{m}{\eta^2} - 0.01919\left(0.45m^2+0.26\right)\frac{m^3}{\eta^2} - 0.07367\left(0.45m^2+0.26\right)\frac{m^2}{\eta} \\
&-0.375\left(0.45m^2+0.26\right)^2\frac{m^3}{\eta^2} - 0.90965\left(0.45m^2+0.26\right)\frac{m^3}{\eta}
\end{aligned}
\tag{147}
$$

So, we upper bound Equation 146 by fixing $\eta = 0.2$, shown below.

$$
\begin{aligned}
&-0.8092m^2 - 0.03888m + 0.45m^4 \\
&+\left(5.12579m^6 + 0.45563m^5 + 3.96496m^4 + 0.43902m^3 + 0.77455m^2 + 0.10156m + 0.00338\right)\sqrt{0.65m + 0.02}
\end{aligned}
\tag{148}
$$

Since the square root term is positive, we replace $\sqrt{0.65m + 0.02}$ with $\sqrt{0.65(0.25) + 0.02}$.

We then upper bound the resulting expression by

$$
\begin{aligned}
&5.12579(0.42721)(0.25)^6 + 0.45563(0.42721)(0.25)^5 + 2.14388(0.25)^2 m^2 \\
&+ 0.43902(0.42721)(0.25)m^2 - 0.47830m^2 + 0.00451m + 0.00338(0.42721),
\end{aligned}
$$

which simplifies (by rounding up after five digits of precision) to $(-0.29741m + 0.00451)m + 0.00217$

Note that we can upper bound $(-0.29741m + 0.00451)m + 0.00217$ by $(-0.29741(0.15) + 0.00451)(0.15) + 0.00217$, which is negative.

This completes the proof. □

Proof of Lemma B.67. To begin the proof, we first show that the coefficients for the $\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{0.1175}\right)\right)$ and $\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{0.1175}\right)\right)$ terms in Equation 45 are negative.

We begin with the coefficient for $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)$. Note that the coefficient can upper bounded (after scaling by η) by

$$-\frac{(0.45)^2}{\sqrt{0.1175}}m^4+\left(-\frac{(2)(0.26)(0.45)}{\sqrt{0.1175}}-\frac{(2)(0.972)(0.45)}{\sqrt{0.1175}}(0.2)+0.225\right)m^2 \\ +\left(\left(0.486+\frac{1}{2\sqrt{0.1175}}-\frac{(2)(0.972)(0.26)}{\sqrt{0.1175}}-\frac{(0.972)^2}{\sqrt{0.1175}}(0.2)\right)(0.2)-\frac{(0.26)^2}{\sqrt{0.1175}}+0.13\right), \tag{149}$$

where all the coefficients of the expression are negative.

Similarly, for the coefficient of $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{0.1175}\right)\right)$, it can be upper bounded by (after scaling by η)

$$-\frac{(0.45)^2}{\sqrt{0.1175}}m^4-\frac{(2)(0.26)(0.45)}{\sqrt{0.1175}}m^2-\frac{(2)(0.45)(0.972)}{\sqrt{0.1175}}\eta m^2-0.225m^2 \\ +\left(-0.13-\frac{(0.26)^2}{\sqrt{0.1175}}+\left(-0.486+\frac{1}{2\sqrt{0.1175}}-\frac{(2)(0.972)(0.26)}{\sqrt{0.1175}}-\frac{(0.972)^2}{\sqrt{0.1175}}(0.2)\right)(0.2)\right), \tag{150}$$

where all the coefficients in the expression are negative.

So, we can lower bound Equation 45 by replacing $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)$ with $\exp\left(\left(\frac{0.45(0.25)^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)$, $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{0.1175}\right)\right)$ with $\exp\left(\left(\frac{0.45(0.15)^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{0.1175}\right)\right)$, and $\exp\left(\frac{0.9m^3}{\eta}+\frac{0.52m}{\eta}+1.944m\right)$ with $\exp\left(\frac{0.9(0.15)^2m}{\eta}+\frac{0.52m}{\eta}+1.944m\right)$.

We will show that this lower bound is decreasing in m. To do so, we split the first partial m into the following two parts.

Part 1

$$\left(-\frac{1.8}{\eta\sqrt{0.1175}}\left(0.45m^3+0.26m+0.972\eta m\right)-\frac{0.45}{\eta}m-\frac{0.45(0.15)^2+0.26+0.972\eta}{\eta^2\sqrt{0.1175}}\left(0.45m^2+0.26+0.972\eta\right)^2 \right. \\ \left. +\frac{0.225(0.15)^2+0.13+0.486\eta}{\eta\sqrt{0.1175}}-\frac{0.225(0.15)^2+0.13+0.486\eta}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)\right) \\ \cdot\exp\left(\left(\frac{0.45(0.15)^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{0.1175}\right)\right) \\ +(0.38)\left(\frac{0.45(0.15)^2+0.26+0.972\eta}{\eta\sqrt{0.1175}}\right)\exp\left(\frac{0.9(0.15)^2m}{\eta}+\frac{0.52m}{\eta}+1.944m\right) \tag{151}$$

Part 2

$$\left(-\frac{1.8m}{\eta\sqrt{0.1175}}\left(0.45m^2+0.26+0.972\eta\right)+\frac{0.45}{\eta}m-\frac{0.45\left(0.25\right)^2+0.26+0.972\eta}{\eta^2\sqrt{0.1175}}\left(0.45m^2+0.26+0.972\eta\right)^2\right.$$
$$\left.+\frac{0.225\left(0.25\right)^2+0.13+0.486\eta}{\eta\sqrt{0.1175}}+\frac{0.225\left(0.25\right)^2+0.13+0.486\eta}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)\right)$$
$$\cdot\exp\left(\left(\frac{0.45\left(0.25\right)^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)$$

$$+0.62\left(\frac{0.45\left(0.15\right)^2+0.26+0.972\eta}{\eta\sqrt{0.1175}}\right)\exp\left(\frac{0.9\left(0.15\right)^2m}{\eta}+\frac{0.52m}{\eta}+1.944m\right) \tag{152}$$

In Equation 151, we upper bound it by replacing $\exp\left(\frac{0.9(0.15)^2m}{\eta}+\frac{0.52m}{n}+1.944m\right)$ with $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)\exp\left(\left(\frac{0.45(0.15)^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{0.1175}\right)\right)$ and factoring out $\exp\left(\left(\frac{0.45(0.15)^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{0.1175}\right)\right)$. We prove the resulting expression (scaled by η^2) is negative in lemma B.68.

Similarly, in Equation 152, we upper bound it by replacing $\exp\left(\frac{0.9(0.15)^2m}{\eta}+\frac{0.52m}{\eta}+1.944m\right)$ with $\exp\left(\left(\frac{0.45(0.25)^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{0.1175}\right)\right)$ and factoring out $\exp\left(\left(\frac{0.45(0.25)^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)$. We prove the resulting expression (scaled by η^2) is negative in lemma B.69.

We then further lower bound Equation 45 by fixing $m=0.25$. Last, we will show that the resulting expression is concave in η. Consider the second partial η below (round up after five digits of precision and scaled by η^5).

$$\left(0.39402\eta^2+0.02661\eta\right)\exp\left(\frac{0.1350625}{\eta}+0.486\right)$$
$$+\left(-0.00286-0.05797\eta-0.17053\eta^2\right)\exp\left(\frac{\left(0.288125\right)\left(0.25+\sqrt{0.1175}\right)}{\eta}+0.972\left(0.25+\sqrt{0.1175}\right)\right)$$
$$+\left(-0.00024+0.03831\eta-0.74106\eta^2\right)\exp\left(\frac{\left(0.270125\right)\left(0.25-\sqrt{0.1175}\right)}{\eta}+0.972\left(0.25-\sqrt{0.1175}\right)\right) \tag{153}$$

To further upper bound the above, note that $-0.00024+\left(0.03831-0.74106\left(0.2\right)\right)\left(0.2\right)$ is negative, so we can create the following upper bound

$$\left(0.39402\eta^2+0.02661\eta\right)\exp\left(\frac{0.1350625}{0.2}+0.486\right)$$
$$+\left(-0.00286-0.05797\eta-0.17053\eta^2\right)\exp\left(\frac{\left(0.288125\right)\left(0.25+\sqrt{0.1175}\right)}{0.3}+0.972\left(0.25+\sqrt{0.1175}\right)\right)$$
$$+\left(-0.00024+0.03831\eta-0.74106\eta^2\right)\exp\left(\frac{\left(0.270125\right)\left(0.25-\sqrt{0.1175}\right)}{0.2}+0.972\left(0.25-\sqrt{0.1175}\right)\right), \tag{154}$$

which simplifies (RU) to

$$0.12501\eta^2 - 0.06638\eta - 0.00918.$$

The above can be upper bounded by

$$(0.12501\,(0.3) - 0.06638)\,(0.2) - 0.00918,$$

which is negative.

So, we evaluate our lower bound for Equation 45 at $\eta = 0.2, 0.3$, both of which are positive. This completes the proof.

□

Proof of Lemma B.68. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we begin by showing that Equation 46 is convex in m. To do, we will show that the second partial m (shown below) of Equation 46 is positive.

$$a_1 + a_2 \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{0.1175}\right)\right), \tag{155}$$

where

$$\begin{aligned} a_1 =& -\frac{(1.8)\,(2.7)}{\sqrt{0.1175}}\eta m - \frac{0.65640375}{\sqrt{0.1175}}m^2 - \frac{(1.35)\,(0.972)\,(1.8)}{\sqrt{0.1175}}\eta m^2 - \frac{0.1264185}{\sqrt{0.1175}} - \frac{0.9275067}{\sqrt{0.1175}}\eta \\ & - \frac{(0.972)^2\,(1.8)}{\sqrt{0.1175}}\eta^2 - 0.12155625 - (0.486)\,(0.9)\,\eta \end{aligned} \tag{156}$$

and

$$\begin{aligned} a_2 =& \frac{0.27714825}{\sqrt{0.1175}}m + \frac{(2.7)\,(0.972)\,(0.38)}{\sqrt{0.1175}}\eta m + 0.09238275 + (0.9)\,(0.972)\,(0.38)\,\eta + \frac{0.1870750687}{\eta\sqrt{0.1175}}m^4 \\ & + \frac{(1.35)^2\,(0.972)\,(0.38)}{\sqrt{0.1175}}m^4 + \frac{0.249433425}{\eta}m^3 + (2)\,(0.9)\,(1.35)\,(0.972)\,(0.38)\,m^3 + \frac{0.08182802081}{\eta\sqrt{0.1175}}m^2 \\ & + \frac{0.528678819}{\sqrt{0.1175}}m^2 + \frac{2\,(1.35)\,(0.972)^2\,(0.38)}{\sqrt{0.1175}}\eta m^2 + (0.9)^2\,(0.972)\,(0.38)\,\sqrt{0.1175}m^2 + \frac{0.04803903}{\eta}m \\ & + 0.352452546m + 2\,(0.9)\,(0.972)^2\,(0.38)\,\eta m + \frac{0.006938971}{\eta\sqrt{0.1175}} \\ & + \frac{0.0768508884}{\sqrt{0.1175}} + \frac{0.283669034}{\sqrt{0.1175}}\eta + \frac{(0.972)^3\,(0.38)}{\sqrt{0.1175}}\eta^2 \end{aligned} \tag{157}$$

To proceed, we naively lower bound the second partial m in η. So, we lower bound a_1 by

$$\begin{aligned} & -\frac{(1.8)\,(2.7)}{\sqrt{0.1175}}(0.3)m - \frac{0.65640375}{\sqrt{0.1175}}m^2 - \frac{(1.35)\,(0.972)\,(1.8)}{\sqrt{0.1175}}(0.3)m^2 \\ & - \frac{0.1264185}{\sqrt{0.1175}} - \frac{0.9275067}{\sqrt{0.1175}}(0.3) - \frac{(0.972)^2\,(1.8)}{\sqrt{0.1175}}(0.3)^2 \\ & - 0.12155625 - (0.486)\,(0.9)\,(0.3). \end{aligned}$$

We also lower bound a_2 by

$$\begin{aligned}a_2 =& \frac{0.27714825}{\sqrt{0.1175}}m + \frac{(2.7)\,(0.972)\,(0.38)}{\sqrt{0.1175}}(0.2)m + 0.09238275 + (0.9)\,(0.972)\,(0.38)\,(0.2) + \frac{0.1870750687}{(0.3)\sqrt{0.1175}}m^4 \\ &+ \frac{(1.35)^2\,(0.972)\,(0.38)}{\sqrt{0.1175}}m^4 + \frac{0.249433425}{(0.3)}m^3 + (2)\,(0.9)\,(1.35)\,(0.972)\,(0.38)\,m^3 + \frac{0.08182802081}{(0.3)\sqrt{0.1175}}m^2 \\ &+ \frac{0.528678819}{\sqrt{0.1175}}m^2 + \frac{2\,(1.35)\,(0.972)^2\,(0.38)}{\sqrt{0.1175}}(0.2)m^2 + (0.9)^2\,(0.972)\,(0.38)\,\sqrt{0.1175}m^2 + \frac{0.04803903}{(0.3)}m \\ &+ 0.352452546m + 2\,(0.9)\,(0.972)^2\,(0.38)\,(0.2)m + \frac{0.006938971}{(0.3)\sqrt{0.1175}} \\ &+ \frac{0.0768508884}{\sqrt{0.1175}} + \frac{0.283669034}{\sqrt{0.1175}}(0.2) + \frac{(0.972)^3\,(0.38)}{\sqrt{0.1175}}(0.2)^2.\end{aligned}$$

Lastly, we lower bound $\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{0.1175}\right)\right)$ by $\exp\left(\left(\frac{0.45(0.15)^2}{0.3} + \frac{0.26}{0.3} + 0.972\right)\left(0.15 + \sqrt{0.1175}\right)\right)$.

The resulting expression simplifies (by rounding down after five digits of precision) to

$$9.51822m^4 + 4.35024m^3 + 3.58161m^2 + 0.85977m - 0.22735$$

that can be lower bounded at $m = 0.15$, which is positive.

So, we evaluate Equation 46 at $m = 0.15, 0.25$.

Case 1: $m = 0.15$. For the $m = 0.15$ case, we will show that it is decreasing in η. Consider the partial η (scaled by $\eta\sqrt{0.1175}$ and rounded up after five digits of precision) below.

$$\begin{aligned}&-2.75499\eta^3 - 1.40799\eta^2 - 0.26378\eta \\ &+ \left(0.73872\eta^2 + 0.05349\eta - 0.01366\right)\exp\left(\frac{(0.270125)\,(0.15) + (0.270125)\,\sqrt{0.1175}}{\eta} + 0.972\left(0.15 + \sqrt{0.1175}\right)\right)\end{aligned} \tag{158}$$

Since $0.73872(0.2)^2 + 0.05349(0.2) - 0.01366$ is positive, we can upper bound $\exp\left(\frac{(0.270125)(0.15)+(0.270125)\sqrt{0.1175}}{\eta} + 0.972\left(0.15 + \sqrt{0.1175}\right)\right)$ by $\exp\left(\frac{(0.270125)(0.15)+(0.270125)\sqrt{0.1175}}{0.2} + 0.972\left(0.15 + \sqrt{0.1175}\right)\right)$.

The resulting simplified expression (RU)

$$-2.75499\eta^3 + 0.91235\eta^2 - 0.09579\eta - 0.04291$$

can be upper bounded by

$$\left(-2.75499\,(0.2)^2 + 0.91235\,(0.3) - 0.09579\right)(0.3) - 0.04291,$$

which is negative.

So, we can upper bound the expression for the $m = 0.15$ case at $\eta = 0.2$, which is negative.

Case 2: $m = 0.25$. For the $m = 0.25$ case, we also show that the first partial η (shown below after scaling by $\eta\sqrt{0.1175}$ and RU) is negative.

$$\begin{aligned}&\left(-2.75499\eta^3 - 1.82593\eta^2 - 0.35815\eta\right) \\ &+ \left(0.73872\eta^2 + 0.03957\eta - 0.01753\right)\exp\left(\frac{(0.288125)\,(0.25) + (0.288125)\,\sqrt{0.1175}}{\eta} + 0.972\left(0.25 + \sqrt{0.1175}\right)\right)\end{aligned} \tag{159}$$

Since $0.73872(0.2)^2 + 0.03957(0.2) - 0.01753$ is positive, we can upper bound $\exp\left(\frac{(0.288125)(0.25)+(0.288125)\sqrt{0.1175}}{\eta} + 0.972\left(0.25 + \sqrt{0.1175}\right)\right)$ by $\exp\left(\frac{(0.288125)(0.25)+(0.288125)\sqrt{0.1175}}{0.2} + 0.972\left(0.25 + \sqrt{0.1175}\right)\right)$.

The resulting expression can be simplified (RU) to

$$-2.75499\eta^3 + 1.26146\eta^2 - 0.19280\eta - 0.07327$$

that can upper bounded by

$$\left(-2.75499\left(0.2\right)^2 + 1.26146\left(0.3\right) - 0.19280\right)\left(0.3\right) - 0.07327,$$

which is negative. So, we can upper bound the expression for the $m = 0.25$ case by $\eta = 0.2$, which is negative.

This completes the proof.

□

Proof of Lemma B.69. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we proceed by showing that the first partial m of Equation 47 is negative. We begin with the following upper bound for the first partial m.

$$\begin{aligned}
&-\frac{0.23338125}{\sqrt{0.1175}}m^3 - \frac{2.43}{\sqrt{0.1175}}\eta m^2 + \left(-\frac{0.468}{\sqrt{0.1175}} + 0.45\right)\eta - \frac{1.7496}{\sqrt{0.1175}}\eta^2 - \frac{0.78732}{\sqrt{0.1175}}\eta m^3 - \frac{0.1348425}{\sqrt{0.1175}}m \\
&+ \left(-\frac{0.9589995}{\sqrt{0.1175}} + 0.4374\right)\eta m - \frac{1.7006112}{\sqrt{0.1175}}\eta^2 m + \left(0.1440625\right)\left(0.9\right)m \\
&\\
&+ \left(\frac{0.226094625}{\sqrt{0.1175}}m^2 + \frac{0.813564}{\sqrt{0.1175}}\left(0.3\right)m^2 - 0.15072975m - 0.542376\eta m + \frac{0.04354415}{\sqrt{0.1175}}\right. \\
&\left. + \frac{0.31947453}{\sqrt{0.1175}}\left(0.3\right) + \frac{\left(0.972\right)^2\left(0.62\right)}{\sqrt{0.1175}}\left(0.3\right)^2\right)\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{0.1175}\right)\right)
\end{aligned} \tag{160}$$

From the above upper bound, we will show that the coefficient for the $\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{0.1175}\right)\right)$ term is positive. To see this, we first replace $-0.542376\eta m$ with $-0.542376(0.3)m$ and group terms into the following.

$$\begin{aligned}
&\left(\frac{0.226094625}{\sqrt{0.1175}} + \frac{0.813564}{\sqrt{0.1175}}\left(0.3\right)\right)m^2 + \left(-0.15072975 - 0.542376(0.3)\right)m \\
&+ \frac{0.04354415 + 0.31947453\left(0.3\right) + \left(0.972\right)^2\left(0.62\right)\left(0.3\right)}{\sqrt{0.1175}}
\end{aligned} \tag{161}$$

The above can then be lower bounded by

$$\begin{aligned}
&\left(\frac{0.226094625}{\sqrt{0.1175}} + \frac{0.813564}{\sqrt{0.1175}}\left(0.3\right)\right)\left(0.15\right)^2 + \left(-0.15072975 - 0.542376(0.3)\right)\left(0.25\right) \\
&+ \frac{0.04354415 + 0.31947453\left(0.3\right) + \left(0.972\right)^2\left(0.62\right)\left(0.3\right)}{\sqrt{0.1175}},
\end{aligned}$$

which is positive.

So, we can upper bound Equation 160 by replacing $\exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{0.1175}\right)\right)$ with $\exp\left(\left(\frac{0.45(0.15)^2}{0.3} + \frac{0.26}{0.3} + 0.972\right)\left(0.25 - \sqrt{0.1175}\right)\right)$. We then simplify the resulting expression (by rounding up after five digits of precision) to the following.

$$-1.14021m^3 - 0.26493m^2 - 1.15209m + 0.08384$$

We evaluate the above at $m = 0.15$, which is negative.

So, we upper bound Equation 47 by fixing $m = 0.15$. After simplifying (RU) the resulting expression, we get the following.

$$-2.67904\eta^3 - 1.15859\eta^2 - 0.10198\eta - 0.02241$$
$$+ \left(0.48859\eta + 1.75809\eta^2\right) \exp\left(\left(\frac{0.45\left(0.15\right)^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(0.15 - \sqrt{0.1175}\right)\right) \tag{162}$$

We will show that Equation 162 is concave in η and show the linearization about $\eta = 0.21$ is negative. For concavity in η, consider the second partial η of Equation 162 below (RU and scaled by η^3).

$$\left(0.00133 + 0.00477\eta + 0.18311\eta^2 + 3.51618\eta^3\right) \exp\left(\frac{\left(0.270125\right)\left(0.15 - \sqrt{0.1175}\right)}{\eta} + 0.972\left(0.15 - \sqrt{0.1175}\right)\right)$$
$$- 16.07424\eta^4 - 2.31718\eta^3 \tag{163}$$

Note that we can upper bound the above by replacing $\exp\left(\frac{\left(0.270125\right)\left(0.15-\sqrt{0.1175}\right)}{\eta} + 0.972\left(0.15 - \sqrt{0.1175}\right)\right)$ with $\exp\left(\frac{\left(0.270125\right)\left(0.15-\sqrt{0.1175}\right)}{0.3} + 0.972\left(0.15 - \sqrt{0.1175}\right)\right)$.

The resulting upper bound can then be simplified (RU) to

$$-16.07424\eta^4 + 0.13360\eta^3 + 0.12763\eta^2 + 0.00333\eta + 0.00093$$

that can be further upper bounded by

$$\left(-16.07424\left(0.2\right)^3 + 0.13360\left(0.3\right)^2 + 0.12763\left(0.3\right) + 0.00333\right)\left(0.2\right) + 0.00093,$$

which is negative.

Using concavity in η, we linearize Equation 162 about $\eta = 0.21$ and evaluate the linearization at $\eta = 0.2$, which is negative.

This completes the proof.

□

Proof of Lemma B.70. We begin the proof by noting the second partial derivative of Equation 48 with respect to c below, after scaling by $4c^{3/2}$. To show the second partial is positive, we first show that it is increasing in c, develop a lower bound that is decreasing in η, and create another lower bound that is decreasing in m.

$$-1 - \exp\left(\frac{0.9m^3}{\eta} + \frac{0.52m}{\eta} + 1.944m\right)$$
$$+ \left(\frac{c}{\eta^2}\left(0.45m^2 + 0.26 + 0.972\eta\right)^2 - 1 - \frac{\sqrt{c}}{\eta}\left(0.45m^2 + 0.26 + 0.972\eta\right) + \frac{2}{n}\left(0.45m^2 + 0.26 + 0.972\eta\right)^2\right.$$
$$\left. + \frac{2\sqrt{c}}{\eta^2}\left(0.45m^2 + 0.26 + 0.972\eta\right)^3\right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m - \sqrt{c}\right)\right)$$
$$+ \left(\frac{c}{\eta^2}\left(0.45m^2 + 0.26 + 0.972\eta\right)^2 - 1 + \frac{\sqrt{c}}{\eta}\left(0.45m^2 + 0.26 + 0.972\eta\right) + \frac{2}{\eta}\left(0.45m^2 + 0.26 + 0.972\eta\right)^2\right.$$
$$\left. - \frac{2\sqrt{c}}{\eta^2}\left(0.45m^2 + 0.26 + 0.972\eta\right)^3\right) \exp\left(\left(\frac{0.45m^2}{\eta} + \frac{0.26}{\eta} + 0.972\right)\left(m + \sqrt{c}\right)\right) \tag{164}$$

To proceed, we show that the second partial derivative above grows in c. The partial derivative with respect to c can be split up into two parts.

Part 1

$$\begin{aligned}
&\left(\frac{1}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^2+\frac{1}{2\eta\sqrt{c}}\left(0.45m^2+0.26+0.972\eta\right)\right.\\
&\left.-\frac{1}{\eta^2\sqrt{c}}\left(0.45m^2+0.26+0.972\eta\right)^3\right)\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{c}\right)\right)\\
&+\left(\frac{c}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^2-1+\frac{\sqrt{c}}{\eta}\left(0.45m^2+0.26+0.972\eta\right)\right.\\
&\left.+\frac{2}{\eta}\left(0.45m^2+0.26+0.972\eta\right)^2-\frac{2\sqrt{x}}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^3\right)\\
&\cdot\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(\frac{1}{2\sqrt{c}}\right)\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{c}\right)\right)
\end{aligned}\tag{165}$$

Part 2

$$\begin{aligned}
&\left(\frac{1}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^2-\frac{1}{2\eta\sqrt{c}}\left(0.45m^2+0.26+0.972\eta\right)\right.\\
&\left.+\frac{1}{\eta^2\sqrt{c}}\left(0.45m^2+0.26+0.972\eta\right)^3\right)\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{c}\right)\right)\\
&+\left(\frac{c}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^2-1-\frac{\sqrt{c}}{\eta}\left(0.45m^2+0.26+0.972\eta\right)\right.\\
&\left.+\frac{2}{\eta}\left(0.45m^2+0.26+0.972\eta\right)^2+\frac{2\sqrt{c}}{\eta^2}\left(0.45m^2+0.26+0.972\eta\right)^3\right)\\
&\cdot\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(-\frac{1}{2\sqrt{c}}\right)\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{c}\right)\right)
\end{aligned}\tag{166}$$

To proceed, first note that Equation 165 is positive by lemma B.74. So, we can lower bound Equation 165 by replacing $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{c}\right)\right)$ with $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{c}\right)\right)$ and show that the sum with Equation 166 is positive.

The resulting expression, with $\left(0.45m^2+0.26+0.972\eta\right)^2\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{c}\right)\right)$ factored out, simplifies to the following.

$$\frac{3}{\eta^2}-\frac{2}{\eta^3}\left(0.45m^2+0.26+0.972\eta\right)^2\tag{167}$$

After scaling by η^3, we can lower bound the resulting expression by the following, which is positive.

$$\begin{aligned}
&-2\left(0.45\right)^2\left(0.25\right)^4-\left(0.52\right)\left(0.9\right)\left(0.25\right)^2-\left(1.944\right)\left(0.9\right)\left(0.25\right)^2\left(0.3\right)-2\left(0.26\right)^2\\
&+\left(-\left(1.944\right)\left(0.52\right)+3\right)\left(0.2\right)-\left(2\right)\left(0.972\right)^2\left(0.3\right)^2
\end{aligned}$$

So, we can lower bound Equation 164 in c by $c = 0.1175$. By lemmas **??** and B.74, we can lower further lower Equation 164 by replacing $\exp\left(\frac{0.9m^3}{\eta}+\frac{0.52m}{\eta}+1.944m\right)$

by $\exp\left(\frac{0.9m^3}{0.2}+\frac{0.52m}{0.2}+1.944m\right)$, $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)$ by $\exp\left(\left(\frac{0.45m^2}{0.3}+\frac{0.26}{0.3}+0.972\right)\left(m+\sqrt{0.1175}\right)\right)$, and $\exp\left(\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)\left(m-\sqrt{0.1175}\right)\right)$ by $\exp\left(\left(\frac{0.45m^2}{0.2}+\frac{0.26}{0.2}+0.972\right)\left(m-\sqrt{0.1175}\right)\right)$.

We next show that the resulting lower bound is decreasing in η. Consider the two parts of the partial derivative with respect to η below.

$$\left(0.235\left(\frac{-0.45m^2}{\eta^2}+\frac{-0.26}{\eta^2}\right)\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)+\sqrt{0.1175}\left(\frac{-0.45m^2}{\eta^2}+\frac{-0.26}{\eta^2}\right)\right.$$
$$+2\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)^2+4\eta\left(\frac{-0.45m^2}{\eta^2}+\frac{-0.26}{\eta^2}\right)\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)$$
$$\left.-2\sqrt{0.1175}\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)^3-6\eta\sqrt{0.1175}\left(\frac{-0.45m^2}{\eta^2}+\frac{-0.26}{\eta^2}\right)\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)^2\right)$$
$$\cdot\exp\left(\left(\frac{0.45m^2}{0.3}+\frac{0.26}{0.3}+0.972\right)\left(m+\sqrt{0.1175}\right)\right) \tag{168}$$

$$\left(0.235\left(\frac{-0.45m^2}{\eta^2}+\frac{-0.26}{\eta^2}\right)\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)-\sqrt{0.1175}\left(\frac{-0.45m^2}{\eta^2}+\frac{-0.26}{\eta^2}\right)\right.$$
$$+2\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)^2+4\eta\left(\frac{-0.45m^2}{\eta^2}+\frac{-0.26}{\eta^2}\right)\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)$$
$$\left.+2\sqrt{0.1175}\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)^3+6\eta\sqrt{0.1175}\left(\frac{-0.45m^2}{\eta^2}+\frac{-0.26}{\eta^2}\right)\left(\frac{0.45m^2}{\eta}+\frac{0.26}{\eta}+0.972\right)^2\right)$$
$$\cdot\exp\left(\left(\frac{0.45m^2}{0.2}+\frac{0.26}{0.2}+0.972\right)\left(m-\sqrt{0.1175}\right)\right) \tag{169}$$

We show that each part (scaled by η^3) is negative in lemmas B.71 and B.72. So, we can use $\eta = 0.3$ to lower bound Equation 164 (lower bounded by $c = 0.1175$ with the replacements for the exponential terms).

We can further lower bound the resulting expression by the following.

$$-1-\exp\left(\frac{0.9m^3}{0.2}+\frac{0.52m}{0.2}+1.944m\right)$$

$$+\left(\frac{0.7175}{(0.3)^2}\left(0.45m^2+0.5516\right)^2-1-\frac{\sqrt{0.1175}}{0.3}\left(0.45m^2+0.5516\right)+\frac{2\sqrt{0.1175}}{(0.3)^2}\left(0.45m^2+0.5516\right)^3\right)$$
$$\cdot\exp\left(\left(\frac{0.45\left(0.25\right)^2+0.4544}{0.2}\right)\left(m-\sqrt{0.1175}\right)\right)$$

$$+\left(\frac{0.7175}{(0.3)^2}\left(0.45m^2+0.5516\right)^2-1+\frac{\sqrt{0.1175}}{0.3}\left(0.45m^2+0.5516\right)-\frac{2\sqrt{0.1175}}{(0.3)^2}\left(0.45m^2+0.5516\right)^3\right)$$
$$\cdot\exp\left(\left(\frac{0.45\left(0.15\right)^2+0.5516}{0.3}\right)\left(m+\sqrt{0.1175}\right)\right) \tag{170}$$

We next show that the above is decreasing in m by proving concavity in m first and showing the first partial m at $m = 0.15$ is negative.

Consider the second partial m below, split into two parts.

Part 1

$$a_{1,1}\exp\left(\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\left(m+\sqrt{0.1175}\right)\right)$$

$$+a_{1,2}\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\exp\left(\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\left(m+\sqrt{0.1175}\right)\right)$$

$$-0.49\left(\left(13.5\right)^{2}m^{4}+2\left(4.544\right)\left(13.5\right)m^{2}+27m+\left(4.544\right)^{2}\right)\exp\left(\frac{0.9m^{3}}{0.2}+\frac{0.52m}{0.2}+1.944m\right), \tag{171}$$

where

$$\begin{aligned}a_{1,1}=&14.35\left(0.45m^{2}+0.5516\right)+14.35\left(0.9m\right)m+3\sqrt{0.1175}-60\sqrt{0.1175}\left(0.45m^{2}+0.5516\right)^{2}\\&-120\left(0.9m\right)m\sqrt{0.1175}\left(0.45m^{2}+0.5516\right)+\frac{0.7175\cdot2\cdot0.9m}{\left(0.3\right)^{2}}\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\left(0.45m^{2}+0.5516\right)\\&+\frac{\sqrt{0.1175}}{0.3}\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\left(0.9m\right)-\frac{6\sqrt{0.1175}}{\left(0.3\right)^{2}}\left(0.9m\right)\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\left(0.45m^{2}+0.5516\right)^{2}\end{aligned}$$

$$\begin{aligned}a_{1,2}=&14.35m\left(0.45m^{2}+0.5516\right)+3m\sqrt{0.1175}-60m\sqrt{0.1175}\left(0.45m^{2}+0.5516\right)^{2}\\&+\frac{0.7175}{\left(0.3\right)^{2}}\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\left(0.45m^{2}+0.5516\right)^{2}-\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\\&+\frac{\sqrt{0.1175}}{0.3}\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\left(0.45m^{2}+0.5516\right)\\&-\frac{2\sqrt{0.1175}}{\left(0.3\right)^{2}}\left(\frac{0.45\left(0.15\right)^{2}+0.5516}{0.3}\right)\left(0.45m^{2}+0.5516\right)^{3}.\end{aligned}$$

Part 2

$$a_{2,1}\exp\left(\left(\frac{0.45\left(0.25\right)^{2}+0.4544}{0.2}\right)\left(m-\sqrt{0.1175}\right)\right)$$

$$+a_{2,2}\left(\frac{0.45\left(0.25\right)^{2}+0.4544}{0.2}\right)\exp\left(\left(\frac{0.45\left(0.25\right)^{2}+0.4544}{0.2}\right)\left(m-\sqrt{0.1175}\right)\right)$$

$$-0.51\left(\left(13.5\right)^{2}m^{4}+2\left(4.544\right)\left(13.5\right)m^{2}+27m+\left(4.544\right)^{2}\right)\exp\left(\frac{0.9m^{3}}{0.2}+\frac{0.52m}{0.2}+1.944m\right), \tag{172}$$

where

$$a_{2,1} = 14.35\left(0.45m^2+0.5516\right)+14.35m\left(0.9m\right)-3\sqrt{0.1175}+60\sqrt{0.1175}\left(0.45m^2+0.5516\right)^2$$
$$+120\left(0.9m\right)m\sqrt{0.1175}\left(0.45m^2+0.5516\right)+\frac{0.7175\cdot 2\cdot 0.9m}{\left(0.3\right)^2}\left(\frac{0.45\left(0.25\right)^2+0.4544}{0.2}\right)\left(0.45m^2+0.5516\right)$$
$$-\frac{\sqrt{0.1175}}{0.3}\left(\frac{0.45\left(0.25\right)^2+0.4544}{0.2}\right)\left(0.9m\right)+\frac{6\sqrt{0.1175}}{\left(0.3\right)^2}0.9m\left(\frac{0.45\left(0.25\right)^2+0.4544}{0.2}\right)\left(0.45m^2+0.5516\right)^2$$

$$a_{2,2} = 14.35m\left(0.45m^2+0.5516\right)-3m\sqrt{0.1175}+60m\sqrt{0.1175}\left(0.45m^2+0.5516\right)^2$$
$$+\frac{0.7175}{\left(0.3\right)^2}\left(\frac{0.45\left(0.25\right)^2+0.4544}{0.2}\right)\left(0.45m^2+0.5516\right)^2-\left(\frac{0.45\left(0.25\right)^2+0.4544}{0.2}\right)$$
$$-\frac{\sqrt{0.1175}}{0.3}\left(\frac{0.45\left(0.25\right)^2+0.4544}{0.2}\right)\left(0.45m^2+0.5516\right)$$
$$+\frac{2\sqrt{0.1175}}{\left(0.3\right)^2}\left(\frac{0.45\left(0.25\right)^2+0.4544}{0.2}\right)\left(0.45m^2+0.5516\right)^3$$

Note that we can upper bound Equation 171 by replacing $\exp\left(\frac{0.9m^3}{0.2}+\frac{0.52m}{0.2}+1.944m\right)$ with $\exp\left(\left(\frac{0.45(0.25)^2}{0.2}+\frac{0.26}{0.2}+0.972\right)\left(0.15-\sqrt{0.1175}\right)\right)\exp\left(\left(\frac{0.45(0.15)^2+0.5516}{0.3}\right)\left(m+\sqrt{0.1175}\right)\right)$ and factoring out $\exp\left(\left(\frac{0.45(0.15)^2+0.5516}{0.3}\right)\left(m+\sqrt{0.1175}\right)\right)$.

The resulting expression (simplified by rounding up after five digits of precision)

$$-2.43359m^6-15.59651m^5-80.20102m^4-14.05347m^3-44.30714m^2+1.74953m-0.94259$$

can be further upper bounded by upper bounding $1.74953m-0.94259$ by $1.74953(0.25)-0.94259$, which is negative.

Similarly, we can upper bound Equation 172 by replacing $\exp\left(\frac{0.9m^3}{0.2}+\frac{0.52m}{0.2}+1.944m\right)$ with $\exp\left(\left(\frac{0.45(0.25)^2+0.4544}{0.2}\right)\left(m-\sqrt{0.1175}\right)\right)\exp\left(\left(\frac{0.45(0.15)^2}{0.2}+\frac{0.26}{0.2}+0.972\right)\left((0.15)+\sqrt{0.1175}\right)\right)$ and factoring out $\exp\left(\left(\frac{0.45(0.25)^2+0.4544}{0.2}\right)\left(x-\sqrt{0.1175}\right)\right)$. The resultling expression (simplified by rounding up after five digits of precision)

$$4.04040m^6+20.09626m^5-246.87170m^4+80.42613m^3-108.27690m^2+20.175351m-7.86031$$

can be upper bounded by

$$\left(4.04040\left(0.25\right)^4+20.09626\left(0.25\right)^3-246.87170\left(0.15\right)^2+80.42613\left(0.25\right)-108.27690\right)\left(0.15\right)^2$$
$$+\left(20.175351\left(0.25\right)-7.86031\right),$$

which is negative.

So, we evaluate the first partial m of Equation 170 at $m=0.15$, which is negative. Then, we evaluate Equation 170 at $m=0.25$, which is positive.

This completes the proof.

□

Proof of Lemma B.71. To proceed, we first show that Equation 49 is convex in η. Consider the second partial η below.

$$12\left(0.972\right)^2\left(1-0.972\sqrt{0.1175}\right)\eta$$

We also show that Equation 49 is convex in m. Consider the second partial m below.

$$10.935\sqrt{0.1175}m^4 + \left(\left(-6 + 17.496\sqrt{0.1175}\right)\eta + \left(9.36\sqrt{0.1175} - 0.705\right)\right)(0.9m)^2 + \left(\left(-1.141578 + 1.829376\sqrt{0.1175}\right)\eta + \left(0.73008\sqrt{0.1175} - 0.10998\right)\right) \tag{173}$$

Note that the above can be lower bounded by

$$10.935\sqrt{0.1175}\,(0.15)^4 + \left(\left(-6 + 17.496\sqrt{0.1175}\right)(0.3) + \left(9.36\sqrt{0.1175} - 0.705\right)\right)(0.9\,(0.15))^2 + \left(\left(-1.141578 + 1.829376\sqrt{0.1175}\right)(0.3) + \left(0.73008\sqrt{0.1175} - 0.10998\right)\right), \tag{174}$$

which is positive.

So, we evaluate Equation 49 at $\eta = 0.2, 0.3$. Then, for $\eta = 0.2, 0.3$, we evaluate at $m = 0.15, 0.25$, all four of which are negative.

This completes the proof.

□

Proof of Lemma B.72. We begin the proof by showing Equation 50 grows in η. Consider the first partial η below.

$$\left(-0.235\,(0.972) + \sqrt{0.1175}\right)\left(0.45m^2 + 0.26\right) + \left(-6\,(0.972)\sqrt{0.1175} - 2\right)\left(0.45m^2 + 0.26\right)^2 + (0.972)^2\left(6\,(0.972)\sqrt{0.1175} + 6\right)\eta^2 \tag{175}$$

Since the above grows in η, we lower bound it at $\eta = 0.2$. We then lower bound the resulting expression in m by the following, which is positive.

$$\left(-0.235\,(0.972) + \sqrt{0.1175} + \left(-6\,(0.972)\sqrt{0.1175} - 2\right)\left(0.45\,(0.25)^2 + 0.26\right)\right)\left(0.45\,(0.25)^2 + 0.26\right) + (0.972)^2\left(6\,(0.972)\sqrt{0.1175} + 6\right)(0.2)^2$$

We then also show that Equation 50 decreases in m. Consider the partial m below (with $0.9m$ factored out).

$$-(0.235)\,(0.972)\,\eta + \eta\sqrt{0.1175} - 12\sqrt{0.1175}\left(0.45m^2 + 0.26\right)^2 + \left(-(2)\,(0.235) - 4\eta - 12\,(0.972)\sqrt{0.1175\eta}\right)\left(0.45m^2 + 0.26\right) \tag{176}$$

We upper bound the above by the following, which is negative.

$$\left(-(2)\,(0.235) - 4\,(0.2) - 12\,(0.972)\sqrt{0.1175}\,(0.2)\right)\left(0.45\,(0.15)^2 + 0.26\right) - 12\sqrt{0.1175}\left(0.45\,(0.15)^2 + 0.26\right)^2 - (0.235)\,(0.972)\,(0.2) + (0.3)\sqrt{0.1175}$$

So, we can upper bound Equation 50 by fixing $\eta = 0.3, m = 0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.73. To begin, we first scale Equation 51 by η^2 and show the resulting expression is growing in c. Consider the first partial derivative with respect to c below, scaled by $\sqrt{c}$.

$$-\frac{\eta}{2} + 0.45m^2\sqrt{c} + 0.26\sqrt{c} + 0.972\eta\sqrt{c} + (0.45)^2 m^4 + (2)\,(0.26)\,(0.45)\,m^2 + (2)\,(0.972)\,(0.45)\,m^2\eta + (2)\,(0.972)\,(0.26)\,\eta + (0.26)^2 + (0.972)^2\eta^2 \tag{177}$$

Note that we can lower bound the above by fixing $c = \sqrt{0.1175}$. Further note that the lower bounded expression is convex in η, so we evaluate the first partial derivative with respect to η at $\eta = 0.2$.

$$-\frac{1}{2} + 0.972\sqrt{0.1175} + (2)(0.972)(0.45)\,m^2 + (2)(0.972)(0.26) + (2)(0.972)^2(0.2) \tag{178}$$

Since the above is increasing in m, we lower bound it at $m = 0.15$, which is positive. So, Equation 177 can be further lower bounded by fixing $\eta = 0.2$. Since the resulting lower bound is convex in m, we calculate the first partial derivative with respect to m evaluated at $m = 0.15$, which is positive. So, Equation 177 can be lower bounded in m by $m = 0.25$, where the result is positive.

So, Equation 51 (scaled by η^2) can lower bounded in c by $c = 0.1175$. We next show that this lower bound is growing in η. Consider the partial derivative of this lower bound with respect to η.

$$\begin{aligned}&\left(\left(1.18098\sqrt{0.1175} + 0.405\right)m^4 + \left(0.914688\sqrt{0.1175} + 0.570789\right)m^2 + \left(0.1342432\sqrt{0.1175} + 0.1945892\right)\right)\\ &+ \left(\left(5.1018336\sqrt{0.1175} + 3.4992\right)m^2 + \left(1.00372608\sqrt{0.1175} + 0.24378424\right)\right)\eta\\ &+ \left(6(0.972)^2 + (6)(0.972)^3\sqrt{0.1175}\right)\eta^2\end{aligned} \tag{179}$$

We lower bound the above in m by fixing $m = 0.15$ and in η by fixing $\eta = 0.2$, where the resulting expression is positive.

So, we can lower bound Equation 51 (scaled by η^2 and lower bounded in c) in η by $\eta = 0.2$. Then, the resutling expression, shown below, can be lower bounded in m by fixing $m = 0.15$, which is positive.

$$\begin{aligned}&\left(0.405\sqrt{0.1175}m^4 + \left(0.232875 + 0.81792\sqrt{0.1175}\right)m^2\right.\\ &\left.+ \left(0.235152 + 0.21295872\sqrt{0.1175}\right)\right)\left(0.45m^2 + 0.4544\right) - 0.04\end{aligned} \tag{180}$$

This completes the proof. □

Proof of Lemma B.74. To begin the proof, we proceed by showing Equation 52 (scaled by η^2) grows in c. Note the first partial derivative with respect to c below (scaled by $\sqrt{c}$).

$$\begin{aligned}\frac{\eta}{2} + 0.45m^2\sqrt{c} + 0.26\sqrt{c} + 0.972n\sqrt{c} - (0.45)^2 m^4 - (2)(0.26)(0.45)\,m^2 - (2)(0.972)(0.45)\,m^2\eta\\ - (0.26)^2 - (2)(0.972)(0.26)\,\eta - (0.972)^2\eta^2\end{aligned} \tag{181}$$

Note that we can lower bound the above in c by $c = 0.1175$. Since the above is concave in η, we split up the proof into two cases.

Case 1: $\eta = 0.2$. Note the resulting expression in m is concave in m. So, we evaluate the resulting expression at $m = 0.15, 0.25$, both of which are positive.

Case 2: $\eta = 0.3$. Similar to case 1, the resulting expression is concave in m, so we evaluate the resulting expression at $m = 0.15, 0.25$, both of which are positive.

So, we can lower bound Equation 52 (scaled by η^2) in c by $c = 0.1175$. We next show that the resulting lower bound grows in η. Consider the first partial derivative with respect to η below.

$$\begin{aligned}&\left(\left(0.405 - 1.18098\sqrt{0.1175}\right)m^4 + \left(-0.914688\sqrt{0.1175} + 0.570789\right)m^2 + \left(-0.1342432\sqrt{0.1175} + 0.1945892\right)\right)\\ &+ \left(0.24378424 - 1.00372608\sqrt{0.1175} + \left(3.4992 - 5.1018336\sqrt{0.1175}\right)m^2\right)\eta\\ &+ \left(5.668704 - 5.509980288\sqrt{0.1175}\right)\eta^2\end{aligned}$$

Note that we can lower bound the above in m by $m = 0.15$. Since the resulting lower bound is convex in η and has a positive first partial η at $\eta = 0.2$, we can further lower bound in η by $\eta = 0.2$.

So, we can lower bound Equation 52 (scaled by η^2 and lower bounded in c) by $\eta = 0.2$. Lastly, we show that the resulting lower bound is increasing in m. Note, the first partial m of the lower bound below.

$$\begin{aligned}&\left(-1.62\sqrt{0.1175}m^3 + \left(0.232875 - 0.81792\sqrt{0.1175}\right)(2)\,m\right)\left(0.45m^2 + 0.4544\right)\\&+\left(-0.405\sqrt{0.1175}m^4 + \left(0.232875 - 0.81792\sqrt{0.1175}\right)m^2\right.\\&\left.+\left(0.235152 - 0.21295872\sqrt{0.1175}\right)\right)(0.9m)\end{aligned} \tag{182}$$

We can lower bound the first partial m naively in m by the following, which is positive.

$$\begin{aligned}&\left(-1.62\sqrt{0.1175}\,(0.25)^3 + \left(0.232875 - 0.81792\sqrt{0.1175}\right)(2)\,(0.25)\right)\left(0.45\,(0.25)^2 + 0.4544\right)\\&+\left(-0.405\sqrt{0.1175}\,(0.25)^4 + \left(0.232875 - 0.81792\sqrt{0.1175}\right)(0.25)^2\right.\\&\left.+\left(0.235152 - 0.21295872\sqrt{0.1175}\right)\right)(0.9\,(0.15))\end{aligned}$$

So, we can lower bound Equation 52 (scaled by η^2 and lower bounded in c, η) in m by $m = 0.15$, which is positive.

This completes the proof.

□

Proofs for Lemmas in Section B.5.1

Proof of Lemma B.75. To prove the lemma, consider the second partial x of Equation 53, $\frac{2(0.672)(0.011)(0.025)}{(0.672x+0.025)^3}$, which is positive.

We then consider the difference $-0.01132x - 0.0114 + \frac{0.011x}{0.672x+0.025}$, which is concave in x and positive for $x = 0.15, 0.25$.

This completes the proof. □

Proof of Lemma B.76. To prove the lemma, consider the second partial m of Equation 54. Note that we first factor out $\exp\left(\frac{0.94}{0.211}x^3 + \frac{0.93365}{0.211}x\right)$ from the second partial m, scale the resulting expression by $(0.672m + 0.025)^3$, and simplify by rounding up after five digits of precision.

$$\begin{aligned}&6.29491m^{10} + 0.70310m^9 + 7.40884m^8 + 57.91120m^7 + 9.25366m^6 + 37.75776m^5 + 12.73141m^4 + 7.11345m^3\\&+ 0.71128m^2 + 0.02401m + 0.00063\\&+\left(\delta_{1,1}\sqrt{0.672m + 0.025} + \delta_{1,2}\right)\exp\left(\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\left(\sqrt{0.672m + 0.025} - m\right)\right)\\&+\left(\delta_{2,1}\sqrt{0.672m + 0.025} + \delta_{2,2}\right)\exp\left(-\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\left(m + \sqrt{0.672m + 0.025}\right)\right),\end{aligned} \tag{183}$$

where

$$\begin{aligned}\delta_{1,1} =& -1.69122m^{10} + 0.87500m^9 - 21.38654m^8 - 10.73028m^7 - 9.84375m^6 - 27.05472m^5 - 4.63493m^4\\&- 6.87112m^3 - 1.41675m^2 + 0.32437m - 0.29411,\end{aligned}$$

$$\delta_{1,2} = -1.01473m^{11} - 0.60389m^{10} - 0.84147m^9 - 23.62276m^8 + 10.53324m^7 - 25.57118m^6 - 1.13379m^5 - 6.00481m^4 - 0.60377m^3 - 1.15740m^2 + 0.37493x + 0.01514,$$

$$\delta_{2,1} = 1.69123m^{10} - 0.87479m^9 + 21.38686m^8 + 10.73075m^7 + 9.84447m^6 + 27.05522m^5 + 4.63531m^4 + 6.87129m^3 + 1.41681m^2 - 0.32436m + 0.29412,$$

and

$$\delta_{2,2} = -1.01473m^{11} - 0.60389m^{10} - 0.84169m^9 - 23.62301m^8 + 10.53287m^7 - 25.57167m^6 - 1.13405m^5 - 6.00502m^4 - 0.60382m^3 - 1.15742m^2 + 0.37493m + 0.01514.$$

To proceed, we first show that $\delta_{1,1}\sqrt{0.672m + 0.025} + \delta_{1,2}$ is negative and $\delta_{2,1}\sqrt{0.672m + 0.025} + \delta_{2,2}$ is positive in lemmas B.81 and B.80.

To proceed, we replace $\exp\left(\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\left(\sqrt{0.672m + 0.025} - m\right)\right)$ by fixing $\eta = 0.25$ (see lemma B.78) and $\exp\left(-\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\left(m + \sqrt{0.672m + 0.025}\right)\right)$ with $-1.21526m + 0.502$ (see lemma B.77).

With the above, we get the following upper bound for Equation 183.

$$\Bigg(1.23317m^{12} - 1.35931m^{11} + 6.07208m^{10} + 27.67530m^9 - 54.12072m^8 + 110.71532m^7 - 42.11690m^6 + 42.71650m^5 + 1.07833m^4 + 7.27453m^3 - 2.13186m^2 + 0.77903m + 0.03187\Bigg)$$

$$+ \Bigg(-2.05528m^{11} - 0.72757m^{10} - 25.06403m^9 - 35.68477m^8 - 23.32468m^7 - 43.30142m^6 - 34.27866m^5 - 13.25772m^4 - 8.99691m^3 - 1.10585m^2 - 0.01398m - 0.31140\Bigg)\sqrt{0.672m + 0.025} \tag{184}$$

From the above, we can replace $\sqrt{0.672m + 0.025}$ with $10\left(\sqrt{0.193} - \sqrt{0.1258}\right)m + 0.227$ (see lemma B.79). So, we can upper bound Equation 184 by

$$-0.50629m^{12} - 2.44163m^{11} - 15.30576m^{10} - 8.21568m^9 - 81.96177m^8 + 68.77290m^7 - 80.95772m^6 + 23.71471m^5 - 9.54561m^4 + 4.29631m^3 - 2.39471m^2 + 0.51231m - 0.03881. \tag{185}$$

To proceed, we first show that the above is concave in m. Note that we can upper bound the second partial m by

$$\Bigg(474.2942\,(0.25) - 114.54732 - 591.52896\,(0.15)^5 - 4589.85912\,(0.15)^4 + \Big(2888.4618\,(0.25) - 2428.7316 - 66.83028\,(0.15)^6 - 268.5793\,(0.15)^5 - 1377.5184\,(0.15)^4\Big)\,(0.15)^2\Bigg)m^2 + 25.77786m - 4.78942, \tag{186}$$

which simplifies (by rounding up after five digits of precision) to

$$-36.75733m^2 + 25.77786m - 4.78942. \tag{187}$$

Since the above is concave in m and the first partial m at $m = 0.25$ is positive, we evaluate it at $m = 0.25$, which is negative.

We then evaluate the first partial m of Equation 185 at $m = 0.15$, which is negative and evaluate Equation 185 at $m = 0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.77. To prove the lemma, we first consider the second partial x of Equation 55 scaled by $(0.672x + 0.025)^{\frac{3}{2}}$ and simplified by rounding down after five digits of precision.

We show the expression below.

$$\begin{aligned}&\left(33.60938x^5 + 2.25062x^4 + 17.84113x^3 - 2.77860x^2 + 1.99824x + 0.32922\right)\\&+ \left(30.00838x^5 + 15.12029x^4 + 20.7040x^3 - 2.66588x^2 + 3.12083x + 0.67498\right)\sqrt{0.672x + 0.025}\end{aligned} \tag{188}$$

Note that the square root coefficient can be lower bounded by

$$30.00838\left(0.15\right)^5 + 15.12029\left(0.15\right)^4 + 20.7040\left(0.15\right)^3 - 2.66588\left(0.25\right)^2 + 3.12083\left(0.15\right) + 0.67498$$

and the other term from the above can be lower bounded by

$$33.60938\left(0.15\right)^5 + 2.25062\left(0.15\right)^4 + 17.84113\left(0.15\right)^3 - 2.77860\left(0.25\right)^2 + 1.99824\left(0.15\right) + 0.32922,$$

which are both positive.

Using convexity of Equation 55, we note that the difference between the linear upper bound and exponential term is concave, so we evaluate the difference at $x = 0.15, 0.25$, which are positive.

This completes the proof.

□

Proof of Lemma B.78. To prove the lemma, note the second partial x of Equation 56 (simplified by rounding up after five digits of precision) below.

$$\frac{3.77212x^2 + 0.22454x - 0.24699}{(0.672x + 0.025)^{\frac{3}{2}}} - 13.36492x \tag{189}$$

We can upper bound the above by

$$\frac{3.77212\left(0.25\right)^2 + 0.22454\left(0.25\right) - 0.24699}{\left(0.672\left(0.15\right) + 0.025\right)^{\frac{3}{2}}} - 13.36492\left(0.15\right),$$

which is negative.

By concavity of Equation 56, we evaluate it at $m = 0.15, 0.25$ and find $m = 0.25$ to be smaller.

This completes the proof.

□

Proof of Lemma B.79. To prove the lemma, note that the difference between the linear lower bound and the square root term is convex in x. So, we evaluate the difference at $x = 0.15, 0.25$, which are negative.

This completes the proof. □

Proof.][Proof of Lemma B.80] To prove the lemma, we first show that $\delta_{2,1}$ as defined Equation 58 is positive.

Note that we can lower bound $\delta_{2,1}$ by

$$\begin{aligned}&1.69123m^{10} - 0.87479\left(0.25\right)^9 + 21.38686m^8 + 10.73075m^7 + 9.84447m^6 + 27.05522m^5 + 4.63531m^4\\&+ 6.87129m^3 + 1.41681m^2 - 0.32436\left(0.25\right) + 0.29412,\end{aligned} \tag{190}$$

which simplifies (by rounding down after five digits of precision) to

$$\begin{aligned}&1.69123m^{10} + 21.38686m^8 + 10.73075m^7 + 9.84447m^6 + 27.05522m^5 + 4.63531m^4\\&+ 6.87129m^3 + 1.41681m^2 + 0.21302.\end{aligned} \tag{191}$$

We can then replace $\delta_{2,1}\sqrt{0.672m+0.025}$ with $\delta_{2,1}\sqrt{0.672(0.15)+0.025}$. Then, Equation 58 can be lower bounded by

$$
\begin{aligned}
&-1.01473\,(0.25)^{11}+\left(1.69123\sqrt{0.1258}-0.60389\right)(0.25)^{10}+\left(-0.87479\sqrt{0.1258}-0.84169\right)(0.25)^{9}\\
&+\left(21.38686\sqrt{0.1258}-23.62301\right)(0.25)^{8}+\left(10.73075\sqrt{0.1258}+10.53287\right)(0.15)^{7}\\
&+\left(9.84447\sqrt{0.1258}-25.57167\right)(0.25)^{6}+\left(27.05522\sqrt{0.1258}-1.13405\right)(0.15)^{5}\\
&+\left(4.63531\sqrt{0.1258}-6.00502\right)(0.15)^{4}+\left(6.87129\sqrt{0.1258}-0.60382\right)(0.15)^{3}\\
&+\left(1.41681\sqrt{0.1258}-1.15742\right)(0.25)^{2}+\left(-0.32436\sqrt{0.1258}+0.37493\right)(0.15)\\
&+\left(0.29412\sqrt{0.1258}+0.01514\right),
\end{aligned}
$$

which is positive.

This completes the proof.

□

Proof of Lemma B.81. To prove the lemma, we first note that $\delta_{1,1}$ as defined in Equation 59 can be upper bounded by

$$
\begin{aligned}
&-1.69122m^{10}+0.87500\,(0.25)^{9}-21.38654m^{8}-10.73028m^{7}-9.84375m^{6}\\
&-27.05472m^{5}-4.63493m^{4}-6.87112m^{3}-1.41675m^{2}+0.32437\,(0.25)-0.29411,
\end{aligned}
\tag{192}
$$

which simplifies (by rounding up after five digits of precision) to the negative expression below.

$$
\begin{aligned}
&-1.69122m^{10}-21.38654m^{8}-10.73028m^{7}-9.84375m^{6}-27.05472m^{5}-4.63493m^{4}\\
&-6.87112m^{3}-1.41675m^{2}-0.21301
\end{aligned}
\tag{193}
$$

We can then replace $\delta_{1,1}\sqrt{0.672m+0.025}$ by $\delta_{1,1}\sqrt{0.672\,(0.15)+0.025}$. Then, Equation 59 can be upper bounded by

$$
\begin{aligned}
&-1.01473\,(0.15)^{11}+\left(-0.60389-1.69122\sqrt{0.1258}\right)(0.15)^{10}+\left(0.87500\sqrt{0.1258}-0.84147\right)(0.15)^{9}\\
&+\left(-21.38654\sqrt{0.1258}-23.62276\right)(0.15)^{8}+\left(-10.73028\sqrt{0.1258}+10.53324\right)(0.25)^{7}\\
&+\left(-9.84375\sqrt{0.1258}-25.57118\right)(0.15)^{6}+\left(-27.05472\sqrt{0.1258}-1.13379\right)(0.15)^{5}\\
&+\left(-4.63493\sqrt{0.1258}-6.00481\right)(0.15)^{4}+\left(-6.87112\sqrt{0.1258}-0.60377\right)(0.15)^{3}\\
&+\left(-1.41675\sqrt{0.1258}-1.15740\right)(0.15)^{2}+\left(0.32437\sqrt{0.1258}+0.37493\right)(0.25)\\
&+\left(-0.29411\sqrt{0.1258}+0.01514\right),
\end{aligned}
$$

which is negative.

This completes the proof. □

Lemma B.102 (Proof of Lemma B.82)**.** *To prove the lemma, we divide up Equation 60 into the following parts and prove they are concave in* m*.*

Part 1

$$\Big(\big(-4.86669m^3 - 0.26392m^2 - 1.41011m - 2.28476\big)\sqrt{0.672m + 0.025} + 4.57470m^5 + 0.24809m^4 + 5.86930m^3 + 0.37310m^2 + 2.31656m + 0.12583\Big)\exp\left(\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\sqrt{0.672m + 0.025}\right) + \big(-7.09418m^4 - 0.26392m^3 - 3.62256m^2 + 0.86524m\big)\exp\left(\frac{0.47}{0.211}m^3 + \frac{0.466825}{0.211}m\right) \tag{194}$$

Part 2

$$\Big(\big(4.86670m^3 + 0.26392m^2 + 1.41012m + 2.28477\big)\sqrt{0.672m + 0.025} + 4.57470m^5 + 0.24809m^4 + 5.86930m^3 + 0.37310m^2 + 2.31656m + 0.12583\Big)\exp\left(-\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\sqrt{0.672m + 0.025}\right) \tag{195}$$

Part 3

$$\big(7.09419m^4 + 0.26393m^3 + 3.62257m^2 + 1.13477m\big)\exp\left(\frac{-0.47}{0.211}m^3 + \frac{-0.466825}{0.211}m\right) \tag{196}$$

We show that Equation 194, 195, and 196 are concave in lemmas B.83, B.85, and B.86.

Since the first partial η of Equation 60 is negative at $m = 0.15$, we evaluate it at $m = 0.15$, which is negative.

This completes the proof.

Proof of Lemma B.83. To prove the lemma, consider the second partial m of Equation 61 with $\exp\left(\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\sqrt{0.672m + 0.025}\right)$ factored out. Note that we replace the $\exp\left(\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\left(m - \sqrt{0.672m + 0.025}\right)\right)$ by linearizing at $m = 0.154$, since the corresponding coefficient,

$$-316.79295m^8 - 11.78543m^7 - 371.53523m^6 - 443.23235m^5 - 155.95023m^4 - 246.51687m^3 - 83.23791m^2 - 29.40705m - 3.41653, \tag{197}$$

is negative and $\exp\left(\left(\frac{0.47}{0.211}m^2 + \frac{0.466825}{0.211}\right)\left(m - \sqrt{0.672m + 0.025}\right)\right)$ is convex in m (see lemma B.84).

We then scale the resulting expression by $(0.672m + 0.025)^{\frac{3}{2}}$, yielding the following.

$$\Big(-45.79860m^8 + 125.38478m^7 - 17.19016m^6 + 107.93983m^5 + 4.90782m^4 + 8.64310m^3 + 0.30780m^2 + 0.41137m + 0.25739\Big) + \Big(-69.90475m^9 - 2.68033m^8 - 49.39472m^7 - 181.02533m^6 - 108.90100m^5 - 49.02459m^4 - 50.57475m^3 - 5.11529m^2 - 1.70107m - 0.59501\Big)\sqrt{0.672m + 0.025} \tag{198}$$

Since the coefficient for the square root term is negative, we replace $\sqrt{0.672m + 0.025}$ with $\sqrt{0.672\,(0.15) + 0.025}$ and show the resulting expression below (simplified by rounding up after five digits of precision).

$$-24.79402m^9 - 46.74926m^8 + 107.86532m^7 - 81.39675m^6 + 69.31451m^5 - 12.48036m^4 - 9.29490m^3 - 1.50650m^2 - 0.19197m + 0.04636. \tag{199}$$

We then upper bound the above as follows.

First, we upper bound Equation 199 by

$$-24.79402m^9 - 46.74926m^8 + 107.86532\,(0.25)\,m^6 - 81.39675m^6 + 69.31451\,(0.25)\,m^4 - 12.48036m^4 - 9.29490m^3 - 1.50650m^2 - 0.19197m + 0.04636, \tag{200}$$

which simplifies (by rounding up after five digits of precision) to

$$-24.79402m^9 - 46.74926m^8 - 54.43042m^6 + 4.84827m^4 - 9.29490m^3 - 1.50650m^2 - 0.19197m + 0.04636. \tag{201}$$

We then upper bound the above by

$$-24.79402m^9 - 46.74926m^8 - 54.43042m^6 + 4.84827\,(0.25)\,m^3 - 9.29490m^3 - 1.50650m^2 - 0.19197m + 0.04636, \tag{202}$$

which simplifies (after rounding up after five digits of precision) to

$$-24.79402m^9 - 46.74926m^8 - 54.43042m^6 - 8.08283m^3 - 1.50650m^2 - 0.19197m + 0.04636. \tag{203}$$

We then upper bound the above by

$$-24.79402\,(0.15)^9 - 46.74926\,(0.15)^8 - 54.43042\,(0.15)^6 - 8.08283\,(0.15)^3 - 1.50650\,(0.15)^2 - 0.19197\,(0.15) + 0.04636,$$

which is negative.

This completes the proof.

□

Proof. To prove the lemma, consider the second partial x of Equation 62 with the exponential term factored out, scaled by $(0.672x + 0.025)^{\frac{3}{2}}$, and simplified by rounding down after five digits of precision.

$$\begin{aligned}&\left(30.00838\sqrt{0.672x + 0.025} - 33.60939\right)x^5 + \left(15.12029\sqrt{0.672x + 0.025} - 2.25063\right)x^4 \\ &+ \left(20.70401\sqrt{0.672x + 0.025} - 17.84114\right)x^3 + \left(15.29658\sqrt{0.672x + 0.025} - 4.76564\right)x^2 \\ &+ \left(3.78907\sqrt{0.672x + 0.025} - 2.44731\right)x + \left(0.67498\sqrt{0.672x + 0.025} + 0.16475\right)\end{aligned} \tag{204}$$

From the above, we note the following expressions are negative.

$$30.00838\sqrt{0.672\,(0.25) + 0.025} - 33.60939$$

$$20.70401\sqrt{0.672\,(0.25) + 0.025} - 17.84114$$

$$3.78907\sqrt{0.672\,(0.25) + 0.025} - 2.44731$$

So, we lower bound Equation 204 by

$$\left(30.00838\sqrt{0.672x+0.025}-33.60939\right)(0.25)^5+\left(15.12029\sqrt{0.672x+0.025}-2.25063\right)(0.15)^4 \\ +\left(20.70401\sqrt{0.672x+0.025}-17.84114\right)(0.25)^3+\left(15.29658\sqrt{0.672x+0.025}-4.76564\right)(0.15)^2 \\ +\left(3.78907\sqrt{0.672x+0.025}-2.44731\right)x+\left(0.67498\sqrt{0.672x+0.025}+0.16475\right), \tag{205}$$

which simplifies (by rounding down after five digits of precision) to

$$3.78907x\sqrt{0.672x+0.025}+1.37961\sqrt{0.672x+0.025}-2.44731x-0.25521. \tag{206}$$

We then show that the above is increasing in m. Consider the partial m below (scaled by $\sqrt{0.672m+0.025}$) that we lower bound by

$$0.672\,(3.78907)\,(0.15)+0.025\,(3.78907)+3.78907\,(0.336)\,(0.15)+1.37961\,(0.336) \\ -2.44731\sqrt{0.672\,(0.25)+0.025},$$

which is positive.

So, we evaluate Equation 206 at $x=0.15$, which is positive.

This completes the proof.

□

Proof of Lemma B.85. To prove the lemma, consider the second partial m of Equation 63 with the exponential term factored out, scaled by $(0.672m+0.025)^{\frac{3}{2}}$ and simplified by rounding up after five digits of precision).

$$\Big(45.79871m^8-125.38474m^7+17.19032m^6-107.93969m^5-4.90774m^4-8.64299m^3-0.30776m^2 \\ -0.41134m-0.25737\Big) \\ +\Big(64.06371m^9+7.28755m^8+107.90909m^7+12.25822m^6-35.97832m^5 \\ +57.67831m^4-11.49612m^3+8.63013m^2+0.20639m-0.54126\Big)\sqrt{0.672m+0.025} \tag{207}$$

To proceed, we show the above is decreasing in m. Consider the first partial m below (simplified by rounding up after five digits of precision).

$$366.38968m^7-877.69318m^6+103.14192m^5-539.69845m^4-19.63096m^3-25.92897m^2 \\ -0.61552m-0.41134 \\ +\frac{408.98273m^9+56.04083m^8+545.31933m^7+72.42800m^6-131.13713m^5+169.92192m^4 \\ -21.27104m^3+13.63641m^2+0.63955m-0.17670}{\sqrt{0.672m+0.025}} \tag{208}$$

From the above note that the square root term is positive. To see, this, note that the numerator of the term can be lower bounded by

$$408.98273\,(0.15)^9+56.04083\,(0.15)^8+545.31933\,(0.15)^7+72.42800\,(0.15)^6-131.13713\,(0.25)^5 \\ +169.92192\,(0.15)^4-21.27104\,(0.25)\,m^2+13.63641m^2+0.63955m-0.17670, \tag{209}$$

which simplifies (by rounding down after five digits of precision) to $8.31865m^2 + 0.63955m - 0.21696$. Since $8.31865m^2 + 0.63955m - 0.21696$ is convex in m, we evaluate the first partial m at $m = 0.15$, which is positive. So, we evaluate $8.31865m^2 + 0.63955m - 0.21696$ at $m = 0.15$, which is positive.

We then replace $\sqrt{0.672m + 0.025}$ in Equation 208 with $\sqrt{0.672\,(0.15) + 0.025}$ and show the resulting expression (simplified by rounding up after five digits of precision) below.

$$1153.09384m^9 + 158.00261m^8 + 1903.87357m^7 - 673.48827m^6 - 266.58864m^5 - 60.61729m^4 - 79.60294m^3 + 12.51779m^2 + 1.18764m - 0.909531 \tag{210}$$

We then upper bound the above by

$$1153.09384\,(0.25)^9 + 158.00261\,(0.25)^8 + 1903.87357\,(0.25)^7 - 673.48827\,(0.15)^6 - 266.58864\,(0.15)^5 - 60.61729\,(0.15)^4 - 79.60294\,(0.15)^3 + 12.51779\,(0.25)^2 + 1.18764\,(0.25) - 0.909531,$$

which is negative.

So, we evaluate Equation 207 at $m = 0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.86. To prove the lemma, consider the second partial m of Equation 64 (with the exponential term factored out and simplified by rounding up after five digits of precision).

$$316.79341m^8 + 11.78589m^7 + 371.53598m^6 - 415.58901m^5 + 127.73187m^4 - 235.96392m^3 + 69.02656m^2 - 24.92072m + 2.22392 \tag{211}$$

We can then upper bound the above by

$$\begin{aligned} &316.79341\,(0.25)^8 + 11.78589\,(0.25)^7 + 371.53598\,(0.25)^6 - 415.58901m^5 \\ &+ (127.73187\,(0.25) - 235.96392)\,(0.15)\,m^2 + 69.02656m^2 - 24.92072m + 2.22392, \end{aligned} \tag{212}$$

which simplifies (by rounding up after five digits of precision) to

$$-415.58901m^5 + 38.42192m^2 - 24.92072m + 2.32019. \tag{213}$$

Note that the above is decreasing in m. To see this, consider the first partial m that can upper bounded by

$$-2077.94505\,(0.15)^4 + 76.84384\,(0.25) - 24.92072.$$

We then evaluate Equation 213 at $m = 0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.87. To prove Equation 65 is concave in η, we split it up into the following parts and show that each part is concave in η.

Part 1

$$\sqrt{2m\eta + 0.25m + 0.025}\exp\left(\frac{0.47m^3}{\eta} + \frac{0.24m}{\eta} + 1.075m\right)$$

$$+ \left(\sqrt{2m\eta + 0.25m + 0.025} - 0.94m^2 - 0.48 - 2.15\eta\right)\exp\left(\left(\frac{0.47m^2}{x} + \frac{0.24}{\eta} + 1.075\right)\sqrt{2m\eta + 0.25m + 0.025}\right) \tag{214}$$

Part 2

$$\left(\sqrt{2m\eta+0.25m+0.025}+0.94m^2+0.48+2.15\eta\right)\exp\left(-\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\sqrt{2m\eta+0.25m+0.025}\right) \tag{215}$$

Part 3

$$\sqrt{2m\eta+0.25m+0.025}\exp\left(\frac{-0.47m^3}{\eta}+\frac{-0.24m}{\eta}-1.075m\right) \tag{216}$$

We prove that Equations 216, 214, and 215 are concave in lemmas B.91, B.88, and B.89.

This completes the proof.

□

Proof of Lemma B.88. To prove the lemma, we consider the second partial η of Equation 66. Note that we factor out $\exp\left(\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\sqrt{2m\eta+0.25m+0.025}\right)$ and subtract out

$$\left(\sqrt{2m\eta+0.25m+0.025}-0.94m^2-0.48-2.15\eta\right)$$
$$\cdot\left(\left(\frac{-0.47m^2}{\eta^2}+\frac{-0.24}{\eta^2}\right)\sqrt{2m\eta+0.25m+0.025}+\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\frac{m}{\sqrt{2m\eta+0.25m+0.025}}\right)^2 \tag{217}$$

(note $\sqrt{2\left(0.25\right)\left(0.3\right)+0.25\left(0.25\right)+0.025}-0.94\left(0.15\right)^2-0.48-2.15\left(0.2\right)$ is negative) from the second partial η. We also upper bound the $\exp\left(\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\left(m-\sqrt{2m\eta+0.25m+0.025}\right)\right)$ term by $\exp\left(\left(\frac{0.47(0.15)^2}{0.3}+\frac{0.24}{0.3}+1.075\right)\left(0.25-\sqrt{2\left(0.15\right)\left(0.2\right)+0.25\left(0.15\right)+0.025}\right)\right)$, since the corresponding coefficient, $\sqrt{2m\eta+0.25m+0.025}\left(\frac{0.94}{\eta^3}m^3+\frac{0.48}{\eta^3}m\right)$, is positive. We then scale the resulting expression by $\eta^3\left(2m\eta+0.25m+0.025\right)^{\frac{3}{2}}$, yielding the following (simplified by rounding up after five digits of precision).

$$\tau_1\sqrt{2m\eta+0.25m+0.025}+\tau_2, \tag{218}$$

where

$$\begin{aligned}\tau_1 =&1.075m^2\eta^3+\left(0.47m^2+0.24\right)m^2\eta^2+\left(0.47m^4+0.047m^3+0.24m^2+0.024m\right)\eta\\&+\left(0.05875m^4+0.01175m^3+0.03059m^2+0.006m+0.0003\right)\end{aligned}$$

and

$$\begin{aligned}\tau_2 =&-6.93375m^2\eta^4+\left(2.021m^4-1.12362m^2-0.11556m\right)\eta^3\\&+\left(-1.3254m^6+3.10622m^5-1.3536m^4+1.58615m^3-0.3456m^2\right)\eta^2\\&+\left(-0.6627m6+0.71029m^5-0.59914m^4+0.32886m^3-0.13314m^2-0.01728m\right)\eta\\&+\left(-0.05522m^6+0.03749m^5-0.04724m^4+0.01399m^3-0.01000m^2-0.00263m-0.00014\right).\end{aligned}$$

Note that τ_1 is positive, so we can replace $\sqrt{2m\eta+0.25m+0.025}$ with $\sqrt{2\left(0.25\right)\eta+0.25\left(0.25\right)+0.025}$, which we linearize at $\eta=0.25$.

We then get the resulting upper bound for Equation 218 (simplified by rounding up after five digits of precision).

$$
\begin{aligned}
&- 6.35074m^2\eta^4 + \left(2.27590m^4 - 0.64366m^2 - 0.11556m\right)\eta^3 \\
&+ \left(-1.3254m^6 + 3.10622m^5 - 0.94576m^4 + 1.61164m^3 - 0.13734m^2 + 0.01302m\right)\eta^2 + \\
&\left(-0.6627m^6 + 0.71029m^5 - 0.41434m^4 + 0.35053m^3 - 0.03846m^2 - 0.00621m + 0.00017\right)\eta \\
&+ \left(-0.05522m^6 + 0.03749m^5 - 0.02812m^4 + 0.01782m^3 - 0.00005m^2 - 0.00067m - 0.00004\right)
\end{aligned} \tag{219}
$$

To show the above is negative, we first show that the second partial η of the above is negative. Consider the second partial η below.

$$
\begin{aligned}
&- 6.35074\,(12)\,m^2\eta^2 + \left(2.27590m^4 - 0.64366m^2 - 0.11556m\right)(6)\,\eta \\
&+ \left(-1.3254m^6 + 3.10622m^5 - 0.94576m^4 + 1.61164m^3 - 0.13734m^2 + 0.01302m\right)(2)
\end{aligned} \tag{220}
$$

From the above, note that $2.27590m^3 - 0.64366m - 0.11556$ can be upper bounded by $2.27590\,(0.25)^3 - 0.64366\,(0.15) - 0.11556$ (negative), so we can upper bound by $\eta = 0.2$. The resulting expression in m (simplified by rounding up after five digits of precision) can then be upper bounded by

$$
-2.6508\,(0.15)^5 + 6.21244\,(0.25)^4 + 0.83956\,(0.25)^3 + 3.22328\,(0.25)^2 - 4.09542\,(0.15) - 0.11263,
$$

which is negative.

So, we consider the first partial η of Equation 219 at $\eta = 0.2$ (simplified by rounding up after five digits of precision) that can be upper bounded by

$$
\begin{aligned}
&- 1.19286\,(0.15)^6 + 1.95279\,(0.25)^5 - 0.51953\,(0.15)^4 + 0.99519\,(0.25)\,(0.15)^2 - 0.37385\,(0.15)^2 \\
&- 0.01486\,(0.15) + 0.00017,
\end{aligned}
$$

which is negative.

We then evaluate Equation 219 at $\eta = 0.2$, which (after simplifying by rounding up after five digits of precision and factoring out m) is shown below.

$$
-0.24077m^5 + 0.30380m^4 - 0.13061m^3 + 0.15240m^2 - 0.02854m - 0.002316 \tag{221}
$$

We then upper bound the above by

$$
((((-0.24077\,(0.15) + 0.30380)\,(0.25) - 0.13061)\,(0.15) + 0.15240)\,(0.25) - 0.02854)\,(0.2) - 0.002316,
$$

which is negative.

This completes the proof.

□

Proof of Lemma B.89. To prove that Equation 67 is concave in η, we divide up the second partial η as follows (with the exponential term factored out). Note for the second part, we divide out a factor of 2.15, scale by $(2m\eta + 0.25m + 0.025)^{\frac{3}{2}}$, and simplify by rounding up after five digits of precision For the third part, we scale by $2m\eta + 0.25m + 0.025$. Note for the fourth part that we scale by $(2m\eta + 0.25m + 0.025)^{\frac{3}{2}}$.

Part 1

$$
\frac{-m^2}{(2m\eta + 0.25m + 0.025)^{\frac{3}{2}}} \tag{222}
$$

Part 2

$$
\begin{aligned}
&-3.225m^2\eta - 0.2975m^2 - 0.05375m + 0.47m^4 \\
&+\Bigg(\frac{1}{\eta}\left(0.2209m^6 - 0.027025m^4 - 0.02526m^3 - 0.0714m^2 - 0.0129m\right) \\
&\quad + \frac{1}{\eta^2}\left(0.11045m^6 + 0.01105m^5 + 0.1128m^4 + 0.01128m^3 + 0.0288m^2 + 0.00288m\right) \\
&\quad + \frac{1}{\eta^3}\left(0.01381m^6 + 0.00277m^5 + 0.01424m^4 + 0.00282m^3 + 0.00375m^2 + 0.00072m + 0.00004\right) \\
&\quad - 1.0105m^4 - 0.516m^2 + 1.15563m^2\eta\Bigg)\sqrt{2m\eta + 0.25m + 0.025}
\end{aligned}
\tag{223}
$$

Part 3

$$
\begin{aligned}
&\frac{1}{\eta^4}\left(\left(0.25\right)^2 m^2 + 0.0125m + \left(0.025\right)^2\right)\left(0.47m^2 + 0.24\right)^2 \\
&+ \frac{1}{\eta^3}\left(0.235m^4 + 0.0235m^3 - 0.005m^2 - 0.013m - 0.00125\right)\left(0.47m^2 + 0.24\right) \\
&+ \frac{1}{\eta^2}\left(0.47m^3 - 1.2975m - 0.15375\right)\left(0.47m^2 + 0.24\right)m \\
&+ \frac{-3.15}{\eta}\left(0.47m^2 + 0.24\right)m^2 + 0.075\left(1.075\right)m^2
\end{aligned}
\tag{224}
$$

Part 4

$$
\begin{aligned}
&\frac{-3m^2}{\eta}\left(0.47m^2 + 0.24\right) + \frac{1}{\eta^2}\left(-1.5m - 0.15\right)\left(0.47m^2 + 0.24\right)m \\
&+ \frac{1}{\eta^3}\left(-0.125m^2 - 0.025m - 0.00125\right)\left(0.47m^2 + 0.24\right) + 1.075m^2 \\
&+\Bigg(\frac{-2.15m^2}{\eta}\left(0.47m^2 + 0.24\right) + \frac{1}{\eta^2}\left(0.47m^3 - 0.2975m - 0.05375\right)\left(0.47m^2 + 0.24\right)m \\
&\quad + \frac{1}{\eta^3}\left(0.5m + 0.05\right)\left(0.47m^2 + 0.24\right)^2 m + \frac{1}{\eta^4}\left(\left(0.25\right)^2 m^2 + 0.0125m + \left(0.025\right)^2\right)\left(0.47m^2 + 0.24\right)^2 \\
&\quad + \left(1.075\right)^2 m^2\Bigg)\sqrt{2m\eta + 0.25m + 0.025}
\end{aligned}
\tag{225}
$$

For Equation 223, we first show the above decreases in η in lemma B.90.

So, we can upper bound Equation 223 in η by $\eta = 0.2$, yielding the following expression.

$$
\begin{aligned}
&0.47m^4 - 0.9425m^2 - 0.05375m \\
&+ \left(5.592m^6 + 0.6225m^5 + 3.45438m^4 + 0.5082m^3 + 0.54688m^2 + 0.0975m + 0.005\right)\sqrt{0.65m + 0.025}
\end{aligned}
\tag{226}
$$

From the above, we replace $\sqrt{0.65m + 0.025}$ by $\sqrt{0.65\,(0.25) + 0.025}$, and the resulting expression can be upper bounded by

$$5.592\sqrt{0.1875}\,(0.25)^6 + 0.6225\sqrt{0.1875}\,(0.25)^5 + \left(3.45438\sqrt{0.1875} + 0.47\right)(0.25)^4 + 0.5082\sqrt{0.1875}\,(0.25)^3 + \left(0.54688\sqrt{0.1875} - 0.9425\right)(0.15)^2 + \left(0.0975\sqrt{0.1875} - 0.05375\right)(0.15) + 0.005\sqrt{0.1875},$$

which is negative.

For Equation 224, we first show that is increases in η in lemma B.103, so we upper bound Equation 224 by $\eta = 0.3$. Note that the resulting expression (simplified by rounding up after five digits of precision),

$$8.24966m^6 + 0.74996m^5 - 6.69786m^4 - 0.47217m^3 - 5.50372m^2 - 0.43666m - 0.00666, \tag{227}$$

can be upper bounded by

$$8.24966\,(0.25)^6 + 0.74996\,(0.25)^5 - 6.69786\,(0.15)^4 - 0.47217\,(0.15)^3 - 5.50372\,(0.15)^2 - 0.43666\,(0.15) - 0.00666,$$

which is negative.

For Equation 225, we first show that the square root coefficient is locally positive. To do so, we show that the coefficient is locally decreasing in η. Consider the first partial η below (scaled by η^5 with $\left(0.47m^2 + 0.24\right)$ factored out).

$$2.15m^2\eta^3 - 2\eta^2\left(0.47m^3 - 0.2975m - 0.05375\right)m - 3\eta\,(0.5m + 0.05)\left(0.47m^2 + 0.24\right)m - 4\left((0.25)^2\,m^2 + 0.0125m + (0.025)^2\right)\left(0.47m^2 + 0.24\right) \tag{228}$$

From the above, note that $0.47m^3 - 0.2975m - 0.05375$ is locally convex in m and negative for $m = 0.15, 0.25$.

So, we upper bound Equation 228 in η by

$$2.15\,(0.3)^3\,m^2 - 0.47\,(2)\,(0.3)^2\,m^4 + 0.2975\,(2)\,(0.3)^2\,m^2 + 0.05375\,(2)\,(0.3)^2\,m - 0.47\,(0.5)\,(3)\,(0.2)\,m^4 - 0.24\,(0.5)\,(3)\,(0.2)\,m^2 - 0.47\,(0.05)\,(3)\,(0.2)\,m^3 - 0.24\,(0.05)\,(3)\,(0.2)\,m - 0.47\,(0.25)\,m^4 - 0.24\,(0.25)\,m^2 - 0.47\,(0.0125)\,(4)\,m^3 - 0.24\,(0.0125)\,(4)\,m - 0.47\,(4)\,(0.025)^2\,m^2 - 0.24\,(4)\,(0.025)^2, \tag{229}$$

which simplifies (by rounding up after five digits of precision) to

$$-0.3431m^4 - 0.0376m^3 - 0.02157m^2 - 0.00952m - 0.0006. \tag{230}$$

So, we can evaluate the square root coefficient of Equation 225 at $\eta = 0.3$, which simplifies (by rounding down after five digits of precision) to

$$8.24966m^6 + 0.74996m^5 + 2.26695m^4 + 0.48523m^3 + 0.17081m^2 + 0.05222m + 0.00444 \tag{231}$$

Since the square root coefficient is positive, we can replace $\sqrt{2m\eta + 0.25m + 0.025}$ by $\sqrt{2\,(0.25)\,(0.3) + 0.25\,(0.25) + 0.025}$ in Equation 225.

We then show that the resulting expression (simplified by rounding up after five digits of precision) grows in η. Consider the first partial η below (scaled by η^5 with $\left(0.47m^2 + 0.24\right)$ factored out).

$$4.04778m^2\eta^3 - 2\eta^2\left(0.22905m^3 - 1.64498m - 0.17619\right)m - 3\eta\left(0.11453m^4 + 0.01146m^3 - 0.06651m^2 - 0.01915m - 0.00125\right) - 4\left(0.03046m^2 + 0.00610m + 0.00031\right)\left(0.47m^2 + 0.24\right) \tag{232}$$

From the above, note that $0.22905m^3 - 1.64498m - 0.17619$ is locally convex in m, and $m = 0.15, 0.25$ are negative.

Also, note that $0.11453m^4+0.01146m^3-0.06651m^2-0.01915m-0.00125$ can be upper bounded by $0.11453\,(0.25)^4 + 0.01146\,(0.25)^3 - 0.06651\,(0.15)^2 - 0.01915\,(0.15) - 0.00125$, which is negative.

So, we can lower bound Equation 232 in η by $\eta = 0.2$, and the resulting expression (rounded down after five digits of precision) can be lower bounded by

$$-0.14431\,(0.25)^4 - 0.01835\,(0.25)^3 + 0.17406\,(0.15)^2 + 0.01972\,(0.15) + 0.00045.$$

Then, we can further upper bound our earlier upper bound for Equation 225 by $\eta = 0.3$. The resulting expression (rounded up after five digits of precision) can be upper bounded by

$$4.02052\,(0.25)^6 + 0.36585\,(0.25)^5 - 13.60407\,(0.15)^4 - 0.98169\,(0.15)^3 - 6.37436\,(0.15)^2 - 0.59668\,(0.15) - 0.00890,$$

which is negative.

□

Lemma B.103. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\begin{aligned}
&\frac{1}{\eta^4}\left((0.25)^2 m^2 + 0.0125m + (0.025)^2\right)\left(0.47m^2 + 0.24\right)^2 \\
&+ \frac{1}{\eta^3}\left(0.235m^4 + 0.0235m^3 - 0.005m^2 - 0.013m - 0.00125\right)\left(0.47m^2 + 0.24\right) \\
&+ \frac{1}{\eta^2}\left(0.47m^3 - 1.2975m - 0.15375\right)\left(0.47m^2 + 0.24\right)m \\
&+ \frac{-3.15}{\eta}\left(0.47m^2 + 0.24\right)m^2 + 0.075\,(1.075)\,m^2
\end{aligned} \tag{233}$$

grows in η.

Proof of Lemma B.103. To prove the lemma, note the first partial η of Equation 233 below (scaled by η^5 with $(0.47m + 0.24)$ factored out).

$$\begin{aligned}
&- 4\left((0.25)^2 m^2 + 0.0125m + (0.025)^2\right)\left(0.47m^2 + 0.24\right) \\
&- 3\eta\left(0.235m^4 + 0.0235m^3 - 0.005m^2 - 0.013m - 0.00125\right) \\
&- 2m\eta^2\left(0.47m^3 - 1.2975m - 0.15375\right) + 3.15m^2\eta^3
\end{aligned} \tag{234}$$

From the above, note that $0.47m^3 - 1.2975m - 0.15375$ is locally convex, so we evaluate at $m = 0.15, 0.25$, which are negative.

For $0.235m^4 + 0.0235m^3 - 0.005m^2 - 0.013m - 0.00125$, note the second partial m can be lower bounded by $2.82\,(0.15)^2 + 0.141\,(0.15) - 0.01$ (positive), so we evaluate $0.235m^4 + 0.0235m^3 - 0.005m^2 - 0.013m - 0.00125$ at $m = 0.15, 0.25$, which are positive.

So, we lower bound Equation 234 by $\eta = 0.2$, where the resulting expression can be lower bounded by

$$-0.2961\,(0.25)^4 - 0.0376\,(0.25)^3 + 0.070825\,(0.15)^2 + 0.0081\,(0.15) + 0.00015,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.90. To prove the lemma, note that we consider the partial η of Equation 68, with term $-3.225m^2$ removed and the remaining expression scaled by $\sqrt{2m\eta + 0.25m + 0.025}$.

We show the resulting expression below.

$$\frac{1}{\eta^4}\Big(-0.01035m^7 - 0.00311m^6 - 0.01088m^5 - 0.00318m^4 - 0.00302m^3 - 0.00082m^2 - 0.00008m - 0.00004\,(0.075)\Big) + \frac{1}{\eta}\left(-0.2209m^7 + 0.027025m^5 + 0.02526m^4 + 0.0714m^3 + 0.0129m^2\right) + \frac{1}{\eta^2}\left(-0.38657m^7 - 0.03867m^6 - 0.33164m^5 - 0.02684m^4 - 0.06791m^3 - 0.00363m^2 + 0.00033m\right) + \frac{1}{\eta^3}\left(-0.12427m^7 - 0.02489m^6 - 0.12815m^5 - 0.02538m^4 - 0.03371m^3 - 0.00648m^2 - 0.00034m\right) + 3.46689m^3\eta - 1.0105m^5 - 0.22709m^3 + 0.02890m^2 \tag{235}$$

From the above, note that

$$-0.38657m^6 - 0.03867m^5 - 0.33164m^4 - 0.02684m^3 - 0.06791m^2 - 0.00363m + 0.00033 \tag{236}$$

can be upper bounded by $m = 0.15$, which is negative.

Also, note that

$$-0.2209m^5 + 0.027025m^3 + 0.02526m^2 + 0.0714m + 0.0129 \tag{237}$$

can be lower bounded by

$$-0.2209\,(0.25)^5 + 0.027025\,(0.15)^3 + 0.02526\,(0.15)^2 + 0.0714\,(0.15) + 0.0129.$$

We can then upper bound Equation 235 in η by

$$\frac{1}{(0.3)^4}\Big(-0.01035m^7 - 0.00311m^6 - 0.01088m^5 - 0.00318m^4 - 0.00302m^3 - 0.00082m^2 - 0.00008m - 0.00004\,(0.075)\Big) + \frac{1}{0.2}\left(-0.2209m^7 + 0.027025m^5 + 0.02526m^4 + 0.0714m^3 + 0.0129m^2\right) + \frac{1}{(0.3)^2}\left(-0.38657m^7 - 0.03867m^6 - 0.33164m^5 - 0.02684m^4 - 0.06791m^3 - 0.00363m^2 + 0.00033m\right) + \frac{1}{(0.3)^3}\left(-0.12427m^7 - 0.02489m^6 - 0.12815m^5 - 0.02538m^4 - 0.03371m^3 - 0.00648m^2 - 0.00034m\right) + 3.46689m^3\,(0.3) - 1.0105m^5 - 0.22709m^3 + 0.02890m^2, \tag{238}$$

which simplifies (after rounding up after five digits of precision) to

$$-11.28009m^7 - 1.73546m^6 - 10.64977m^5 - 1.50451m^4 - 1.20593m^3 - 0.28816m^2 - 0.01880m - 0.00037. \tag{239}$$

This completes the proof.

□

Proof. To prove the lemma, consider the second partial η of Equation 69 below, simplified by rounding up after five digits of precision and scaling by η^4.

$$-m^2\eta^4 - \left(1.88m^5 + 0.96m^3\right)\eta^3 + \left(0.8836m^8 + 0.9024m^6 - 0.705m^5 + 0.1599m^4 - 0.36m^3 - 0.036m^2\right)\eta^2 + \left(0.2209m^8 + 0.02209m^7 + 0.2256m^6 - 0.03619m^5 + 0.04585m^4 - 0.02482m^3 - 0.006m^2 - 0.0003m\right)\eta + \left(0.01381m^8 + 0.00277m^7 + 0.01424m^6 + 0.00282m^5 + 0.00375m^4 + 0.00072m^3 + 0.00004m^2\right) \tag{240}$$

To proceed, we show that the first partial η of Equation 240 is negative. To see this, first note that

$$\left(0.8836m^6 + 0.9024m^4 - 0.705m^3 + 0.1599m^2 - 0.36m - 0.036\right) \tag{241}$$

can be upper bounded by

$$0.8836\left(0.25\right)^6 + 0.9024\left(0.25\right)^4 - 0.705\left(0.15\right)^3 + 0.1599\left(0.25\right)^2 - 0.36\left(0.15\right) - 0.036,$$

which is negative.

So, we can upper bound the partial η by $\eta = 0.2$, resulting in the following upper bound (with m factored out).

$$0.57434m^7 + 0.02209m^6 + 0.58656m^5 - 0.54379m^4 + 0.10981m^3 - 0.28402m^2 - 0.0524m - 0.0003 \tag{242}$$

We can upper bound the above by

$$0.57434\left(0.25\right)^7 + 0.02209\left(0.25\right)^6 + 0.58656\left(0.25\right)^5 - 0.54379\left(0.15\right)^4 + 0.10981\left(0.25\right)^3 - 0.28402\left(0.15\right)^2 - 0.0524\left(0.15\right) - 0.0003,$$

which is negative.

So, we can upper bound Equation 240 in η by $\eta = 0.2$, resulting in the following expression (simplified by rounding up after five digits of precision with m factored out).

$$0.09334m^7 + 0.00719m^6 + 0.09546m^5 - 0.04765m^4 + 0.01932m^3 - 0.02632m^2 - 0.0042m - 0.00006 \tag{243}$$

We upper bound the above by

$$\Bigg(\Bigg(\Bigg(\Bigg(\Bigg(\Bigg(0.09334\left(0.25\right) + 0.00719\Bigg)\left(0.25\right) + 0.09546\Bigg)\left(0.25\right) - 0.04765\Bigg)\left(0.15\right) + 0.01932\Bigg)\left(0.25\right) - 0.02632\Bigg)\left(0.15\right) - 0.0042\Bigg)\left(0.15\right) - 0.00006,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.92. To prove the lemma, we first show that the coefficients for the $\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.15}\right)\right)$ and $\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)$ terms (scaled by $\eta\sqrt{0.15}$) are negative in lemmas **??** and B.93.

We then lower bound Equation 70 by replacing $\exp\left(\frac{0.94m^3}{\eta} + \frac{0.48m}{\eta} + 2.15m\right)$ with $\exp\left(\frac{0.94m^3}{0.3} + \frac{0.48m}{0.3} + 2.15m\right)$, $\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.15}\right)\right)$ with $\exp\left(\left(\frac{0.47m^2}{0.2} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.15}\right)\right)$, and $\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)$ with $\exp\left(\left(\frac{0.47m^2}{0.3} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)$.

We then show that the resulting lower bound is concave in η. Consider the second partial η that we split into two parts (shown below) based on the exponential term and show both are negative. The first part is the coefficient for the $\exp\left(\left(\frac{0.47m^2}{0.2} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.15}\right)\right)$ term and the second part is the coefficient for $\exp\left(\left(\frac{0.47m^2}{0.3} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)$ term. Note that both parts are scaled by η^5 and simplified by rounding up after five digits of precision.

Part 1

$$\left(-1.14072m^4 - 1.25236m^3 - 1.35190m^2 + 0.10505m + 0.00902\right)\eta^2$$
$$+ \left(-0.54754m^5 - 0.36234m^4 - 0.45000m^3 - 0.12316m^2 - 0.00543m - 0.00639\right)\eta$$
$$+ \left(-0.03285m^6 - 0.02544m^5 - 0.02494m^4 - 0.01550m^3 - 0.00465m^2 - 0.00128m - 0.00024\right) \tag{244}$$

Part 2

$$\left(-1.14072m^4 - 1.25236m^3 - 1.32182m^2 - 0.14470m - 0.45562\right)\eta^2$$
$$+ \left(-0.54754m^5 + 0.06178m^4 - 0.66838m^3 + 0.09935m^2 - 0.11694m + 0.07261\right)\eta$$
$$+ \left(-0.03285m^6 + 0.02545m^5 - 0.05201m^4 + 0.03648m^3 - 0.02254m^2 + 0.01199m - 0.00232\right) \tag{245}$$

To see that Equation 244 is negative, note that we can upper bound it by

$$\left(-1.14072\,(0.2)^4 - 1.25236\,(0.2)^3 - 1.35190\,(0.2)^2 + 0.10505\,(0.25) + 0.00902\right)(0.2)^2$$
$$+ \left(-0.54754\,(0.2)^5 - 0.36234\,(0.2)^4 - 0.45000\,(0.2)^3 - 0.12316\,(0.2)^2 - 0.00543\,(0.2) - 0.00639\right)(0.2)$$
$$+ \Big(-0.03285\,(0.2)^6 - 0.02544\,(0.2)^5 - 0.02494\,(0.2)^4 - 0.01550\,(0.2)^3 - 0.00465\,(0.2)^2$$
$$- 0.00128\,(0.2) - 0.00024\Big),$$

which is negative.

To see that Equation 245 is negative, note that we can upper bound it by

$$\left(-1.14072\,(0.2)^4 - 1.25236\,(0.2)^3 - 1.32182\,(0.2)^2 - 0.14470\,(0.2) - 0.45562\right)(0.2)^2$$
$$+ \left(-0.54754\,(0.2)^5 + 0.06178\,(0.25)^4 - 0.66838\,(0.2)^3 + 0.09935\,(0.25)^2 - 0.11694\,(0.2) + 0.07261\right)(0.2)$$
$$+ \Big(-0.03285\,(0.2)^6 + 0.02545\,(0.25)^5 - 0.05201\,(0.2)^4 + 0.03648\,(0.25)^3 - 0.02254\,(0.2)^2$$
$$+ 0.01199\,(0.25) - 0.00232\Big),$$

which is negative.

After showing concavity in η, we consider the cases where $\eta = 0.2, 0.3$.

Case 1: $\eta = 0.2$. In this case, we first replace $\exp\left(\frac{0.94m^3}{0.3} + \frac{0.48m}{0.3} + 2.15m\right)$ by linearizing at $m = 0.25$. We replace $\exp\left(\left(\frac{0.47m^2}{0.3} + \frac{0.24}{(0.2)} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)$ with $\exp\left(\left(\frac{0.47(0.2)m}{0.3} + \frac{0.24}{(0.2)} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)$, which we then replace with local linear upper bound $1.56934m + 0.33157$. Last, we replace $\exp\left(\left(\frac{0.47m^2}{0.2} + \frac{0.24}{(0.2)} + 1.075\right)\left(m + \sqrt{0.15}\right)\right)$ with $\exp\left(\left(\frac{0.47(0.25)m}{0.2} + \frac{0.24}{(0.2)} + 1.075\right)\left(m + \sqrt{0.15}\right)\right)$, which we then replace with local linear upper bound $12.09526m + 1.65693$. To see that the local linear upper bounds are valid, consider the difference between the linear bound and the exponential term at $m = 0.15, 0.25$, since the difference is concave in m.

The resulting expression (simplified by rounding down after five digits of precision),

$$-38.96880m^5 - 5.67082m^4 - 63.08228m^3 - 9.42237m^2 + 8.10981m - 0.24102, \tag{246}$$

is concave in m, so we evaluate at $m = 0.2, 0.25$, which are positive.

Case 2: $\eta = 0.3$. In this case, we first replace $\exp\left(\frac{0.94m^3}{0.3} + \frac{0.48m}{0.3} + 2.15m\right)$ by linearizing at $m = 0.25$. We then replace $\exp\left(\left(\frac{0.47m^2}{0.3} + \frac{0.24}{(0.3)} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)$ by $\exp\left(\left(\frac{0.47(0.2)m}{0.3} + \frac{0.24}{(0.3)} + 1.075\right)\left(m - \sqrt{0.15}\right)\right)$, which we replace with local linear upper bound $1.38245m + 0.41916$. Last, we replace $\exp\left(\left(\frac{0.47m^2}{0.2} + \frac{0.24}{(0.3)} + 1.075\right)\left(m + \sqrt{0.15}\right)\right)$ with $\exp\left(\left(\frac{0.47(0.25)m}{0.2} + \frac{0.24}{(0.3)} + 1.075\right)\left(m + \sqrt{0.15}\right)\right)$, which we replace with local linear upper bound $8.09734m + 1.60315$.

The resulting expression (simplified by rounding down after five digits of precision),

$$-18.02302m^5 - 3.84483m^4 - 37.88021m^3 - 8.27558m^2 + 7.73506m - 0.78749, \tag{247}$$

is concave in m, so we evaluate at $m = 0.2, 0.25$, which are positive.

This completes the proof.

□

Proof of Lemma B.93. To prove the lemma, we upper bound Equation 71 by

$$-(1.075)^2(0.2)^2 + \left(\frac{1}{2} - (1.075)(0.94)(0.2)^2 - (1.075)(0.48) - 0.5375\sqrt{0.15}\right)(0.2)$$
$$+ \left(-0.235\sqrt{0.15}(0.2)^2 - 0.12\sqrt{0.15} - (0.47)^2(0.2)^4 - (0.48)(0.47)(0.2)^2 - (0.24)^2\right),$$

which is negative.

This completes the proof. □

Proof of Lemma B.94. To prove the lemma, we first note that the coefficients for the $\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.1175}\right)\right)$ and $\exp\left(\left(\frac{0.47m^2}{\eta} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.1175}\right)\right)$ terms (scaled by η) are negative by lemmas B.96 and B.95.

So, we can lower bound Equation 72 by

$$\frac{1}{2\sqrt{0.1175}} + \frac{1}{2\sqrt{0.1175}}\exp\left(\frac{0.94m^3}{0.3} + \frac{0.48m}{0.3} + 2.15m\right)$$

$$+\left(\frac{1}{2\sqrt{0.1175}} + \frac{1}{2}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)\right.$$
$$\left. - \frac{\eta}{\sqrt{0.1175}}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)^2\right)\exp\left(\left(\frac{0.47m^2}{0.2} + \frac{0.24}{\eta} + 1.075\right)\left(m + \sqrt{0.1175}\right)\right)$$

$$+\left(\frac{1}{2\sqrt{0.1175}} - \frac{1}{2}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)\right.$$
$$\left. - \frac{\eta}{\sqrt{0.1175}}\left(\frac{0.47m^2 + 0.24}{\eta} + 1.075\right)^2\right)\exp\left(\left(\frac{0.47m^2}{0.3} + \frac{0.24}{\eta} + 1.075\right)\left(m - \sqrt{0.1175}\right)\right) \tag{248}$$

We then proceed by showing that the above lower bound is concave in η in lemma B.97. So, we consider the cases $\eta = 0.2, 0.3$.

Case 1: $\eta = 0.2$. In this case, we first replace $\exp\left(\frac{0.94m^3}{0.3} + \frac{0.48m}{0.3} + 2.15m\right)$ with $\exp\left(\frac{0.94(0.15)^2m}{0.3} + \frac{0.48m}{0.3} + 2.15m\right)$ and linearize it at $m = 0.2$.

Then, we replace $\exp\left(\left(\frac{0.47x^2}{0.3}+\frac{0.24}{(0.2)}+1.075\right)\left(x-\sqrt{0.1175}\right)\right)$ with $\exp\left(\left(\frac{0.47(0.15)^2}{0.3}+\frac{0.24}{(0.2)}+1.075\right)\left(m-\sqrt{0.1175}\right)\right)$, which we then replace by linear upper bound $1.56877m+0.40527$. To see that the linear bound is valid, consider the difference between the linear bound and the exponential term at $m=0.15, 0.2$, since the difference is concave in m. Last, we replace $\exp\left(\left(\frac{0.47m^2}{0.2}+\frac{0.24}{(0.2)}+1.075\right)\left(m+\sqrt{0.1175}\right)\right)$ by $\exp\left(\left(\frac{0.47(0.2)^2}{0.2}+\frac{0.24}{(0.2)}+1.075\right)\left(m+\sqrt{0.1175}\right)\right)$, which we then replace by linear upper bound $8.08233m+2.00129$. To see that the linear bound is valid, consider the difference between the linear bound and the exponential term at $m=0.15, 0.2$, since the difference is concave in m.

Equation 248 for $\eta=0.2$ can then be lower bounded by

$$-31.09736m^5-7.75431m^4-52.5563m^3-13.13836m^2+4.30772m+0.25598, \tag{249}$$

which is concave in m. So, we evaluate at $m=0.15, 0.2$, both of which are positive.

Case 2: $\eta=0.3$. In this case, we first replace $\exp\left(\left(\frac{0.47m^2}{0.3}+\frac{0.24}{(0.3)}+1.075\right)\left(m-\sqrt{0.1175}\right)\right)$ by $\exp\left(\left(\frac{0.47(0.15)}{0.3}m+\frac{0.24}{(0.3)}+1.075\right)\left(m-\sqrt{0.1175}\right)\right)$, which we then replace with local linear upper bound $1.36143x+0.48773$. To see that the linear bound is valid, consider the difference between the linear bound and the exponential term at $m=0.15, 0.2$, since the difference is concave in m.

We also replace $\exp\left(\left(\frac{0.47m^2}{0.2}+\frac{0.24}{(0.3)}+1.075\right)\left(m+\sqrt{0.1175}\right)\right)$ with $\exp\left(\left(\frac{0.47(0.2)m}{0.2}+\frac{0.24}{(0.3)}+1.075\right)\left(m+\sqrt{0.1175}\right)\right)$, which we then replace with linear lower bound $6.06760x+1.69819$. To see that the linear bound is valid, consider the difference between the linear bound and the exponential term at $m=0.15, 0.2$, since the difference is concave in m.

We also linearize $\exp\left(\frac{0.94m^3}{0.3}+\frac{0.48m}{0.3}+2.15m\right)$ at $m=0.2$.

The resulting expression simplifies (by rounding down after five digits of precision) to

$$-15.95833m^5-4.69558m^4-34.51164m^3-10.29124m^2+5.45480m-0.39030 \tag{250}$$

Note that the above is concave in m, so we evaluate it at $m=0.15, 0.2$, which are positive.

This completes the proof.

□

Proof of Lemma B.95. To prove the lemma, note that Equation 73 can be upper bounded by

$$-\frac{(1.075)^2}{\sqrt{0.1175}}(0.2)^2+\left(\frac{1}{2\sqrt{0.1175}}-0.5375-\frac{(1.075)(0.94)}{\sqrt{0.1175}}(0.15)^2-\frac{(1.075)(0.48)}{\sqrt{0.1175}}\right)(0.2)$$
$$+\left(-0.235(0.15)^2-0.12-\frac{(0.47)^2}{\sqrt{0.1175}}(0.15)^4-\frac{(0.48)(0.47)}{\sqrt{0.1175}}(0.15)^2-\frac{(0.24)^2}{\sqrt{0.1175}}\right),$$

which is negative.

This completes the proof. □

Proof of Lemma B.96. To prove the lemma, we note that Equation 74 can be upper bounded by

$$-\frac{(1.075)^2}{\sqrt{0.1175}}(0.2)^2+\left(\frac{1}{2\sqrt{0.1175}}+0.5375-\frac{(2.15)(0.47)}{\sqrt{0.1175}}(0.15)^2-\frac{(1.075)(0.48)}{\sqrt{0.1175}}\right)(0.3)$$
$$+\left(0.235(0.2)^2+0.12-\frac{(0.47)^2}{\sqrt{0.1175}}(0.15)^4-\frac{(0.24)(0.94)}{\sqrt{0.1175}}(0.15)^2-\frac{(0.24)^2}{\sqrt{0.1175}}\right),$$

which is negative.

This completes the proof. □

Proof of Lemma B.97. To prove the lemma, we consider the second partial η, which we split into two parts (shown below) based on the exponential term and show both are negative. The first part (Equation 251) is the coefficient for the $\exp\left(\left(\frac{0.47m^2}{0.2}+\frac{0.24}{\eta}+1.075\right)\left(m+\sqrt{0.1175}\right)\right)$ term. The second part (Equation 252) is the coefficient for the $\exp\left(\left(\frac{0.47m^2}{0.3}+\frac{0.24}{\eta}+1.075\right)\left(m-\sqrt{0.1175}\right)\right)$ term. Both parts are scaled by η^5 and simplified by rounding up after five digits of precision.

Part 1

$$\begin{aligned}&\left(-0.03711m^6-0.02544m^5-0.02873m^4-0.01670m^3-0.00563m^2-0.00189m-0.00032\right)\\&+\left(-0.61865m^5-2.95959m^4-0.52262m^3-2.76349m^2-0.02673m-0.68463\right)\eta\\&+\left(1.28887m^4-1.41500m^3+1.10706m^2+0.10247m+0.63402\right)\eta^2\end{aligned} \tag{251}$$

Part 2

$$\begin{aligned}&\left(-1.28886m^4-1.41500m^3-1.49543m^2-0.14727m-0.50277\right)\eta^2\\&+\left(-0.61865m^5+0.04227m^4-0.74100m^3+0.24031m^2-0.25344m+0.09084\right)\eta\\&+\left(-0.03711m^6+0.02545m^5-0.05580m^4+0.03527m^3-0.02263m^2+0.01138m-0.00194\right)\end{aligned} \tag{252}$$

Note that Equation 251 can be upper bounded by

$$\begin{aligned}&\Big(-0.03711\left(0.15\right)^6-0.02544\left(0.15\right)^5-0.02873\left(0.15\right)^4-0.01670\left(0.15\right)^3-0.00563\left(0.15\right)^2\\&\qquad\qquad-0.00189\left(0.15\right)-0.00032\Big)\\&+\Big(-0.61865\left(0.15\right)^5-2.95959\left(0.15\right)^4-0.52262\left(0.15\right)^3-2.76349\left(0.15\right)^2-0.02673\left(0.15\right)-0.68463\Big)\left(0.2\right)\\&+\Big(1.28887\left(0.2\right)^4-1.41500\left(0.15\right)^3+1.10706\left(0.2\right)^2+0.10247\left(0.2\right)+0.63402\Big)\left(0.3\right)^2,\end{aligned}$$

which is negative.

Further note that Equation 252 can be upper bounded by

$$\begin{aligned}&\Big(-1.28886\left(0.15\right)^4-1.41500\left(0.15\right)^3-1.49543\left(0.15\right)^2-0.14727\left(0.15\right)-0.50277\Big)\left(0.2\right)^2\\&+\Big(-0.61865\left(0.15\right)^5+0.04227\left(0.2\right)^4-0.74100\left(0.15\right)^3+0.24031\left(0.2\right)^2-0.25344\left(0.15\right)+0.09084\Big)\left(0.3\right)\\&+\Big(-0.03711\left(0.15\right)^6+0.02545\left(0.2\right)^5-0.05580\left(0.15\right)^4+0.03527\left(0.2\right)^3-0.02263\left(0.15\right)^2\\&\qquad\qquad+0.01138\left(0.2\right)-0.00194\Big),\end{aligned}$$

which is negative.

This completes the proof.

□

Proof of Lemma B.98. To prove the lemma, we start with the second partial c of Equation 76, shown below (scaled by $4c^{\frac{3}{2}}$).

$$-1-\exp\left(\frac{0.94m^3}{\eta}+\frac{0.48m}{\eta}+2.15m\right)+k_1\exp\left(\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\left(m-\sqrt{c}\right)\right)$$
$$+k_2\exp\left(\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\left(m+\sqrt{c}\right)\right), \tag{253}$$

where

$$k_1=-1-\left(\frac{\sqrt{c}}{\eta}\right)\left(0.47m^2+0.24+1.075\eta\right)+\left(\frac{c}{\eta^2}\right)\left(0.47m^2+0.24+1.075\eta\right)^2$$
$$+\left(\frac{2}{\eta}\right)\left(0.47m^2+0.24+1.075\eta\right)^2+\left(\frac{2\sqrt{c}}{\eta^2}\right)\left(0.47m^2+0.24+1.075\eta\right)^3$$

and

$$k_2=-1+\left(\frac{\sqrt{c}}{\eta}\right)\left(0.47m^2+0.24+1.075\eta\right)+\left(\frac{c}{\eta^2}\right)\left(0.47m^2+0.24+1.075\eta\right)^2$$
$$+\left(\frac{2}{\eta}\right)\left(0.47m^2+0.24+1.075\eta\right)^2-\left(\frac{2\sqrt{c}}{\eta^2}\right)\left(0.47m^2+0.24+1.075\eta\right)^3.$$

To proceed, we will show that Equation 253 grows in c. We first show that k_1 is positive and grows in c in lemmas B.100.

So, we can replace $\exp\left(\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\left(m+\sqrt{c}\right)\right)$ in the partial c for Equation 253 by $\exp\left(\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\left(m-\sqrt{c}\right)\right)$.

The resulting lower bound for the partial derivative (with the exponent and $\left(0.47m^2+0.24+1.075\eta\right)^2$ term factored out and scaled by η^3) is shown below.

$$-(2)(0.47)^2m^4-(0.48)(0.94)m^2-(2.15)(0.94)m^2\eta-(2)(0.24)^2$$
$$+(-(2.15)(0.48)+3)\eta-(2)(1.075)^2\eta^2 \tag{254}$$

Note that we lower bound the above by

$$-(2)(0.47)^2(0.25)^4-(0.48)(0.94)(0.25)^2-(2.15)(0.94)(0.25)^2(0.3)-(2)(0.24)^2$$
$$+(-(2.15)(0.48)+3)(0.2)-(2)(1.075)^2(0.3)^2,$$

which is positive.

So, we lower bound Equation 253 in c by $c=\sqrt{0.1175}$.

Since we know that k_2 is positive by lemma B.99, we can further lower bound Equation 253 by replacing $\exp\left(\frac{0.94m^3}{\eta}+\frac{0.48m}{\eta}+2.15m\right)$ with $\exp\left(\frac{0.94m^3}{0.2}+\frac{0.48m}{0.2}+2.15m\right)$, $\exp\left(\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\left(m+\sqrt{0.1175}\right)\right)$ with $\exp\left(\left(\frac{0.47m^2}{0.3}+\frac{0.24}{0.3}+1.075\right)\left(m+\sqrt{0.1175}\right)\right)$, and $\exp\left(\left(\frac{0.47m^2}{\eta}+\frac{0.24}{\eta}+1.075\right)\left(m-\sqrt{0.1175}\right)\right)$ with $\exp\left(\left(\frac{0.47m^2}{0.2}+\frac{0.24}{0.2}+1.075\right)\left(m-\sqrt{0.1175}\right)\right)$.

We then show that the resulting expression is convex in η and first partial η at $\eta=0.3$ is negative.

Consider the second partial η below (scaled by η^4), that we split into two parts with the exponential terms and $\left(0.47m^2+0.24\right)$ factored out. We will show that both parts are positive.

$$1.46525\eta+5.096\sqrt{0.1175}\eta+6.063\sqrt{0.1175}m^2\eta+13.8675\sqrt{0.1175}\eta^2-12\sqrt{0.1175}\left(0.47m^2+0.24+1.075\eta\right)^2$$
$$+1.88m^2\eta+0.33135m^2+0.1692 \tag{255}$$

$$- 2\sqrt{0.1175}\eta + 0.705\left(0.47m^2 + 0.24 + 1.075\eta\right) - 0.235\left(1.075\right)\eta - 4\left(0.47m^2\eta + 0.24\eta + 1.075\eta^2\right)$$
$$+ 8\left(0.47m^2\eta + 0.24\eta + 1.075\eta^2\right) - 4\left(1.075\right)\eta^2 - 6\sqrt{0.1175}\left(0.47m^2 + 0.24 + 1.075\eta\right)^2$$
$$+ 18\sqrt{0.1175}\left(0.47m^2 + 0.24 + 1.075\eta\right)^2 - 12\left(1.075\right)\sqrt{0.1175}\eta\left(0.47m^2 + 0.24 + 1.075\eta\right) \tag{256}$$

To proceed, we first show that Equation 255 grows in η. Consider the partial η below.

$$\left(1.46525 - 1.096\sqrt{0.1175}\right) + \left(-6.063\sqrt{0.1175} + 1.88\right)m^2 \tag{257}$$

We lower bound the above by fixing $m = 0.25$, which is positive, so we lower bound Equation 255 in η by $\eta = 0.2$. Note that the resulting expression is decreasing in m, so we lower bound by 0.25, which is positive.

For Equation 256, we show that it grows in η. Consider the partial η below.

$$\left(12\left(1.075\right)\sqrt{0.1175} + 4\right)\left(0.47m^2 + 0.24\right) + \left(\left(0.47\right)\left(1.075\right) - 2\sqrt{0.1175}\right) \tag{258}$$

We lower bound the above by fixing $m = 0.15$, which is positive. So, we lower bound Equation 256 by $\eta = 0.2$. Note that the resulting expression grows in m, so we lower bound it by $m = 0.15$, which is positive.

After showing convexity of our lower bound for Equation 253, we show the first partial η at $\eta = 0.3$ is negative. Consider the first partial η at $\eta = 0.3$ (simplified by rounding up after five digits of precision) below.

$$\left(5.27240m^6 + 6.67198m^4 - 0.41977m^2 + 0.20751\right)\exp\left(\left(\frac{0.47m^2}{0.3} + \frac{0.24}{0.3} + 1.075\right)\left(m + \sqrt{0.1175}\right)\right)$$
$$+ \left(-5.27240m^6 - 20.33506m^4 - 16.17253m^2 - 0.49501\right)\exp\left(\left(\frac{0.47m^2}{0.2} + \frac{0.24}{0.2} + 1.075\right)\left(m - \sqrt{0.1175}\right)\right) \tag{259}$$

From the above, note that, for $5.27240m^6 + 6.67198m^4 - 0.41977m^2 + 0.20751$, $\left(-0.41977\left(0.25\right)^2 + 0.20751\right)$ is positive.

So, we can upper bound Equation 259 by replacing $\frac{0.47m^2}{0.2} + \frac{0.24}{0.2} + 1.075$ in the exponential terms by $\frac{0.47(0.25)^2}{0.2} + \frac{0.24}{0.2} + 1.075$. Furthermore, we can replace $\exp\left(\left(\frac{0.47(0.25)^2}{0.2} + \frac{0.24}{0.2} + 1.075\right)\left(x - \sqrt{0.1175}\right)\right)$ by linearizing at $m = 0.15$ and replace $\exp\left(\left(\frac{0.47(0.25)^2}{0.3} + \frac{0.24}{0.3} + 1.075\right)\left(m + \sqrt{0.1175}\right)\right)$ with local linear upper bound $5.76612m + 1.77889$. To see that the local linear upper bound is valid, consider the difference between the linear bound and the exponential function at $m = 0.15, 0.25$ (which are positive), since the difference is concave.

Note the resulting upper bound below.

$$22.39579m^7 + 7.27439m^6 + 7.59509m^5 + 3.75137m^4 - 26.9764m^3 - 7.20247m^2 + 0.44492m + 0.17154 \tag{260}$$

We can then upper bound the above by

$$\Bigg(\Bigg(\Bigg(\Bigg(\Bigg(\Bigg(22.39579\left(0.25\right) + 7.27439\Bigg)\left(0.25\right) + 7.59509\Bigg)\left(0.25\right) + 3.75137\Bigg)\left(0.25\right) - 26.9764\Bigg)\left(0.15\right)$$
$$- 7.20247\Bigg)\left(0.15\right) + 0.44492\Bigg)\left(0.15\right) + 0.17154, \tag{261}$$

which is negative.

So, we can lower bound Equation 253 in η by $\eta = 0.3$. We then lower bound the resulting expression by the following.

$$
\begin{aligned}
&-1-\exp\left(\frac{0.94\left(0.25\right)^3}{0.2}+\frac{0.48m}{0.2}+2.15\left(0.25\right)\right)\\
&+\left(-1-\sqrt{0.1175}\left(\frac{0.47m^2+0.5625}{0.3}\right)+0.7175\left(\frac{0.47m^2+0.5625}{0.3}\right)^2\right.\\
&\qquad\left.+0.6\sqrt{0.1175}\left(\frac{0.47m^2+0.5625}{0.3}\right)^3\right)\exp\left(\left(\frac{0.47\left(0.25\right)^2}{0.2}+\frac{0.24}{0.2}+1.075\right)\left(m-\sqrt{0.1175}\right)\right)\\
&+\left(-1+\sqrt{0.1175}\left(\frac{0.47m^2+0.5625}{0.3}\right)+0.7175\left(\frac{0.47m^2+0.5625}{0.3}\right)^2\right.\\
&\qquad\left.-0.6\sqrt{0.1175}\left(\frac{0.47m^2+0.5625}{0.3}\right)^3\right)\exp\left(\left(\frac{0.47\left(0.15\right)^2}{0.3}+\frac{0.24}{0.3}+1.075\right)\left(m+\sqrt{0.1175}\right)\right)
\end{aligned}
\tag{262}
$$

To proceed, we first upper bound $\exp\left(\frac{0.94(0.25)^3}{0.2}+\frac{0.48m}{0.2}+2.15\left(0.25\right)\right)$ with local linear upper bound $7.16212m+1.56611$. To verify the upper bound is valid, note that the difference between the linear bound and exponential term is positive for $m = 0.15, 0.25$, since the difference is concave in m. We then linearize the remaining exponential terms at $m = 0.15$. From this, we get the following lower bound Equation 262.

$$
-2.67182m^7-1.13071m^6+1.70449m^5-0.13592m^4+17.37510m^3+5.30112m^2+0.19583m-0.193366
\tag{263}
$$

The above can be lower bounded by

$$
\begin{aligned}
&\left(\left(\left(\left(\left(\left(-2.67182\left(0.25\right)-1.13071\right)\left(0.25\right)+1.70449\right)\left(0.15\right)-0.13592\right)\left(0.15\right)+17.37510\right)\left(0.15\right)\right.\right.\\
&\qquad\left.\left.+5.30112\right)\left(0.15\right)+0.19583\right)\left(0.15\right)-0.193366,
\end{aligned}
$$

which is positive.

This completes the proof.

□

Proof of Lemma B.99. To begin the lemma, we first scale Equation B.99 by η^2 and show that the resulting expression grows in c.

Consider the first partial c below (scaled by $\sqrt{c}$ with $\left(0.47m^2+0.24+1.075n\right)$ factored out).

$$
\frac{n}{2}+\sqrt{c}\left(0.47m^2+0.24+1.075n\right)-\left(0.47m^2+0.24+1.075n\right)^2
\tag{264}
$$

Since the above grows in c, we lower bound it by $c = 0.1175$. Note that the resulting expression is concave in η. So, we consider two cases. Case 1: $\eta = 0.2$. Note that the resulting expression (shown below) is decreasing in m.

$$
\begin{aligned}
&-\left(0.47\right)^2m^4+\left(0.47\sqrt{0.1175}-\left(0.48\right)\left(0.47\right)-\left(2.15\right)\left(0.47\right)\left(0.2\right)\right)m^2-\left(0.24\right)^2+0.24\sqrt{0.1175}\\
&-\left(1.075\right)^2\left(0.2\right)^2+\left(\frac{1}{2}+1.075\sqrt{0.1175}-\left(1.075\right)\left(0.48\right)\right)\left(0.2\right)
\end{aligned}
\tag{265}
$$

So, we lower bound the above by $m = 0.25$, which is positive.

Case 2: $\eta = 0.3$. Similar to case 1, the resulting expression (shown below) is decreasing in m.

$$-(0.47)^2 m^4 + \left(0.47\sqrt{0.1175} - (0.48)(0.47) - (2.15)(0.47)(0.3)\right) m^2 - (0.24)^2 + 0.24\sqrt{0.1175} - (1.075)^2 (0.3)^2 + \left(\frac{1}{2} + 1.075\sqrt{0.1175} - (1.075)(0.48)\right)(0.3) \tag{266}$$

So, we lower bound it by $m = 0.25$, which is positive.

So, we can lower bound Equation 77 in c by $c = 0.1175$, yielding the following.

$$\left((2) - (2.15)\sqrt{0.1175}\right)(1.075)^2 \eta^3 + \sigma_1 \eta^2 + \sigma_3 \eta \sigma_4 \tag{267}$$

where

$$\begin{aligned}\sigma_1 = &-1 + 1.075\sqrt{0.1175} + (1.075)(0.48)(2) + (1.075)^2 (0.1175) \\ &+ \left(-6(0.47)(1.075)^2 \sqrt{0.1175} + (2.15)(0.47)(2)\right) m^2 \\ &- 2(0.48)(1.075)^2 \sqrt{0.1175} - (0.48)(1.075)^2 \sqrt{0.1175},\end{aligned}$$

$$\begin{aligned}\sigma_2 = &\left((0.47)^2 (2) - (3)(2.15)(0.47)^2 \sqrt{0.1175}\right) m^4 \\ &+ \left(0.47\sqrt{0.1175} - (3)(0.94)(0.48)(1.075)\sqrt{0.1175} + (0.48)(0.47)(2) + (2.15)(0.47)(0.1175)\right) m^2 \\ &+ 0.24\sqrt{0.1175} + (2)(0.24)^2 + (1.075)(0.48)(0.1175) - (2.15)(0.24)^2 \sqrt{0.1175} - (0.48)^2 (1.075)\sqrt{0.1175},\end{aligned}$$

and

$$\begin{aligned}\sigma_3 = &-(2)(0.47)^3 \sqrt{0.1175} m^6 - (3)(0.48)(0.47)^2 \sqrt{0.1175} m^4 + (0.47)^2 (0.1175) m^4 \\ &- 6(0.47)(0.24)^2 \sqrt{0.1175} m^2 + (0.48)(0.47)(0.1175) m^2 + (0.1175)(0.24)^2 - 2(0.24)^3 \sqrt{0.1175}.\end{aligned}$$

We can then lower bound Equation 267 in m by replacing σ_1 with

$$\begin{aligned}&-1 + 1.075\sqrt{0.1175} + (1.075)(0.48)(2) + (1.075)^2 (0.1175) \\ &+ \left(-6(0.47)(1.075)^2 \sqrt{0.1175} + (2.15)(0.47)(2)\right)(0.15)^2 \\ &- 2(0.48)(1.075)^2 \sqrt{0.1175} - (0.48)(1.075)^2 \sqrt{0.1175},\end{aligned}$$

σ_2 with

$$\begin{aligned}&\left((0.47)^2 (2) - (3)(2.15)(0.47)^2 \sqrt{0.1175}\right)(0.25)^4 \\ &+ \left(0.47\sqrt{0.1175} - (3)(0.94)(0.48)(1.075)\sqrt{0.1175} + (0.48)(0.47)(2) + (2.15)(0.47)(0.1175)\right)(0.15)^2 \\ &+ 0.24\sqrt{0.1175} + (2)(0.24)^2 + (1.075)(0.48)(0.1175) - (2.15)(0.24)^2 \sqrt{0.1175} - (0.48)^2 (1.075)\sqrt{0.1175},\end{aligned}$$

and σ_3 with

$$\begin{aligned}&-(2)(0.47)^3 \sqrt{0.1175}(0.25)^6 + \left(-(3)(0.48)(0.47)^2 \sqrt{0.1175} + (0.47)^2 (0.1175)\right)(0.25)^4 \\ &+ \left(-6(0.47)(0.24)^2 \sqrt{0.1175} + (0.48)(0.47)(0.1175)\right)(0.25)^2 + (0.1175)(0.24)^2 - 2(0.24)^3 \sqrt{0.1175},\end{aligned}$$

which simplifies (by rounding down after five digits of precision) to

$$1.45957\eta^3 - 0.01381\eta^2 + 0.13579\eta - 0.00488. \tag{268}$$

We lower bound the above by

$$1.45957(0.2)^3 - 0.01381(0.3)^2 + 0.13579(0.2) - 0.00488,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.100. To prove the lemma, we begin by scaling Equation 78 by η^2 and show that the expression grows in c.

Consider the first partial c below (scaled by $\sqrt{c}$ with $\left(0.47m^2 + 0.24 + 1.075\eta\right)$ factored out).

$$-\frac{\eta}{2} + \sqrt{c}\left(0.47m^2 + 0.24 + 1.075\eta\right) + \left(0.47m^2 + 0.24 + 1.075\eta\right)^2 \tag{269}$$

Note that we can lower bound the above by

$$-\frac{0.3}{2} + \sqrt{0.1175}\left(0.47\left(0.15\right)^2 + 0.24 + 1.075\left(0.2\right)\right) + \left(0.47\left(0.15\right)^2 + 0.24 + 1.075\left(0.2\right)\right)^2,$$

which is positive.

So, we lower bound Equation 78 in c by $c = 0.1175$. Then, we show that the resulting expression grows in η. Consider the partial η below.

$$\begin{aligned}
&\left(6\left(1.075\right)\sqrt{0.1175} + 6\right)\left(1.075\right)^2\eta^2 \\
&+\Bigg(\left(\left(2.15\right)\left(0.47\right)\left(6\left(1.075\right)\sqrt{0.1175} + 2\right) + \left(4\right)\left(1.075\right)\left(0.47\right)\right)m^2 \\
&\quad + \left(2.15\right)\left(0.24\right)\left(6\left(1.075\right)\sqrt{0.1175} + 2\right) + \left(4\right)\left(1.075\right)\left(0.24\right) + \left(2\right)\left(1.075\right)^2\left(0.1175\right) - 2 - 2.15\sqrt{0.1175}\Bigg)\eta \\
&+\Bigg(\left(0.47\right)^2\left(6\left(1.075\right)\sqrt{0.1175} + 2\right)m^4 \\
&\quad + \left(\left(2\right)\left(1.075\right)\left(0.1175\right)\left(0.47\right) - 0.47\sqrt{0.1175} + \left(0.24\right)\left(0.94\right)\left(6\left(1.075\right)\sqrt{0.1175} + 2\right)\right)m^2 \\
&\quad + \left(2\right)\left(1.075\right)\left(0.1175\right)\left(0.24\right) - 0.24\sqrt{0.1175} + \left(0.24\right)^2\left(6\left(1.075\right)\sqrt{0.1175} + 2\right)\Bigg)
\end{aligned} \tag{270}$$

Note that we lower bound the above by

$$\begin{aligned}
&\left(6\left(1.075\right)\sqrt{0.1175} + 6\right)\left(1.075\right)^2\left(0.2\right)^2 \\
&+\Bigg(\left(\left(2.15\right)\left(0.47\right)\left(6\left(1.075\right)\sqrt{0.1175} + 2\right) + \left(4\right)\left(1.075\right)\left(0.47\right)\right)\left(0.15\right)^2 \\
&\quad + \left(2.15\right)\left(0.24\right)\left(6\left(1.075\right)\sqrt{0.1175} + 2\right) + \left(4\right)\left(1.075\right)\left(0.24\right) + \left(2\right)\left(1.075\right)^2\left(0.1175\right) - 2 - 2.15\sqrt{0.1175}\Bigg)\left(0.2\right) \\
&+\Bigg(\left(0.47\right)^2\left(6\left(1.075\right)\sqrt{0.1175} + 2\right)\left(0.15\right)^4 \\
&\quad + \left(\left(2\right)\left(1.075\right)\left(0.1175\right)\left(0.47\right) - 0.47\sqrt{0.1175} + \left(0.24\right)\left(0.94\right)\left(6\left(1.075\right)\sqrt{0.1175} + 2\right)\right)\left(0.15\right)^2 \\
&\quad + \left(2\right)\left(1.075\right)\left(0.1175\right)\left(0.24\right) - 0.24\sqrt{0.1175} + \left(0.24\right)^2\left(6\left(1.075\right)\sqrt{0.1175} + 2\right)\Bigg),
\end{aligned}$$

which is positive.

So, we further lower bound Equation 78 in η by $\eta = 0.2$. Since the terms for the resulting expression in m are all positive, this completes the proof.

□

B.5.2 Complex-Valued System Jacobian Eigenvalues

In this section, we prove Theorem B.18, where we will we will show that the eigenvalues of the 2D dynamical system Jacobian evaluated at the fixed point of interest $(\tilde{q}, \tilde{c})$ are complex-valued for the parameter range $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$. To do so, we consider the characteristic equation for the Jacobian matrix (a quadratic equation) and show that the discriminant for the root solution (eigenvalue solution) is negative at $(\tilde{q}, \tilde{c})$.

Consider the equation for the discriminant below.

$$t_1(m, \eta, q, c)^2 - t_2(m, \eta, q, c)^2$$

where

$$\begin{aligned} t_1(m, \eta, q, c) = 2 - \frac{\eta q}{2}\Bigg(&\left(1 + \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\left(m + \sqrt{c}\right)^2 \\ &+ \left(1 + \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\left(m - \sqrt{c}\right)^2\Bigg) \\ &- \eta q^2\Bigg(\left(1 + \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right) \\ &\qquad + \left(1 + \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\Bigg) \end{aligned}$$

$$\begin{aligned} t_2(m, \eta, q, c) = \eta\sqrt{2q}\Bigg(&2\left(\left(1 + \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\right)^{-1} - \left(1 + \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\right)^{-1}\right) \\ &+ q\Bigg(\left(1 + \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\left(m + \sqrt{c}\right) \\ &\qquad - \left(1 + \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\left(m - \sqrt{c}\right)\Bigg)\Bigg) \end{aligned}$$

Note that since the fixed point satisfies the following condition

$$\sqrt{\tilde{c}} - \eta\tilde{q}\left[\left(1 + \exp\left(\frac{1}{2}\tilde{q}\left(m - \sqrt{\tilde{c}}\right)\right)\right)^{-1} - \left(1 + \exp\left(\frac{1}{2}\tilde{q}\left(m + \sqrt{\tilde{c}}\right)\right)\right)^{-1}\right] = 0$$

by definition, we rewrite $t_2(m, \eta, q, c)$ as follows:

$$\begin{aligned} t_2(m, \eta, q, c) = \eta\sqrt{2q}\Bigg(&\frac{2\sqrt{c}}{\eta q} + q\Bigg(\left(1 + \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\left(m + \sqrt{c}\right) \\ &\qquad - \left(1 + \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\left(m - \sqrt{c}\right)\Bigg)\Bigg) \end{aligned}$$

Note that we already proved that the fixed point $(\tilde{q}, \tilde{c})$ exists for $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$ and is contained within a rectangle, where $\tilde{q} \in \left[\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944, \frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15\right]$ and $\tilde{c} \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025]$.

Generally (relaxing the m and η dependency) , this means that $\tilde{q} \in [3.74483, 4.84375]$ and $\tilde{c} \in [0.1175, 0.2375]$.

We will show that the discriminant is negative for any point in the relaxed fixed point containment rectangle ($[3.74483, 4.84375] \times [0.1175, 0.2375]$).

To prove the theorem, we leverage the fact that the discriminant expression (after applying the fixed point condition) is the difference of two squares and split it into two parts, $t_1(m, \eta, q, c) -$

$t_2(m,\eta,q,c)$ and $t_1(m,\eta,q,c)+t_2(m,\eta,q,c)$. After scaling and reparameterization, we show that $t_1(m,\eta,q,c)+t_2(m,\eta,q,c)$ is negative and $t_1(m,\eta,q,c)+t_2(m,\eta,q,c)$ is positive. To show that the scaled and reparameterized version of $t_1(m,\eta,q,c)-t_2(m,\eta,q,c)$ is negative, we first upper bound the expression in η and divide it up further into parts, where one part contains a sigmoid squared where the inner expression is positive (positive sigmoid squared) and the other part contains a sigmoid squared with an inner expression that is negative (negative sigmoid squared). We show that these subparts are negative. Similar to the above, we show that the scaled and reparameterized version of $t_1(m,\eta,q,c)+t_2(m,\eta,q,c)$ is positive by first upper bounding the expression in η and dividing it up further into two parts, where where one part contains a sigmoid squared where the inner expression is negative (negative sigmoid squared) and the other part contains a sigmoid squared with an inner expression that is positive (positive sigmoid squared). We show that these subparts are positive.

We present the form version of the proof for the theorem below.

Proof of Theorem **??**. To prove the theorem, we define the following change of variable $p=\sqrt{q}, \nu=\sqrt{c}$ and $k_1(m,\eta,p,\nu), k_2(m,\eta,p,\nu)$ below.

$$\begin{aligned}k_1(m,\eta,p,\nu) = 2p - \frac{\eta p^3}{2}\Bigg(&\left(\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(m+\nu\right)^2\right.\\ &\left.+\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(m-\nu\right)^2\right)\\ &-\eta p^5\Bigg(\left(\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right.\\ &\left.+\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)\end{aligned}$$

$$\begin{aligned}k_2(m,\eta,p,\nu) = \eta\sqrt{2}p\Bigg(&\frac{2\nu}{\eta p}+p^3\Bigg(\left(\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(m+\nu\right)\right.\\ &\left.-\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(m-\nu\right)\right)\Bigg).\end{aligned}$$

Note that in addition to the change of variable, we also incorporated scaling by p into the definition of $k_1(m,\eta,p,\nu), k_2(m,\eta,p,\nu)$. So, the analogue for $k_1(m,\eta,p,\nu), k_2(m,\eta,p,\nu)$ under the q,c variables would be $\sqrt{q}\cdot t_1(m,\eta,q,c), \sqrt{q}\cdot t_2(m,\eta,q,c)$ respectively.

We will show that $k_1(m,\eta,p,\nu)^2 - k_2(m,\eta,p,\nu)^2$ is negative for $m\in[0.15,0.25], \eta\in[0.2,0.3], p\in\left[\sqrt{3.74483},\sqrt{4.84375}\right], \nu\in\left[\sqrt{0.1175},\sqrt{0.2375}\right]$.

We decompose $k_1(m,\eta,p,\nu)^2 - k_2(m,\eta,p,\nu)^2$ into $\left(k_1(m,\eta,q,\nu)-k_2(m,\eta,p,\nu)\right)\cdot\left(k_1(m,\eta,p,\nu)+k_2(m,\eta,p,\nu)\right)$.

We will show that $k_1(m,\eta,p,\nu)-k_2(m,\eta,p,\nu)$ is negative and $k_1(m,\eta,p,\nu)+k_2(m,\eta,p,\nu)$ is positive for the specified range of m,η,p,ν.

First, we will show that $k_1(m,\eta,p,\nu)-k_2(m,\eta,p,\nu)$, shown below, is negative.

$$
\begin{aligned}
&2p - 2\sqrt{2}\nu - \frac{\eta p^3}{2}\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(m+\nu\right)^2\\
&-\frac{\eta p^3}{2}\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(m-\nu\right)^2\\
&-\eta p^5\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\\
&-\eta p^5\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\\
&-\sqrt{2}\eta p^4\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(m+\nu\right)\\
&+\sqrt{2}\eta p^4\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(m-\nu\right)
\end{aligned}
$$

First, we upper bound the above by fixing $\eta = 0.2$, since partial η of $k_1(m,\eta,p,\nu) - k_2(m,\eta,p,\nu)$ is negative.

To see this, we'll show that

$$\left(1+\exp\left(\frac{1}{2}p^2\left(m-u\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(m-\nu\right)$$

is negative.

Since $\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)$ are positive terms, consider $m-\nu$. We can upper bound $m-\nu$ by $0.25-\sqrt{0.1175}$, which is negative. So, all the terms with an η factor are negative.

Then, we divide the upper bound into two parts, shown below, and show that both are negative.

$$
\begin{aligned}
&\left(-0.1p^3\left(m+\nu\right)^2 - 0.2p^5 - 0.2\sqrt{2}p^4\left(m+\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\\
&\qquad + (0.51)\left(2p - 2\sqrt{2}\nu\right) \quad (271)
\end{aligned}
$$

$$
\begin{aligned}
&\left(-0.1p^3\left(m-\nu\right)^2 - 0.2p^5 + 0.2\sqrt{2}p^4\left(m-\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\\
&\qquad + (0.49)\left(2p - 2\sqrt{2}\nu\right) \quad (272)
\end{aligned}
$$

From lemmas B.105 and B.104, we know Equations 272 and 271 are negative.

Next, we will show that $k_1(m,\eta,p,\nu) + k_2(m,\eta,p,\nu)$ is positive. We first lower bound $k_1(m,\eta,p,\nu) + k_2(m,\eta,p,\nu)$ in η by fixing $\eta = 0.3$, since the partial derivative with respect to η is negative, which we proved in lemma B.108.

After upper bounding in η, we split up $k_1(m,\eta,p,\nu) + k_2(m,\eta,p,\nu)$ into the following two parts and prove that each part is positive.

$$
\begin{aligned}
&\left(-0.15p^3\left(m+\nu\right)^2 - 0.3p^5 + 0.3\sqrt{2}p^4\left(m+\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\\
&\qquad + (-0.7m + 0.485)\left(2p + 2\sqrt{2}\nu\right) \quad (273)
\end{aligned}
$$

$$\left(-0.15p^3\left(m-\nu\right)^2-0.3p^5-0.3p^4\sqrt{2}\left(m-\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}$$
$$+\left(0.7m+0.515\right)\left(2p+2\sqrt{2}\nu\right) \quad (274)$$

For clarity, note that $(0.7m + 0.515) = 1 - (-0.7m + 0.485)$, where $0.31 \leq (-0.7m + 0.485) \leq 0.38$ for our choice of m.

From lemmas B.107 and B.106, we know Equations 274 and 273 are positive.

This completes the proof.

□

Next, we introduce the supporting lemmas for the theorem.

Supporting Lemmas for Theorem B.18 In this section, we introduce the lemmas used to prove that each of the subparts described above are positive/negative. Lemmas B.104 and B.105 are related to proving $t_1(m, \eta, q, c) - t_2(m, \eta, q, c)$ is negative. Lemmas B.106 and B.107 are related to proving $t_1(m, \eta, q, c) + t_2(m, \eta, q, c)$ is positive. Lemma B.108 shows that the scaled and reparameterized version of $t_1(m, \eta, q, c) + t_2(m, \eta, q, c)$ decreases in η.

Lemma B.104. *For* $m \in [0.15, 0.25], p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.1p^3\left(m+\nu\right)^2-0.2p^5-0.2\sqrt{2}p^4\left(m+\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}$$
$$+\left(0.51\right)\left(2p-2\sqrt{2}\nu\right) \quad (271)$$

is negative.

Lemma B.105. *For* $m \in [0.15, 0.25], p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.1p^3\left(m-\nu\right)^2-0.2p^5+0.2\sqrt{2}p^4\left(m-\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}$$
$$+\left(0.49\right)\left(2p-2\sqrt{2}\nu\right) \quad (272)$$

is negative.

Lemma B.106. *For* $m \in [0.15, 0.25], p \in [\sqrt{3.74483}, \sqrt{4.84375}], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.15p^3\left(m+\nu\right)^2-0.3p^5+0.3\sqrt{2}p^4\left(m+\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}$$
$$+\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\nu\right) \quad (273)$$

is positive.

Lemma B.107. *For* $m \in [0.15, 0.25], p \in [\sqrt{3.74483}, \sqrt{4.84375}], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.15p^3\left(m-\nu\right)^2-0.3p^5-0.3p^4\sqrt{2}\left(m-\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}$$
$$+\left(0.7m+0.515\right)\left(2p+2\sqrt{2}\nu\right) \quad (274)$$

is positive.

Lemma B.108. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], p \in [\sqrt{3.74483}, \sqrt{4.84375}], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\begin{aligned}
&2p + 2\sqrt{2}\nu - \frac{\eta p^3}{2}\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(m+\nu\right)^2 \\
&- \frac{\eta p^3}{2}\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(\nu-m\right)^2 \\
&- \eta p^5\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right) \\
&- \eta p^5\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right) \\
&+ \sqrt{2}\eta p^4\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(m+\nu\right) \\
&+ \sqrt{2}\eta p^4\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(\nu-m\right)
\end{aligned} \tag{275}$$

decreases in η.

Proof of Lemma B.104. We begin the proof by rescaling Equation 271 by $\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^2$, shown below

$$\begin{aligned}
&\left(-0.1p^3\left(m+\nu\right)^2 - 0.2p^5 - 0.2\sqrt{2}p^4\left(m+\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right) \\
&\qquad + (0.51)\left(2p - 2\sqrt{2}\nu\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^2
\end{aligned} \tag{23}$$

From lemma B.109, we know that Equation 23 decreases in ν.

So, we can upper bound it by $\nu\ \sqrt{0.1175}$. Further note that the upper bound decreases in p by lemma B.114, so we can upper bound by $p\ \sqrt{3.74483}$.

This yields the following expression (rounded up after five digits of precision).

$$\begin{aligned}
&\left(-0.72468m^2 - 4.46333m - 3.91364\right)\exp\left(\frac{1}{2}\left(\sqrt{3.74483}\right)^2\left(m+\sqrt{0.1175}\right)\right) \\
&\qquad + 1.47940\exp\left(\left(\sqrt{3.74483}\right)^2\left(m+\sqrt{0.1175}\right)\right) + 1.47940
\end{aligned}$$

To proceed, we upper bound the above by replacing the exponent terms with linear functions. We replace $\exp\left(\frac{1}{2}\left(\sqrt{3.74483}\right)^2\left(m+\sqrt{0.1175}\right)\right)$ by a linearization at m 0.17, $\left(\exp\left(\frac{1}{2}\left(\sqrt{3.74483}\right)^2\left(0.17+\sqrt{0.1175}\right)\right) + \frac{1}{2}\left(\sqrt{3.74483}\right)^2\exp\left(\frac{1}{2}\left(\sqrt{3.74483}\right)^2\left(0.17+\sqrt{0.1175}\right)\right)\left(x-0.17\right)\right)$, and $\exp\left(\left(\sqrt{3.74483}\right)^2\left(m+\sqrt{0.1175}\right)\right)$ by $28.75587m + 2.02$.

To see that $28.75587m + 2.02$ is a local upper bound for $\exp\left(\left(\sqrt{3.74483}\right)^2\left(m+\sqrt{0.1175}\right)\right)$, note that at the boundary points of m (i.e., $0.15, 0.25$), $28.75587m + 2.02 - \exp\left(\left(\sqrt{3.74483}\right)^2\left(m+\sqrt{0.1175}\right)\right)$ (concave in m) is positive.

This yields the following (rounded up after five digits of precision).

$$-3.54431m^3 - 23.11993m^2 + 15.45288m - 2.50087$$

Since the above is concave in m and its partial derivative at m 0.25 is positive, we can upper bound the above by m 0.25, which is negative.

This completes the proof.

□

Supporting Lemmas for Lemma B.104 In this section, we introduce the supporting lemmas used to show that the positive sigmoid squared component from the scaled and reparamterized version of $t_1(m,\eta,q,c) - t_2(m,\eta,q,c)$ is negative. For the proofs of these lemmas, please refer to Section B.5.2.

Lemma B.109. *For* $m \in [0.15, 0.25], p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.1m^2p^3 - 0.2mp^3\nu - 0.1p^3\nu^2 - 0.2p^5 - 0.2\sqrt{2}mp^4 - 0.2\sqrt{2}p^4\nu + 2.04p - 2.04\sqrt{2}\nu\right)\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right) + \left(1.02p - 1.02\sqrt{2}\nu\right)\exp\left(p^2\left(m+\nu\right)\right) + 1.02p - 1.02\sqrt{2}\nu \quad (23)$$

decreases in ν.

Lemma B.110. *For* $p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.05p^3 - 0.2p^3\nu - 0.2\sqrt{2}p^4 - 2.04\sqrt{2} - 0.003125p^5 - 0.025p^5\nu - 0.05p^5\nu^2 - 0.1p^7 - 0.025\sqrt{2}p^6 - 0.1\sqrt{2}p^6\nu + 1.02p^3 - 1.02\sqrt{2}p^2\nu\right)\exp\left(-\frac{1}{2}p^2\left(0.25+\nu\right)\right) - 1.02\sqrt{2} + 1.02p^3 - 1.02\sqrt{2}p^2\nu \quad (24)$$

is concave in ν.

Lemma B.111. *For* $p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(0.05p^9 + 0.0285\sqrt{2}p^8 + 0.0081225p^7 - 0.51p^5 + 0.1632\sqrt{2}p^4 - 0.2p^3\right)\exp\left(-0.285p^2\right) - 1.02\sqrt{2}p^2 \quad (25)$$

is positive.

Lemma B.112. *For* $p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right]$,

$$0.00211p^9 + 0.00170p^8 - 0.02480p^7 - 0.02027p^6 - 0.02981p^5 - 0.06141p^4 + 1.23941p^3 - 0.81905p^2 - 2.16794 \quad (26)$$

decreases in p.

Lemma B.113. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.2mp^3 - 0.2p^3\nu - 0.2\sqrt{2}p^4 - 2.04\sqrt{2} - 0.05m^2p^5 - 0.1mp^5\nu - 0.05p^5\nu^2 - 0.1p^7 - 0.1\sqrt{2}mp^6 - 0.1\sqrt{2}p^6\nu + 1.02p^3 - 1.02\sqrt{2}p^2\nu\right)\exp\left(-\frac{1}{2}p^2\left(m+\nu\right)\right) - 1.02\sqrt{2} + 1.02p^3 - 1.02\sqrt{2}p^2\nu \quad (27)$$

grows in m.

Lemma B.114. *For* $m \in [0.15, 0.25], p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right]$,

$$\left(-0.2p^5 + \left(-0.2\sqrt{2}m - 0.2\sqrt{0.235}\right)p^4 + \left(-0.1m^2 - 0.2\sqrt{0.1175}m - 0.01175\right)p^3 + 2.04x - 2.04\sqrt{0.235}\right)\exp\left(\frac{1}{2}p^2\left(m+\sqrt{0.1175}\right)\right) + \left(1.02p - 1.02\sqrt{0.235}\right)\exp\left(p^2\left(m+\sqrt{0.1175}\right)\right) + 1.02p - 1.02\sqrt{0.235} \quad (28)$$

decreases in p.

Supporting Lemmas for Lemma B.105 In this section, we introduce the supporting lemmas used to show that the negative sigmoid squared component from the scaled and reparamterized version of $t_1(m,\eta,q,c) - t_2(m,\eta,q,c)$ is negative. For the proofs of these lemmas, please refer to Section B.5.2.

Lemma B.115. *For* $m \in [0.15, 0.25], p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.1p^3\left(m-\nu\right)^2 - 0.2p^5 + 0.2\sqrt{2}p^4\left(m-\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2} + (0.49)\left(2p - 2\sqrt{2}\nu\right) \quad (272)$$

is convex in ν.

Lemma B.116. *For* $m \in [0.15, 0.25], p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right]$,

$$\begin{aligned}&\left(-0.2p^5 + \left(0.2m\sqrt{2} - 0.2\sqrt{0.475}\right)p^4 + \left(-0.1m^2 + 0.2\sqrt{0.2375}m - 0.1\left(0.2375\right)\right)p^3\right.\\ &\left. + 1.96p - 1.96\sqrt{0.475}\right)\exp\left(\frac{1}{2}p^2\left(m - \sqrt{0.2375}\right)\right)\\ &+ \left(0.98p - 0.98\sqrt{0.475}\right)\exp\left(p^2\left(m-\sqrt{0.2375}\right)\right)\\ &+ 0.98p - 0.98\sqrt{0.475}\end{aligned} \quad (29)$$

is decreasing in p.

Lemma B.117. *For* $m \in [0.15, 0.25], p \in \left[\sqrt{3.74483}, \sqrt{4.84375}\right]$,

$$\begin{aligned}&\left(-0.2p^5 + \left(0.2m\sqrt{2} - 0.2\sqrt{0.235}\right)p^4 + \left(-0.1m^2 + 0.2\sqrt{0.1175}m - 0.1\left(0.1175\right)\right)p^3\right.\\ &\left. + 1.96p - 1.96\sqrt{0.235}\right)\exp\left(\frac{1}{2}p^2\left(m - \sqrt{0.1175}\right)\right)\\ &+ \left(0.98p - 0.98\sqrt{0.235}\right)\exp\left(p^2\left(m-\sqrt{0.1175}\right)\right) + 0.98p - 0.98\sqrt{0.235}\end{aligned} \quad (30)$$

is decreasing in p.

Supporting Lemmas for Lemma B.106 In this section, we introduce the supporting lemmas used to show that the positive sigmoid squared component from the scaled and reparamterized version of $t_1(m,\eta,q,c) + t_2(m,\eta,q,c)$ is positive. For the proofs of these lemmas, please refer to Section B.5.2.

Lemma B.118. *For* $m \in [0.15, 0.25], p \in [\sqrt{3.74483}, \sqrt{4.84375}], \nu \in [\sqrt{0.1175}, \sqrt{0.2375}]$,

$$\left(-0.15p^3\left(m+\nu\right)^2 - 0.3p^5 + 0.3\sqrt{2}p^4\left(m+\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right) + \left(-0.7m + 0.485\right)\left(2p + 2\sqrt{2}\nu\right)\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^2 \quad (31)$$

grows in ν.

Lemma B.119. *For* $m \in [0.15, 0.25], p \in [\sqrt{3.74483}, \sqrt{4.84375}]$,

$$\begin{aligned}&\left(-0.7m + 0.485\right)\left(2p + 2\sqrt{0.235}\right) + \left(-0.7m + 0.485\right)\left(2p + 2\sqrt{0.235}\right)\exp\left(p^2\left(m + \sqrt{0.1175}\right)\right)\\ &+ \left(-0.15p^3\left(m + \sqrt{0.1175}\right)^2 - 0.3p^5 + 0.3\sqrt{2}p^4\left(m + \sqrt{0.1175}\right)\right.\\ &\left. + 0.62\left(2\sqrt{3.74483} + 2\sqrt{0.235}\right)\right)\exp\left(\frac{1}{2}p^2\left(m + \sqrt{0.1175}\right)\right)\end{aligned} \quad (32)$$

Supporting Lemmas for Lemma B.107 In this section, we introduce the supporting lemmas used to show that the negative sigmoid squared component from the scaled and reparamterized version of $t_1(m,\eta,q,c)+t_2(m,\eta,q,c)$ is positive. For the proofs of these lemmas, please refer to Section B.5.2.

Lemma B.120. *For* $m\in[0.15,0.25], p\in[\sqrt{3.74483},\sqrt{4.84375}], \nu\in[\sqrt{0.1175},\sqrt{0.2375}]$,

$$\left(-0.15p^3\left(m-\nu\right)^2-0.3p^5-0.3p^4\sqrt{2}\left(m-\nu\right)+2\left(0.7m+0.515\right)\left(2p+2\sqrt{2}\nu\right)\right)\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)$$
$$+\left(0.7m+0.515\right)\left(2p+2\sqrt{2}\nu\right)+\left(0.7m+0.515\right)\left(2p+2\sqrt{2}\nu\right)\exp\left(p^2\left(m-\nu\right)\right) \quad (33)$$

is increasing in ν.

Lemma B.121. *For* $m\in[0.15,0.25], p\in[\sqrt{3.74483},\sqrt{4.84375}]$,

$$\left(-0.15p^3\left(m-\sqrt{0.1175}\right)^2-0.3p^5-0.3p^4\sqrt{2}\left(m-\sqrt{0.1175}\right)\right.$$
$$\left.+2\left(0.7m+0.515\right)\left(2p+2\sqrt{0.235}\right)\right)\exp\left(\frac{1}{2}p^2\left(m-\sqrt{0.1175}\right)\right)$$
$$+\left(0.7m+0.515\right)\left(2p+2\sqrt{0.235}\right)+\left(0.7m+0.515\right)\left(2p+2\sqrt{0.235}\right)\exp\left(p^2\left(m-\sqrt{0.1175}\right)\right) \quad (34)$$

is decreasing in p.

Proofs for Lemmas Section B.5.2

Proof of Lemma B.107. To begin the proof, we first rescale Equation 274 by $\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^2$. Then, we lower bound the rescaled equation by fixing $\nu=\sqrt{0.1175}$m. since the rescaled equation grows in ν, as seen in lemma B.120.

Then, we further lower bound the rescaled equation by fixing $p=\sqrt{4.84375}$, since it decreases in p, which we prove in lemma B.121. We show the resulting lower bound below.

$$\left(-0.15\left(\sqrt{4.84375}\right)^3\left(m-\sqrt{0.1175}\right)^2-0.3\left(\sqrt{4.84375}\right)^5-0.3\left(4.84375\right)^2\sqrt{2}\left(m-\sqrt{0.1175}\right)\right.$$
$$\left.+2\left(0.7m+0.515\right)\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)\right)\exp\left(2.421875\left(m-\sqrt{0.1175}\right)\right)$$
$$+\left(0.7m+0.515\right)\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)\exp\left(4.84375\left(m-\sqrt{0.1175}\right)\right)$$
$$+\left(0.7m+0.515\right)\left(2\sqrt{4.84375}+2\sqrt{0.235}\right) \quad (35)$$

To further lower bound the above, we replace the exponential terms with linear functions. Note that, since $\left(0.7m+0.515\right)\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)$ is positive, we replace $\exp\left(4.84375\left(m-\sqrt{0.1175}\right)\right)$ with a linearization at m 0.23 (which we round down to $(2.80499m-0.06605)$), since $\exp\left(4.84375\left(m-\sqrt{0.1175}\right)\right)$ is convex in m.

To replace $\exp\left(2.421875\left(m-\sqrt{0.1175}\right)\right)$ with a linear function, we first show that the corresponding coefficient is negative. Consider the following upper bound the coefficient (which is negative).

$$\left(-0.15\left(\sqrt{4.84375}\right)^3\left(\sqrt{0.1175}-0.25\right)+0.3\left(4.84375\right)^2\sqrt{2}\right)\left(\sqrt{0.1175}-0.15\right)-0.3\left(\sqrt{4.84375}\right)^5$$
$$+2\left(0.7\left(0.25\right)+0.515\right)\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)$$

So, we replace $\exp\left(2.421875\left(m-\sqrt{0.1175}\right)\right)$ with $1.71804m + 0.37$. To see that $1.71804m + 0.37$ is a local upper bound for the exponential function, note that $1.71804m + 0.37 - \exp\left(2.421875\left(m-\sqrt{0.1175}\right)\right)$ is a concave function. So, we evaluate $1.71804m + 0.37 - \exp\left(2.421875\left(m-\sqrt{0.1175}\right)\right)$ at both m boundary points, which are positive.

After plugging in these linear function and some algebraic manipulation, we further lower bound Equation 35 in m by the following,

$$\left(-\left(1.71804\right)\left(-0.15\left(\sqrt{4.84375}\right)^3\left(\sqrt{0.1175}-0.15\right)+0.3\left(4.84375\right)^2\sqrt{2}\right)\right.$$
$$\left.+\left(1.71804\right)\left(0.7\right)\left(4\sqrt{4.84375}+4\sqrt{0.235}\right)+\left(2.80499\right)\left(0.7\right)\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)\right)m^2$$

$$+\left(\left(\left(1.71804\right)\sqrt{0.1175}-\left(0.37\right)\right)\left(-0.15\left(\sqrt{4.84375}\right)^3\left(\sqrt{0.1175}-0.15\right)+0.3\left(\sqrt{4.84375}\right)^4\sqrt{2}\right)\right.$$
$$+\left(0.37\right)\left(0.7\right)\left(4\left(\sqrt{4.84375}\right)+4\sqrt{2\left(0.1175\right)}\right)+\left(1.71804\right)\left(0.515\right)\left(4\left(\sqrt{4.84375}\right)+4\sqrt{2\left(0.1175\right)}\right)$$
$$-\left(0.06605\right)\left(0.7\right)\left(2\left(\sqrt{4.84375}\right)+2\sqrt{2\left(0.1175\right)}\right)+\left(2.80499\right)\left(0.515\right)\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)$$
$$\left.-\left(0.3\right)\left(1.71804\right)\left(\sqrt{4.84375}\right)^5+0.7\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)\right)m$$

$$+\left(\left(0.37\right)\sqrt{0.1175}\left(-0.15\left(\sqrt{4.84375}\right)^3\left(\sqrt{0.1175}-0.15\right)+0.3\left(\sqrt{4.84375}\right)^4\sqrt{2}\right)-\left(0.3\right)\left(0.37\right)\left(\sqrt{4.84375}\right)^5\right.$$
$$-\left(0.06605\right)\left(0.515\right)\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)+\left(0.37\right)\left(0.515\right)\left(4\sqrt{4.84375}+4\sqrt{0.235}\right)$$
$$\left.+0.515\left(2\sqrt{4.84375}+2\sqrt{0.235}\right)\right),$$

which, by rounding down terms after five digits of precision, simplifies to the following

$$6.89377m^2 - 0.94452m + 0.12221$$

The above expression is positive (i.e., we can lower bound it by $6.89377\left(0.15\right)^2 - 0.94452\left(0.25\right) + 0.12221$, which is positive). This completes the proof.

□

Proof of Lemma B.106. We begin the lemma by first scaling Equation 273 by $\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}$ and showing that the rescaled equation grows in ν in lemma B.118.

Lower bounding the rescaled equation by $\nu\sqrt{0.1175}$ yields the following expression.

$$\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\sqrt{0.1175}\right)+\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\sqrt{0.1175}\right)\exp\left(p^2\left(m+\sqrt{0.1175}\right)\right)$$

$$+\left(-0.15p^3\left(m+\sqrt{0.1175}\right)^2-0.3p^5+0.3\sqrt{2}p^4\left(m+\sqrt{0.1175}\right)\right.$$
$$\left.+2\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\sqrt{0.1175}\right)\right)\exp\left(\frac{1}{2}p^2\left(m+\sqrt{0.1175}\right)\right)$$

We then further lower bound the above by lower bounding $\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\sqrt{0.1175}\right)$ by $\left(-0.7(0.25)+0.485\right)\left(2\sqrt{3.74483}+2\sqrt{2}\sqrt{0.1175}\right)$.

From lemma B.119, we know that our lower bound decreases in p, so we can fix p $\sqrt{4.84375}$ to lower bound the above, yielding the following expression.

$$\left(-0.15\left(\sqrt{4.84375}\right)^3\left(m+\sqrt{0.1175}\right)^2-0.3\left(\sqrt{4.84375}\right)^5+0.3\sqrt{2}\left(4.84375\right)^2\left(m+\sqrt{0.1175}\right)\right.$$
$$\left.+\left(-0.7m+0.485\right)\left(4\sqrt{4.84375}+4\sqrt{2}\sqrt{0.1175}\right)\right)\exp\left(2.421875\left(m+\sqrt{0.1175}\right)\right)$$

$$+\left(0.62\sqrt{3.74483}+0.62\sqrt{2}\sqrt{0.1175}\right)$$

$$+\left(-0.7m+0.485\right)\left(2\sqrt{4.84375}+2\sqrt{2}\sqrt{0.1175}\right)\exp\left(4.84375\left(m+\sqrt{0.1175}\right)\right)$$

To show that the above is positive, we first replace the exponential terms by linear functions.

Since $\left(-0.7x+0.485\right)\left(2\sqrt{4.84375}+2\sqrt{2}\sqrt{0.1175}\right)$ is positive, we replace the corresponding exponential function by linearizing at m 0.17 as a lower bound (which we further round down to $(58.06020m+2.11638)$), since the exponential function is convex in m.

Before replacing $\exp\left(2.421875\left(m+\sqrt{0.1175}\right)\right)$ with a linear function, we first will show that the corresponding coefficient is negative.

To see that the coefficient is negative, note the upper bound for the coefficient below, which is negative.

$$\left(-0.15\left(\sqrt{4.84375}\right)^3\left(0.15+\sqrt{0.1175}\right)+0.3\sqrt{2}\left(4.84375\right)^2\right)\left(0.25+\sqrt{0.1175}\right)-0.3\left(\sqrt{4.84375}\right)^5$$
$$+\left(-0.7\left(0.15\right)+0.485\right)\left(4\sqrt{4.84375}+4\sqrt{2}\sqrt{0.1175}\right)$$

So, we replace $\exp\left(2.421875\left(m+\sqrt{0.1175}\right)\right)$ with the following linear function: $9.03890m+1.95$. To see that $9.03890m+1.95$ is a local upper bound for $\exp\left(2.421875\left(m+\sqrt{0.1175}\right)\right)$, note that $9.03890m+1.95-\exp\left(2.421875\left(m+\sqrt{0.1175}\right)\right)$, so we evaluate the expression at both m boundary points, which are positive.

Rounding down terms after five digits of precision, yields the following lower bound,

$$-204.86312m^2+81.16874m-7.01390,$$

which is concave in m. So, we evaluate the above at both boundary points for m and see that they are positive.

This completes the proof.

□

Proof of Lemma B.108. To prove the lemma, consider the partial derivative of Equation 275 with respect to η below.

$$
\begin{aligned}
&- \frac{p^3}{2}\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(m+\nu\right)^2 \\
&- \frac{p^3}{2}\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(\nu-m\right)^2 \\
&- p^5\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right) \\
&- p^5\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right) \\
&+ \sqrt{2}p^4\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\left(m+\nu\right) \\
&+ \sqrt{2}p^4\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\left(\nu-m\right)
\end{aligned}
$$

We split the above partial derivative into the following two parts.

$$
\left(\left(\sqrt{2}p-\frac{1}{2}\left(m+\nu\right)\right)\left(m+\nu\right)-p^2\right)p^3\left(1+\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)
$$

$$
\left(\left(\sqrt{2}p-\frac{1}{2}\left(\nu-m\right)\right)\left(\nu-m\right)-p^2\right)p^3\left(1+\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)\right)^{-2}\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)
$$

Note that the inner expression for both parts $\left(\left(\sqrt{2}p-\frac{1}{2}\left(m+\nu\right)\right)\left(m+\nu\right)-p^2\right.$ and $\left.\left(\sqrt{2}p-\frac{1}{2}\left(\nu-m\right)\right)\left(\nu-m\right)-p^2\right)$ are negative (i.e., they can upper bounded by $\left(\sqrt{2}\sqrt{4.84375}-\frac{1}{2}\left(0.15+\sqrt{0.1175}\right)\right)\left(0.25+\sqrt{0.2375}\right)-3.74483$ and $\left(\sqrt{2}\sqrt{4.84375}-\frac{1}{2}\left(\sqrt{0.1175}-0.25\right)\right)\left(\sqrt{0.2375}-0.15\right)-3.74483$, which are both negative).

This completes the proof.

□

Proof of Lemma B.105. To prove the lemma, we first show Equation 272 is convex in ν. Then, we show that Equation 272 evaluated at both ν boundary points (i.e., $\sqrt{0.1175}, \sqrt{0.2375}$) is decreasing in p, which we use to upper bound Equation 272 evaluated at each ν boundary point by $p\sqrt{3.74483}$. We then further upper each of the equations in m to show both are negative.

From lemma B.115, we know that Equation 272 is convex in ν.

From lemmas B.117 and B.116, we see that Equation 272 evaluated at both ν boundary points (i.e., $\sqrt{0.1175}, \sqrt{0.2375}$) is decreasing in p. Upper bounding Equation 272 evaluated at both ν boundary points by $p\sqrt{3.74483}$ yields the following expressions.

Note first the $\nu\ \sqrt{0.1175}$ case.

$$\begin{aligned}
&\left(-0.2\left(\sqrt{3.74483}\right)^5+\left(0.2m\sqrt{2}-0.2\sqrt{0.235}\right)\left(3.74483\right)^2\right.\\
&+\left(-0.1m^2+0.2\sqrt{0.1175}m-0.1\left(0.1175\right)\right)\left(\sqrt{3.74483}\right)^3\\
&\left.+1.96\sqrt{3.74483}-1.96\sqrt{0.235}\right)\exp\left(1.872415\left(m-\sqrt{0.1175}\right)\right)\\
&+\left(0.98\sqrt{3.74483}-0.98\sqrt{0.235}\right)\exp\left(3.74483\left(m-\sqrt{0.1175}\right)\right)\\
&+0.98\sqrt{3.74483}-0.98\sqrt{0.235}
\end{aligned}\tag{36}$$

Next, consider the $\nu\ \sqrt{0.2375}$ case.

$$\begin{aligned}
&\left(-0.2\left(\sqrt{3.74483}\right)^5+\left(0.2m\sqrt{2}-0.2\sqrt{0.475}\right)\left(3.74483\right)^2\right.\\
&+\left(-0.1m^2+0.2\sqrt{0.2375}m-0.1\left(0.2375\right)\right)\left(\sqrt{3.74483}\right)^3\\
&\left.+1.96\sqrt{3.74483}-1.96\sqrt{0.475}\right)\exp\left(1.872415\left(m-\sqrt{0.2375}\right)\right)\\
&+\left(0.98\sqrt{3.74483}-0.98\sqrt{0.475}\right)\exp\left(3.74483\left(m-\sqrt{0.2375}\right)\right)\\
&+0.98\sqrt{3.74483}-0.98\sqrt{0.475}
\end{aligned}\tag{37}$$

To show the first case ($\nu\ \sqrt{0.1175}$) is negative, we first show that the partial derivative with respect to m is positive, so we can use $m\ 0.25$ as an upper bound.

Consider the partial derivative with respect to m below (after dividing out a factor of $\exp\left(1.872415\left(x-\sqrt{0.1175}\right)\right)$).

$$\begin{aligned}
&\left(0.2\left(3.74483\right)^2\sqrt{2}+\left(-0.2m+0.2\sqrt{0.1175}\right)\left(\sqrt{3.74483}\right)^3\right)\\
&+\left(-0.2\left(\sqrt{3.74483}\right)^5+\left(0.2m\sqrt{2}-0.2\sqrt{0.235}\right)\left(3.74483\right)^2\right.\\
&+\left(-0.1m^2+0.2\sqrt{0.1175}m-0.1\left(0.1175\right)\right)\left(\sqrt{3.74483}\right)^3\\
&\left.+1.96\sqrt{3.74483}-1.96\sqrt{0.235}\right)\left(1.872415\right)\\
&+\left(0.98\sqrt{3.74483}-0.98\sqrt{0.235}\right)\left(3.74483\right)\exp\left(1.872415\left(m-\sqrt{0.1175}\right)\right)
\end{aligned}$$

We upper bound the above with the following

$$\left(0.2\left(3.74483\right)^2\sqrt{2}+\left(-0.2\left(0.25\right)+0.2\sqrt{0.1175}\right)\left(\sqrt{3.74483}\right)^3\right)$$

$$+\left(-0.2\left(\sqrt{3.74483}\right)^5+\left(0.2\left(0.15\right)\sqrt{2}-0.2\sqrt{0.235}\right)\left(3.74483\right)^2\right.$$
$$+\left(-0.1\left(0.25\right)^2+0.2\sqrt{0.1175}\left(0.15\right)-0.1\left(0.1175\right)\right)\left(\sqrt{3.74483}\right)^3$$
$$\left.+1.96\sqrt{3.74483}-1.96\sqrt{0.235}\right)\left(1.872415\right)$$

$$+\left(0.98\sqrt{3.74483}-0.98\sqrt{0.235}\right)\left(3.74483\right)\exp\left(1.872415\left(0.15-\sqrt{0.1175}\right)\right),$$

which is positive.

So, we can evaluate Equation 36 at m 0.25, which is negative.

To show the second case (ν $\sqrt{0.2375}$) is negative, we upper bound Equation 37 with the following

$$\left(-0.2\left(\sqrt{3.74483}\right)^5+\left(0.2\left(0.25\right)\sqrt{2}-0.2\sqrt{0.475}\right)\left(3.74483\right)^2\right.$$
$$+\left(-0.1\left(0.15\right)^2+0.2\sqrt{0.2375}\left(0.25\right)-0.1\left(0.2375\right)\right)\left(\sqrt{3.74483}\right)^3$$
$$\left.+1.96\sqrt{3.74483}-1.96\sqrt{0.475}\right)\exp\left(1.872415\left(0.15-\sqrt{0.2375}\right)\right)$$

$$+\left(0.98\sqrt{3.74483}-0.98\sqrt{0.475}\right)\exp\left(3.74483\left(0.25-\sqrt{0.2375}\right)\right)$$

$$+0.98\sqrt{3.74483}-0.98\sqrt{0.475},$$

which is negative.

This completes the proof. □

Proofs for Lemmas Section B.5.2

Proof of Lemma B.116. To prove the lemma, we first show that Equation 29 is concave in p locally. Then, we show that the first partial of Equation 29 with respect to p evaluated at p $\sqrt{3.74483}$ is negative.

To show that Equation 29 is concave in p locally, we divide Equation 29 into three subexpressions, which we show below.

$$0.98p-0.98\sqrt{0.475} \tag{38}$$

$$\left(0.98p-0.98\sqrt{0.475}\right)\exp\left(p^2\left(m-\sqrt{0.2375}\right)\right) \tag{39}$$

$$\left(-0.2p^5+\left(0.2m\sqrt{2}-0.2\sqrt{0.475}\right)p^4+\left(-0.1m^2+0.2\sqrt{0.2375}m-0.1\left(0.2375\right)\right)p^3\right. \tag{40}$$
$$\left.+1.96p-1.96\sqrt{0.475}\right)\exp\left(\frac{1}{2}p^2\left(m-\sqrt{0.2375}\right)\right)$$

Note that the first subexpression ($0.98p-0.98\sqrt{0.475}$) is linear in p, so it is concave.

For the second subexpression, we show that second partial derivative with respect to p below (after dividing out a factor of $\left(m-\sqrt{0.2375}\right)$ and $\exp\left(x^2\left(m-\sqrt{0.2375}\right)\right)$).

$$\left(3.92p-3.92\sqrt{0.475}\right)p^2\left(m-\sqrt{0.2375}\right)-1.96\sqrt{0.475}+5.88p$$

Note that $\left(m-\sqrt{0.2375}\right)$ is negative (i.e., we can upper bound it by $\left(0.25-\sqrt{0.2375}\right)$, which is negative), so we will prove the above is positive.

We can lower bound the above by the following

$$\left(3.92\left(\sqrt{4.84375}\right)-3.92\sqrt{0.475}\right)\left(\sqrt{4.84375}\right)^2\left(0.15-\sqrt{0.2375}\right)-1.96\sqrt{0.475}+5.88\left(\sqrt{3.74483}\right),$$

which is positive.

For the third subexpression, consider the second partial derivative with respect to p below (after dividing out a factor of $\exp\left(\frac{1}{2}x^2\left(m-\sqrt{0.2375}\right)\right)$).

$$\begin{aligned}
&-0.2\left(\sqrt{0.2375}-m\right)^2p^7-0.2\sqrt{2}\left(\sqrt{0.2375}-m\right)^3p^6+\left(0.1\left(\sqrt{0.2375}-m\right)^3-2.2\right)\left(m-\sqrt{0.2375}\right)p^5\\
&+1.8\sqrt{2}\left(m-\sqrt{0.2375}\right)^2p^4+\left(1.96\left(\sqrt{0.2375}-m\right)^2+0.7\left(\sqrt{0.2375}-m\right)^3-4\right)p^3\\
&+\left(1.96\sqrt{0.475}\left(\sqrt{0.2375}-m\right)+2.4\sqrt{2}\right)\left(m-\sqrt{0.2375}\right)p^2\\
&+\left(0.6\left(\sqrt{0.2375}-m\right)+5.88\right)\left(m-\sqrt{0.2375}\right)p+1.96\sqrt{0.475}\left(\sqrt{0.2375}-m\right)
\end{aligned}$$

We upper bound the above in m, shown below.

$$\begin{aligned}
&-0.2\left(\sqrt{0.2375}-0.25\right)^2p^7-0.2\sqrt{2}\left(\sqrt{0.2375}-0.25\right)^3p^6-\left(0.1\left(\sqrt{0.2375}-0.25\right)^4-2.2\left(\sqrt{0.2375}-0.15\right)\right)p^5\\
&+1.8\sqrt{2}\left(\sqrt{0.2375}-0.15\right)^2p^4+\left(1.96\left(\sqrt{0.2375}-0.15\right)^2+0.7\left(\sqrt{0.2375}-0.15\right)^3-4\right)p^3\\
&-\left(1.96\sqrt{0.475}\left(\sqrt{0.2375}-0.25\right)^2+2.4\sqrt{2}\left(\sqrt{0.2375}-0.25\right)\right)p^2\\
&-\left(0.6\left(\sqrt{0.2375}-0.25\right)^2+5.88\left(\sqrt{0.2375}-0.25\right)\right)p+1.96\sqrt{0.475}\left(\sqrt{0.2375}-0.15\right)
\end{aligned}$$

For convenience, we simplify the above by rounding up after five digits of precision, shown below.

$$-0.01126p^7-0.00378p^6+\left(-3.75008+0.74184p^2\right)p^3+\left(-0.88165p-1.42935+0.28969p^3\right)p+0.45570$$

Note that $\left(-3.75008+0.74184p^2\right)$ and $\left(-0.88165p-1.42935+0.28969p^3\right)$ are locally negative (i.e., we can upper bound them by $\left(0.28969\left(\sqrt{4.84375}\right)^2-0.88165\right)\sqrt{4.84375}-1.42935$ and $-3.75008+0.74184\left(\sqrt{4.84375}\right)^2$, which are negative).

So, we upper bound the above by the following

$$\begin{aligned}
&-0.01126\left(\sqrt{3.74483}\right)^7-0.00378\left(\sqrt{3.74483}\right)^6+\left(-3.75008+0.74184\left(\sqrt{4.84375}\right)^2\right)\left(\sqrt{3.74483}\right)^3\\
&+\left(\left(0.28969\left(\sqrt{4.84375}\right)^2-0.88165\right)\sqrt{4.84375}-1.42935\right)\left(\sqrt{3.74483}\right)+0.45570,
\end{aligned}$$

which is negative.

So, the third subexpression is concave.

Next, we show that the first partial derivative of Equation 29 with respect to p, evauated at p $\sqrt{3.74483}$, is negative.

Consider the first partial derivative below.

$$0.98 + \Bigg(0.98 + 1.96\left(3.74483\right)m - 1.96\left(3.74483\right)\sqrt{0.2375} - 1.96\sqrt{0.475}\sqrt{3.74483}m$$
$$+ 0.4655\sqrt{2}\sqrt{3.74483}\Bigg)\exp\left(3.74483\left(m - \sqrt{0.2375}\right)\right)$$

$$+ \Bigg(-\left(3.74483\right)^2 + 0.8\left(\sqrt{3.74483}\right)^3\sqrt{2}\left(m - \sqrt{0.2375}\right) - 0.3\left(3.74483\right)m\left(m - \sqrt{0.2375}\right)$$
$$+ 0.3\left(3.74483\right)\sqrt{0.2375}\left(m - \sqrt{0.2375}\right) + 1.96 - 0.2\left(3.74483\right)^3\left(m - \sqrt{0.2375}\right)$$
$$+ 0.2\left(\sqrt{3.74483}\right)^5\sqrt{2}\left(m - \sqrt{0.2375}\right)^2 - 0.1\left(3.74483\right)^2 m\left(m - \sqrt{0.2375}\right)^2$$
$$+ 0.1\left(3.74483\right)^2\sqrt{0.2375}\left(m - \sqrt{0.2375}\right)^2 + 1.96\left(3.74483\right)\left(m - \sqrt{0.2375}\right)$$
$$- 1.96\sqrt{0.475}\sqrt{3.74483}\left(m - \sqrt{0.2375}\right)\Bigg)\exp\left(1.872415\left(m - \sqrt{0.2375}\right)\right)$$

We upper bound the above by the following

$$0.98 + \Bigg(0.98 + 1.96\left(0.25\right)\left(3.74483\right) - 1.96\left(3.74483\right)\sqrt{0.2375} - 1.96\left(0.15\right)\sqrt{0.475}\sqrt{3.74483}$$
$$+ 0.4655\sqrt{2}\sqrt{3.74483}\Bigg)\exp\left(\left(\sqrt{3.74483}\right)^2\left(0.25 - \sqrt{0.2375}\right)\right)$$

$$+ \Bigg(-\left(3.74483\right)^2 - 0.8\left(\sqrt{3.74483}\right)^3\sqrt{2}\left(\sqrt{0.2375} - 0.25\right) + 0.3\left(3.74483\right)\left(0.25\right)\left(\sqrt{0.2375} - 0.15\right)$$
$$- 0.3\left(3.74483\right)\sqrt{0.2375}\left(\sqrt{0.2375} - 0.25\right) + 1.96 + 0.2\left(3.74483\right)^3\left(\sqrt{0.2375} - 0.15\right)$$
$$+ 0.2\sqrt{2}\left(\sqrt{3.74483}\right)^5\left(\sqrt{0.2375} - 0.15\right)^2 - 0.1\left(0.15\right)\left(3.74483\right)^2\left(\sqrt{0.2375} - 0.25\right)^2$$
$$+ 0.1\sqrt{0.2375}\left(3.74483\right)^2\left(\sqrt{0.2375} - 0.15\right)^2 - 1.96\left(3.74483\right)\left(\sqrt{0.2375} - 0.25\right)$$
$$+ 1.96\sqrt{0.475}\sqrt{3.74483}\left(\sqrt{0.2375} - 0.15\right)\Bigg)\exp\left(1.872415\left(0.15 - \sqrt{0.2375}\right)\right),$$

which is negative.

This completes the proof. □

Proof of Lemma B.117. To prove the lemma, we will first show that Equation 30 is locally concave in p. Then, using the concavity of the expression, we will show that the first partial derivative of Equation 30 with respect to p evaluated at p $\sqrt{3.74483}$ is negative.

To show that Equation 30 is concave in p, we will divide up Equation 30 into three subexpressions (shown below) and show that each is concave.

$$0.98p - 0.98\sqrt{0.235} \tag{41}$$

$$\left(0.98p - 0.98\sqrt{0.235}\right) \exp\left(p^2\left(m - \sqrt{0.1175}\right)\right) \tag{42}$$

$$\left(-0.2p^5 + \left(0.2m\sqrt{2} - 0.2\sqrt{0.235}\right)p^4 + \left(-0.1m^2 + 0.2\sqrt{0.1175}m - 0.1\,(0.1175)\right)p^3 \right. \tag{43}$$
$$\left. + 1.96p - 1.96\sqrt{0.235}\right) \exp\left(\frac{1}{2}p^2\left(m - \sqrt{0.1175}\right)\right)$$

Note that since the first subexpression ($0.98p - 0.98\sqrt{0.235}$) is linear in p, it is concave.

For the second subexpression, we show the second partial derivative with respect to p below (after dividing out a factor of $\left(m - \sqrt{0.1175}\right)$ and $\exp\left(p^2\left(m - \sqrt{0.1175}\right)\right)$).

$$5.88p - 1.96\sqrt{0.235} + \left(-3.92p + 3.92\sqrt{0.235}\right)p^2\left(\sqrt{0.1175} - m\right)$$

Note that since $\left(m - \sqrt{0.1175}\right)$ is negative for our range of m (i.e. we can upper bound $\left(m - \sqrt{0.1175}\right)$ by $\left(0.25 - \sqrt{0.1175}\right)$, which is negative), we will show that the expression above is positive.

We can lower bound the expression with the following

$$5.88\left(\sqrt{3.74483}\right) - 1.96\sqrt{0.235} + \left(-3.92\left(\sqrt{4.84375}\right) + 3.92\sqrt{0.235}\right)\left(\sqrt{4.84375}\right)^2\left(\sqrt{0.1175} - 0.15\right),$$

which is positive.

So, subexpression 2 is locally concave.

For the third subexpression, we show the second partial derivative with respect to p below (after dividing out a factor of $\exp\left(\frac{1}{2}p^2\left(m - \sqrt{0.1175}\right)\right)$).

$$\begin{aligned}
&- 0.2\left(\sqrt{0.1175} - m\right)^2 p^7 - 0.2\sqrt{2}\left(\sqrt{0.1175} - m\right)^3 p^6 \\
&+ \left(1.2\left(\sqrt{0.1175} - m\right) - 0.1\left(\sqrt{0.1175} - m\right)^4 + \left(\sqrt{0.1175} - m\right)\right)p^5 + 1.8\sqrt{2}\left(\sqrt{0.1175} - m\right)^2 p^4 \\
&+ \left(1.96\left(\sqrt{0.1175} - m\right)^2 - 4 + 0.7\left(\sqrt{0.1175} - m\right)^3\right)p^3 \\
&+ \left(-2.4\sqrt{2}\left(\sqrt{0.1175} - m\right) - 1.96\sqrt{0.235}\left(\sqrt{0.1175} - m\right)^2\right)p^2 \\
&+ \left(-0.6\left(\sqrt{0.1175} - m\right)^2 - 5.88\left(\sqrt{0.1175} - m\right)\right)p + 1.96\sqrt{0.235}\left(\sqrt{0.1175} - m\right)
\end{aligned}$$

We upper bound the above with the following

$$\begin{aligned}
&- 0.2\left(\sqrt{0.1175} - 0.25\right)^2 p^7 - 0.2\sqrt{2}\left(\sqrt{0.1175} - 0.25\right)^3 p^6 \\
&+ \left(1.2\left(\sqrt{0.1175} - 0.15\right) - 0.1\left(\sqrt{0.1175} - 0.25\right)^4 + \left(\sqrt{0.1175} - 0.15\right)\right)p^5 \\
&+ 1.8\sqrt{2}\left(\sqrt{0.1175} - 0.15\right)^2 p^4 + \left(1.96\left(\sqrt{0.1175} - 0.15\right)^2 - 4 + 0.7\left(\sqrt{0.1175} - 0.15\right)^3\right)p^3 \\
&+ \left(-2.4\sqrt{2}\left(\sqrt{0.1175} - 0.25\right) - 1.96\sqrt{0.235}\left(\sqrt{0.1175} - 0.25\right)^2\right)p^2 \\
&+ \left(-0.6\left(\sqrt{0.1175} - 0.25\right)^2 - 5.88\left(\sqrt{0.1175} - 0.25\right)\right)p + 1.96\sqrt{0.235}\left(\sqrt{0.1175} - 0.15\right),
\end{aligned}$$

which we simplify (by rounding up after five digits of precision) to the following

$$-0.00172p^7-0.00022p^6+\left(-3.92214+0.42412p^2\right)p^3+\left(0.09461p^3-0.32309p-0.55072\right)p+0.18318$$

From the above. note that $\left(-3.92214+0.42412p^2\right)$ and $\left(0.09461p^3-0.32309p-0.55072\right)$ are negative (i.e., we can upper bound them by $-3.92214+0.42412\left(\sqrt{4.84375}\right)^2$ and $\left(\left(0.09461\left(\sqrt{4.84375}\right)^2-0.32309\right)\left(\sqrt{4.84375}\right)-0.55072\right)$, which are both negative).

So, we can upper bound the above by the following

$$-0.00172\left(\sqrt{3.74483}\right)^7-0.00022\left(\sqrt{3.74483}\right)^6+\left(-3.92214+0.42412\left(\sqrt{4.84375}\right)^2\right)\left(\sqrt{3.74483}\right)^3$$
$$+\left(\left(\left(0.09461\left(\sqrt{4.84375}\right)^2-0.32309\right)\left(\sqrt{4.84375}\right)-0.55072\right)\right)\left(\sqrt{3.74483}\right)+0.18318,$$

which is negative.

So, subexpression 3 is locally concave.

Next, we consider the first partial derivative of Equation 30 with respect to p evaluated at $p\sqrt{3.74483}$, shown below.

$$\left(0.98+1.96\left(3.74483\right)\left(m-\sqrt{0.1175}\right)-1.96\sqrt{0.235}\sqrt{3.74483}\left(m-\sqrt{0.1175}\right)\right)\exp\left(3.74483\left(m-\sqrt{0.1175}\right)\right)$$
$$+\left(-\left(3.74483\right)^2+0.8\left(\sqrt{3.74483}\right)^3\sqrt{2}\left(m-\sqrt{0.1175}\right)-0.3\left(3.74483\right)m\left(m-\sqrt{0.1175}\right)\right.$$
$$+0.3\left(3.74483\right)\sqrt{0.1175}\left(m-\sqrt{0.1175}\right)+1.96-0.2\left(3.74483\right)^3\left(m-\sqrt{0.1175}\right)$$
$$+0.2\left(\sqrt{3.74483}\right)^5\sqrt{2}\left(m-\sqrt{0.1175}\right)^2-0.1\left(3.74483\right)^2m\left(m-\sqrt{0.1175}\right)^2$$
$$+0.1\left(3.74483\right)^2\sqrt{0.1175}\left(m-\sqrt{0.1175}\right)^2+1.96\left(3.74483\right)\left(m-\sqrt{0.1175}\right)$$
$$\left.-1.96\sqrt{3.74483}\sqrt{0.235}\left(m-\sqrt{0.1175}\right)\right)\exp\left(1.872415\left(m-\sqrt{0.1175}\right)\right)$$

$$+0.98$$

We upper bound the above in m with the following,

$$\left(0.98 - 1.96\left(3.74483\right)\left(\sqrt{0.1175} - 0.25\right)\right.$$
$$\left. + 1.96\sqrt{0.235}\sqrt{3.74483}\left(\sqrt{0.1175} - 0.15\right)\right)\exp\left(3.74483\left(0.25 - \sqrt{0.1175}\right)\right)$$

$$+ \left(-\left(3.74483\right)^2 - 0.8\left(\sqrt{3.74483}\right)^3\sqrt{2}\left(\sqrt{0.1175} - 0.25\right) + 0.3\left(3.74483\right)\left(0.25\right)\left(\sqrt{0.1175} - 0.15\right)\right.$$
$$- 0.3\left(3.74483\right)\sqrt{0.1175}\left(\sqrt{0.1175} - 0.25\right) + 1.96 + 0.2\left(3.74483\right)^3\left(\sqrt{0.1175} - 0.15\right)$$
$$+ 0.2\left(\sqrt{3.74483}\right)^5\sqrt{2}\left(\sqrt{0.1175} - 0.15\right)^2 - 0.1\left(3.74483\right)^2\left(0.15\right)\left(\sqrt{0.1175} - 0.25\right)^2$$
$$+ 0.1\left(3.74483\right)^2\sqrt{0.1175}\left(\sqrt{0.1175} - 0.15\right)^2 - 1.96\left(3.74483\right)\left(\sqrt{0.1175} - 0.25\right)$$
$$\left. + 1.96\sqrt{3.74483}\sqrt{0.235}\left(\sqrt{0.1175} - 0.15\right)\right)\exp\left(1.872415\left(0.15 - \sqrt{0.1175}\right)\right)$$

$$+ 0.98,$$

which is negative. This completes the proof.

□

Proof of Lemma B.115. Consider the second partial derivative of Equation 272 with respect to ν below (after dividing out a factor of p^2 and $\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)$).

$$- 0.025p^5\nu^2 + \left(0.2p^3 + 0.05mp^5 - 0.05\sqrt{2}p^6 - 0.49\sqrt{2}p^2\right)\nu$$
$$+ \left(-0.2p - 0.2mp^3 + 0.2\sqrt{2}p^4 + 1.96\sqrt{2} - 0.025m^2p^5 - 0.05p^7 + 0.05\sqrt{2}mp^6 + 0.49p^3\right)$$

$$+ \left(1.96\sqrt{2} + 0.98p^3 - 0.98\sqrt{2}p^2\nu\right)\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)$$

To proceed, we first show that coefficient of the exponent term, $\left(1.96\sqrt{2} + \left(0.98p - 0.98\sqrt{2}\nu\right)p^2\right)$, is positive. To do so, we lower bound $\left(0.98p - 0.98\sqrt{2}\nu\right)$ by $\left(0.98\left(\sqrt{3.74483}\right) - 0.98\sqrt{2}\left(\sqrt{0.2375}\right)\right)$, which is positive.

So, we can lower bound the exponent term $\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)$ by $\exp\left(\frac{1}{2}\left(\sqrt{4.84375}\right)^2\left(0.15 - \sqrt{0.2375}\right)\right)$ (which we round down to 0.44175).

This yields the following lower bound for the second partial derivative.

$$- 0.025p^5\nu^2 + \left(-0.05\sqrt{2}p^6 + 0.05mp^5 + 0.2p^3 - 0.922915\sqrt{2}p^2\right)\nu$$
$$+\left(-0.05p^7 + 0.05\sqrt{2}mp^6 - 0.025m^2p^5 + 0.2\sqrt{2}p^4 + \left(0.922915 - 0.2m\right)p^3 - 0.2p + 2.82583\sqrt{2}\right)$$

To further lower bound the expression, we will first show that the term linear in ν, $\left(-0.05\sqrt{2}p^6 + 0.05mp^5 + 0.2p^3 - 0.922915\sqrt{2}p^2\right)$ is negative. After dividing out a factor of p^2, we upper bound the linear term by $-0.05\sqrt{2}\left(\sqrt{3.74483}\right)^4 + 0.05\left(0.25\right)\left(\sqrt{4.84375}\right)^3 + 0.2\left(\sqrt{4.84375}\right) - 0.922915\sqrt{2}$, which is negative.

Then, we see that our lower bound for the second partial derivative is decreasing in ν, so we further lower bound it by evaluating at $\nu\ \sqrt{0.2375}$.

Then, we further lower bound it in m by the following

$$-0.05p^7+\left(-0.05\left(\sqrt{0.2375}\right)\sqrt{2}+0.05\sqrt{2}\,(0.15)\right)p^6$$
$$+\left(-0.025\left(\sqrt{0.2375}\right)^2+0.05\left(\sqrt{0.2375}\right)(0.15)-0.025\,(0.25)^2\right)x^5+0.2\sqrt{2}p^4$$
$$+\left(0.922915-0.2\,(0.25)+0.2\left(\sqrt{0.2375}\right)\right)p^3-0.922915\left(\sqrt{0.2375}\right)\sqrt{2}p^2-0.2p+2.82583\sqrt{2}$$

To proceed, we first round down the terms after five digits of precision and show the resulting expression (shown below) is concave in p.

$$-0.05p^7-0.02386p^6-0.00385p^5+0.28284p^4+0.97038p^3-0.63608p^2-0.2p+3.99632$$

Consider the second partial derivative of the above expression, shown below.

$$\left(-2.1p^4+3.39408p-0.7158p^3+5.82228\right)p-0.7158p^4-0.077p^3-1.27216$$

Note that $\left(-2.1p^4+3.39408p-0.7158p^3+5.82228\right)$ is negative (i.e., it can upper bounded by $\left(-2.1\left(\sqrt{3.74483}\right)^3+3.39408\right)\left(\sqrt{3.74483}\right)-0.7158\left(\sqrt{3.74483}\right)^3+5.82228$, which is negative).

So, we have that

$$-0.05p^7-0.02386p^6-0.00385p^5+0.28284p^4+0.97038p^3-0.63608p^2-0.2p+3.99632,$$

is concave in p. Evaluating the above at both boundary points of p (i.e., $\sqrt{3.74483}$, $\sqrt{4.84375}$), we see it is positive, which shows convexity in ν.

This completes the proof. □

Proofs for Lemmas Section B.5.2

Proof of Lemma B.114. We begin the proof by showing that the second partial derivative of Equation 28 is locally concave in p. Then, leveraging the concavity property, we show that the first partial derivative of Equation 28 evaluated at $p\sqrt{3.74483}$ is negative.

Consider the second partial derivative of Equation 28 shown below, after dividing out a factor of $\exp\left(\frac{1}{2}x^2\left(m+\sqrt{0.1175}\right)\right)$ and rounding terms up after five digits of precision.

$$\left(-0.2m^2-0.13711m-0.0235\right)p^7+\left(-0.28284m^3-0.29086m^2-0.09970m-0.01139\right)p^6$$
$$+\left(-0.1m^4-0.13711m^3-0.0705m^2-2.21611m-0.75550\right)p^5+\left(-2.54558m^2-1.74516m-0.29910\right)p^4$$
$$+\left(-0.7m^3+1.32016m^2+1.15181m-3.78849\right)p^3+\left(-0.98892m^2-4.07208m-1.27964\right)p^2$$
$$+\left(-0.6m^2+5.70867m+2.02734\right)p+\left(-0.98892m-0.33898\right)$$

$$+\left(\left(4.08m^2+2.79711m+0.4794\right)p^3+\left(-1.97785m^2-1.35594m-0.23239\right)p^2+\left(6.12m+2.09784\right)p\right.$$
$$\left.+\left(-0.98892m-0.33898\right)\right)\exp\left(\frac{1}{2}p^2\left(m+\sqrt{0.1175}\right)\right)$$

To upper bound the second partial derivative above, we will first show that the coefficient of the exponential term is positive.

Consider the coefficient below.

$$\left(\left(4.08m^2+2.79711m+0.4794\right)p+\left(-1.97785m^2-1.35594m-0.23239\right)\right)p^2$$
$$+\left(\left(6.12m+2.09784\right)p+\left(-0.98892m-0.33898\right)\right)$$

We will show that $\left(\left(4.08m^2 + 2.79711m + 0.4794\right)p + \left(-1.97785m^2 - 1.35594m - 0.23239\right)\right)$ and $\left(\left(6.12m + 2.09784\right)p + \left(-0.98892m - 0.33898\right)\right)$ are positive.

First, consider $\left(\left(4.08m^2 + 2.79711m + 0.4794\right)p + \left(-1.97785m^2 - 1.35594m - 0.23239\right)\right)$. We can lower bound it in p at $\sqrt{3.74483}$. Then, rewriting the equation (shown below), we see that all the terms are positive.

$$\left(4.08\left(\sqrt{3.74483}\right) - 1.97785\right)m^2 + \left(2.79711\left(\sqrt{3.74483}\right) - 1.35594\right)m + \left(0.4794\left(\sqrt{3.74483}\right) - 0.23239\right)$$

Next, we consider $\left(\left(6.12m + 2.09784\right)p + \left(-0.98892m - 0.33898\right)\right)$. We similarly lower bound it in p at $\sqrt{3.74483}$ and rewrite it (shown below) to see that all the terms are positive.

$$\left(6.12\left(\sqrt{3.74483}\right) - 0.98892\right)m + \left(2.09784\left(\sqrt{3.74483}\right) - 0.33898\right)$$

Now that we've shown both terms are positive, we can upper bound the second partial derivative with the following

$$\begin{aligned}
&\left(-0.2m^2 - 0.13711m - 0.0235\right)p^7 + \left(-0.28284m^3 - 0.29086m^2 - 0.09970m - 0.01139\right)p^6 \\
&+ \left(-0.1m^4 - 0.13711m^3 - 0.0705m^2 - 2.21611m - 0.75550\right)p^5 + \left(-2.54558m^2 - 1.74516m - 0.29910\right)p^4 \\
&+ \left(-0.7m^3 + 1.32016m^2 + 1.15181m - 3.78849\right)p^3 + \left(-0.98892m^2 - 4.07208m - 1.27964\right)p^2 \\
&+ \left(-0.6m^2 + 5.70867m + 2.02734\right)p + \left(-2.04\sqrt{0.235}m - 0.2397\sqrt{2}\right) \\
&+ \left(\left(4.08m^2 + 2.79711m + 0.4794\right)p^3 + \left(-1.97785m^2 - 1.35594m - 0.23239\right)p^2 + \left(6.12m + 2.09784\right)p \right. \\
&\left. + \left(-0.98892m - 0.33898\right)\right)\exp\left(\frac{1}{2}p^2\left(0.25 + \sqrt{0.1175}\right)\right)
\end{aligned}$$

We further locally upper bound the the exponent term with the following linear function: $4.39675x - 5.47$. To see that the linear function is a local upper bound, note that the difference between the linear and exponent function (i.e., $4.39675p - 5.47 - \exp\left(\frac{1}{2}p^2\left(0.25 + \sqrt{0.1175}\right)\right)$) is concave in p. Evaluating the difference between the linear and exponent functions at the two boundary points for p (i.e., $\sqrt{3.74483}, \sqrt{4.84375}$), we see both are positive.

Plugging in the linear equation for the exponent term (and simplifying by rounding up after five digits of precision), we further upper bound for the second partial derivative.

$$\begin{aligned}
&\left(-0.2m^2 - 0.13711m - 0.0235\right)p^7 + \left(-0.28284m^3 - 0.29086m^2 - 0.09970m - 0.01139\right)p^6 \\
&+ \left(-0.1m^4 - 0.13711m^3 - 0.0705m^2 - 2.21611m - 0.75550\right)p^5 + \left(15.39316m^2 + 10.55304m + 1.80871\right)p^4 \\
&+ \left(-0.7m^3 - 29.69355m^2 - 20.11011m - 7.43256\right)p^3 + \left(9.82992m^2 + 30.25303m + 9.21522\right)p^2 \\
&+ \left(-0.6m^2 - 32.11576m - 10.93825\right)p + \left(4.42048m + 1.51525\right) \quad (44)
\end{aligned}$$

To proceed, we will show that the partial derivative of our upper bound with respect to p is negative.

Consider the partial derivative below.

$$\begin{aligned}
&\left(\left(-1.4m^2 - 0.95977m - 0.1645\right)p^4 + \left(61.57264m^2 + 42.21216m + 7.23484\right)p \right. \\
&\left. + \left(-2.1m^3 - 89.08065m^2 - 60.33033m - 22.29768\right)\right)p^2 \\
&+ \left(-1.69704m^3 - 1.74516m^2 - 0.5982m - 0.06834\right)x^5 \\
&+ \left(\left(-0.5m^4 - 0.68555m^3 - 0.3525m^2 - 11.08055m - 3.7775\right)p^3 + \left(19.65984m^2 + 60.50606m + 18.43044\right)\right)p \\
&+ \left(-0.6m^2 - 32.11576m - 10.93825\right)
\end{aligned}$$

To prove the above is negative, we will first show that the following two expressions are negative.

$$\left(-1.4m^2 - 0.95977m - 0.1645\right)p^4 + \left(61.57264m^2 + 42.21216m + 7.23484\right)p + \left(-2.1m^3 - 89.08065m^2 - 60.33033m - 22.29768\right)$$

$$\left(-0.5m^4 - 0.68555m^3 - 0.3525m^2 - 11.08055m - 3.7775\right)p^3 + \left(19.65984m^2 + 60.50606m + 18.43044\right)$$

First, we consider

$$\left(-1.4m^2 - 0.95977m - 0.1645\right)p^4 + \left(61.57264m^2 + 42.21216m + 7.23484\right)p + \left(-2.1m^3 - 89.08065m^2 - 60.33033m - 22.29768\right).$$

We first upper bound the expression in p by the following

$$\left(-1.4m^2 - 0.95977m - 0.1645\right)\left(\sqrt{3.74483}\right)^4 + \left(61.57264m^2 + 42.21216m + 7.23484\right)\left(\sqrt{4.84375}\right) + \left(-2.1m^3 - 89.08065m^2 - 60.33033m - 22.29768\right)$$

Then, we further upper bound the above in m with the following

$$-2.1\left(0.15\right)^3 + \left(-1.4\left(\sqrt{3.74483}\right)^4 + 61.57264\left(\sqrt{4.84375}\right) - 89.08065\right)\left(0.25\right)^2 + \left(-0.95977\left(\sqrt{3.74483}\right)^4 + 42.21216\left(\sqrt{4.84375}\right) - 60.33033\right)\left(0.25\right) + \left(7.23484\left(\sqrt{4.84375}\right) - 22.29768 - 0.1645\left(\sqrt{3.74483}\right)^4\right),$$

which is negative.

For the second expression,

$$\left(-0.5m^4 - 0.68555m^3 - 0.3525m^2 - 11.08055m - 3.7775\right)p^3 + \left(19.65984m^2 + 60.50606m + 18.43044\right),$$

we first upper bound it in p with the following

$$\left(-0.5m^4 - 0.68555m^3 - 0.3525m^2 - 11.08055m - 3.7775\right)\left(\sqrt{3.74483}\right)^3 + \left(19.65984m^2 + 60.50606m + 18.43044\right).$$

We then upper bound the above in m with the following

$$-0.1\left(5\right)\left(\sqrt{3.74483}\right)^3\left(0.15\right)^4 - 0.13711\left(5\right)\left(\sqrt{3.74483}\right)^3\left(0.15\right)^3 + \left(-0.0705\left(5\right)\left(\sqrt{3.74483}\right)^3 + 9.82992\left(2\right)\right)\left(0.25\right)^2 + \left(30.25303\left(2\right) - 2.21611\left(5\right)\left(\sqrt{3.74483}\right)^3\right)\left(0.15\right) + 9.21522\left(2\right) - 0.75550\left(5\right)\left(\sqrt{3.74483}\right)^3,$$

which is negative.

So, the partial derivative is negative.

Then, we can further upper bound Equation 44 (rounded up after five digits of precision) by $p\sqrt{3.74483}$, shown below.

$$-2.71381m^4 - 23.64747m^3 - 1.17777m^2 - 21.48866m - 37.12944$$

Note that each term above is negative, so this shows that Equation 28 is concave in p.

Next, we will show that the first partial derivative of Equation 28 in p evaluated at $p\sqrt{3.74483}$ is negative.

We upper bound the first partial derivative (rounded up after five digits of precision) by the following

$$(5.72573\,(0.25) + 2.98268)\exp\left(\left(\sqrt{3.74483}\right)^2\left(0.25 + \left(\sqrt{0.1175}\right)\right)\right)$$

$$+\left(-1.40237\,(0.15)^3 - 10.24140\,(0.15)^2 - 19.50325\,(0.15) - 17.52225\right)\exp\left(\frac{1}{2}\left(\sqrt{3.74483}\right)^2\left(0.15 + \sqrt{0.1175}\right)\right)$$

$$+ 1.02,$$

which is negative. This completes the proof.

□

Proof of Lemma B.109. Note that $1.02p - 1.02\sqrt{2}\nu$ is already decreasing in ν, so we will show that the below is decreasing in ν.

$$\left(-0.1m^2p^3 - 0.2mp^3\nu - 0.1p^3\nu^2 - 0.2p^5 - 0.2\sqrt{2}mp^4 - 0.2\sqrt{2}p^4\nu + 2.04p - 2.04\sqrt{2}\nu\right)\exp\left(\frac{1}{2}p^2\,(m+\nu)\right)$$
$$+\left(1.02p - 1.02\sqrt{2}\nu\right)\exp\left(p^2\,(m+\nu)\right)$$

So, we will show that the first partial derivative of the above with respect ν, shown below after dividing out a factor of $\exp\left(p^2\,(m+\nu)\right)$, is negative.

$$\left(-0.2mp^3 - 0.2p^3\nu - 0.2\sqrt{2}p^4 - 2.04\sqrt{2} - 0.05m^2p^5 - 0.1mp^5\nu - 0.05p^5\nu^2 - 0.1p^7 - 0.1\sqrt{2}mp^6 - 0.1\sqrt{2}p^6\nu\right.$$
$$\left.+ 1.02p^3 - 1.02\sqrt{2}p^2\nu\right)\exp\left(-\frac{1}{2}p^2\,(m+\nu)\right)$$

$$- 1.02\sqrt{2} + 1.02p^3 - 1.02\sqrt{2}p^2\nu \tag{45}$$

By lemma B.113, we know that the partial derivative of the above with respect to m is positive. So, we can upper bound the partial derivative above by m 0.25.

To further upper bound the partial derivative, we will leverage the fact that it is concave in ν by lemma B.110. We upper bound the partial derivative by linearization at ν 0.32. We evaluate the linearization at ν $\sqrt{0.2375}$, since the first partial derivative of Equation 45 evaluated at ν 0.32 is positive, according to lemma B.111.

Note the linearization below, where terms are rounded up after five digits of precision.

$$\left(\left(0.00837p^5 - 0.08061p^2 - 0.24422\right)p^4 + \left(0.00675p^3 - 0.11858\right)p^5 + \left(-0.09864p^4 + 0.87254\right)p^3\right.$$
$$\left.- 0.46159p^2 - 2.88499\right)\exp\left(-0.285p^2\right)$$

$$+ 1.02p^3 - 0.70298p^2 - 1.44249$$

To further upper bound the above linearization, we first show that the coefficient of the exponential term is negative.

Consider the coefficient below.

$$\left(0.00837p^5 - 0.08061p^2 - 0.24422\right)p^4 + \left(0.00675p^3 - 0.11858\right)p^5 + \left(-0.09864p^4 + 0.87254\right)p^3 - 0.46159p^2 - 2.88499$$

Note that $\left(0.00837p^5 - 0.08061p^2 - 0.24422\right)$, $\left(0.00675p^3 - 0.11858\right)$, and $\left(-0.09864p^4 + 0.87254\right)$ are negative for our range of p (i.e., they can be upper bounded by $\left(\left(0.00837\left(\sqrt{4.84375}\right)^3 - 0.08061\right)\left(\sqrt{4.84375}\right)^2 - 0.24422\right)$, $\left(0.00675\left(\sqrt{4.84375}\right)^3 - 0.11858\right)$, and $\left(-0.09864\left(\sqrt{3.74483}\right)^4 + 0.87254\right)$ respectively, which are all negative).

Since the coefficient of the exponential term is negative, we further upper bound the linearization by the following

$$\left(0.00837p^9 + 0.00675p^8 - 0.09864p^7 - 0.08061xp^6 - 0.11858p^5 - 0.24422xp^4 + 0.87254p^3 - 0.46159p^2 - 2.88499\right)\exp\left(-0.285\left(\sqrt{4.84375}\right)^2\right) + 1.02p^3 - 0.70298p^2 - 1.44249.$$

After simplifying the above, and rounding up after five digits of precision, we have the following expression.

$$0.00211p^9 + 0.00170p^8 - 0.02480p^7 - 0.02027p^6 - 0.02981p^5 - 0.06141p^4 + 1.23941p^3 - 0.81905p^2 - 2.16794$$

By lemma B.112, we know that the above is locally decreasing in p, so we can upper bound by evaluating at $p\ \sqrt{3.74483}$, which is negative.

This completes the proof.

□

Proof of Lemma B.112. To prove the lemma, we first show that Equation 26 is locally concave and use that to show the equation decreases in p by evaluating the first partial derivative at $p\ \left(\sqrt{3.74483}\right)$.

Consider the second partial derivative below.

$$\left(0.13293p^2 - 0.868\right)p^4 + \left(0.0816p^2 - 0.48648\right)p^3 - 0.44715p^2 - 0.49128p + 3.71823$$

Note that $\left(0.13293p^2 - 0.868\right)$ and $\left(0.0816x^2 - 0.48648\right)$ are locally negative (i.e., they can upper bounded by $\left(0.13293\left(\sqrt{4.84375}\right)^2 - 0.868\right)$ and $\left(0.0816\left(\sqrt{4.84375}\right)^2 - 0.48648\right)$ respectively, which are negative).

So we can upper bound the second partial derivative by the following,

$$\left(0.13293\left(\sqrt{4.84375}\right)^2 - 0.868\right)\left(\sqrt{3.74483}\right)^4 + \left(0.0816\left(\sqrt{4.84375}\right)^2 - 0.48648\right)\left(\sqrt{3.74483}\right)^3 - 0.44715\left(\sqrt{3.74483}\right)^2 - 0.49128\left(\sqrt{3.74483}\right) + 3.71823,$$

which is negative.

So, we evaluate the first partial derivative of Equation 26 at $p\ \left(\sqrt{3.74483}\right)$, which is negative.

This completes the proof. □

Proof of Lemma B.111. We begin the proof by dividing out a factor of p^2 from Equation 25.

Consider the coefficient of the exponent term (with a factor of p divided out) below, which we will show is positive.

$$\left(0.05p^4 - 0.51\right)p^2 + 0.0285\sqrt{2}p^5 + 0.0081225p^4 + 0.1632\sqrt{2}p - 0.2$$

Since $\left(0.05x^4 - 0.51\right)$ is positive (i.e., can be lower bounded by $\left(0.05\left(\sqrt{3.74483}\right)^4 - 0.51\right)$), we can lower bound the exponential coefficient by evaluating it at $p\left(\sqrt{3.74483}\right)$, which is positive.

Then, we can lower bound Equation 25 by linearizing the the exponential term $\exp\left(-0.285p^2\right)$ at $p\sqrt{3.74483}$, since $\exp\left(-0.285p^2\right)$ is locally convex in p. To see this, consider the second partial derivative of $\exp\left(-0.285p^2\right)$ with respect to p, $\left(-1 + 0.57p^2\right)(0.57)\exp\left(-0.285p^2\right)$, where we focus on the inner term $\left(-1 + 0.57p^2\right)$ that can be lower bounded at $p\sqrt{3.74483}$, which is positive.

After rounding down coefficients after five digits of precision, we lower bound Equation 25 with the following:

$$-0.01897p^8 + 0.03861p^7 + 0.04037p^6 + 0.00875p^5 + 0.19348p^4 - 0.63740p^3 + 0.32470p^2 - 0.21563p - 1.44250.$$

To further lower bound the above, we show that the partial derivative with respect to p is positive. Consider the partial derivative below (note that we split up the 5th order term of the partial derivative to show all terms of degree at least one are positive).

$$\left(-0.15176p^2 + 0.27027p + 0.24222\left(\frac{3}{4}\right)\right)p^5 + 0.04375p^4 + \left(\left(\frac{1}{4}\right)0.24222p^3 + 0.77392p - 1.9122\right)p^2 + 0.6494x - 0.21563$$

Note that $\left(-0.15176p^2 + 0.27027p + 0.24222\left(\frac{3}{4}\right)\right)$ is positive, we can be seen by leveraging the concavity of the expression. So, we evaluate $\left(-0.15176x^2 + 0.27027p + 0.24222\left(\frac{3}{4}\right)\right)$ at $p \quad \sqrt{3.74483}, \sqrt{4.84375}$, which are both positive and $\left(-0.15176\left(\sqrt{3.74483}\right)^2 + 0.27027\left(\sqrt{3.74483}\right) + 0.24222\left(\frac{3}{4}\right)\right) > \left(-0.15176\left(\sqrt{4.84375}\right)^2 + 0.27027\left(\sqrt{4.84375}\right) + 0.24222\left(\frac{3}{4}\right)\right)$.

Consider $\left(\left(\frac{1}{4}\right)0.24222x^3 + 0.77392x - 1.9122\right)$, which we can lower bound by evaluating at $p\sqrt{3.74483}$. So, we can lower bound the partial derivative with the following

$$\left(-0.15176\left(\sqrt{4.84375}\right)^2 + 0.27027\left(\sqrt{4.84375}\right) + 0.24222\left(\frac{3}{4}\right)\right)\left(\sqrt{3.74483}\right)^5 + 0.04375\left(\sqrt{3.74483}\right)^4 + \left(\left(\frac{1}{4}\right)0.24222\left(\sqrt{3.74483}\right)^3 + 0.77392\left(\sqrt{3.74483}\right) - 1.9122\right)\left(\sqrt{3.74483}\right)^2 + 0.6494\left(\sqrt{3.74483}\right) - 0.21563,$$

which is positive.

So, we can lower bound

$$-0.01897p^8 + 0.03861p^7 + 0.04037p^6 + 0.00875p^5 + 0.19348p^4 - 0.63740p^3 + 0.32470p^2 - 0.21563p - 1.44250$$

by evaluating at $p\sqrt{3.74483}$, which is positive.

This completes the proof.

□

Proof of Lemma B.110. We begin the proof by considering the second partial derivative of Equation 24 with respect to ν below, after dividing out a factor of p^4 and $\exp\left(-\frac{1}{2}p^2\left((0.25) + \nu\right)\right)$.

$$-0.0125p^5\nu^2 + \left(-0.025\sqrt{2}p^4 - 0.00625p^3 + 0.05p - 0.255\sqrt{2}\right)p^2\nu + \left(-0.025p^7 - 0.00625\sqrt{2}p^6 + 0.255p^3 + 0.0125p^3 + 0.05\sqrt{2}p^4 + 0.51\sqrt{2} + 0.1p - 0.0125\left(0.25\right)^2 p^5\right)$$

First, note that $\left(-0.025\sqrt{2}p^4 - 0.00625p^3 + 0.05p - 0.255\sqrt{2}\right)$ is negative (i.e. $\left(-0.025\sqrt{2}p^4 - 0.00625p^3 + 0.05p - 0.255\sqrt{2}\right)$ can be upper bounded $-0.05\left(\frac{1}{2}\right)\sqrt{2}\left(\sqrt{3.74483}\right)^4 - 0.05\left(0.25\right)\left(\frac{1}{2}\right)\left(\sqrt{3.74483}\right)^3 + 0.05\left(\sqrt{4.84375}\right) - 0.51\left(\frac{1}{2}\right)\sqrt{2}$, which is negative).

So, we can upper bound the second partial derivative by evaluating at $\nu\ \sqrt{0.1175}$.

After rounding up the coefficients after five digits of precision, we are left with the following polynomial in p.

$$\left(-0.025p^4 + 0.27214\right)p^3 + \left(-0.02095p^2 + 0.07072\right)p^4 - 0.00439p^5 + \left(-0.12361p + 0.1\right)p + 0.72204$$

Note that $\left(-0.025x^4 + 0.27214\right), \left(-0.02095p^2 + 0.07072\right), \left(-0.12361p + 0.1\right)$ are negative (i.e., they can be upper bounded by the following $\left(-0.025\left(\sqrt{3.74483}\right)^4 + 0.27214\right), \left(-0.02095\left(\sqrt{3.74483}\right)^2 + 0.07072\right), \left(-0.12361\left(\sqrt{3.74483}\right) + 0.1\right)$, which are all negative).

So, we upper bound the polynomial by evaluating it at $p\ \sqrt{3.74483}$, which is negative. This completes the proof.

□

Proof of Lemma B.113. Consider the partial derivative of Equation 45 with respect to m below, after dividing out a factor of p^2 and $\exp\left(-\frac{1}{2}p^2\left(m + \nu\right)\right)$.

$$0.025p^5\nu^2 + \left(0.05\sqrt{2}p^6 + 0.05mp^5 + 0.51\sqrt{2}p^2\right)\nu + \left(0.05p^7 + 0.05\sqrt{2}mp^6 - 0.51p^3 - 0.2p + 1.02\sqrt{2} + 0.025m^2p^5\right)$$

Since the above increases in ν and m, we can lower bound it with $\nu\ \sqrt{0.1175}$ and $m\ 0.15$, which yields the following.

$$\left(0.05p^4 - 0.51\right)p^3 + \left(0.05\sqrt{2}\sqrt{0.1175} + 0.05\sqrt{2}\left(0.15\right)\right)p^6 + \left(0.025\left(0.1175\right) + 0.05\left(0.15\right)\sqrt{0.1175} + 0.025\left(0.15\right)^2\right)p^5 + \left(0.51\sqrt{2}\sqrt{0.1175}p - 0.2\right)p + 1.02\sqrt{2}$$

Note that $\left(0.05p^4 - 0.51\right)$ and $\left(0.51\sqrt{2}\sqrt{0.1175}p - 0.2\right)$ are positive for our parameter range (i.e., lower bound $\left(0.05p^4 - 0.51\right)$ and $\left(0.51\sqrt{2}\sqrt{0.1175}p - 0.2\right)$ by $\left(0.05\left(\sqrt{3.74483}\right)^4 - 0.51\right)$ and $\left(0.51\sqrt{2}\sqrt{0.1175}\left(\sqrt{3.74483}\right) - 0.2\right)$).

So, Equation 45 grows in m. This completes the proof. □

Proofs for Lemmas Section B.5.2

Proof of Lemma B.119. To prove the lemma, we will start by proving that Equation32 is concave in p, and then show that the first partial derivative of Equation 32 with respect to p at $p\ \sqrt{3.74483}$ is negative.

To prove concavity in p, first note that second partial derivative of Equation 32 with respect to p below (after dividing out a factor of $\exp\left(\frac{1}{2}p^2\left(m + \sqrt{0.1175}\right)\right)$).

$$
\begin{aligned}
&- 6p^3 - 0.15p^5 \left(m + \sqrt{0.1175}\right)^4 + \left(0.3\sqrt{2}p^6 - 1.05p^3\right) \left(m + \sqrt{0.1175}\right)^3 \\
&+ \left(-0.3p^7 + 2.7\sqrt{2}p^4 + 2\left(-0.7m + 0.485\right)\left(2p^3 + 2\sqrt{0.235}p^2\right) - 0.9p\right)\left(m + \sqrt{0.1175}\right)^2 \\
&+ \left(-1.8p^5 + 2\left(-0.7m + 0.485\right)\left(4p + 2\sqrt{0.235}\right) - 1.5p^5 + 4p\left(-0.7m + 0.485\right) + 3.6\sqrt{2}p^2\right)\left(m + \sqrt{0.1175}\right) \\
&+ \left(12p + 4\sqrt{0.235} + \left(8p + 8\sqrt{0.235}\right)p^2\left(m + \sqrt{0.1175}\right)\right)\left(m + \sqrt{0.1175}\right) \\
&\cdot \left(-0.7m + 0.485\right)\exp\left(\frac{1}{2}p^2\left(m + \sqrt{0.1175}\right)\right)
\end{aligned}
$$

We further upper bound the above by first upper bounding $\exp\left(\frac{1}{2}p^2\left(m + \sqrt{0.1175}\right)\right)$ by $\exp\left(\frac{1}{2}p^2\left(0.25 + \sqrt{0.1175}\right)\right)$, which we replace with the linear function: $4.39675p - 5.47$. Since $4.39675p - 5.47 - \exp\left(\frac{1}{2}p^2\left(0.25 + \sqrt{0.1175}\right)\right)$ is concave in p, we evaluate the expression at both boundary points of p, which are positive.

After applying the upper bound, we then show that the bound is decreasing in p.

Consider the partial derivative of our upper bound with respect to p below.

$$
\begin{aligned}
&- 2.1\left(m + \sqrt{0.1175}\right)^2 p^6 + 1.8\sqrt{2}\left(m + \sqrt{0.1175}\right)^3 p^5 + \left(-0.75\left(m + \sqrt{0.1175}\right)^3 - 16.5\right)\left(m + \sqrt{0.1175}\right)p^4 \\
&+ \left(10.8\sqrt{2} + 140.696\left(-0.7m + 0.485\right)\right)\left(m + \sqrt{0.1175}\right)^2 p^3 \\
&+ \left(\left(-119.28 + 105.522\sqrt{2}\sqrt{0.1175}\right)\left(-0.7m + 0.485\right)\left(m + \sqrt{0.1175}\right)^2 - 3.15\left(m + \sqrt{0.1175}\right)^3 - 18\right)p^2 \\
&+ \left(7.2\sqrt{2} + 105.522\left(-0.7m + 0.485\right) - 79.52\sqrt{2}\sqrt{0.1175}\left(-0.7m + 0.485\right)\left(m + \sqrt{0.1175}\right)\right)\left(m + \sqrt{0.1175}\right)p \\
&+ \left(\left(-53.64 + 17.587\sqrt{2}\sqrt{0.1175}\right)\left(-0.7m + 0.485\right) - 0.9\left(m + \sqrt{0.1175}\right)\right)\left(m + \sqrt{0.1175}\right)
\end{aligned}
$$

We divide the above into three subexpressions, shown below.

$$
\begin{aligned}
&\left(-2.1\left(m + \sqrt{0.1175}\right)p^3 + \left(-0.75\left(m + \sqrt{0.1175}\right)^3 - 16.5\right)p \right. \\
&\left. + \left(10.8\sqrt{2} + 140.696\left(-0.7m + 0.485\right)\right)\left(m + \sqrt{0.1175}\right)\right)\left(m + \sqrt{0.1175}\right)p^3
\end{aligned}
$$

$$
\begin{aligned}
&\left(\left(\left(-119.28 + 105.522\sqrt{2}\sqrt{0.1175}\right)\left(-0.7m + 0.485\right)\left(m + \sqrt{0.1175}\right)^2 - 3.15\left(m + \sqrt{0.1175}\right)^3 - 18\right)p \right. \\
&+ \left(7.2\sqrt{2} + 105.522\left(-0.7m + 0.485\right) - 79.52\sqrt{2}\sqrt{0.1175}\left(-0.7m + 0.485\right)\left(m + \sqrt{0.1175}\right)\right)\left(m + \sqrt{0.1175}\right) \\
&\left. + 1.8\sqrt{2}\left(m + \sqrt{0.1175}\right)^3 p^4\right)p
\end{aligned}
$$

$$
\left(\left(-53.64 + 17.587\sqrt{2}\sqrt{0.1175}\right)\left(-0.7m + 0.485\right) - 0.9\left(m + \sqrt{0.1175}\right)\right)\left(m + \sqrt{0.1175}\right)
$$

We will show that each subexpression is negative locally.

For the first subexpression, we divide out $\left(m+\sqrt{0.1175}\right)p^3$ and lower bound the remaining expression by the following, which is negative.

$$-0.75\left(\sqrt{3.74483}\right)\left(0.15+\sqrt{0.1175}\right)^3+140.696\left(-0.7\left(0.15\right)+0.485\right)\left(0.25+\sqrt{0.1175}\right)$$
$$+\left(-2.1\left(\sqrt{3.74483}\right)^3+10.8\sqrt{2}\right)\left(0.25+\sqrt{0.1175}\right)-16.5\left(\sqrt{3.74483}\right)$$

For the second subexpression, after dividing out a factor of p, we upper bound the remaining expression by the following, which is negative.

$$\left(1.8\sqrt{2}\left(0.25+\sqrt{0.1175}\right)^3\left(\sqrt{4.84375}\right)^3\right.$$

$$+\left(\left(-119.28+105.522\sqrt{2}\sqrt{0.1175}\right)\left(-0.7\left(0.25\right)+0.485\right)\left(0.15+\sqrt{0.1175}\right)^2-3.15\left(0.15+\sqrt{0.1175}\right)^3-18\right)\Big)$$
$$\cdot\left(\sqrt{3.74483}\right)$$

$$+\left(7.2\sqrt{2}+105.522\left(-0.7\left(0.15\right)+0.485\right)-79.52\sqrt{2}\sqrt{0.1175}\left(-0.7\left(0.25\right)+0.485\right)\left(0.15+\sqrt{0.1175}\right)\right)$$
$$\cdot\left(0.25+\sqrt{0.1175}\right)$$

For the third subexpression, after dividing out a factor $\left(m+\sqrt{0.1175}\right)$, the remaining expression can be upper bounded by the following, which is negative.

$$\left(\left(-53.64+17.587\sqrt{2}\sqrt{0.1175}\right)\left(-0.7\left(0.25\right)+0.485\right)-0.9\left(0.15+\sqrt{0.1175}\right)\right)\left(0.15+\sqrt{0.1175}\right)$$

Since the three subexpressions are negative, we can further upper bound our linearized second partial derivative for Equation 32 with respect to p by fixing p $\sqrt{3.74483}$, which yields the following expression (terms are rounded up after five digits of precision).

$$-4.07072\left(m+\sqrt{0.1175}\right)^4+14.67172\left(m+\sqrt{0.1175}\right)^3+21.31800\left(m+\sqrt{0.1175}\right)^2$$
$$-70.49033\left(m+\sqrt{0.1175}\right)+256.52556\left(-0.7m+0.485\right)\left(m+\sqrt{0.1175}\right)^2$$
$$+101.60994\left(m+\sqrt{0.1175}\right)\left(-0.7m+0.485\right)-43.48098$$

We then upper bound the above with the following.

$$\begin{aligned}&-4.07072\left(m+\sqrt{0.1175}\right)^4+14.67172\left(m+\sqrt{0.1175}\right)^3\\&+\left(21.31800+256.52556\left(-0.7\left(0.15\right)+0.485\right)\right)\left(m+\sqrt{0.1175}\right)^2\\&+\left(-70.49033+101.60994\left(-0.7\left(0.15\right)+0.485\right)\right)\left(m+\sqrt{0.1175}\right)-43.48098\end{aligned}\tag{46}$$

Next, we show that the above grows in m. Consider the partial derivative of the above with respect to m below,

$$-16.28288\left(m+\sqrt{0.1175}\right)^3+44.01516\left(m+\sqrt{0.1175}\right)^2+237.59544\left(m+\sqrt{0.1175}\right)-31.87855,$$

which we lower bound with the following (that is positive)

$$-16.28288\left(0.25+\sqrt{0.1175}\right)^3+44.01516\left(0.15+\sqrt{0.1175}\right)^2+237.59544\left(0.15+\sqrt{0.1175}\right)-31.87855.$$

So, we upper bound Equation 46 by evaluating at m 0.25, which is negative. So, this prove that Equation 32 is concave in p.

So, we evaluate the first partial derivative of Equation 32 with respect to p at p $\sqrt{3.74483}$ and show it is negative.

Consider the first partial derivative evaluated at p $\sqrt{3.74483}$ below (after dividing out a factor of $\exp\left(1.872415\left(x+\sqrt{0.1175}\right)\right)$).

$$\left(2+\left(4\left(3.74483\right)+4\sqrt{2}\sqrt{0.1175}\sqrt{3.74483}\right)\left(m+\sqrt{0.1175}\right)\right)\left(-0.7m+0.485\right)\exp\left(1.872415\left(m+\sqrt{0.1175}\right)\right)$$

$$-0.45\left(3.74483\right)\left(m+\sqrt{0.1175}\right)^{2}+1.2\sqrt{2}\left(\sqrt{3.74483}\right)^{3}\left(m+\sqrt{0.1175}\right)+4\left(-0.7m+0.485\right)$$
$$-0.15\left(3.74483\right)^{2}\left(m+\sqrt{0.1175}\right)^{3}-0.3\left(3.74483\right)^{3}\left(m+\sqrt{0.1175}\right)+0.3\sqrt{2}\left(\sqrt{3.74483}\right)^{5}\left(m+\sqrt{0.1175}\right)^{2}$$
$$+\left(4\left(3.74483\right)+4\sqrt{2}\sqrt{0.1175}\sqrt{3.74483}\right)\left(-0.7m+0.485\right)\left(m+\sqrt{0.1175}\right)-1.5\left(3.74483\right)^{2}$$

Then, we upper bound the above by upper bounding $\exp\left(1.872415\left(m+\sqrt{0.1175}\right)\right)$ by $\exp\left(1.872415\left(0.25+\sqrt{0.1175}\right)\right)$ (which we round to (3.03417)).

Rounding terms up after five digits of precision yields the following expression

$$-2.10356\left(x+\sqrt{0.1175}\right)^{3}+9.82857\left(x+\sqrt{0.1175}\right)^{2}-3.45668\left(x+\sqrt{0.1175}\right)$$
$$+75.56698\left(-0.7x+0.485\right)\left(x+\sqrt{0.1175}\right)+10.06834\left(-0.7x+0.485\right)-21.03562,$$

which we upper bound with the following (that is negative)

$$-2.10356\left(0.15+\sqrt{0.1175}\right)^{3}+9.82857\left(0.25+\sqrt{0.1175}\right)^{2}-3.45668\left(0.15+\sqrt{0.1175}\right)$$
$$+75.56698\left(\left(-0.7\left(0.15\right)+\left(-0.7\sqrt{0.1175}+0.485\right)\right)\left(0.25\right)+0.485\sqrt{0.1175}\right)$$
$$+10.06834\left(-0.7\left(0.15\right)+0.485\right)-21.03562.$$

This completes the proof.

□

Proof of LemmaB.118. To prove the lemma, we start by showing that Equation 31 is convex in ν. Then, we show that the first partial derivative of Equation 31 at ν $\sqrt{0.1175}$ is positive.

To show that Equation 31 is convex in ν, we split Equation 31 into three subexpression (shown below) and show each is convex in ν.

$$\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\nu\right) \tag{47}$$

$$\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\nu\right)\exp\left(p^{2}\left(m+\nu\right)\right) \tag{48}$$

$$\left(-0.15p^{3}\left(m+\nu\right)^{2}-0.3p^{5}+0.3\sqrt{2}p^{4}\left(m+\nu\right)+2\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\nu\right)\right)\exp\left(\frac{1}{2}p^{2}\left(m+\nu\right)\right) \tag{49}$$

Note that the first subexpression, $\left(-0.7m+0.485\right)\left(2p+2\sqrt{2}\nu\right)$, is linear in ν, so it is convex in ν.

For the second subexpression, we show the second partial derivative with respect to ν below (after dividing out a factor of p^2, $(-0.7m + 0.485)$, and $\exp\left(p^2\left(m+\nu\right)\right)$).

$$2\sqrt{2}x + 2\sqrt{2} + 2p^3 + 2\sqrt{2}xp^2$$

The above is positive.

For the third subexpression, we also show the second partial derivative with respect to ν below (after dividing out a factor of $\exp\left(\frac{1}{2}p^2\left(m+\nu\right)\right)$ and p^2).

$$-0.0375p^5\left(m+\nu\right)^2 + \left(0.075\sqrt{2}p^3 - 0.3\right)p^3\left(m+\nu\right) + \left(-0.7m + 0.485\right)\left(p + \sqrt{2}\nu\right)p^2 + \left(0.3\sqrt{2}p^4 + 4\sqrt{2}\left(-0.7m + 0.485\right) - 0.075p^7 - 0.3p\right) \quad (20)$$

Note that the above is concave in ν, so we'll show that the partial derivative with respect to ν at ν $\sqrt{0.2375}$ is positive.

Consider the partial derivative below (after factoring out p^2).

$$-0.075\left(m + \sqrt{0.2375}\right)p^3 + \left(0.075\sqrt{2}p^3 - 0.3\right)p + \left(-0.7m + 0.485\right)\sqrt{2}$$

Note that the above is decreasing in m, so, in addition to fixing m to 0.25, we also lower bound it in p (which is positive), shown below.

$$0.075\sqrt{2}\left(\sqrt{3.74483}\right)^4 - 0.075\left(0.25 + \sqrt{0.2375}\right)\left(\sqrt{4.84375}\right)^3 - 0.3\left(\sqrt{4.84375}\right) + \left(-0.7\left(0.25\right) + 0.485\right)\sqrt{2}$$

So we can lower bound our second partial derivative inner expression by evaluating at u $\sqrt{0.1175}$.

Also note that our second partial derivative inner expression (Equation 20) is concave in m, so we will show that the partial derivative with respect to m at m 0.15 is negative.

To do so, we upper bound the partial derivative with respect to m by the following

$$0.075\sqrt{2}\left(\sqrt{4.84375}\right)^6 - 0.075\left(0.15 + \sqrt{0.1175}\right)\left(\sqrt{3.74483}\right)^5 - \left(\sqrt{3.74483}\right)^3 - 0.7\sqrt{2}\sqrt{0.1175}\left(\sqrt{3.74483}\right)^2 - 2.8\sqrt{2},$$

which is negative.

So, we can further lower bound Equation 20 by evaluating at m 0.25, shown below (rounded down after five digits of precision).

$$-0.075p^7 + 0.06287p^6 - 0.01318p^5 + 0.42426p^4 + 0.13216p^3 + 0.15027p^2 - 0.3p + 1.75362$$

To proceed, we will first show that the above is concave in p. Note the second partial derivative below.

$$\left(\left(-3.15p + 1.8861\right)p^2 + 5.09112\right)p^2 + \left(-0.2636p^2 + 0.79296\right)p + 0.30054$$

We upper bound the above with the following

$$\left(\left(-3.15\left(\sqrt{3.74483}\right) + 1.8861\right)\left(\sqrt{3.74483}\right)^2 + 5.09112\right)\left(\sqrt{3.74483}\right)^2 + \left(-0.2636\left(\sqrt{3.74483}\right)^2 + 0.79296\right)\left(\sqrt{3.74483}\right) + 0.30054,$$

which is negative.

So, we evaluate the first partial derivative of our lower bound (in ν and m) for Equation 20 at p $\sqrt{3.74483}$, which is negative.

So, we evaluate

$$-0.075p^7 + 0.06287p^6 - 0.01318p^5 + 0.42426p^4 + 0.13216p^3 + 0.15027p^2 - 0.3p + 1.75362$$

at p $\sqrt{4.84375}$, which is positive.

So, our third subexpression is convex in ν.

Next, we show that the first partial derivative of Equation 31 with respect to ν is positive at ν $\sqrt{0.1175}$, shown below.

$$2\sqrt{2}\left(-0.7m + 0.485\right) + \left(2p^3 + 2\sqrt{0.235}p^2 + 2\sqrt{2}\right)\left(-0.7m + 0.485\right)\exp\left(p^2\left(m + \sqrt{0.1175}\right)\right)$$

$$+ \left(-0.15p^7 + 0.15\sqrt{2}\left(m + \sqrt{0.1175}\right)p^6 - 0.075\left(m + \sqrt{0.1175}\right)^2 p^5 + 0.3\sqrt{2}p^4\right.$$
$$+ \left(2\left(-0.7m + 0.485\right) - 0.3\left(m + \sqrt{0.1175}\right)\right)p^3 + 2\left(-0.7m + 0.485\right)\sqrt{0.235}p^2$$
$$\left. + 4\sqrt{2}\left(-0.7m + 0.485\right)\right)\exp\left(\frac{1}{2}p^2\left(m + \sqrt{0.1175}\right)\right)$$

We can further lower bound the above by subtracting out $2\sqrt{2}\left(-0.7m + 0.485\right)$ and simplify by dividing out the remaining terms by $\exp\left(\frac{1}{2}x^2\left(m + \sqrt{0.1175}\right)\right)$, which yields the following expression.

$$\left(2x^3 + 2\sqrt{0.235}p^2 + 2\sqrt{2}\right)\left(-0.7m + 0.485\right)\exp\left(\frac{1}{2}p^2\left(m + \sqrt{0.1175}\right)\right) - 0.15p^7 + 0.15\sqrt{2}\left(m + \sqrt{0.1175}\right)p^6$$
$$- 0.075\left(m + \sqrt{0.1175}\right)^2 p^5 + 0.3\sqrt{2}p^4 + \left(2\left(-0.7m + 0.485\right) - 0.3\left(m + \sqrt{0.1175}\right)\right)p^3$$
$$+ 2\left(-0.7m + 0.485\right)\sqrt{0.235}p^2 + 4\sqrt{2}\left(-0.7m + 0.485\right)$$

We further lower bound the above by lower bounding $\exp\left(\frac{1}{2}p^2\left(m + \sqrt{0.1175}\right)\right)$ by $\exp\left(\frac{1}{2}(\sqrt{3.74483})^2\left(0.15 + \sqrt{0.1175}\right)\right)$ (which we round down to 2.51605), subtracting out the subsequent constant and quadratic terms (in p), and lower bounding in m to yield the following expression (after dividing out a factor of p^2).

$$\left(-0.15p + 0.15\sqrt{2}\left(0.15 + \sqrt{0.1175}\right)\right)p^3 + \left(-0.075\left(0.25 + \sqrt{0.1175}\right)^2 p + 0.3\sqrt{2}\right)p$$
$$+ \left(7.0321\left(-0.7\left(0.25\right) + 0.485\right) - 0.3\left(0.25 + \sqrt{0.1175}\right)\right)$$

We can then lower bound the above by the following

$$\left(-0.15\sqrt{4.84375} + 0.15\sqrt{2}\left(0.15 + \sqrt{0.1175}\right)\right)\left(\sqrt{4.84375}\right)^3$$
$$+\left(-0.075\left(0.25 + \sqrt{0.1175}\right)^2\sqrt{4.84375} + 0.3\sqrt{2}\right)\sqrt{3.74483} + \left(7.0321\left(0.31\right) - 0.3\left(0.25 + \sqrt{0.1175}\right)\right),$$

which is positive.

This completes the proof.

□

Proofs for Lemmas Section B.5.2

Proof of Lemma B.121. To prove the lemma, we first show that Equation 34 is concave in p. Then, we show that the first partial derivative of Equation 34 with respect to p at $p = \sqrt{3.74483}$ is negative.

To show that Equation 34 is concave in p, we split it up into three subexpressions and show that each is concave in p. Consider the three subexpressions below.

$$(0.7m + 0.515)\left(2p + 2\sqrt{0.235}\right) \tag{21}$$

$$(0.7m + 0.515)\left(2p + 2\sqrt{0.235}\right)\exp\left(p^2\left(m - \sqrt{0.1175}\right)\right) \tag{22}$$

$$\begin{aligned}&\left(-0.15p^3\left(m - \sqrt{0.1175}\right)^2 - 0.3p^5 - 0.3p^4\sqrt{2}\left(m - \sqrt{0.1175}\right)\right. \\ &\left. + 2\,(0.7m + 0.515)\left(2p + 2\sqrt{0.235}\right)\right)\exp\left(\frac{1}{2}p^2\left(m - \sqrt{0.1175}\right)\right)\end{aligned} \tag{23}$$

Note that the first subexpression, $(0.7m + 0.515)\left(2p + 2\sqrt{0.235}\right)$, is linear in p, so we know it is concave. To show that the second subexpression is concave in p, consider its second partial derivative with respect to p below (after dividing out a factor of $\exp\left(p^2\left(m - \sqrt{0.1175}\right)\right)$, $(0.7m + 0.515)$, and $\left(m - \sqrt{0.1175}\right)$).

$$12p + 4\sqrt{0.235} + \left(8p^3 + 8p^2\sqrt{0.235}\right)\left(m - \sqrt{0.1175}\right)$$

Since $\left(m - \sqrt{0.1175}\right)$ is negative (i.e., we can upper bound it by $\left(0.25 - \sqrt{0.1175}\right)$, which is negative), we will show that our expression for the second partial derivative is positive.

Note that we can lower bound

$$12p + 4\sqrt{0.235} + \left(8p^3 + 8p^2\sqrt{0.235}\right)\left(m - \sqrt{0.1175}\right)$$

by the following (which is positive)

$$\left(12\left(\sqrt{3.74483}\right) + 4\sqrt{0.235} + \left(8\left(\sqrt{4.84375}\right)^3 + 8\,(4.84375)\sqrt{0.235}\right)\left(0.15 - \sqrt{0.1175}\right)\right).$$

So, the second subexpression is concave in p

Next, we will show that the third subexpression is concave in p. Consider the second partial derivative with respect to p of the third subexpression below (after dividing out a factor of $\exp\left(\frac{1}{2}p^2\left(m - \sqrt{0.1175}\right)\right)$)

$$\begin{aligned}&-0.3\left(\sqrt{0.1175} - m\right)^2 p^7 + 0.3\sqrt{2}\left(\sqrt{0.1175} - m\right)^3 p^6 + \left(3.3 - 0.15\left(\sqrt{0.1175} - m\right)^3\right)\left(\sqrt{0.1175} - m\right)p^5 \\ &- 2.7\sqrt{2}\left(\sqrt{0.1175} - m\right)^2 p^4 + \left(4\,(0.7m + 0.515)\left(\sqrt{0.1175} - m\right)^2 - 6 + 1.05\left(\sqrt{0.1175} - m\right)^3\right)p^3 \\ &+ \left(4\sqrt{0.235}\,(0.7m + 0.515)\left(\sqrt{0.1175} - m\right) + 3.6\sqrt{2}\right)\left(\sqrt{0.1175} - m\right)p^2 \\ &- \left(0.9\left(\sqrt{0.1175} - m\right) + 12\,(0.7m + 0.515)\right)\left(\sqrt{0.1175} - m\right)p \\ &- 4\sqrt{0.235}\,(0.7m + 0.515)\left(\sqrt{0.1175} - m\right)\end{aligned}$$

We upper bound the above by the following (which is negative)

$$
\begin{aligned}
&-0.3\left(\sqrt{0.1175}-0.25\right)^2\left(\sqrt{3.74483}\right)^7+0.3\sqrt{2}\left(\sqrt{0.1175}-0.15\right)^3(4.84375)^3\\
&+\left(3.3-0.15\left(\sqrt{0.1175}-0.25\right)^3\right)\left(\sqrt{0.1175}-0.15\right)\left(\sqrt{4.84375}\right)^5-2.7\sqrt{2}\left(\sqrt{0.1175}-0.25\right)^2(3.74483)^2\\
&+\left(4\left(0.7\left(0.25\right)+0.515\right)\left(\sqrt{0.1175}-0.15\right)^2-6+1.05\left(\sqrt{0.1175}-0.15\right)^3\right)\left(\sqrt{3.74483}\right)^3\\
&+\left(4\sqrt{0.235}\left(0.7\left(0.25\right)+0.515\right)\left(\sqrt{0.1175}-0.15\right)+3.6\sqrt{2}\right)\left(\sqrt{0.1175}-0.15\right)(4.84375)\\
&-\left(0.9\left(\sqrt{0.1175}-0.25\right)+12\left(0.7\left(0.15\right)+0.515\right)\right)\left(\sqrt{0.1175}-0.25\right)\sqrt{3.74483}\\
&-4\sqrt{0.235}\left(0.7\left(0.15\right)+0.515\right)\left(\sqrt{0.1175}-0.25\right).
\end{aligned}
$$

So, the third subexpression is concave in p.

So, Equation 34 is concave in p.

Next, we show that the first partial derivative of Equation 34 with respect to p at $p=\sqrt{3.74483}$ is negative. Consider the first partial derivative below.

$$
\begin{aligned}
&2\left(0.7m+0.515\right)\\
&+\left(2-\left(14.97932+4\sqrt{3.74483}\sqrt{0.235}\right)\left(\sqrt{0.1175}-m\right)\right)\left(0.7m+0.515\right)\exp\left(3.74483\left(m-\sqrt{0.1175}\right)\right)\\
&+\left(-0.45\left(3.74483\right)\left(\sqrt{0.1175}-m\right)^2-1.5\left(3.74483\right)^2+1.2\left(\sqrt{3.74483}\right)^3\sqrt{2}\left(\sqrt{0.1175}-m\right)+4\left(0.7m+0.515\right)\right.\\
&+0.15\left(3.74483\right)^2\left(\sqrt{0.1175}-m\right)^3+0.3\left(3.74483\right)^3\left(\sqrt{0.1175}-m\right)-0.3\left(\sqrt{3.74483}\right)^5\sqrt{2}\left(\sqrt{0.1175}-m\right)^2\\
&\left.-\left(4\left(3.74483\right)+4\sqrt{3.74483}\sqrt{0.235}\right)\left(0.7m+0.515\right)\left(\sqrt{0.1175}-m\right)\right)\exp\left(1.872415\left(m-\sqrt{0.1175}\right)\right)
\end{aligned}
$$

We can upper bound the above with the following (which is negative)

$$
\begin{aligned}
&2\left(0.7\left(0.25\right)+0.515\right)\\
&+\left(2-\left(14.97932+4\sqrt{3.74483}\sqrt{0.235}\right)\left(\sqrt{0.1175}-0.25\right)\right)\left(0.7\left(0.25\right)+0.515\right)\exp\left(3.74483\left(0.25-\sqrt{0.1175}\right)\right)\\
&+\left(-0.45\left(3.74483\right)\left(\sqrt{0.1175}-0.25\right)^2-1.5\left(3.74483\right)^2+1.2\left(\sqrt{3.74483}\right)^3\sqrt{2}\left(\sqrt{0.1175}-0.15\right)\right.\\
&+4\left(0.7\left(0.25\right)+0.515\right)+0.15\left(3.74483\right)^2\left(\sqrt{0.1175}-0.15\right)^3+0.3\left(3.74483\right)^3\left(\sqrt{0.1175}-0.15\right)\\
&-0.3\left(\sqrt{3.74483}\right)^5\sqrt{2}\left(\sqrt{0.1175}-0.25\right)^2\\
&\left.-\left(4\left(3.74483\right)+4\sqrt{3.74483}\sqrt{0.235}\right)\left(0.7\left(0.15\right)+0.515\right)\left(\sqrt{0.1175}-0.25\right)\right)\exp\left(1.872415\left(0.15-\sqrt{0.1175}\right)\right).
\end{aligned}
$$

This completes the proof.

□

Proof of Lemma B.120. To prove the lemma, we first show that Equation 33 is concave in ν. Using the concavity property, we show that the first partial derivative of Equation 33 with respect to ν evaluated at $\nu\ \sqrt{0.2375}$ is positive.

To show Equation 33 is concave in ν, consider the second partial derivative with respect to ν below (after dividing out a factor of p^2 and $\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)$).

$$\left(2\sqrt{2}p^2\nu + 2p^3 - 4\sqrt{2}\right)\left(0.7m + 0.515\right)\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)$$

$$-0.3p - 0.3p^3\left(m-\nu\right) - 0.3\sqrt{2}p^4 - 4\sqrt{2}\left(0.7m+0.515\right) - 0.0375p^5\left(m-\nu\right)^2 - 0.075p^7$$
$$-0.075\sqrt{2}p^6\left(m-\nu\right) + \left(0.7m+0.515\right)\left(p+\sqrt{2}\nu\right)p^2 \tag{18}$$

To upper bound the above, note that $\left(2\sqrt{2}p^2\nu + 2p^3 - 4\sqrt{2}\right)$ is positive (i.e., we can lower bound it by $\left(2\sqrt{2}\left(3.74483\right)\sqrt{0.1175} + 2\left(\sqrt{3.74483}\right)^3 - 4\sqrt{2}\right)$, which is positive). So, we can upper bound $\exp\left(\frac{1}{2}p^2\left(m-\nu\right)\right)$ by $\exp\left(1.872415\left(0.25-\sqrt{0.1175}\right)\right)$ (which we round up to (0.84053)).

This yields the following expression.

$$\left(2.68106\sqrt{2}p^2\nu + 2.68106p^3 - 7.36212\sqrt{2}\right)\left(0.7m+0.515\right) - 0.0375p^5\left(m-\nu\right)^2$$
$$+\left(-0.075\sqrt{2}p^6 - 0.3p^3\right)\left(m-\nu\right) + \left(-0.075p^7 - 0.3p - 0.3\sqrt{2}p^4\right)$$

To proceed, note that the above is concave in ν. So, we will show that the first partial derivative with respect to ν of the above at $\nu\ \sqrt{0.2375}$ is positive.

Consider the partial derivative below evaluated at $\nu\ \sqrt{0.2375}$, after dividing out a factor of p^2.

$$0.075\sqrt{2}p^4 + 0.075x^3\left(m - \sqrt{0.2375}\right) + 0.3p + 2.68106\sqrt{2}\left(0.7m+0.515\right)$$

We can lower bound the above by the following (which is positive).

$$\left(0.075\sqrt{2}\sqrt{3.74483} + 0.075\left(0.15-\sqrt{0.2375}\right)\right)\left(\sqrt{3.74483}\right)^3 + 0.3\left(\sqrt{3.74483}\right) + 2.68106\sqrt{2}\left(0.7\left(0.15\right)+0.515\right)$$

So, we can upper bound Equation 18 by fixing $\nu\ \sqrt{0.2375}$.

To further upper bound Equation 18, note that $\left(2.68106\sqrt{0.475}p^2 + 2.68106p^3 - 7.36212\sqrt{2}\right)$ is positive (i.e., can be lower bounded by fixing $p = \sqrt{3.74483}$, which is positive). So, we create the following upper bound,

$$-0.0375p^5\left(\sqrt{0.2375}-0.25\right)^2 + \left(0.075\sqrt{2}p^6 + 0.3p^3\right)\left(\sqrt{0.2375}-0.15\right)$$
$$+\left(2.68106\sqrt{0.475}p^2 + 2.68106p^3 - 7.36212\sqrt{2}\right)\left(0.7\left(0.25\right)+0.515\right) + \left(-0.075p^7 - 0.3p - 0.3\sqrt{2}p^4\right),$$

which we simply by rounding up after five digits of precision to the following.

$$-0.075p^7 + 0.03579p^6 - 0.00211p^5 - 0.42426p^4 + 1.95114p^3 + 1.27498p^2 - 0.3p - 7.18401$$

To upper bound the above, first note that it is decreasing in p.

To see this, consider the partial derivative with respect to p below (upper bounded by removing the constant and quartic terms with the remaining expression divided by a factor of p).

$$-0.525p^5 + 0.21474p^4 - 1.69704p^2 + 5.85342p + 2.54996$$

Note that the partial derivative can be upper bounded by the following (which is negative).

$$\left(\left(-0.525\sqrt{3.74483} + 0.21474\right)\left(\sqrt{3.74483}\right)^3 + 5.85342\right)\sqrt{4.84375} - 1.69704\left(3.74483\right) + 2.54996$$

So, we can upper bound

$$-0.075p^7 + 0.03579p^6 - 0.00211p^5 - 0.42426p^4 + 1.95114p^3 + 1.27498p^2 - 0.3p - 7.18401$$

by setting $p = \sqrt{3.74483}$, which is negative.

So, Equation 33 is concave in ν.

Next, we evaluate the first partial derivative of Equation 33 with respect to ν at $\nu = \sqrt{0.2375}$, shown below.

$$\begin{aligned}
&2\sqrt{2}\left(0.7m + 0.515\right) \\
&+ \left(-2p^3 - 2\sqrt{0.475}p^2 + 2\sqrt{2}\right)\left(0.7m + 0.515\right)\exp\left(p^2\left(m - \sqrt{0.2375}\right)\right) \\
&+ \left(0.15p^7 + \left(-0.15\sqrt{2}p^6 + 0.075\left(\sqrt{0.2375} - m\right)p^5 - 0.3p^3\right)\left(\sqrt{0.2375} - m\right) + 0.3\sqrt{2}p^4\right. \\
&\left. + \left(-2p^3 - 2\sqrt{0.475}p^2 + 4\sqrt{2}\right)\left(0.7m + 0.515\right)\right)\exp\left(\frac{1}{2}p^2\left(m - \sqrt{0.2375}\right)\right)
\end{aligned}$$

To show the above is positive, first note that $\left(-2p^3 - 2\sqrt{0.475}p^2 + 4\sqrt{2}\right)$. $\left(-2p^3 - 2\sqrt{0.475}p^2 + 2\sqrt{2}\right)$, and $\left(-0.15\sqrt{2}p^6 + 0.075\left(\sqrt{0.2375} - m\right)p^5 - 0.3p^3\right)$ are negative. To see this, we first upper bound $\left(-2p^3 - 2\sqrt{0.475}p^2 + 4\sqrt{2}\right)$ by $\left(-2\left(\sqrt{3.74483}\right)^3 - 2\sqrt{0.475}\left(\sqrt{3.74483}\right)^2 + 4\sqrt{2}\right)$, which is negative. This implies that $\left(-2p^3 - 2\sqrt{0.475}p^2 + 2\sqrt{2}\right)$ is negative. For $\left(-0.15\sqrt{2}p^6 + 0.075\left(\sqrt{0.2375} - m\right)p^5 - 0.3p^3\right)$, we rewrite it as $\left(\left(-0.15\sqrt{2}p + 0.075\left(\sqrt{0.2375} - m\right)\right)p^2 - 0.3\right)p^3$ and upper bound $\left(\left(-0.15\sqrt{2}p + 0.075\left(\sqrt{0.2375} - m\right)\right)p^2 - 0.3\right)$ by $\left(-0.15\sqrt{2}\sqrt{3.74483} + 0.075\left(\sqrt{0.2375} - 0.15\right)\right)\left(3.74483\right) - 0.3$, which is negative.

So, we create the following lower bound.

$$\begin{aligned}
&2\sqrt{2}\left(0.7\left(0.15\right) + 0.515\right) \\
&+ \left(-2p^3 - 2\sqrt{0.475}p^2 + 2\sqrt{2}\right)\left(0.7\left(0.25\right) + 0.515\right)\exp\left(p^2\left(0.25 - \sqrt{0.2375}\right)\right) \\
&+ \left(\left(0.15p^4 - 0.15\sqrt{2}\left(\sqrt{0.2375} - 0.15\right)p^3 - \left(0.3\left(\sqrt{0.2375} - 0.15\right) + 1.38\right)\right)p^3\right. \\
&+ 0.075\left(\sqrt{0.2375} - 0.25\right)\left(\sqrt{0.2375} - 0.15\right)p^5 \\
&\left. + \left(0.3\sqrt{2}p^2 - 1.38\sqrt{0.475}\right)p^2 + 2.76\sqrt{2}\right)\exp\left(\frac{1}{2}x^2\left(m - \sqrt{0.2375}\right)\right)
\end{aligned}$$

To further lower bound the above, we first show that the coefficient for $\exp\left(\frac{1}{2}p^2\left(m - \sqrt{0.2375}\right)\right)$ is positive. To see this, we show that $0.15p^4 - 0.15\sqrt{2}\left(\sqrt{0.2375} - 0.15\right)p^3 - \left(0.3\left(\sqrt{0.2375} - 0.15\right) + 1.38\right)$ and $0.3\sqrt{2}p^2 - 1.38\sqrt{0.475}$ are positive. Note that we can lower bound $0.15p^4 - 0.15\sqrt{2}\left(\sqrt{0.2375} - 0.15\right)p^3 - \left(0.3\left(\sqrt{0.2375} - 0.15\right) + 1.38\right)$ by $\left(0.15\left(\sqrt{3.74483}\right) - 0.15\sqrt{2}\left(\sqrt{0.2375} - 0.15\right)\right)\left(\sqrt{3.74483}\right)^3 - \left(0.3\left(\sqrt{0.2375} - 0.15\right) + 1.38\right)$ and $0.3\sqrt{2}p^2 - 1.38\sqrt{0.475}$ by $0.3\sqrt{2}\left(3.74483\right) - 1.38\sqrt{0.475}$.

So, we can further lower bound the first partial derivative of Equation 33 by the following.

$$1.24\sqrt{2}$$

$$+\left(-1.38p^3 - 1.38\sqrt{0.475}p^2 + 1.38\sqrt{2}\right)\exp\left(\left(\sqrt{3.74483}\right)^2\left(0.25 - \sqrt{0.2375}\right)\right)$$

$$+\left(\left(0.15p^7 - 0.15\sqrt{2}\left(\sqrt{0.2375} - 0.15\right)p^6 - \left(0.3\left(\sqrt{0.2375} - 0.15\right) + 1.38\right)p^3\right)\right.$$
$$+ 0.075\left(\sqrt{0.2375} - 0.25\right)\left(\sqrt{0.2375} - 0.15\right)p^5$$
$$\left.+\left(0.3\sqrt{2}p^4 - 1.38\sqrt{0.475}p^2\right) + 2.76\sqrt{2}\right)\exp\left(\frac{1}{2}\left(\sqrt{4.84375}\right)^2\left(0.15 - \sqrt{0.2375}\right)\right)$$

Note that we round $\exp\left(\left(\sqrt{3.74483}\right)^2\left(0.25 - \sqrt{0.2375}\right)\right)$ up to (0.411151) and round $\exp\left(\frac{1}{2}\left(\sqrt{4.84375}\right)^2\left(0.15 - \sqrt{0.2375}\right)\right)$ down to (0.44175) before further simplifying the above by rounding terms down after five digits of precision, yielding the following.

$$0.06626p^7 - 0.03162p^6 + 0.00265p^5 + 0.18741p^4 - 1.22171p^3 - 0.811194p^2 + 4.28028$$

To lower bound the above, we show that the partial derivative with respect to p of the above is positive. Consider the partial derivative below (after dividing out a factor of p).

$$\left(\left(0.46382p - 0.18972\right)p^3 - 3.66513\right)p + 0.01325p^3 + 0.74964p^2 - 1.622388$$

To see that the above is positive, note that we can lower bound the above by the following (which is positive)

$$\left(\left(0.46382\sqrt{3.74483} - 0.18972\right)\left(\sqrt{3.74483}\right)^3 - 3.66513\right)\left(\sqrt{3.74483}\right) + 0.01325\left(\sqrt{3.74483}\right)^3$$
$$+ 0.74964\left(3.74483\right) - 1.622388$$

So, we lower bound

$$0.06626p^7 - 0.03162p^6 + 0.00265p^5 + 0.18741p^4 - 1.22171p^3 - 0.811194p^2 + 4.28028$$

by setting $p = \sqrt{3.74483}$, which is positive.

This completes the proof.

□

B.5.3 System Jacobian Eigenvalues Crossing Unit Circle

In this section, we will prove Theorem B.19, where we will show that the eigenvalues of the asymptotic 2D dynamical system Jacobian evaluated at the fixed point of interest $(\tilde{q}, \tilde{c})$ cross the unit circle in the complex plane for each $m \in [0.15, 0.25]$ at some $\eta_{c,m}$, where $\eta_{c,m} \in [0.2, 0.3]$. First, we introduce the proof of Theorem B.19 followed by the supporting lemmas.

To show that $\eta_{c,m}$ exists, we will show that, for all $m \in [0.15, 0.25]$, the magnitude of the eigenvalues of the system Jacobian (Jacobian determinant) at $(\tilde{q}, \tilde{c})$ is greater than one for $\eta = 0.3$ and less than one for $\eta = 0.2$, where we we have previously proven that $\tilde{q} \in \left[\frac{0.9}{n}m^2 + \frac{0.52}{\eta} + 1.944, \frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15\right]$ and $\tilde{c} \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025]$.

Note the formula for the determinant of the Jacobian, evaluated at $(\tilde{q}, \tilde{c})$, below:

$$d_1\left(m, \eta, \tilde{q}, \tilde{c}\right) \cdot d_2\left(m, \eta, \tilde{q}, \tilde{c}\right) + d_3\left(m, \eta, \tilde{q}, \tilde{c}\right) - 1$$

, where we define $d_1(m,\eta,\tilde{q},\tilde{c})$, $d_2(m,\eta,\tilde{q},\tilde{c})$, and $d_3(m,\eta,\tilde{q},\tilde{c})$ as follows.

$$d_1(m,\eta,\tilde{q},\tilde{c}) = \left(1 - \frac{\eta\tilde{q}}{2}\left(\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\left(\sqrt{\tilde{c}}+m\right)^2 + \left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\left(\sqrt{\tilde{c}}-m\right)^2\right)\right)$$

$$d_2(m,\eta,\tilde{q},\tilde{c}) = \left(-1 + \eta\tilde{q}^2\left(\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right) + \left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)\right)$$

$$d_3(m,\eta,\tilde{q},\tilde{c}) = \frac{\eta^2\tilde{q}}{2}\left(2\left(\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-1} - \left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-1}\right) + \tilde{q}\left(\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\left(\sqrt{\tilde{c}}+m\right) + \left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\left(\sqrt{\tilde{c}}-m\right)\right)^2\right)$$

Note that in our proof, we replace $\left(\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{x}\right)\right)\right)^{-1} - \left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{x}\right)\right)\right)^{-1}\right)$ in $d_3(m,\eta,q,c)$ with $\frac{\sqrt{c}}{\eta q}$ by definition of the fixed point $(\tilde{q},\tilde{c})$.

To prove the theorem, we consider two parts of the theorem: existence and uniqueness. For existence, we note that the equation for the determinant of the system Jacobian at the fixed point $(\tilde{q},\tilde{c})$ is continuous in η (for $\eta \in [0.2, 0.3]$). So, by intermediate value theorem, if the determinant is less than one for $\eta = 0.2$ and greater than one for $\eta = 0.3$, then there exists some $\eta_{c,m} \in [0.2, 0.3]$ such that the determinant is one. For simplicity, we add a -1 term to the system Jacobian determinant to transform the problem into proving that the -1 shifted determinant is negative at $\eta = 0.2$ and positive at $\eta = 0.3$. In both cases, we split up the -1 shifted determinant by grouping together terms with a $\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-2}$ factor and grouping terms with a $\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-2}$ factor. We split the remaining terms among two expression for the specific sigmoid square factors. We then show that the resulting split expressions are negative/positive for $\eta = 0.2/0.3$.

In the case of uniqueness, we similarly split up the terms of the determinant based on those that share $\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m\pm\sqrt{\tilde{c}}\right)\right)\right)^{-2}$ factor and split the remaining terms. Then, we show that the partial derivative with respect to η of both split expressions is positive.

Proof of Theorem B.19. Consider some $m \in [0.15, 0.25]$. To prove the theorem, we first prove existence of $\eta_{c,m}$, then we show that this $\eta_{c,m}$ is unique (within the range $[0.2, 0.3]$).

To prove existence, we first note that the determinant of the system Jacobian evaluated at the fixed point

$$d_1(m,\eta,\tilde{q},\tilde{c}) \cdot d_2(m,\eta,\tilde{q},\tilde{c}) + d_3(m,\eta,\tilde{q},\tilde{c}) - 1$$

is continuous in η for $\eta \in [0.2, 0.3]$.

So, we use Intermediate Value Theorem to prove existence of $\eta_{c,m}$ by showing that the determinant at the fixed point is less than 1 at $\eta = 0.2$ and greater than 1 at $\eta = 0.3$.

Note that we add a -1 term to the determinant to transform the problem into showing that this shifted determinant expression is negative at $\eta = 0.2$ and positive at $\eta = 0.3$. Furthermore, we apply the change of variable $\tilde{u} = \sqrt{\tilde{c}}$.

First, we will show that the shifted determinant is negative at $\eta = 0.2$. To prove that the expression is negative, we first upper bound it by leaving out

$$-2\eta^2\tilde{q}^3\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}\exp\left(\tilde{q}m\right)m^2$$

and divide the remaining expression into two parts, shown below, and prove that each part is negative.

$$\begin{aligned}
&0.1\tilde{q}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\left(\tilde{u}+m\right)^2\\
&+0.2\tilde{q}^2\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\\
&+0.4\tilde{q}\tilde{u}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\left(\tilde{u}+m\right)\\
&-2\left(-0.52m+0.536\right)+\frac{2\left(0.75\right)\tilde{u}^2}{\tilde{q}}
\end{aligned} \tag{19}$$

$$\begin{aligned}
&0.1\tilde{q}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\left(\tilde{u}-m\right)^2\\
&+0.2\tilde{q}^2\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\\
&+0.4\tilde{q}\tilde{u}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\left(\tilde{u}-m\right)\\
&-2\left(1-\left(-0.52m+0.536\right)\right)+\frac{2\left(0.25\right)\tilde{u}^2}{\tilde{q}}
\end{aligned} \tag{20}$$

In lemma B.122, we show that Equation 19 is negative (after scaling by $\tilde{q}\exp\left(-\frac{1}{2}q\left(m+\tilde{u}\right)\right)\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^2$).

In lemma B.123, we show that Equation 20 is negative (after scaling by $\tilde{q}\exp\left(-\frac{1}{2}q\left(m-\tilde{u}\right)\right)\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^2$).

Next, we will show that the shifted determinant is positive at $\eta = 0.3$. To prove that the expression is positive, we further divide it into two parts, shown below, and prove that each part is positive.

$$\begin{aligned}
&\frac{\eta\tilde{q}}{2}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\left(u+m\right)^2\\
&+\eta\tilde{q}^2\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\\
&+2\eta\tilde{q}\tilde{u}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\left(\tilde{u}+m\right)\\
&-2\eta^2\tilde{q}^3m^2\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}\exp\left(\tilde{q}m\right)\\
&-2(0.477)+\frac{2\tilde{u}^2}{\tilde{q}}
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\frac{\eta\tilde{q}}{2}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\left(\tilde{u}-m\right)^{2}\\
&+\eta\tilde{q}^{2}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\\
&+2\eta\tilde{q}\tilde{u}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\left(\tilde{u}-m\right)\\
&-2\left(1-0.477\right)
\end{aligned} \tag{22}$$

In lemma B.124, we show that Equation 21 is negative (after scaling by $\tilde{q}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{2}$).

In lemma B.125, we show that Equation 22 is negative (after scaling by $\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{2}$).

After showing the existence of $\eta_{c,m}$, we show uniqueness of $\eta_{c,m}$ in the range $[0.2, 0.3]$ by showing that the first partial η of the determinant evaluated at the fixed point is positive. To do so, we split the determinant into two parts, shown below, and show that both parts grows in η.

$$\begin{aligned}
&\left(\frac{\eta\tilde{q}}{2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\left(\tilde{u}+m\right)^{2}+\eta\tilde{q}^{2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right.\\
&+2\eta\tilde{q}\tilde{u}\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\left(\tilde{u}+m\right)-2\eta^{2}\tilde{q}^{3}m^{2}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}\exp\left(\tilde{q}m\right)\\
&\left.+\frac{2\tilde{u}^{2}}{\tilde{q}}\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{2}\right)\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\tilde{u}\right)\right)\right)^{-2}
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\left(\frac{\eta\tilde{q}}{2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\left(\tilde{u}-m\right)^{2}+\eta\tilde{q}^{2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)+2\eta\tilde{q}\tilde{u}\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\left(\tilde{u}-m\right)\right.\\
&\left.-\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{2}\right)\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\tilde{u}\right)\right)\right)^{-2}
\end{aligned} \tag{24}$$

In lemma B.126, we show that Equation 23 grows in η. In lemma B.127, we show that Equation 24 grows in η.

This completes the proof.

□

Now, we introduce the supporting lemmas for Theorem B.19.

Lemmas For Existence Of $\eta_{c,m}$ In this section, we introduce the supporting lemmas used in Theorem B.19, regarding the system Jacobian determinant being less than one $\eta = 0.2$ and greater than one at $\eta = 0.3$. Note that we split the determinant evaluated at $\eta = 0.2/0.3$ into two parts and show that each part is positive/negative. Lemmas B.122 and B.123 handle the parts in the case of $\eta = 0.2$. We prove lemma B.122 by showing that Equation 25 is decreasing in $\sqrt{c}$ and an upper bound for the equation is concave in m. We prove lemma B.123 by showing that Equation 26 is decreasing in $\sqrt{c}$ and an upper bound the equation is convex in m. Lemmas B.124 and B.125 handle the parts in the case of $\eta = 0.3$. We prove lemma B.124 by showing that a lower bond for Equation 27 grows in $\sqrt{c}$, grows in q, and is convex in m. We prove lemma B.125 by showing that Equation 28 is decreasing in $\sqrt{c}$. The proofs of the below lemmas can be found in Section B.5.3.

Lemma B.122. *Consider* $m \in [0.15, 0.25]$, $q \in [4.5m^2 + 4.544, 4.7m^2 + 4.55]$, $u \in [\sqrt{0.65m + 0.02}, \sqrt{0.65m + 0.025}]$. *Then,*

$$\begin{aligned}&\left(0.5q^2u^2 + 0.6mq^2u + 0.1m^2q^2 + 0.2q^3 + 2.08mq - 2.144q + 3u^2\right)\\&+ \left(1.04mq - 1.072q + 1.5u^2\right)\exp\left(-\frac{1}{2}q\left(m+u\right)\right)\\&+ \left(1.04mq - 1.072q + 1.5u^2\right)\exp\left(\frac{1}{2}q\left(m+u\right)\right)\end{aligned} \tag{25}$$

is negative.

Lemma B.123. *Consider* $m \in [0.15, 0.25]$, $q \in [4.5m^2 + 4.544, 4.7m^2 + 4.55]$, $u \in [\sqrt{0.65m + 0.02}, \sqrt{0.65m + 0.025}]$. *Then,*

$$\begin{aligned}&\left(0.5q^2u^2 - 0.6mq^2u + 0.1m^2q^2 + 0.2q^3 - 2.08mq - 1.856q + u^2\right)\\&+ \left(-1.04mq - 0.928q + 0.5u^2\right)\exp\left(-\frac{1}{2}q\left(m-u\right)\right)\\&+ \left(-1.04mq - 0.928q + 0.5u^2\right)\exp\left(\frac{1}{2}q\left(m-u\right)\right)\end{aligned} \tag{26}$$

is negative.

Lemma B.124. *Consider* $m \in [0.15, 0.25]$, $q \in [3m^2 + 3.67733, 3.13334m^2 + 3.75]$, $u \in [\sqrt{0.85m + 0.02}, \sqrt{0.85m + 0.025}]$. *Then,*

$$\begin{aligned}&0.15q^2\exp\left(\frac{1}{2}q\left(m+u\right)\right)\left(u+m\right)^2 + 0.3q^3\exp\left(\frac{1}{2}q\left(m+u\right)\right) + 0.6q^2u\exp\left(\frac{1}{2}q\left(m+u\right)\right)\left(u+m\right)\\&- 0.18q^4m^2\left(1+\exp\left(\frac{1}{2}q\left(m-u\right)\right)\right)^{-2}\exp\left(qm\right) + \left(2u^2 - 0.954q\right)\left(1+\exp\left(\frac{1}{2}q\left(m+u\right)\right)\right)^2\end{aligned} \tag{27}$$

is positive.

Lemma B.125. *Consider* $m \in [0.15, 0.25]$, $q \in [3m^2 + 3.67733, 3.13334m^2 + 3.75]$, $u \in [\sqrt{0.85m + 0.02}, \sqrt{0.85m + 0.025}]$. *Then,*

$$\left(0.15q\left(u-m\right)^2 + 0.3q^2 + 0.6qu\left(u-m\right) - 2.092\right)\exp\left(\frac{1}{2}q\left(m-u\right)\right) - 1.046 - 1.046\exp\left(q\left(m-u\right)\right) \tag{28}$$

is positive.

Lemmas For Uniqueness Of $\eta_{c,m}$ In this section, we introduce the supporting lemmas used in Theorem **??**, regarding the uniqueness of $\eta_{c,m}$, the point where the system Jacobian determinant evaluated at fixed point $(\tilde{q}, \tilde{c})$ equals one. Note that we split the determinant into two parts and show that each part is growing in η in lemmas B.126 and B.127. Refer to Section B.5.3 for the proofs of the below lemmas.

Lemma B.126. *Consider* $m \in [0.15, 0.25]$, $q \in [3.74483, 4.84375]$, $c \in [0.1175, 0.2375]$. *Then,*

$$\begin{aligned}&\left(\frac{\eta q}{2}\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\left(\sqrt{c}+m\right)^2 + \eta q^2\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\right.\\&+ 2\eta q\sqrt{c}\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\left(\sqrt{c}+m\right) - 2\eta^2q^3m^2\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\right)^{-2}\exp\left(qm\right)\\&\left.+ \frac{2c}{q}\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\right)^2\right)\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\right)^{-2}\end{aligned} \tag{29}$$

grows in η.

Lemma B.127. *Consider* $m \in [0.15, 0.25]$, $q \in [3.74483, 4.84375]$, $c \in [0.1175, 0.2375]$. *Then,*

$$\left(\frac{\eta q}{2}\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\left(\sqrt{c}-m\right)^2+\eta q^2\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\right.$$
$$\left.+2\eta q\sqrt{c}\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\left(\sqrt{c}-m\right)-\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\right)^2\right)\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\right)^{-2} \tag{30}$$

grows in η.

Supporting Lemmas For Lemma B.122 (Positive Sigmoid Square for $\eta = 0.2$) In this section, we include supporting lemmas for lemma B.122 about Equation 25 being negative for our specified range. This includes Equation 25 being concave in u (lemmas B.128 and B.129), locally decreasing in u (lemma B.130), and a upper bound for Equation 25 being concave in m (lemma B.131). Refer to Section B.5.3 for the proofs of the below lemmas.

Lemma B.128. *Consider* $m \in [0.15, 0.25]$, $q \in [4.5m^2 + 4.544, 4.7m^2 + 4.55]$, $u \in [\sqrt{0.65m + 0.02}, \sqrt{0.65m + 0.025}]$. *Then,*

$$\left(1.04mq - 1.072q + 1.5u^2\right)\exp\left(-\frac{1}{2}q\left(m+u\right)\right) \tag{31}$$

is concave in u.

Lemma B.129. *Consider* $m \in [0.15, 0.25]$, $q \in \left[4.5m^2 + 4.544, 4.7m^2 + 4.55\right]$, $u \in [\sqrt{0.65m + 0.02}, \sqrt{0.65m + 0.025}]$. *Then,*

$$\left(0.5q^2u^2 + 0.6mq^2u + 0.1m^2q^2 + 0.2q^3 + 2.08mq - 2.144q + 3u^2\right)+\left(1.04mq - 1.072q + 1.5u^2\right)\exp\left(\frac{1}{2}q\left(m+u\right)\right) \tag{32}$$

is concave in u.

Lemma B.130. *Consider* $m \in [0.15, 0.25]$, $q \in \left[4.5m^2 + 4.544, 4.7m^2 + 4.55\right]$. *Then,*

$$q^2\sqrt{0.65m + 0.02} + 0.6mq^2 + 6\sqrt{0.65m + 0.02}$$
$$+\left(-0.52mq^2 + 0.536q^2 - 0.75q\left(0.65m + 0.02\right) + 3\sqrt{0.65m + 0.02}\right)\exp\left(-\frac{1}{2}q\left(m+\sqrt{0.65m + 0.02}\right)\right)$$
$$+\left(0.52mq^2 - 0.536q^2 + 0.75q\left(0.65m + 0.02\right) + 3\sqrt{0.65m + 0.02}\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{0.65m + 0.02}\right)\right) \tag{33}$$

is negative.

Lemma B.131. *For* $m \in [0.15, 0.25]$,

$$\left(33.05703m^6 + 10.32069m^5 + 84.32684m^4 + 29.75861m^3 + 60.25228m^2 + 21.08644m + 9.3511\right)$$
$$+\left(4.68m^3 - 4.824m^2 + 5.70076m - 4.84116\right)\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m+\sqrt{0.65m + 0.02}\right)\right)$$
$$+\left(4.68m^3 - 4.824m^2 + 5.70076m - 4.84116\right)\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m+\sqrt{0.65m + 0.02}\right)\right) \tag{34}$$

is convex in m.

Supporting Lemmas For Lemma B.123 (Negative Sigmoid Square for $\eta = 0.2$) In this section, we include supporting lemmas for lemma B.123 about Equation 26 being negative for our specified range. This includes Equation 26 being concave in u (lemmas B.132 and B.133) and locally decreasing in u (lemma B.134). Refer to Section B.5.3 for the proofs of the below lemmas.

Lemma B.132. *Consider* $m \in [0.15, 0.25]$, $q \in [4.5m^2 + 4.544, 4.7m^2 + 4.55]$, $u \in [\sqrt{0.65m + 0.02}, \sqrt{0.65m + 0.025}]$. *Then,*

$$\left(-1.04mq - 0.928q + 0.5u^2\right)\exp\left(\frac{1}{2}q\left(m-u\right)\right) \tag{35}$$

is concave in u.

Lemma B.133. *Consider* $m \in [0.15, 0.25]$, $q \in [4.5m^2 + 4.544, 4.7m^2 + 4.55]$, $u \in [\sqrt{0.65m + 0.02}, \sqrt{0.65m + 0.025}]$. *Then,*

$$\begin{aligned} &0.5q^2u^2 - 0.6mq^2u + 0.1m^2q^2 + 0.2q^3 - 2.08mq - 1.856q + u^2 \\ &+ \left(-1.04mq - 0.928q + 0.5u^2\right) \exp\left(-\frac{1}{2}q\left(m - u\right)\right) \end{aligned} \tag{36}$$

is concave in u.

Lemma B.134. *Consider* $m \in [0.15, 0.25]$, $q \in [4.5m^2 + 4.544, 4.7m^2 + 4.55]$. *Then,*

$$\begin{aligned} &q^2\sqrt{0.65m + 0.02} - 0.6mq^2 + 2\sqrt{0.65m + 0.02} \\ &+ \left(-0.52mq^2 - 0.464q^2 + 0.1625mq + 0.005q + \sqrt{0.65m + 0.02}\right) \exp\left(-\frac{1}{2}q\left(m - \sqrt{0.65m + 0.02}\right)\right) \\ &+ \left(0.52mq^2 + 0.464q^2 - 0.1625mq - 0.005q + \sqrt{0.65m + 0.02}\right) \exp\left(\frac{1}{2}q\left(m - \sqrt{0.65m + 0.02}\right)\right) \end{aligned} \tag{37}$$

is negative.

Supporting Lemmas For Lemma B.124 (Positive Sigmoid Square for $\eta = 0.3$) In this section, we include supporting lemmas for lemma B.124 about Equation 27 being positive for our specified range. This includes Equation 27 being convex in u (lemmas B.135 and B.136) and locally growing in u (lemma B.137). Refer to Section B.5.3 for the proofs of the below lemmas.

Lemma B.135. *Consider* $m \in [0.15, 0.25]$, $q \in [3m^2 + 3.67733, 3.13334m^2 + 3.75]$, $u \in [\sqrt{0.85m + 0.02}, \sqrt{0.85m + 0.025}]$. *Then,*

$$\begin{aligned} &\left(0.15q^2\left(u + m\right)^2 + 0.3q^3 + 0.6q^2u\left(u + m\right) + 4u^2 - 1.908q\right) \exp\left(\frac{1}{2}q\left(m + u\right)\right) \\ &+ \left(2u^2 - 0.954q - 0.21059q^4m^2\right) + \left(2u^2 - 0.954q\right) \exp\left(q\left(m + u\right)\right) \end{aligned} \tag{38}$$

is convex in u.

Lemma B.136. *For* $m \in [0.15, 0.25]$,

$$\begin{aligned} &14.23426m^8 + 6.26326m^7 + 59.75275m^6 - 92.20747m^5 + 163.74914m^4 - 239.93551m^3 + 259.39145m^2 \\ &- 103.87329m + 13.55991 \end{aligned}$$

is positive.

Lemma B.137. *Consider* $m \in [0.15, 0.25]$, $q \in [3m^2 + 3.67733, 3.13334m^2 + 3.75]$. *Then,*

$$\begin{aligned} &4\sqrt{0.85m + 0.02} + \left(4\sqrt{0.85m + 0.02} + \left(1.7m + 0.04 - 0.954q\right)q\right) \exp\left(q\left(m + \sqrt{0.85m + 0.02}\right)\right) \\ &+ \Bigg(\Bigg(0.15q^3 + 0.075m^2q^2 + 0.31875mq^2 + 0.0075q^2 + 0.45mq^2\sqrt{0.85m + 0.02} - 0.954q \\ &+ 1.5q\sqrt{0.85m + 0.02} + 0.9mq + 1.7m + 0.04\Bigg)q + 8\sqrt{0.85m + 0.02}\Bigg) \exp\left(\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right) \end{aligned} \tag{39}$$

is positive.

Lemma B.138. *Consider* $m \in [0.15, 0.25]$, $q \in [3m^2 + 3.67733, 3.13334m^2 + 3.75]$. *Then,*

$$\begin{aligned} &\left(0.3q^3 + 1.03312m^2q^2 + 0.85062mq^2 + 0.015q^2 - 1.908q + 3.4m + 0.08\right) \\ &+ \left(-40.53797m^4 - 51.50529m^2 + 1.7m - 3.5375\right) \exp\left(\left(-0.99062m - 0.1184\right)q\right) \\ &+ \left(-2.98921m^2 + 1.7m - 3.5375\right) \exp\left(\left(0.94071m + 0.13092\right)q\right). \end{aligned} \tag{40}$$

grows in q.

Supporting Lemmas For Lemma B.125 (Negative Sigmoid Square for $\eta = 0.3$) In this section, we include supporting lemmas for lemma B.125 about Equation 28 being positive for our specified range. This includes Equation 28 being concave in u (lemma B.139) and locally decreasing in u (lemma B.140). Refer to Section B.5.3 for the proofs of the below lemmas.

Lemma B.139. *Consider* $m \in [0.15, 0.25]$, $q \in [3m^2 + 3.67733, 3.13334m^2 + 3.75]$, $u \in [\sqrt{0.85m + 0.02}, \sqrt{0.85m + 0.025}]$. *Then,*

$$\left(0.15q\left(u-m\right)^2 + 0.3q^2 + 0.6qu\left(u-m\right) - 2.092\right)\exp\left(\frac{1}{2}q\left(m-u\right)\right) - 1.046 - 1.046\exp\left(q\left(m-u\right)\right) \tag{28}$$

is concave in u.

Lemma B.140. *Consider* $m \in [0.15, 0.25]$, $q \in [3m^2 + 3.67733, 3.13334m^2 + 3.75]$. *Then,*

$$\begin{aligned}&\left(1.5\sqrt{0.85m + 0.02} - 0.9m - 0.31875mq - 0.0075q + 0.45mq\sqrt{0.85m + 0.02}\right.\\ &\left. - 0.075m^2q - 0.15q^2 + 1.046\right) + 1.046\exp\left(\frac{1}{2}q\left(m - \sqrt{0.85m + 0.02}\right)\right)\end{aligned} \tag{41}$$

is negative.

Proofs for Lemmas in Section B.5.3

Proof of Lemma B.122. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we start by showing that Equation 25 is concave in u. Then, we show that the first partial u of Equation 25 is negative at $u = \sqrt{0.65m + 0.02}$. After upper bound Equation 25 in u by $u = \sqrt{0.65m + 0.02}$, we upper bound the resulting expression in q. Then, we show that the our expression m is concave in m and show that the linearization of the expression at $m = 0.192$ is negative locally.

First, we prove Equation 25 is concave in u. To do so, we prove

$$\left(1.04mq - 1.072q + 1.5u^2\right)\exp\left(-\frac{1}{2}q\left(m+u\right)\right) \tag{31}$$

and

$$\left(0.5q^2u^2 + 0.6mq^2u + 0.1m^2q^2 + 0.2q^3 + 2.08mq - 2.144q + 3u^2\right) + \left(1.04mq - 1.072q + 1.5u^2\right)\exp\left(\frac{1}{2}q\left(m+u\right)\right) \tag{32}$$

are concave in u. We prove this in lemmas B.128 and B.129.

Next, we will show that the first partial u of Equation 25 is negative at $u = \sqrt{0.65m + 0.02}$. We prove this in lemma B.134.

So, we can upper bound Equation 25 in u by $u = \sqrt{0.65m + 0.02}$, shown below.

$$\begin{aligned}&\left(0.325mq^2 + 0.01q^2 + 0.6mq^2\sqrt{0.65m + 0.02} + 0.1m^2q^2 + 0.2q^3 + 2.08mq - 2.144q + 1.95m + 0.06\right)\\ &+ \left(\left(1.04m - 1.072\right)q + 0.975m + 0.03\right)\left(\exp\left(-\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right) + \exp\left(\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right)\right)\end{aligned} \tag{42}$$

From the above, we can upper bound $0.6mq^2\sqrt{0.65m + 0.02}$ by linearizing $\sqrt{0.65m + 0.02}$ at $m = 0.25$. We also upper bound $\left(1.04m - 1.072\right)q$ by $\left(1.04m - 1.072\right)\left(4.5m^2 + 4.544\right)$, where $\left(1.04m - 1.072\right)\left(4.5m^2 + 4.544\right) + 0.975m + 0.03$ simplifies (RU) to $4.68m^3 - 4.824m^2 + 5.70076m - 4.84116$.

Note that $4.68m^3 - 4.824m^2 + 5.70076m - 4.84116$ is negative (i.e., can be upper bounded by $\left(4.68\left(0.25\right)^3 - 4.824\left(0.15\right)^2 + 5.70076\left(0.25\right) - 4.84116\right)$). Also note that

$$\left(\exp\left(-\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right) + \exp\left(\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right)\right) \tag{43}$$

grows in q and can be lower bounded by $q = 4.5m^2 + 4.544$.

With the above, Equation 42 can be upper bounded (RU) by

$$\begin{aligned}&\left(0.2q^3+\left(0.55647m^2+0.46721m+0.01\right)q^2+\left(2.08m-2.144\right)q+1.95m+0.06\right)\\&+\left(4.68m^3-4.824m^2+5.70076m-4.84116\right)\left(\exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)\right.\\&\left.+\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)\right).\end{aligned}\tag{44}$$

We then rewrite

$$0.2q^3+\left(0.55647m^2+0.46721m+0.01\right)q^2+\left(2.08m-2.144\right)q+1.95m+0.06\tag{45}$$

as

$$\left(\left(0.2q+0.55647m^2+0.46721m+0.01\right)q+2.08m-2.144\right)q+1.95m+0.06\tag{46}$$

and iteratively upper bound the above in q.

First, we upper bound $\left(0.2q+0.55647m^2+0.46721m+0.01\right)q$ by $\left(0.2\left(4.7m^2+4.55\right)+0.55647m^2+0.46721m+0.01\right)\left(4.7m^2+4.55\right)$, where

$$\left(0.2\left(4.7m^2+4.55\right)+0.55647m^2+0.46721m+0.01\right)\left(4.7m^2+4.55\right)+2.08m-2.144\tag{47}$$

can be simplified (RU) to $7.03341m^4+2.19589m^3+11.13294m^2+4.20581m+2.042$.

Then, we can upper bound

$$\left(7.03341m^4+2.19589m^3+11.13294m^2+4.20581m+2.042\right)q+1.95m+0.06\tag{48}$$

by

$$\left(7.03341m^4+2.19589m^3+11.13294m^2+4.20581m+2.042\right)\left(4.7m^2+4.55\right)+1.95m+0.06,\tag{49}$$

which simplifies (RU) to

$$33.05703m^6+10.32069m^5+84.32684m^4+29.75861m^3+60.25228m^2+21.08644m+9.3511\tag{50}$$

Next, we will show that the resulting expression

$$\begin{aligned}&\left(33.05703m^6+10.32069m^5+84.32684m^4+29.75861m^3+60.25228m^2+21.08644m+9.3511\right)\\&+\left(4.68m^3-4.824m^2+5.70076m-4.84116\right)\exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)\\&+\left(4.68m^3-4.824m^2+5.70076m-4.84116\right)\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)\end{aligned}\tag{34}$$

is negative.

To proceed, we show that the above is concave in m in lemma B.131. Then, we linearize the above at $m=0.192$ (which has positive partial m) and evaluate the linearization at $m=0.25$, which is negative.

This completes the proof.

□

Proof of Lemma B.123. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we first show that Equation 26 is concave in u and that the first partial u at $u=\sqrt{0.65m+0.02}$ is negative, so we can upper bound Equation 26 at $u=\sqrt{0.65m+0.02}$. Then, we upper bound the resulting expression in q, then upper bound it in m.

We begin by showing that Equation 26 is concave in u. To do so, we'll show that

$$\left(-1.04mq - 0.928q + 0.5u^2\right) \exp\left(\frac{1}{2}q\left(m - u\right)\right) \tag{35}$$

and

$$\begin{aligned} &0.5q^2u^2 - 0.6mq^2u + 0.1m^2q^2 + 0.2q^3 - 2.08mq - 1.856q + u^2 \\ &+ \left(-1.04mq - 0.928q + 0.5u^2\right) \exp\left(-\frac{1}{2}q\left(m - u\right)\right) \end{aligned} \tag{36}$$

are concave in u.

We show this in lemmas B.132 and B.133.

Next, we show that the first partial u of Equation 26 at $u = \sqrt{0.65m + 0.02}$ is negative. We prove in this lemma B.134. So, we can upper bound Equation 26 in u by $u = \sqrt{0.65m + 0.02}$. From the resulting expression (shown below in Equation 51), we can upper bound $-0.6mq^2\sqrt{0.65m + 0.02}$ by $-0.6mq^2\left(0.84417m + 0.2161\right)$ (note lemma B.241.

$$\begin{aligned} &\left(0.2q^3 + 0.1m^2q^2 + 0.325mq^2 + 0.01q^2 - 0.6mq^2\sqrt{0.65m + 0.02} - 2.08mq - 1.856q + 0.65m + 0.02\right) \\ &+ \left(-1.04mq - 0.928q + 0.325m + 0.01\right) \exp\left(-\frac{1}{2}q\left(m - \sqrt{0.65m + 0.02}\right)\right) \\ &+ \left(-1.04mq - 0.928q + 0.325m + 0.01\right) \exp\left(\frac{1}{2}q\left(m - \sqrt{0.65m + 0.02}\right)\right) \end{aligned} \tag{51}$$

Furthermore, from Equation 51, note that $-1.04mq - 0.928q + 0.325m + 0.01$ is negative. To see this, we first upper bound $-1.04mq - 0.928q + 0.325m + 0.01$ in q by $q = 4.5m^2 + 4.544$, where the resulting expression in m, $-4.68m^3 - 4.176m^2 - 4.40076m - 4.206832$, is negative.

If we also consider that $\exp\left(-\frac{1}{2}q\left(m - \sqrt{0.65m + 0.02}\right)\right) + \exp\left(\frac{1}{2}q\left(m - \sqrt{0.65m + 0.02}\right)\right)$ grows in q, we can upper bound

$$\left(-1.04mq - 0.928q + 0.325m + 0.01\right)\left(\exp\left(-\frac{1}{2}q\left(m - \sqrt{0.65m + 0.02}\right)\right) + \exp\left(\frac{1}{2}q\left(m - \sqrt{0.65m + 0.02}\right)\right)\right) \tag{52}$$

by

$$\begin{aligned} &\left(-1.04mq - 0.928q + 0.325m + 0.01\right)\Bigg(\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right) \\ &\qquad + \exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)\Bigg). \end{aligned} \tag{53}$$

Then, we lower bound $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ by linearizing about $m = 0.25$, since it is convex in m (note lemma B.244). Furthermore, we lower bound $\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ by $-0.31361m + 1.6118$. To see that $-0.31361m + 1.6118$ is a viable lower bound, note that $-0.31361m + 1.6118 - \exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ is convex in m by lemma B.245, so we evaluate the difference at $m = 0.15, 0.25$, which are negative.

We then write the resulting expression as follows (RU).

$$\left(\left(0.2q - 0.40650m^2 + 0.1954m + 0.01\right)q + 0.06991m^2 - 4.30803m - 3.89975\right)q - 0.02184m^2 + 1.36509m + 0.04203 \tag{54}$$

We then proceed to iteratively upper bound the above as follows. First, we upper bound $0.2q$ by $0.2\left(4.7m^2 + 4.55\right)$. Then, $\left(0.2\left(4.7m^2 + 4.55\right) - 0.40650m^2 + 0.1954m + 0.01\right)q$ simplifies to $\left(0.5335m^2 + 0.1954m + 0.92\right)q$, which we upper bound by $\left(0.5335m^2 + 0.1954m + 0.92\right)\left(4.7m^2 + 4.55\right)$. So,

$$\left(0.5335m^2 + 0.1954m + 0.92\right)\left(4.7m^2 + 4.55\right) + 0.06991m^2 - 4.30803m - 3.89975 \tag{55}$$

simplifies (RU) to $2.50745m^4 + 0.91838m^3 + 6.82134m^2 - 3.41896m + 0.28625$. Note that $2.50745m^4+0.91838m^3+6.82134m^2-3.41896m+0.28625$ is negative. To see this, note that we can upper bound $2.50745m^4+0.91838m^3+6.82134m^2-3.41896m+0.28625$ by $2.50745\left(0.25\right)^4+0.91838\left(0.25\right)^3+6.82134m^2-3.41896m+0.28625$, which simplifies (RU) to $6.82134m^2-3.41896m+0.31040$. Since $6.82134m^2-3.41896m+0.31040$ is convex in m, we evaluate at $m=0.15,0.25$, both of which are negative.

So, we can upper bound

$$\left(2.50745m^4+0.91838m^3+6.82134m^2-3.41896m+0.28625\right)q \tag{56}$$

by

$$\left(2.50745m^4+0.91838m^3+6.82134m^2-3.41896m+0.28625\right)\left(4.5m^2+4.544\right), \tag{57}$$

which simplifies (RU) to

$$11.28353m^6+4.13271m^5+42.08989m^4-11.21220m^3+32.26246m^2-14.17066m+1.34275. \tag{58}$$

The above can then be upper bounded by

$$11.28353\left(0.25\right)^6+4.13271\left(0.25\right)^5+42.08989\left(0.25\right)^2m^2-11.21220\left(0.15\right)m^2+32.26246m^2-14.17066m+1.34275, \tag{59}$$

which we simplify (RU) to $33.21125m^2-14.17066m+1.34955$. Since $33.21125m^2-14.17066m+1.34955$ is convex in m, we evaluate it at $m=0.15,0.25$, both of which are negative.

This completes the proof.

□

Proof of Lemma B.124. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we begin by lower bounding Equation 27 by lower bounding

$$-0.18q^4m^2\left(1+\exp\left(\frac{1}{2}q\left(m-u\right)\right)\right)^{-2}\exp\left(qm\right).$$

To do so, we upper bound $\left(1+\exp\left(\frac{1}{2}q\left(m-u\right)\right)\right)^{-2}$ by

$$\left(1+\exp\left(\frac{1}{2}\left(3.13334\left(0.25\right)^2+3.75\right)\left(0.15-\sqrt{0.85\left(0.25\right)+0.025}\right)\right)\right)^{-2}$$

and upper bound $\exp\left(qm\right)$ by $\exp\left(\left(3.13334\left(0.25\right)^2+3.75\right)\left(0.25\right)\right)$. The resulting lower bound simplifies (RD) to $-0.21059m^2q^4$.

Consider the resulting lower bounded expression below.

$$\begin{aligned}&\left(0.15q^2\left(u+m\right)^2+0.3q^3+0.6q^2u\left(u+m\right)+4u^2-1.908q\right)\exp\left(\frac{1}{2}q\left(m+u\right)\right)\\&+\left(2u^2-0.954q-0.21059q^4m^2\right)+\left(2u^2-0.954q\right)\exp\left(q\left(m+u\right)\right)\end{aligned} \tag{60}$$

To show the above is positive, we proceed as follows. First, we show that Equation 60 is convex in u, and the first partial u at $u=\sqrt{0.85m+0.02}$ is positive. Then, we can lower bound in u at $u=\sqrt{0.85m+0.02}$. After further lower bounding the resulting expression, we show that it grows in q, so we can lower bound in q at $q=3m^2+3.67733$. Last, we show that the resulting expression is convex in m, linearize the expression at $m=0.235$, and show that the linearization evaluated at $m=0.25$ is positive.

By lemma B.135, we know that Equation 60 is convex in u.

After showing convexity in u, we consider the first partial u of Equation 60 at $u = \sqrt{0.85m + 0.02}$. We know that it is positive by lemma B.137. So, we lower bound Equation 60 at $u = \sqrt{0.85m + 0.02}$. Factoring out $\exp\left(\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right)$, we get the following expression.

$$\begin{aligned}
&\left(0.3q^3 + 0.15m^2q^2 + 0.6375mq^2 + 0.9mq^2\sqrt{0.85m + 0.02} + 0.015q^2 + 3.4m + 0.08 - 1.908q\right) \\
&+ \left(1.7m + 0.04 - 0.954q - 0.21059m^2q^4\right)\exp\left(-\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right) \\
&+ (1.7m + 0.04 - 0.954q)\exp\left(\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right)
\end{aligned} \tag{61}$$

We can then lower bound $0.9mq^2\sqrt{0.85m + 0.02}$ from the above by $0.9mx^2\left(0.98125m + 0.2368\right)$ by lemma B.243 and simplify the result (RD) to get the following lower bound for Equation 61.

$$\begin{aligned}
&\left(0.3q^3 + 1.03312m^2q^2 + 0.85062mq^2 + 0.015q^2 - 1.908q + 3.4m + 0.08\right) \\
&+ \left(1.7m + 0.04 - 0.954q - 0.21059m^2q^4\right)\exp\left(-\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right) \\
&+ (1.7m + 0.04 - 0.954q)\exp\left(\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right)
\end{aligned} \tag{62}$$

To further lower bound Equation 62, note that $1.7m + 0.04 - 0.954q$ can be upper bounded by $1.7\left(0.25\right) + 0.04 - 0.954\left(3\left(0.15\right)^2 + 3.67733\right)$, which is negative. With this, we can lower bound Equation 62 by upper bounding $\exp\left(-\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right)$ by replacing $\sqrt{0.85m + 0.02}$ by $\left(0.98125m + 0.2368\right)$ (note lemma B.243) and upper bounding $\exp\left(\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right)$ by linearizing $\sqrt{0.85m + 0.02}$ at $m = 0.25$.

This results in the following expression.

$$\begin{aligned}
&\left(0.3q^3 + 1.03312m^2q^2 + 0.85062mq^2 + 0.015q^2 - 1.908q + 3.4m + 0.08\right) \\
&+ \left(1.7m + 0.04 - 0.954q - 0.21059m^2q^4\right)\exp\left(\left(-0.99062m - 0.1184\right)q\right) \\
&+ (1.7m + 0.04 - 0.954q)\exp\left(\left(0.94071m + 0.13092\right)q\right)
\end{aligned} \tag{63}$$

To proceed, we can lower bound $\left(1.7m + 0.04 - 0.954q - 0.21059m^2q^4\right)$ by

$$\left(1.7m + 0.04 - 0.954\left(3.13334m^2 + 3.75\right) - 0.21059m^2\left(3.13334\left(0.25\right)^2 + 3.75\right)^3\left(3.13334m^2 + 3.75\right)\right)$$

and lower bound $(1.7m + 0.04 - 0.954q)$ by

$$\left(1.7m + 0.04 - 0.954\left(3.13334m^2 + 3.75\right)\right).$$

We then simplify the resulting expression (RD) to

$$\begin{aligned}
&\left(0.3q^3 + 1.03312m^2q^2 + 0.85062mq^2 + 0.015q^2 - 1.908q + 3.4m + 0.08\right) \\
&+ \left(-40.53797m^4 - 51.50529m^2 + 1.7m - 3.5375\right)\exp\left(\left(-0.99062m - 0.1184\right)q\right) \\
&+ \left(-2.98921m^2 + 1.7m - 3.5375\right)\exp\left(\left(0.94071m + 0.13092\right)q\right).
\end{aligned} \tag{64}$$

Note that the above grows in q by lemma B.138, so we can lower bound it in q by $q = 3m^2 + 3.67733$. Since $-2.98921m^2 + 1.7m - 3.5375$ is negative (i.e., can be upper bounded by $-2.98921\left(0.15\right)^2 + 1.7\left(0.25\right) - 3.5375$), we can simplify the resulting expression to

$$\begin{aligned}
&\left(17.39808m^6 + 7.65558m^5 + 52.71611m^4 + 18.76806m^3 + 45.08903m^2 + 14.90272m + 8.18478\right) \\
&+ \left(-40.53797m^4 - 51.50529m^2 + 1.7m - 3.5375\right)\exp\left(-2.97186m^3 - 0.3552m^2 - 3.64283m - 0.43539\right) \\
&+ \left(-2.98921m^2 + 1.7m - 3.5375\right)\exp\left(2.82213m^3 + 0.39276m^2 + 3.45931m + 0.48144\right)
\end{aligned} \tag{65}$$

We can then replace $\exp\left(-2.97186m^3 - 0.3552m^2 - 3.64283m - 0.43539\right)$ and $\exp\left(2.82213m^3 + 0.39276m^2 + 3.45931m + 0.48144\right)$ with the following local linear upper bounds respectively: $-1.24949m + 0.5554$, $13.46533m + 0.75$.

To see that these local linear upper bounds are valid, note that the differences $-1.24949m + 0.5554 - \exp\left(-2.97186m^3 - 0.3552m^2 - 3.64283m - 0.43539\right)$ and $13.46533m + 0.75 - \exp\left(2.82213m^3 + 0.39276m^2 + 3.45931m + 0.48144\right)$ are concave in m (note lemmas B.250), so we evaluate the differences at $m = 0.15, 0.25$, which are positive.

With these local linear upper bounds, we get the following lower bound for Equation 65.

$$17.39808m^6 + 58.30736m^5 + 30.20132m^4 + 42.87270m^3 + 35.00801m^2 - 26.09164m + 3.56692 \tag{66}$$

Since the above is convex in m, we linearize it at $m = 0.235$, and evaluate the linearization at $m = 0.25$ (note that first partial m of Equation 66 is negative at $m = 0.235$), which is positive.

This completes the proof.

□

Proof of Lemma B.125. To prove the lemma, we begin showing that Equation 28 is concave in u and first partial u is negative at $u = \sqrt{0.85m + 0.02}$. Then, we lower bound Equation 28 in u at $u = \sqrt{0.85m + 0.025}$. Last, we proceed by further lower bounding the resulting expression.

From lemmas B.139 and B.140, we know that Equation 28 is concave in u and first partial u is negative at $u = \sqrt{0.85m + 0.02}$.

So, we can lower bound Equation 28 in u by $u = \sqrt{0.85m + 0.025}$. After factoring $\exp\left(\frac{1}{2}q\left(m - \sqrt{0.85m + 0.025}\right)\right)$, we get the following.

$$\begin{aligned}&\left(0.3q^2 + 0.15m^2q + 0.6375mq + 0.01875q - 0.9mq\sqrt{0.85m + 0.025} - 2.092\right)\\ &- 1.046\exp\left(-\frac{1}{2}q\left(m - \sqrt{0.85m + 0.025}\right)\right) - 1.046\exp\left(\frac{1}{2}q\left(m - \sqrt{0.85m + 0.025}\right)\right)\end{aligned} \tag{67}$$

From the above, we lower bound $-0.9mq\sqrt{0.85m + 0.025}$ by linearizing $\sqrt{0.85m + 0.025}$ at $m = 0.25$. Furthermore, note that

$$\exp\left(-\frac{1}{2}q\left(m - \sqrt{0.85m + 0.025}\right)\right) + \exp\left(\frac{1}{2}q\left(m - \sqrt{0.85m + 0.025}\right)\right)$$

grows in q, so we can upper bound by $q = 3.13334m^2 + 3.75$.

Using lemma B.253, we can upper bound $\exp\left(-\frac{1}{2}q\left(m - \sqrt{0.85m + 0.025}\right)\right)$ by linearizing at $m = 0.15$. We can also replace $\exp\left(\frac{1}{2}(3.13334m^2 + 3.75)\left(m - \sqrt{0.85m + 0.025}\right)\right)$ by a local linear upper bound, $-0.05541m + 0.64$. To see this, consider the difference $-0.05541m + 0.64 - \exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m - \sqrt{0.85m + 0.025}\right)\right)$, which is concave in m (note lemma B.252). We evaluate the difference at $m = 0.15, 0.25$, which are positive.

With the above, Equation 67 can then be lower bounded by

$$0.3q^2 + \left(-0.63488m^2 + 0.39511m + 0.01875\right)q - 0.40861m - 4.34747 \tag{68}$$

From Equation 68, note that $-0.63488m^2 + 0.39511m + 0.01875$ can lower bounded by $-0.63488\left(0.25\right)^2 + 0.39511\left(0.15\right) + 0.01875$, which is positive.

So, we can lower bound Equation 68 in q by $q = 3m^2 + 3.67733$. The resulting expression (simplified by rounding down after five digits of precision) can be lower bounded by

$$0.79536\left(0.15\right)^4 + 1.18533\left(0.15\right)^3 + 4.34078\left(0.15\right)^2 + 1.04433\left(0.15\right) - 0.22170,$$

which is positive.

This completes the proof.

□

Proofs for Lemmas in Section B.5.3

Proof of Lemma B.126. Consider the first partial η of Equation 29 below, with $\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\right)^{-2}$ factored out.

$$\frac{q}{2}\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\left(\sqrt{c}+m\right)^2+q^2\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)$$
$$+2q\sqrt{c}\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)\left(\sqrt{c}+m\right)-4\eta q^3m^2\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\right)^{-2}\exp\left(qm\right) \tag{69}$$

From the above, note that we can upper bound

$$4\eta q^3m^2\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\right)^{-2}\exp\left(qm\right)$$

by

$$4\left(0.3\right)\left(1+\exp\left(\frac{1}{2}\left(4.84375\right)\left(0.15-\sqrt{0.2375}\right)\right)\right)^{-2}q^3m^2\exp\left(qm\right),$$

which we simplify (by rounding up after five digits of precision) to $0.57730m^2q^3\exp\left(qm\right)$.

From the lower bounded expression, we factor out $q\exp\left(\frac{1}{2}q\left(m+\sqrt{c}\right)\right)$, resulting in the following.

$$\frac{5}{2}c+3m\sqrt{c}+\frac{1}{2}m^2+q-0.57730m^2q^2\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)$$

We can then upper bound the above by

$$\frac{5}{2}\left(0.1175\right)+3m\sqrt{0.1175}+\frac{1}{2}m^2+q-0.57730m^2q^2\exp\left(\frac{1}{2}\left(3.74483\right)\left(0.25-\sqrt{0.1175}\right)\right),$$

which is concave in q.

So, we evaluate the above at both q ends $3.74483, 4.84375$. For $q=3.74483$, we can lower bound the resulting expression by

$$\frac{5}{2}\left(0.1175\right)+3\sqrt{0.1175}\left(0.15\right)+\left(3.74483\right)+\left(\frac{1}{2}-0.57730\left(3.74483\right)^2\exp\left(\frac{1}{2}\left(3.74483\right)\left(0.25-\sqrt{0.1175}\right)\right)\right)\left(0.25\right)^2,$$

which is positive.

For $q=4.84375$, we can lower bound the resulting expression by

$$\frac{5}{2}\left(0.1175\right)+3\sqrt{0.1175}\left(0.15\right)+\left(4.84375\right)+\left(\frac{1}{2}-0.57730\left(4.84375\right)^2\exp\left(\frac{1}{2}\left(3.74483\right)\left(0.25-\sqrt{0.1175}\right)\right)\right)\left(0.25\right)^2,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.127. Consider the partial η of Equation 30 below, with $\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\right)^{-2}$ factored out.

$$\frac{q}{2}\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\left(\sqrt{c}-m\right)^2+q^2\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)+2q\sqrt{c}\exp\left(\frac{1}{2}q\left(m-\sqrt{c}\right)\right)\left(\sqrt{c}-m\right) \tag{70}$$

Since $\left(\sqrt{c}-m\right)\geq 0.1175-0.25>0$, Equation 70 is positive.

This completes the proof. □

Proofs for Lemmas in Section B.5.3

Proof of Lemma B.128. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we will show that the second partial u of Equation 31 is negative. Consider the second partial u below, with $\exp\left(-\frac{1}{2}q\left(m+u\right)\right)$ factored out.

$$0.26mq^3 - 0.268q^3 + 0.375q^2u^2 - 3qu + 3 \tag{71}$$

We first upper bound the above in u by

$$0.26mq^3 - 0.268q^3 + 0.375q^2\left(0.65m + 0.025\right) - 3q\sqrt{0.65m + 0.02} + 3, \tag{72}$$

where we also upper bound $-3q\sqrt{0.65m+0.02}$ by $-3q\left(0.84417m + 0.2161\right)$ (note lemma B.241).

We write the resulting expression as follows (RU).

$$\left(\left(\left(0.26m - 0.268\right)q + \left(0.24375m + 0.00938\right)\right)q + \left(-2.53251m - 0.648\right)\right)q + 3 \tag{73}$$

Next, we iteratively upper bound the above. First, we upper $\left(0.26m - 0.268\right)q$ by $\left(0.26m - 0.268\right)\left(4.5m^2 + 4.544\right)$. So, $\left(0.26m - 0.268\right)\left(4.5m^2 + 4.544\right) + \left(0.24375m + 0.00938\right)$ simplifies (RU) to $1.17m^3 - 1.206m^2 + 1.42519m - 1.20841$.

Note that $1.17m^3 - 1.206m^2 + 1.42519m - 1.20841$ can be upper bounded by $1.17\left(0.25\right)^3 - 1.206\left(0.15\right)^2 + 1.42519\left(0.25\right) - 1.20841$, which is negative.

So, we upper bound

$$\left(\left(1.17m^3 - 1.206m^2 + 1.42519m - 1.20841\right)q + \left(-2.53251m - 0.648\right)\right)q + 3 \tag{74}$$

by

$$\left(\left(1.17m^3 - 1.206m^2 + 1.42519m - 1.20841\right)\left(4.5m^2 + 4.544\right) + \left(-2.53251m - 0.648\right)\right)\left(4.5m^2 + 4.544\right) + 3, \tag{75}$$

which simplifies (RU) to

$$23.6925m^7 - 24.4215m^6 + 76.70844m^5 - 73.79083m^4 + 71.04642m^3 - 77.23648m^2 + 17.91954m - 24.89566. \tag{76}$$

We can then upper bound the above by

$$23.6925\left(0.25\right)^7 - 24.4215\left(0.15\right)^6 + 76.70844\left(0.25\right)^5 - 73.79083\left(0.15\right)^4 + 71.04642\left(0.25\right)^3 - 77.23648\left(0.15\right)^2 + 17.91954\left(0.25\right) - 24.89566, \tag{77}$$

which is negative.

This completes the proof.

□

Proof of Lemma B.129. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we will show that the second partial u of Equation 32 is negative. Consider the second partial u below, scaled by $\exp\left(-\frac{1}{2}q\left(m+u\right)\right)$.

$$\left(q^2 + 6\right)\exp\left(-\frac{1}{2}q\left(m+u\right)\right) + 0.26mq^3 - 0.268q^3 + 0.375q^2u^2 + 3qu + 3 \tag{78}$$

We can then upper bound the above by

$$\begin{aligned}&\left(q^2 + 6\right)\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)\\ &+ 0.26mq^3 - 0.268q^3 + 0.375q^2\left(0.65m + 0.025\right) + 3q\sqrt{0.65m + 0.025} + 3.\end{aligned} \tag{79}$$

From the above, we can upper bound $3q\sqrt{0.65m+0.025}$ by linearizing $\sqrt{0.65m+0.025}$ at $m = 0.25$. We also upper bound $\exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ by $-1.23186m + 0.5032$. To see that this upper bound is valid, note the difference $-1.23186m + 0.5032 - \exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ is concave in m (by lemma B.246), so we evaluate the difference at $m = 0.15, 0.25$, which are both positive.

With the above, we can upper bound Equation 79 (RU) by

$$(0.26m-0.268)\,q^3+(-0.98811m+0.51258)\,q^2+(2.25167m+0.73613)\,q-7.39116m+6.0192. \tag{80}$$

We rewrite the above as

$$(((0.26m-0.268)\,q+(-0.98811m+0.51258))\,q+(2.25167m+0.73613))\,q-7.39116m+6.0192 \tag{81}$$

and upper bound iteratively in q.

First, we upper bound $(0.26m-0.268)\,q$ by $(0.26m-0.268)\left(4.5m^2+4.544\right)$. We can then simplify $(0.26m-0.268)\left(4.5m^2+4.544\right)+(-0.98811m+0.51258)$ to (RU) $1.17m^3-1.206m^2+0.19333m-0.70521$. Note that $1.17m^3-1.206m^2+0.19333m-0.70521$ is negative (i.e., can be upper bounded by $1.17\,(0.25)^3-1.206\,(0.15)^2+0.19333\,(0.25)-0.70521$), so we upper bound $\left(1.17m^3-1.206m^2+0.19333m-0.70521\right)q$ by

$$\left(1.17m^3-1.206m^2+0.19333m-0.70521\right)\left(4.5m^2+4.544\right). \tag{82}$$

Then,

$$\left(1.17m^3-1.206m^2+0.19333m-0.70521\right)\left(4.5m^2+4.544\right)+(2.25167m+0.73613) \tag{83}$$

can be simplified (RU) to

$$5.265m^5-5.427m^4+6.18647m^3-8.65350m^2+3.13017m-2.46834. \tag{84}$$

Note that the above can be upper bounded by

$$5.265\,(0.25)^5-5.427\,(0.15)^4+6.18647\,(0.25)^3-8.65350\,(0.15)^2+3.13017\,(0.25)-2.46834,$$

which is negative.

So, we can upper bound

$$\left(5.265m^5-5.427m^4+6.18647m^3-8.65350m^2+3.13017m-2.46834\right)q \tag{85}$$

by

$$\left(5.265m^5-5.427m^4+6.18647m^3-8.65350m^2+3.13017m-2.46834\right)\left(4.5m^2+4.544\right) \tag{86}$$

and

$$\left(5.265m^5-5.427m^4+6.18647m^3-8.65350m^2+3.13017m-2.46834\right)\left(4.5m^2+4.544\right)-7.39116m+6.0192 \tag{87}$$

can be simplified (RU) to

$$23.6925m^7-24.4215m^6+51.76328m^5-63.60103m^4+42.19709m^3-50.42903m^2+6.83234m-5.19693. \tag{88}$$

Note that the above can be upper bounded by

$$23.6925\,(0.25)^7-24.4215\,(0.15)^6+51.76328\,(0.25)^5-63.60103\,(0.15)^4+42.19709\,(0.25)^3-50.42903\,(0.15)^2+6.83234\,(0.25)-5.19693,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.130. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To begin the proof, we first note that the terms with a $\sqrt{0.65m + 0.02}$ factor in Equation 33 are positive. So, we can upper bound them by linearizing $\sqrt{0.65m + 0.02}$ at $m = 0.25$. Next, we upper bound the resulting expression in q. Then, we linearize the exponential terms and upper bound the resulting expression in m.

After linearizing the terms with a $\sqrt{0.65m + 0.02}$ factor in Equation 33, we get the following upper bound for Equation 33 (RU).

$$\begin{aligned}
&1.36077mq^2 + 0.23701q^2 + 4.56461m + 1.42205 \\
&+ \left(-0.52mq^2 + 0.536q^2 - 0.4875mq - 0.015q + 2.28231m + 0.71103\right) \exp\left(-\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right) \\
&+ \left(0.52mq^2 - 0.536q^2 + 0.4875mq + 0.015q + 2.28231m + 0.71103\right) \exp\left(\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right)
\end{aligned} \tag{89}$$

We know upper bound the resulting expression in q. First, we start with the non-exponential terms as follows.

$$\begin{aligned}
&1.36077m\left(4.7m^2 + 4.55\right)^2 + 0.23701\left(4.7m^2 + 4.55\right)^2 + 4.56461m + 1.42205 \\
&+\Bigg(\left(-0.52m + 0.536\right)\left(4.7m^2 + 4.55\right)^2 - 0.4875m\left(4.5m^2 + 4.544\right) \\
&\qquad - 0.015\left(4.5m^2 + 4.544\right) + 2.28231m + 0.71103\Bigg) \exp\left(-\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right) \\
&+\Bigg(\left(0.52m - 0.536\right)\left(4.5m^2 + 4.544\right)^2 + 0.4875m\left(4.7m^2 + 4.55\right) \\
&\qquad + 0.015\left(4.7m^2 + 4.55\right) + 2.28231m + 0.71103\Bigg) \exp\left(\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right)
\end{aligned} \tag{90}$$

Note that we simplify (RU) Equation 90 to the following.

$$\begin{aligned}
&30.05941m^5 + 5.23556m^4 + 58.20014m^3 + 10.13692m^2 + 32.73596m + 6.32875 \\
&+ \left(10.53m^5 - 10.854m^4 + 23.55717m^3 - 21.84975m^2 + 15.23737m - 10.28801\right) \exp\left(\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right) \\
&+\Bigg(-11.4868m^5 + 11.84024m^4 - 24.43415m^3 + 22.85722m^2 \\
&\qquad - 10.69819m + 11.73941\Bigg) \exp\left(-\frac{1}{2}q\left(m + \sqrt{0.65m + 0.02}\right)\right)
\end{aligned} \tag{91}$$

From the above, note that

$$-11.4868m^5 + 11.84024m^4 - 24.43415m^3 + 22.85722m^2 - 10.69819m + 11.73941 \tag{92}$$

can be lower bounded by

$$-11.4868\left(0.25\right)^5 + 11.84024\left(0.15\right)^4 - 24.43415\left(0.25\right)^3 + 22.85722\left(0.15\right)^2 - 10.69819\left(0.25\right) + 11.73941,$$

which is positive.

Furthermore,

$$10.53m^5 - 10.854m^4 + 23.55717m^3 - 21.84975m^2 + 15.23737m - 10.28801 \tag{93}$$

can be upper bounded by

$$10.53\,(0.25)^5 - 10.854\,(0.15)^4 + 23.55717\,(0.25)^3 - 21.84975\,(0.15)^2 + 15.23737\,(0.25) - 10.28801,$$

which is negative.

So, we can upper bound Equation 91 by upper bounding $\exp\left(-\frac{1}{2}q\left(m+\sqrt{0.65m+0.02}\right)\right)$ and lower bounding $\exp\left(\frac{1}{2}q\left(m+\sqrt{0.65m+0.02}\right)\right)$ in q by $q = 4.5m^2 + 4.544$.

Then, $\exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ can be upper bounded $-1.23186m + 0.5032$. To see this, note that the difference

$$-1.23186m + 0.5032 - \exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right) \tag{94}$$

is concave in m (note lemma B.246). So, we evaluate the difference at $m = 0.15, 0.25$, both of which are positive.

And, $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ can be lower bounded by by linearizing at $m = 0.15$ (note lemma B.247).

With the above, we can upper bound (RU) Equation 91 by

$$174.80813m^6 - 146.93139m^5 + 391.45639m^4 - 295.53629m^3 + 248.67089m^2 - 131.08565m + 3.46611 \tag{95}$$

Note that the above can be upper bounded by

$$174.80813\,(0.25)^6 - 146.93139\,(0.15)^5 + 391.45639\,(0.25)^4 - 295.53629\,(0.15)^3 + 248.67089\,(0.25)^2 - 131.08565\,(0.15) + 3.46611,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.131. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU. To prove the lemma, consider the second partial m of Equation 34 below (RU), scaled by $(0.65m+0.02)^{\frac{3}{2}}$,

$$a\sqrt{0.65m+0.02} + \left(b_1\sqrt{0.65m+0.02}+b_2\right)\exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right) + \left(c_1+c_2\sqrt{0.65m+0.02}\right)\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right), \tag{96}$$

where

$$a = 644.61209m^5 + 154.00319m^4 + 661.87763m^3 + 136.29703m^2 + 81.89900m + 2.41010$$

$$\begin{aligned} b_1 =& 138.60113m^8 - 76.03811m^7 + 196.33194m^6 - 297.40967m^5 + 157.49425m^4 \\ & - 216.78020x^3 + 72.48326m^2 - 33.55357m - 3.85040 \end{aligned}$$

$$\begin{aligned} b_2 =& 150.15123m^8 - 146.45518m^7 + 230.03642m^6 - 283.69821m^5 + 120.51123m^4 \\ & - 116.35300m^3 + 26.15418m^2 - 8.10804m - 1.4838 \end{aligned}$$

$$\begin{aligned} c_1 =& 150.15123m^8 - 146.45518m^7 + 255.30632m^6 - 141.51139m^5 + 37.89751m^4 \\ & - 33.02052m^3 - 14.19391m^2 - 7.21490m + 0.99659 \end{aligned}$$

$$\begin{aligned} c_2 =& 138.60113m^8 - 76.03811m^7 + 196.33194m^6 + 31.12633m^5 - 86.38055m^4 \\ & + 58.43869m^3 - 60.76304m^2 - 4.24589m - 2.81423 \end{aligned}$$

To proceed, we upper bound $a\sqrt{0.65m + 0.02}$ by linearizing $\sqrt{0.65m + 0.02}$ at $m = 0.15$.

We also note the following. First, b_1 can be upper bounded by

$$\begin{aligned} & 138.60113\,(0.25)^8 - 76.03811\,(0.15)^7 + 196.33194\,(0.25)^6 - 297.40967\,(0.15)^5 \\ & + 157.49425\,(0.25)^4 - 216.78020\,(0.15)^3 + 72.48326\,(0.25)^2 - 33.55357\,(0.15) - 3.85040, \end{aligned}$$

which is negative.

Second, c_2 can be upper bounded by

$$\begin{aligned} & 138.60113\,(0.25)^8 - 76.03811\,(0.15)^7 + 196.33194\,(0.25)^6 + 31.12633\,(0.25)^5 \\ & - 86.38055\,(0.15)^4 + 58.43869\,(0.25)^3 - 60.76304\,(0.15)^2 - 4.24589\,(0.15) - 2.81423, \end{aligned}$$

which is negative.

So, we can upper bound

$$b_1\sqrt{0.65m + 0.02} \quad (97)$$

and

$$c_2\sqrt{0.65m + 0.02} \quad (98)$$

by lower bounding $\sqrt{0.65m + 0.02}$ by $(0.84417m + 0.2161)$ (note lemma B.241).

The upper bound for $b_1\sqrt{0.65m + 0.02}$ simplifies (RU) to

$$\begin{aligned} & 117.00292m^9 + 115.89999m^8 + 2.85813m^7 + 21.37980m^6 - 214.98677m^5 - 28.46935m^4 \\ & - 101.98932m^3 + 13.48565m^2 - 18.60600m - 2.31555. \quad (99) \end{aligned}$$

Also, the upper bound for $c_2\sqrt{0.65m + 0.02}$ simplifies (RU) to

$$\begin{aligned} & 117.00292m^9 + 115.89999m^8 + 2.85813m^7 + 323.98994m^6 - 207.70797m^5 + 68.57151m^4 \\ & - 71.69209m^3 - 30.90297m^2 - 10.50770m + 0.38872. \quad (100) \end{aligned}$$

Note that the two upper bounds above are negative. To see this, note that Equation 99 can be upper bounded by

$$\begin{aligned} & 117.00292\,(0.25)^9 + 115.89999\,(0.25)^8 + 2.85813\,(0.25)^7 + 323.98994\,(0.25)^6 \\ & - 207.70797\,(0.15)^5 + 68.57151\,(0.25)^4 - 71.69209\,(0.15)^3 - 30.90297\,(0.15)^2 \\ & - 10.50770\,(0.15) + 0.38872, \end{aligned}$$

which is negative.

Also, Equation 100 can be upper bounded by

$$117.00292\,(0.25)^9 + 115.89999\,(0.25)^8 + 2.85813\,(0.25)^7 + 21.37980\,(0.25)^6 - 214.98677\,(0.15)^5 - 28.46935\,(0.15)^4 - 101.98932\,(0.15)^3 + 13.48565\,(0.25)^2 - 18.60600\,(0.15) - 2.31555,$$

which is negative.

So, we can further upper bound Equation 96 by linearizing $\exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ and $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ at $m = 0.15$ (note lemmas B.247 and B.246).

From the above upper bounds, we can upper bound Equation 96 by the following (RU).

$$1604.19545m^{10}+1753.20166m^9+201.76808m^8+4914.11632m^7-1937.44766m^6+1070.16598m^5 - 234.87802m^4 - 347.60750m^3 - 45.47807m^2 + 9.02659m - 0.45955 \quad (101)$$

Note that we rewrite the above as

$$\Big(\Big(\Big(\Big(\Big(\Big(\Big(\Big(1604.19545m + 1753.20166\Big)m + 201.76808\Big)m + 4914.11632\Big)m - 1937.44766\Big)m + 1070.16598\Big)m - 234.87802\Big)m - 347.60750\Big)m - 45.47807\Big)m + 9.02659\Big)m - 0.45955$$

and upper bound iteratively, resulting in the following

$$\Big(\Big(\Big(\Big(\Big(\Big(\Big(\Big(1604.19545\,(0.25) + 1753.20166\Big)\,(0.25) + 201.76808\Big)\,(0.25) + 4914.11632\Big)\,(0.25) - 1937.44766\Big)\,(0.15) + 1070.16598\Big)\,(0.25) - 234.87802\Big)\,(0.25) - 347.60750\Big)\,(0.15) - 45.47807\Big)\,(0.15) + 9.02659\Big)\,(0.15) - 0.45955,$$

which is negative.

This completes the proof.

□

Proofs for Lemmas in Section B.5.3

Proof of Lemma B.132. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we will show that the second partial u of Equation 35 is negative.

Consider the second partial derivative below, with $\exp\left(\frac{1}{2}q\,(m-u)\right)$ factored out.

$$-0.26mq^3 - 0.232q^3 + 0.125q^2x^2 - qx + 1 \quad (102)$$

We first upper bound in u with the following.

$$-0.26mq^3 - 0.232q^3 + 0.125q^2(0.65m + 0.025) - q\sqrt{0.65m + 0.02} + 1 \quad (103)$$

From the above, we can upper bound $-q\sqrt{0.65m+0.02}$ by $-q\left(0.84417m+0.2161\right)$ (note lemma B.241). Then, we can write the resulting expression as follows (RU).

$$\left(\left(\left(-0.26m-0.232\right)q+0.08125m+0.00313\right)q-0.84417m-0.2161\right)q+1 \tag{104}$$

From the above, we first upper bound $\left(-0.26m-0.232\right)q$ by $\left(-0.26m-0.232\right)\left(4.5m^2+4.544\right)$. With this substitution, we get that $\left(-0.26m-0.232\right)q+0.08125m+0.00313$ simplifies (RU) to $\left(-1.17m^3-1.044m^2-1.10019m-1.05107\right)$. So, the overall resulting expression

$$\left(\left(-1.17m^3-1.044m^2-1.10019m-1.05107\right)q-0.84417m-0.2161\right)q+1 \tag{105}$$

can be upper bounded in q by $q=\left(4.5m^2+4.544\right)$. This simplifies (RU) to

$$-23.6925m^7-21.141m^6-70.12714m^5-63.97954m^4-72.95018m^3-65.51339m^2-26.55254m-21.68437 \tag{106}$$

This completes the proof. □

Proof of Lemma B.133. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we will show that the second partial u of Equation 36 is negative.

Consider the second partial derivative below (after scaling by $\exp\left(\frac{1}{2}q\left(m-u\right)\right)$).

$$\left(q^2+2\right)\exp\left(\frac{1}{2}q\left(m-u\right)\right)+\left(-0.26mq^3-0.232q^3+0.125q^2u^2+qu+1\right) \tag{107}$$

We first upper bound with the following.

$$\begin{aligned}&\left(q^2+2\right)\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)\\&+\left(-0.26mq^3-0.232q^3+0.125q^2\left(0.65m+0.025\right)+q\sqrt{0.65m+0.025}+1\right)\end{aligned} \tag{108}$$

Then, we upper bound $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$ by $0.13070m+0.6195$. To see that $0.13070m+0.6195$ is a local upper bound, note that the difference $0.13070m+0.6195-\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$ is concave in m (note lemma B.244). So, we evaluate the difference at $m=0.15,0.25$, both of which are positive.

With the above substitution, we write the resulting expression (RU) as follows.

$$\left(\left(\left(-0.26m-0.232\right)q+\left(0.21195m+0.62263\right)\right)q+\sqrt{0.65m+0.025}\right)q+0.2614m+2.239 \tag{109}$$

Now, we iteratively upper bound the above in q. We start by upper bounding $\left(-0.26m-0.232\right)q$ by $\left(-0.26m-0.232\right)\left(4.5m^2+4.544\right)$. Then, we simplify

$$\left(-0.26m-0.232\right)\left(4.5m^2+4.544\right)+\left(0.21195m+0.62263\right)$$

to $-1.17m^3-1.044m^2-0.96949m-0.43157$. Then, $\left(-1.17m^3-1.044m^2-0.96949m-0.43157\right)q$ can be upper bounded by

$$\left(-1.17m^3-1.044m^2-0.96949m-0.43157\right)\left(4.5m^2+4.544\right).$$

We also linearize $\sqrt{0.65m+0.025}$ in Equation 109 at $m=0.25$, yielding the resulting expression (RU).

$$\left(-5.265m^5-4.698m^4-9.67918m^3-6.68600m^2-3.65480m-1.71568\right)q+0.2614m+2.239 \tag{110}$$

We then upper bound the above by $q=\left(4.5m^2+4.544\right)$, resulting in the following negative expression in m (RU).

$$-23.6925m^7-21.141m^6-67.48047m^5-51.43471m^4-60.42879m^3-38.10174m^2-16.34601m-5.55704$$

This completes the proof.

□

Proof of Lemma B.134. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we first note that the $\sqrt{0.65m + 0.02}$ terms are positive. So, we linearize $\sqrt{0.65m + 0.02}$ at $m = 0.25$, resulting in the following (RU).

$$\begin{aligned}
&(0.16077m + 0.23701)\, q^2 + 1.52154m + 0.47402 \\
&+ \left((-0.52m - 0.464)\, q^2 + (0.1625m + 0.005)\, q + 0.76077m + 0.23701\right) \exp\left(-\frac{1}{2} q \left(m - \sqrt{0.65m + 0.02}\right)\right) \\
&+ \left((0.52m + 0.464)\, q^2 + (-0.1625m - 0.005)\, q + 0.76077m + 0.23701\right) \exp\left(\frac{1}{2} q \left(m - \sqrt{0.65m + 0.02}\right)\right)
\end{aligned} \tag{111}$$

We can further upper bound the above as follows. First, we upper bound $(0.16077m + 0.23701)\, q^2$ by $(0.16077m + 0.23701)\left(4.7m^2 + 4.55\right)^2$.

Then, we rewrite $(-0.52m - 0.464)\, q^2 + (0.1625m + 0.005)\, q$ as $((-0.52m - 0.464)\, q + (0.1625m + 0.005))\, q$ and $(0.52m + 0.464)\, q^2 + (-0.1625m - 0.005)\, q$ as $((0.52m + 0.464)\, q + (-0.1625m - 0.005))\, q$.

We start with $((-0.52m - 0.464)\, q + (0.1625m + 0.005))\, q$, which we upper bound by $\left((-0.52m - 0.464)\left(4.5m^2 + 4.544\right) + (0.1625m + 0.005)\right) q$. Then, the resulting expression (RU) $\left(-2.34m^3 - 2.088m^2 - 2.20038m - 2.10341\right) q$ can be upper bounded by $\left(-2.34m^3 - 2.088m^2 - 2.20038m - 2.10341\right)\left(4.5m^2 + 4.544\right)$.

For $((0.52m + 0.464)\, q + (-0.1625m - 0.005))\, q$, we start with the upper bound $\left((0.52m + 0.464)\left(4.7m^2 + 4.55\right) + (-0.1625m - 0.005)\right) q$. The resulting expression (RU) $\left(2.444m^3 + 2.1808m^2 + 2.2035m + 2.1062\right) q$ can be further upper bounded by $\left(2.444m^3 + 2.1808m^2 + 2.2035m + 2.1062\right)\left(4.7m^2 + 4.55\right)$.

We can then upper bound the resulting expression by the following (RU).

$$\begin{aligned}
&3.55141m^5 + 5.23556m^4 + 6.87614m^3 + 10.13692m^2 + 4.84989m + 5.38072 \\
&+ \Bigg(-10.53m^5 - 9.396m^4 - 20.53467m^3 - 18.95321m^2 \\
&\qquad - 9.23775m - 9.32088\Bigg) \exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right) \\
&+ \Bigg(11.4868m^5 + 10.24976m^4 + 21.47665m^3 + 19.82178m^2 \\
&\qquad + 10.78670m + 9.82022\Bigg) \exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)
\end{aligned} \tag{112}$$

To proceed, we upper bound $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ by $0.13070m + 0.6195$ and lower bound $\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ by $-0.31361m + 1.6118$. To see that these upper/lower bounds are valid, consider the difference of the upper/lower bound with the original exponential expression, which is concave/convex. Evaluate the concave/convex difference at $m = 0.15, 0.25$, which are both positive/negative.

The resulting expression with these linear upper/lower bounds is as follows (RU).

$$4.80364m^6 - 2.01844m^5 + 5.68769m^4 - 4.38223m^3 - 3.82539m^2 + 0.84947m - 3.55904 \tag{113}$$

The above can be upper bounded by

$$4.80364\,(0.25)^6 - 2.01844\,(0.15)^5 + 5.68769\,(0.25)^4 - 4.38223\,(0.15)^3 - 3.82539\,(0.15)^2 + 0.84947\,(0.25) - 3.55904,$$

which is negative.

This completes the proof.

□

Proofs for Lemmas in Section B.5.3

Proof of Lemma B.135. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, consider the second partial u below (with $\exp\left(\frac{1}{2}q\left(m+u\right)\right)$ factored out).

$$\begin{aligned}&\left(0.1875q^4+q^2\right)u^2+\left(0.225mq^4+1.5q^3+8q\right)u\\&\quad+\left(0.0375m^2q^4+0.075q^5-0.477q^3+0.9mq^3+1.5q^2+8\right)\\&\quad+\left(2q^2u^2+8qu-0.954q^3+4\right)\exp\left(\frac{1}{2}q\left(m+u\right)\right)\end{aligned}\tag{114}$$

To prove the above is positive, note that $2q^2u^2+8qu-0.954q^3+4$ can be upper bounded by

$$\begin{aligned}&\left(\left(-0.954\left(3\left(0.15\right)^2+3.67733\right)+2\left(\sqrt{0.85\left(0.25\right)+0.025}\right)^2\right)\left(3\left(0.15\right)^2+3.67733\right)\right.\\&\quad\left.+8\left(\sqrt{0.85\left(0.25\right)+0.025}\right)\right)\left(3\left(0.15\right)^2+3.67733\right)+4,\end{aligned}$$

which is negative.

Then, we can lower bound Equation 114 by

$$\left(0.1875q^4+q^2\right)u^2+\left(0.225mq^4+1.5q^3+8q\right)u+\left(0.0375m^2q^4+0.075q^5-0.477q^3+0.9mq^3+1.5q^2+8\right)$$
$$+\left(2q^2u^2+\right.$$

(115)

Furthermore, we locally upper bound

$$\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m+\sqrt{0.85m+0.025}\right)\right)$$

by $14.75125m+0.596$. To see that the linear function is a local upper bound, note that

$$\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m+\sqrt{0.85m+0.025}\right)\right)$$

is convex in m (by lemma B.251), so

$$14.75125m+0.596-\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m+\sqrt{0.85m+0.025}\right)\right)$$

is concave in m. Evaluated at both m ends $0.15, 0.25$, the difference is positive.

Using the linear function, we can lower bound Equation 115 by

$$\begin{aligned}&\left(0.1875q^4+29.5025mq^2+2.192q^2\right)u^2+\left(0.225mq^4+1.5q^3+118.01mq+12.768q\right)u\\&+\left(0.0375m^2q^4+0.075q^5-1.04559q^3-13.17270mq^3+1.5q^2+59.005m+10.384\right)\end{aligned}\tag{116}$$

Note that the above is growing in u, so we can lower bound by $u=\sqrt{0.85m+0.02}$, resulting in the following (RD).

$$\begin{aligned}&\left(0.075q^5+0.0375m^2q^4+0.15937mq^4+0.00375q^4-1.04559q^3-13.17270mq^3\right.\\&\quad\left.+25.07712m^2q^2+2.45325mq^2+1.54384q^2+59.005m+10.384\right)\\&+\left(0.225mq^4+1.5q^3+118.01mq+12.768q\right)\sqrt{0.85m+0.02}\end{aligned}\tag{117}$$

From the above, we can create a local lower bound for $\sqrt{0.85m + 0.02}$, $0.98125m + 0.2368$, by lemma B.243.

With the lower bound for $\sqrt{0.85m + 0.02}$, we can lower bound Equation 117 by

$$0.075q^5 + \left(0.25828m^2 + 0.21265m + 0.00375\right) q^4 + \left(-0.69039 - 11.70083m\right) q^3 + \left(25.07712m^2 + 2.45325m + 1.54384\right) q^2 + \left(115.79731m^2 + 40.47336m + 3.02346\right) q + 59.005m + 10.384 \tag{118}$$

Note that, we rewrite the above as follows

$$\Bigg(\Bigg(\Bigg(\Bigg(0.075q + \left(0.25828m^2 + 0.21265m + 0.00375\right)\Bigg) q + \left(-0.69039 - 11.70083m\right)\Bigg) q + \left(25.07712m^2 + 2.45325m + 1.54384\right)\Bigg) q + \left(115.79731m^2 + 40.47336m + 3.02346\right)\Bigg) q + 59.005m + 10.384 \tag{119}$$

to further lower bound it.

We begin by lower bounding $\Bigg(0.075q + \left(0.25828m^2 + 0.21265m + 0.00375\right)\Bigg) q$ by

$$\left(0.075\left(3m^2 + 3.67733\right) + \left(0.25828m^2 + 0.21265m + 0.00375\right)\right)\left(3m^2 + 3.67733\right),$$

where we simplify (RD) the above by

$$1.44984m^4 + 0.63795m^3 + 2.61580m^2 - 10.91885m + 0.33757.$$

Note that

$$1.44984m^4 + 0.63795m^3 + 2.61580m^2 - 10.91885m + 0.33757$$

can be upper bounded by

$$\left(1.44984\left(0.25\right)^4 + 0.63795\left(0.25\right)^3 + 2.61580\left(0.25\right)^2 - 10.91885\left(0.15\right) + 0.33757\right),$$

which is negative.

So, we can lower bound

$$\left(1.44984m^4 + 0.63795m^3 + 2.61580m^2 - 10.91885m + 0.33757\right) q$$

by

$$\left(1.44984m^4 + 0.63795m^3 + 2.61580m^2 - 10.91885m + 0.33757\right)\left(3.13334m^2 + 3.75\right),$$

where we can then simplify (RD)

$$\left(\left(1.44984m^4 + 0.63795m^3 + 2.61580m^2 - 10.91885m + 0.33757\right)\left(3.13334m^2 + 3.75\right) + \left(25.07712m^2 + 2.45325m + 1\right.\right.$$

by

$$4.54284m^6 + 1.99891m^5 + 13.63309m^4 - 31.82016m^3 + 35.94409m^2 - 38.49244m + 2.80972.$$

Note that

$$4.54284m^6 + 1.99891m^5 + 13.63309m^4 - 31.82016m^3 + 35.94409m^2 - 38.49244m + 2.80972$$

can be upper bounded by

$$\Bigg(4.54284\left(0.25\right)^6 + 1.99891\left(0.25\right)^5 + 13.63309\left(0.25\right)^4 - 31.82016\left(0.15\right)^3 + 35.94409\left(0.25\right)^2 - 38.49244\left(0.15\right) + 2.80972\Bigg),$$

which is negative.

So,

$$\left(4.54284m^6 + 1.99891m^5 + 13.63309m^4 - 31.82016m^3 + 35.94409m^2 - 38.49244m + 2.80972\right) q$$

can be lower bounded by

$$\left(4.54284m^6 + 1.99891m^5 + 13.63309m^4 - 31.82016m^3 + 35.94409m^2 - 38.49244m + 2.80972\right) \left(3.13334m^2 + 3.75\right),$$

where we can then simplify (RD)

$$\begin{aligned}&\left(115.79731m^2 + 40.47336m + 3.02346\right)\\ &+\left(4.54284m^6 + 1.99891m^5 + 13.63309m^4 - 31.82016m^3 + 35.94409m^2 - 38.49244m + 2.80972\right) \left(3.13334m^2 + 3.75\right)\end{aligned}$$

by

$$14.23426m^8+6.26326m^7+59.75275m^6-92.20747m^5+163.74914m^4-239.93551m^3+259.39145m^2 - 103.87329m + 13.55991$$

Note that

$$14.23426m^8+6.26326m^7+59.75275m^6-92.20747m^5+163.74914m^4-239.93551m^3+259.39145m^2 - 103.87329m + 13.55991$$

is positive by lemma B.136.

Then, we lower bound

$$\left(14.23426m^8+6.26326m^7+59.75275m^6-92.20747m^5+163.74914m^4-239.93551m^3+259.39145m^2 - 103.87329m + 13.55991\right) q + 59.005m + 10.384$$

by

$$\left(14.23426m^8+6.26326m^7+59.75275m^6-92.20747m^5+163.74914m^4-239.93551m^3+259.39145m^2 - 103.87329m + 13.55991\right) \left(3(0.15)^2 + 3.67733\right) + 59.005m + 10.384,$$

where we can simplify (RD)

$$53.30488m^8+23.45484m^7+223.76389m^6-345.30130m^5+613.21269m^4-898.51770m^3+971.37688m^2 - 329.98282m + 61.16355.$$

We can then lower bound

$$53.30488m^8+23.45484m^7+223.76389m^6-345.30130m^5+613.21269m^4-898.51770m^3+971.37688m^2 - 329.98282m + 61.16355.$$

by

$$53.30488\,(0.15)^8+23.45484\,(0.15)^7+223.76389\,(0.15)^6-345.30130\,(0.25)^5+613.21269\,(0.15)^4 + \left(-898.51770\,(0.25) + 971.37688\right) x^2 - 329.98282x + 61.16355,$$

which simplifies (RD) to $746.74745m^2 - 329.98282m + 61.13938$.

Since the above is convex, we find it attains its minimum at $m = \frac{329.98282}{746.74745(2)}$, where $746.74745m^2 - 329.98282m + 61.13938$ evaluated at the minimum is positive.

This completes the proof. □

Proof of Lemma B.136. To prove the lemma, we first lower bound

$$14.23426m^8+6.26326m^7+59.75275m^6-92.20747m^5+163.74914m^4-239.93551m^3+259.39145m^2 - 103.87329m + 13.55991$$

by

$$14.23426\,(0.15)^8+6.26326\,(0.15)^7+59.75275\,(0.15)^6-92.20747\,(0.25)^5+(163.74914\,(0.15) - 239.93551)\,m^3 + 259.39145m^2 - 103.87329m + 13.55991,$$

which simplifies (by rounding down after five digits of precision) to $-215.37314m^3 + 259.39145m^2 - 103.87329m + 13.47055$.

We proceed by lower bounding $-215.37314m^3 + 259.39145m^2$ by $(-215.37314\,(0.25) + 259.39145)\,m^2$. Then, $-215.37314m^3 + 259.39145m^2 - 103.87329m + 13.47055$ can be lower bounded by $205.54816m^2 - 103.87329m + 13.47055$.

Note that the above is convex in m and attains its minimum at $m = \frac{103.87329}{205.54816(2)}$. Evaluating $205.54816m^2 - 103.87329m + 13.47055$ is positive.

This completes the proof. □

Proof of Lemma B.137. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we begin by lower bounding Equation 39.

First, note that the $\sqrt{0.85m + 0.02}$ terms, excluding the exponential terms, are positive. So, by lemma B.243, we can lower bound those terms by replacing $\sqrt{0.85m + 0.02}$ with $0.98125m + 0.2368$.

Then, we factor out a $\exp\left(-\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right)$ term, resulting in the following lower bound.

$$\begin{aligned}&(3.925m + 0.9472)\exp\left(-\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right)\\ &+ (3.925m + 0.9472 + (1.7m + 0.04 - 0.954q)\,q)\exp\left(\frac{1}{2}q\left(m + \sqrt{0.85m + 0.02}\right)\right)\\ &+ \left(7.85m + 1.8944 + \left(0.15q^3 + 0.51656m^2q^2 + 0.42531mq^2 + 0.0075q^2 + 2.37187mq - 0.5988q + 1.7m + 0.04\right)q\right)\end{aligned} \tag{120}$$

From the above, note that $3.925m + 1.7mq + 0.9472 + 0.04q - 0.954q^2$ is negative. To see this, consider the upper bound

$$3.925\,(0.25) + 1.7\,(0.25)\left(3.13334\,(0.25)^2 + 3.75\right) + 0.9472 + 0.04\left(3.13334\,(0.25)^2 + 3.75\right) - 0.954\left(3\,(0.15)^2 + 3.67733\right)^2,$$

which is negative.

So, we can lower bound Equation 120 by

$$(3.925m + 0.9472) \exp\left(-\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$$
$$+ \left(3.925m + 0.9472 + (1.7m + 0.04 - 0.954q)\, q\right) \exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$$
$$+ \left(7.85m + 1.8944 + \left(0.15q^3 + 0.51656m^2q^2 + 0.42531mq^2 + 0.0075q^2 + 2.37187mq - 0.5988q + 1.7m + 0.04\right) q\right). \tag{121}$$

From the above, note that $\exp\left(-\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ and $\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ are convex by lemmas B.248 and B.249.

To lower bound Equation 121, we lower bound $\exp\left(-\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ by linearizing at $m = 0.25$ and upper bounding $\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ locally by $14.66180m + 0.575$.

To see that $14.66180m + 0.575$ is a local upper bound, note that $14.66180m + 0.575 - \exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ is concave in m, so we evaluate the difference at $m = 0.15, 0.25$, which are both positive.

With these bounds applied to $\exp\left(-\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ and $\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$, we obtain the following lower bound for Equation 121.

$$53.5804m^2 + 24.95467m + 2.90178 + \Big(0.15q^3 + 0.51656m^2q^2 + 0.42531mq^2 + 0.0075q^2 - 11.61549mq$$
$$- 1.14735q + 24.92506m^2 + 3.26397m + 0.063\Big) q \tag{122}$$

To further the above, we begin by rewriting it as

$$53.5804m^2 + 24.95467m + 2.90178 + \Big(\Big(\Big(\Big(0.15q + 0.51656m^2 + 0.42531m + 0.0075\Big) q$$
$$- 11.61549m - 1.14735\Big) q + 24.92506m^2 + 3.26397m + 0.063\Big) q.$$

We begin by lower bounding $\Big(0.15q + 0.51656m^2 + 0.42531m + 0.0075\Big) q$ by

$$\left(0.15\left(3m^2 + 3.67733\right) + 0.51656m^2 + 0.42531m + 0.0075\right)\left(3m^2 + 3.67733\right).$$

So, after simplifying the resulting expression (RD), we get the following.

$$53.5804m^2 + 24.95467m + 2.90178 + \Big(\Big(2.89968m^4 + 1.27593m^3 + 5.23163m^2 - 10.05149m + 0.9086\Big) q$$
$$+ 24.92506m^2 + 3.26397m + 0.063\Big) q \tag{123}$$

From the above, note that $2.89968m^4 + 1.27593m^3 + 5.23163m^2 - 10.05149m + 0.9086$ can be upper bounded by

$$2.89968\left(0.25\right)^4 + 1.27593\left(0.25\right)^3 + 5.23163\left(0.25\right)^2 - 10.05149\left(0.15\right) + 0.9086,$$

which is negative.

So, we can lower bound $\left(2.89968m^4 + 1.27593m^3 + 5.23163m^2 - 10.05149m + 0.9086\right) q$ by

$$\left(2.89968m^4 + 1.27593m^3 + 5.23163m^2 - 10.05149m + 0.9086\right)\left(3.13334m^2 + 3.75\right).$$

With this lower bound, the resulting expression (RD) is shown below.

$$53.5804m^2 + 24.95467m + 2.90178 + \Big(9.08568m^6 + 3.99792m^5 + 27.26627m^4 - 26.71000m^3 + 47.39062m^2 - 34.42912m + 3.47025\Big)q \tag{124}$$

From the above, note that

$$9.08568m^6 + 3.99792m^5 + 27.26627m^4 - 26.71000m^3 + 47.39062m^2 - 34.42912m + 3.47025$$

can be upper bounded by

$$9.08568\,(0.25)^6 + 3.99792\,(0.25)^5 + 27.26627\,(0.25)^4 - 26.71000\,(0.15)^3 + 47.39062m^2 - 34.42912m + 3.47025.$$

Simplifying the above (RD) yields $47.39062m^2 - 34.42912m + 3.49274$, which is convex in m. Evaluating $47.39062m^2 - 34.42912m + 3.49274$ at both m ends is negative.

Then, we can lower bound Equation 124 by

$$\Big(9.08568m^6 + 3.99792m^5 + 27.26627m^4 - 26.71000m^3 + 47.39062m^2 - 34.42912m + 3.47025\Big)\big(3.13334m^2 + 3.75\big) + 53.5804m^2 + 24.95467m + 2.90178 \tag{125}$$

which we can simplify (RD) to

$$28.46852m^8 + 12.52684m^7 + 119.50579m^6 - 68.69932m^5 + 250.73943m^4 - 208.04064m^3 + 242.16869m^2 - 104.15454m + 15.91521. \tag{126}$$

We can lower bound the above by

$$28.46852\,(0.15)^8 + 12.52684\,(0.15)^7 + 119.50579\,(0.15)^6 - 68.69932\,(0.25)^5 + 250.73943\,(0.15)^4 - 208.04064\,(0.25)^3 + 242.16869m^2 - 104.15454m + 15.91521, \tag{127}$$

which simplifies (RD) to $242.16869m^2 - 104.15454m + 12.72581$.

Note that $242.16869m^2 - 104.15454m + 12.72581$ is convex in m, so we can lower bound at the minimum point $m_{min} = \left(\frac{104.15454}{(2)242.16869}\right)$, which is positive.

This completes the proof.

□

Proof of Lemma B.138. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we will show that the first partial q of Equation 40 is positive. Consider the first partial q below (RD).

$$\begin{aligned}&\big(0.9q^2 + 2.06624m^2q + 1.70124mq + 0.03q - 1.908\big)\\ &+ \big(40.15772m^5 + 4.79969m^4 + 51.02217m^3 + 4.41417m^2 + 3.30303m + 0.41884\big)\exp\left((-0.99062m - 0.1184)\,q\right)\\ &+ \big(-2.81198m^3 + 1.20785m^2 - 3.10520m - 0.46313\big)\exp\left((0.94071m + 0.13092)\,q\right)\end{aligned} \tag{128}$$

To proceed, we lower bound the above as follows. We first lower bound

$$0.9q^2 + 2.06624m^2q + 1.70124mq + 0.03q - 1.908 \tag{129}$$

in m by $m = 0.15$,the resulting expression is lower bounded in q by $q = 3m^2 + 3.67733$, and subsequently lower bounded in m by $m = 0.15$, which simplifies (RD) to 11.95539.

Next, we lower bound

$$\left(40.15772m^5 + 4.79969m^4 + 51.02217m^3 + 4.41417m^2 + 3.30303m + 0.41884\right) \tag{130}$$

in m by $m = 0.15$.

Then, we lower bound

$$\left(-2.81198m^3 + 1.20785m^2 - 3.10520m - 0.46313\right) \tag{131}$$

in m by

$$-2.81198\left(0.25\right)^3 + 1.20785\left(0.15\right)^2 - 3.10520\left(0.25\right) - 0.46313,$$

which simplifies (RD) to -1.25620.

To further lower bound Equation 128, we replace $\exp\left(\left(0.94071m + 0.13092\right)q\right)$ by $\exp\left(\left(0.94071\left(0.25\right) + 0.13092\right)\left(3.13334\left(0.25\right)^2 + 3.75\right)\right)$ and $\exp\left(\left(-0.99062m - 0.1184\right)q\right)$ by $\exp\left(\left(-0.99062\left(0.25\right) - 0.1184\right)\left(3.13334\left(0.25\right)^2 + 3.75\right)\right)$.

The resulting expression of these lower bounds is positive.

This completes the proof

□

Proofs for Lemmas in Section B.5.3

Proof of Lemma B.139. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, consider the second partial u of Equation 28 below, after factoring out $q\exp\left(\frac{1}{2}q\left(m-u\right)\right)$.

$$\begin{aligned}\Bigg(&\left(0.075q + 0.1875u^2 - 0.225mu + 0.0375m^2\right)q - 1.5u - 0.523\\&+ 0.9m - 1.046\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m-u\right)\right)\Bigg)q + 1.5\end{aligned} \tag{132}$$

To upper bound the above, we begin by lower bonding $\exp\left(\frac{1}{2}q\left(m-u\right)\right)$ by $\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m-u\right)\right)$.

Next, note that $0.1875u^2 - 0.225mu + 0.0375m^2$ is positive. To see this, we lower bound $0.1875u^2 - 0.225mu + 0.0375m^2$ in u by $u = \sqrt{0.85m + 0.02}$. The resulting expression (simplified by rounding down after five digits of precision) can be lower bounded by

$$0.0375\left(0.15\right)^2 + \left(0.15937 - 0.225\sqrt{0.85\left(0.25\right) + 0.025}\right)\left(0.15\right) + 0.00375,$$

which is positive.

So, we can upper bound $\left(0.075q + 0.1875u^2 - 0.225mu + 0.0375m^2\right)q$ by

$$\left(0.075\left(3.13334m^2 + 3.75\right) + 0.1875x^2 - 0.225mx + 0.0375m^2\right)\left(3.13334m^2 + 3.75\right).$$

After simplifying (RU), we get the following expression.

$$\begin{aligned}\Bigg(&0.58751m^2u^2 + 0.70313u^2 - 0.70500m^3u - 0.84375mu - 1.5u + 0.85387m^4\\&+ 1.90317m^2 + 0.9m + 0.53169 - 1.046\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m-u\right)\right)\Bigg)q + 1.5\end{aligned} \tag{133}$$

We can then upper bound Equation 133 in u with

$$\Bigg(0.58751m^2\left(\sqrt{0.85m+0.025}\right)^2+0.70313\left(\sqrt{0.85m+0.025}\right)^2-0.70500m^3\left(\sqrt{0.85m+0.02}\right)$$
$$-0.84375m\left(\sqrt{0.85m+0.02}\right)-1.5\left(\sqrt{0.85m+0.02}\right)+0.85387m^4+1.90317m^2$$
$$+0.9m+0.53169-1.046\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m-\sqrt{0.85m+0.025}\right)\right)\Bigg)q+1.5, \tag{134}$$

that we can further upper bound by replacing the $\sqrt{0.85m+0.025}$ terms by $(0.98125m+0.2368)$ (note lemma B.243), which results in the following (RU).

$$\Bigg(0.16209m^4+0.33244m^3+1.08993m^2-0.17401m+0.19407$$
$$-1.046\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m-\sqrt{0.85m+0.025}\right)\right)\Bigg)q+1.5 \tag{135}$$

To proceed, note that $\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m-\sqrt{0.85m+0.025}\right)\right)$ is convex in m (by lemma B.252), so we can upper bound Equation 135 by linearizing $\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m-\sqrt{0.85m+0.025}\right)\right)$ at $m=0.25$.

Applying the above linearization results in the following upper bound.

$$\left(0.16209m^4+0.33244m^3+1.08993m^2-0.21753m-0.44994\right)q+1.5 \tag{136}$$

From the above, note that $0.16209m^4+0.33244m^3+1.08993m^2-0.21753m-0.44994$ can be upper bounded by

$$0.16209\left(0.25\right)^4+0.33244\left(0.25\right)^3+1.08993\left(0.25\right)^2-0.21753\left(0.15\right)-0.44994,$$

which is negative.

So, we can upper bound Equation 136 by

$$\left(0.16209m^4+0.33244m^3+1.08993m^2-0.21753m-0.44994\right)\left(3m^2+3.67733\right)+1.5, \tag{137}$$

which we can simplify (RU) to

$$0.48627m^6+0.99732m^5+3.86585m^4+0.56991m^3+2.65822m^2-0.79992m-0.15457. \tag{138}$$

The above can be upper bounded by

$$0.48627\left(0.25\right)^6+0.99732\left(0.25\right)^5+3.86585\left(0.25\right)^4+0.56991\left(0.25\right)^3+2.65822\left(0.25\right)^2-0.79992\left(0.15\right)-0.15457,$$

which is negative.

This completes the proof. □

Proof of Lemma B.140. To show Equation 41 is negative, we first upper bound the $\sqrt{0.85m+0.02}$ terms by linearizing $\sqrt{0.85m+0.02}$ at $m=0.25$, upper bound $\exp\left(\frac{1}{2}q\left(m-\sqrt{0.85m+0.02}\right)\right)$ by $\exp\left(\frac{1}{2}\left(3m^2+3.67733\right)\left(m-\sqrt{0.85m+0.02}\right)\right)$, and replace $\exp\left(\frac{1}{2}\left(3m^2+3.67733\right)\left(m-\sqrt{0.85m+0.02}\right)\right)$ with a local linear upper bound, $-0.06688m+0.6552$. To see that $-0.06688m+0.6552$ is a local upper bound for $\exp\left(\frac{1}{2}\left(3m^2+3.67733\right)\left(m-\sqrt{0.85m+0.02}\right)\right)$, note that the difference $-0.06688m+0.6552-\exp\left(\frac{1}{2}\left(3m^2+3.67733\right)\left(m-\sqrt{0.85m+0.02}\right)\right)$ is concave in m (by lemma B.254). We evaluate the difference at $m=0.15,0.25$, which are positive.

Applying the above to Equation 41 yields the following upper bound (simplified by rounding up after five digit of precision).

$$-0.15q^2+\left(0.32164m^2-0.20092m-0.0075\right)q+0.35217m+2.12410 \tag{139}$$

From the above, note that $0.32164m^2 - 0.20092m - 0.0075$ can be upper bounded by $0.32164\,(0.25)^2 - 0.20092\,(0.15) - 0.0075$, which is negative.

So, we can upper bound Equation 139 in q by $q = \left(3m^2 + 3.67733\right)$.

The resulting expression (simplified by rounding up after five digits of precision) can then be upper bounded by

$$-0.38508\,(0.15)^4 - 0.60276\,(0.15)^3 - 2.14929\,(0.15)^2 - 0.38667\,(0.15) + 0.06815,$$

which is negative.

This completes the proof. □

B.5.4 Sufficient Condition for Perpetual Oscillation in Loss and Sharpness

In this section, we prove Theorem 5.4, where we provide a sufficient condition for the perpetual loss and sharpness oscillation.

We first show regions in the local phase space surrounding $(\tilde{q}, \tilde{c})$ where the sharpness/loss will increase/decrease.

To see this, consider the phase space about the fixed point $(\tilde{q}, \tilde{c})$ that we divide up into four subregions based on where Equations 7 and 8 are positive/negative, shown in Figure 23

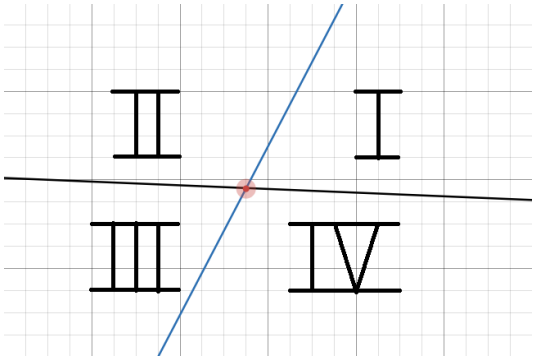


Figure 23: In this figure, we show the phase space about the fixed point of interest $(\tilde{q}, \tilde{c})$ (denoted by the red point in the center) divided up based on where Equations 7 (black line) and 8 (blue line) equal zero. We define region 1 where Equations 7 and 8 are negative. In region 2, Equation 7 is negative, but Equation 8 is positive. In region 3, both Equations 7 and 8 are positive. In region 4, Equation 7 is positive, but Equation 8 is negative.

To understand how the sharpness behaves, first note that we have an exact formula for the sharpness in the asymptotic regime (sharpness approximation 3 derived in Appendix B.2.2), given in Equation 140,

$$\frac{1}{4}\left(\psi_1 q\left(m+\sqrt{c}\right)^2 + \psi_2 q\left(m-\sqrt{c}\right)^2 - \phi_{1,t}\left(m+\sqrt{c}\right) - \phi_{2,t}\left(m-\sqrt{c}\right) + 2\left(\psi_1+\psi_2\right)q^2\right) + \frac{1}{2}\sqrt{\begin{array}{l}\left(\frac{1}{4}\left(\psi_1 q\left(m+\sqrt{c}\right)^2 + \psi_{2,t} q\left(m-\sqrt{c}\right)^2 - \phi_{1,t}\left(m+\sqrt{c}\right) - \phi_{2,t}\left(m-\sqrt{c}\right) + 2\left(\psi_1+\psi_2\right)q^2\right)\right)^2 \\ - 2q^2\left(\psi_1+\psi_2\right)\left(\psi_{1,t} q\left(m+\sqrt{c}\right)^2 + \psi_{2,t} q\left(m-\sqrt{c}\right)^2 - \phi_1\left(m+\sqrt{c}\right) - \phi_2\left(m-\sqrt{c}\right)\right) \\ + 2q\left(\psi_1 q\left(m+\sqrt{c}\right) - \psi_2 q\left(m-\sqrt{c}\right) - 2\phi_1 + 2\phi_2\right)^2\end{array}} \tag{140}$$

, where

$$\psi_1 = \left(1 + \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)$$
$$\psi_2 = \left(1 + \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\right)^{-2} \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)$$
$$\phi_1 = \left(1 + \exp\left(\frac{1}{2}q\left(m + \sqrt{c}\right)\right)\right)^{-1}$$
$$\phi_1 = \left(1 + \exp\left(\frac{1}{2}q\left(m - \sqrt{c}\right)\right)\right)^{-1}$$

For technical convenience, our analysis uses $(\psi_1 + \psi_2) q^2$ (approximation 4 derived in Appendix B.2.2) as an approximation for the true sharpness (Equation 140), where the error bound for the approximation (calculated using Weyl's inequality) is shown in Equation 141.

$$\frac{1}{4}\left(\psi_1 q\left(m + \sqrt{c}\right)^2 + \psi_2 q\left(m - \sqrt{c}\right)^2 - \phi_{1,t}\left(m + \sqrt{c}\right) - \phi_{2,t}\left(m - \sqrt{c}\right)\right) + \frac{1}{2}\sqrt{\begin{array}{l}\frac{1}{4}\left(\psi_1 q\left(m + \sqrt{c}\right)^2 + \psi_2 q\left(m - \sqrt{c}\right)^2 - \phi_{1,t}\left(m + \sqrt{c}\right) - \phi_{2,t}\left(m - \sqrt{c}\right)\right) \\ + 2q\left(\psi_1 q\left(m + \sqrt{c}\right) - \psi_2 q\left(m - \sqrt{c}\right) - 2\phi_1 + 2\phi_2\right)^2\end{array}} \tag{141}$$

From lemma B.150, we know that a sufficient condition for our sharpness approximation to increase is for the iterate to enter region 3, and the iterate entering region 1 is a sufficient condition for the sharpness approximation to decrease.

For the loss, we see from lemma B.158 that the iterate entering region 3 is sufficient condition for the loss to decrease and entering region 1 is sufficient for the loss to increase, under our assumption that the invariant curve is very close to the fixed point.

To see perpetual oscillation in the loss and sharpness in our perturbation setting, it is sufficient show that if the trajectory iterate is in region 1, then it will eventually enter region 3 and vice versa.

To show this, we first show in lemma B.161 that a sufficient condition for region 1 is if system iterate lies between 0 and 0.245π radians (relative to the fixed point), and a sufficient condition for region 3 is if the system iterate lies between π and 1.245π radians (relative to the fixed point).

After defining these regions where the sharpness/loss will increase/decrease, we analyze dynamics on the invariant leading to the oscillations in sharpness and loss as follows. We first introduce preliminary theory to convert our original system into a canonical normal form for the Neimark Sacker Bifurcation, introduced in Kuznetsov (1998) and Devaney (2008).

Preliminary Theory on Neimark Sacker Bifurcation Consider a system

$$x \to f(x, \alpha),\ x = (x_1, x_2) \in \mathbb{R}^2\, \alpha \in \mathbb{R}$$

with a smooth function f, which has a fixed point at $x = 0$ for $\alpha = 0$ with complex-valued eigenvalues $\mu_{1,2} = \exp(i\phi_0),\ 0 < \phi_0 < \pi$. Note that, under the application of a parameter-dependent coordinate shift, we can assume WLOG that $x = 0$ is the fixed point of the system for $|\alpha|$ sufficiently small. Then, for all sufficiently small $|\alpha|$, the system can be written as follows,

$$x \to A(\alpha) x + F(x, \alpha), \tag{142}$$

where $A(\alpha)$ is the Jacobian matrix of the system evaluated at fixed point $x = 0$ and $F = (F_1, F_2)$ is a smooth vector function whose components F_1, F_2 have Taylor expansions in x starting with at least quadratic terms.

The Jacobian matrix $A(\alpha)$ has eigenvalues $\mu_{1,2}(\alpha) = r(\alpha)\exp(\pm i\phi(\alpha))$, where $r(0) = 1$, $\phi(0) = \phi_0$. Then, we can write $r(\alpha) = 1 + \beta(\alpha)$ for some smooth function $\beta(\alpha)$, where $\beta(0) = 0$. Assume that $\beta'(0) \neq 0$. Then, the eigenvalues for Jacobian matrix A can be written in terms of new parameter β where $\mu_1(\beta) = \mu(\beta)$, $\mu_2 = \bar{\mu}(\beta)$, $\mu(\beta) = (1 + \beta)\exp(i\psi(\beta))$ with smooth function $\psi(\beta)$ such that $\psi(0) = \phi_0$.

Lemma B.141. *By the introduction of a complex variable and a new parameter β (as described in the above), system 142 can be transformed for all sufficiently small $|\alpha|$ into the form*

$$z \to \mu(\beta) z + g(z, \bar{z}, \beta), \tag{143}$$

where $\beta \in \mathbb{R}$, $z \in \mathbb{C}$, $\mu(\beta) = (1+\beta)\exp(i\psi(\beta))$, and g is a complex-valued smooth function of $z, \bar{z}, \beta$ whose Taylor expansion with respect to $(z, \bar{z})$ contains quadratic and higher-order terms:

$$g(z, \bar{z}, \beta) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(\beta) z^k \bar{z}^l,$$

with $k, l = 0, 1, ...$

Remarks.

1. *Since the smooth parameter transformation $\beta(\alpha)$ satisfies $\beta'(0) \neq 0$, then, for sufficiently small $|\alpha|$, $\beta(\alpha)$ is a diffeomorphism (i.e., there exists a smooth inverse parameter transformation) by the Inverse Function Theorem.*

2. *The coefficients g_{kl} can be obtained as follows. Consider the Jacobian matrix $A(\alpha)$ from system 142. Let $q(\alpha) \in \mathbb{C}^2$ be an eigenvector of $A(\alpha)$ corresponding to eigenvalue $r(\alpha)\exp(i\phi(\alpha))$ and $p(\alpha) \in \mathbb{C}^2$ be an eigenvector of $A(\alpha)^T$ corresponding to eigenvalue $r(\alpha)\exp(-i\phi(\alpha))$, where $\langle p(\alpha) q(\alpha) \rangle = 1$. Then, for small $|\alpha|$ any vector $x \in \mathbb{R}^2$ can be written as $x = zq(\alpha) + \bar{z}\bar{q}(\alpha)$. Using new parameter β, we can write $g(z, \bar{z}, \beta) = \langle p(\beta), F(zq(\beta) + \bar{z}\bar{q}(\beta)) \rangle$. Therefore,*

$$g_{kl}(\beta) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \langle p(\beta), F(zq(\beta) + \bar{z}\bar{q}(\beta)) \rangle \Big|_{z=0}$$

Lemma B.142. *The map*

$$z \to \mu z + \frac{g_{20}}{2} z^2 + g_{11} z\bar{z} + \frac{g_{02}}{2} \bar{z}^2 + O\left(|z|^3\right), \tag{144}$$

where $\mu = \mu(\beta) = (1+\beta)\exp(i\psi(\beta))$, $g_{ij} = g_{ij}(\beta)$, can be transformed by an invertible parameter-dependent change of complex coordinate

$$z = \gamma + \frac{h_{20}}{2}\gamma^2 + h_{11}\gamma\bar{\gamma} + \frac{h_{02}}{2}\bar{\gamma}^2,$$

for all sufficiently small $|\beta|$, into a map without quadratic terms:

$$\gamma \to \mu\gamma + O\left(|\gamma|^3\right),$$

provided that $\exp(i\phi_0) \neq 1$ and $\exp(3i\phi_0) \neq 1$.

Remarks.

1. *Setting $h_{20} = \frac{g_{20}}{\mu^2 - \mu}$, $h_{11} = \frac{g_{11}}{|\mu|^2 - \mu}$, $h_{02} = \frac{g_{02}}{\bar{\mu}^2 - \mu}$ are valid substitutions if the denominators are nonzero for all sufficiently small $|\beta|$ including $\beta = 0$, which is achieved by the restrictions on ϕ_0*

2. *The smooth coordinate transformation, which we will denote as L_1, is near-identity for some neighborhood of the origin, since $DL_1(0) = I$. Since L_1 is smooth, there exists some neighborhood of the origin such that L_1 is a diffeomorophism (i.e., the inverse is smooth) by applying the Inverse Function Theorem.*

Lemma B.143. *The map*

$$\gamma \to \mu\gamma + \frac{t_{30}}{6}\gamma^3 + \frac{t_{21}}{2}\gamma^3\bar{\gamma} + \frac{t_{12}}{2}\gamma\bar{\gamma}^2 + \frac{t_{03}}{6}\bar{\gamma}^3 + O\left(|\gamma|^4\right),$$

where $\mu = \mu(\beta) = (1+\beta)\exp(i\psi(\beta))$, $t_{ij} = t_{ij}(\beta)$, can be transformed by an invertible parameter-dependent change of coordinates

$$\gamma = \omega + \frac{h_{30}}{6}\omega^3 + \frac{h_{21}}{2}\omega^2\bar{\omega} + \frac{h_{12}}{2}\omega\bar{\omega}^2 + \frac{h_{03}}{6}\bar{\omega}^3,$$

for all sufficiently small $|\beta|$, into the map with only one cubic term:

$$\omega \to \mu\omega + c_1\omega^2\bar{\omega} + O\left(|\omega|^4\right)$$

provided that $\exp(2i\phi_0) \neq 1$ and $\exp(4i\phi_0) \neq 1$.

Remarks.

1. *Setting $h_{30} = \frac{t_{30}}{\mu^3 - \mu}$, $h_{12} = \frac{t_{12}}{\bar{\mu}|\mu|^2 - \mu}$, $h_{03} = \frac{t_{03}}{\bar{\mu}^3 - \mu}$ are valid substitutions as long as the denominators are nonzero for all sufficiently small β due to the restrictions on β.*

2. *The $\omega^2\bar{\omega}$ terms could also be eliminated by setting $h_{21} = \frac{t_{21}}{\mu\left(1-|\mu|^2\right)}$ for small $\beta \neq 0$, but the denominator vanished at $\beta = 0$ for all ϕ_0. So, $h_{21} = 0$ to ensure the transformation is smoothly dependent on β, resulting in $c_1 = \frac{t_{21}}{2}$*

3. *Similar to the coordinate transformation in lemma B.142 (denoted as L_1), the above smooth coordinate transformation, denoted as L_2, is also near-identity and a diffeomorphism for some neighborhood of the origin (i.e. $DL_2(0) = I$).*

The above results can be summarized in the following theorem.

Theorem B.144. *Suppose a two-dimensional discrete-time system generated by the map*

$$x \to f(x, \alpha),\ x \in \mathbb{R}^2,\ \alpha \in \mathbb{R}$$

with smooth f has, for sufficiently small $|\alpha|$, the fixed point $x = 0$ with eigenvalues

$$\mu_{1,2}(\alpha) = r(\alpha) \exp\left(\pm i\phi(\alpha)\right),$$

where $r(0) = 1$, $\phi(0) = \phi_0$. Let the following generic conditions be satisfied:

1. $r'(0) \neq 0$

2. $\exp(ik\phi_0) \neq 1$ *for* $k = 1, 2, 3, 4$.

Then, there are smooth invertible coordinate and parameter changes transforming the system into

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to (1+\beta) \begin{pmatrix} \cos(\psi(\beta)) & -\sin(\psi(\beta)) \\ \sin(\psi(\beta)) & \cos(\psi(\beta)) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \left(y_1^2 + y_2^2\right) \begin{pmatrix} \cos(\psi(\beta)) & -\sin(\psi(\beta)) \\ \sin(\psi(\beta)) & \cos(\psi(\beta)) \end{pmatrix} \begin{pmatrix} a(\beta) & -b(\beta) \\ b(\beta) & a(\beta) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O\left(\|y\|^4\right). \quad (145)$$

with $\psi(0) = \phi_0$ and $a(0) = Re\left(\exp(-i\phi_0)\, c_1(0)\right)$.

Remarks.

1. *The conditions $\exp(ik\phi_0) \neq 1$ for $k = 1, 2, 3, 4$ are not merely technical. If there are not satisfied, the closed invariant curve may not appear at all or there might be several invariant curves bifurcating from the fixed point*

2. *The coefficient $a(0)$ (i.e, the first Lyapunov coefficient) determines the direction of the appearance of the invariant curve and can be calculated via*

$$a(0) = Re\left(\frac{\exp(-i\phi_0)\, g_{21}}{2}\right) - Re\left(\frac{(1 - 2\exp(i\phi_0))\exp(-2i\phi_0)}{2(1 - \exp(i\phi_0))} g_{20} g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2, \quad (146)$$

where

$$\begin{aligned} g_{11} &= \langle p, B[q, \bar{q}] \rangle \\ g_{02} &= \langle p, B[\bar{q}, \bar{q}] \rangle \\ g_{20} &= \langle p, B[q, q] \rangle \\ g_{21} &= \langle p, C[q, q, \tilde{q}] \rangle \end{aligned}$$

and $B = f_{xx}(0,0)$, $C = f_{xxx}(0,0)$, and $q, p \in \mathbb{C}^2$ satisfy $f_x(0,0)\, q = \exp(i\phi_0)\, q$, $f_x^T(0,0)\, p = \exp(-i\phi_0)\, p$, and $\langle p, q \rangle = 1$.

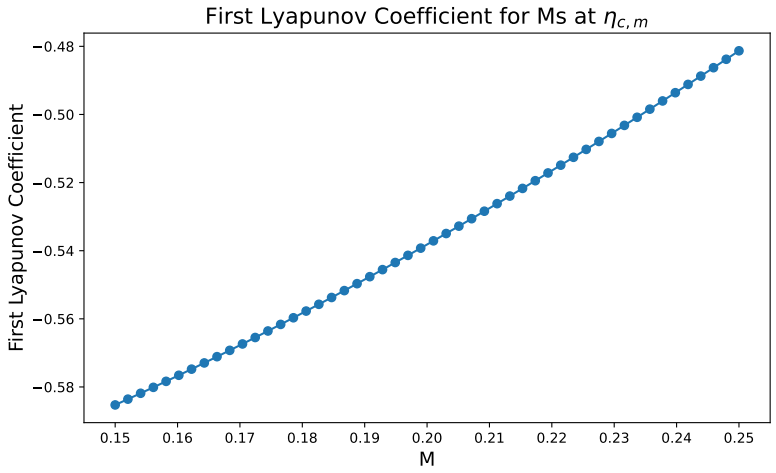


Figure 24: In this figure, we show the First Lyapunov Coefficient for the asymptotic dynamical system over an evenly spaced subset of $m \in [0.15, 0.25]$, which are negative.

EOS Theorem With the above preliminary theory on the coordinate and parameter transformations to convert a dynamical system into the Neimark-Sacker canonical normal form, we apply it to our system. We first fix some $m \in [0.15, 0.25]$, making our dynamical system a one-parameter system in η. To satisfy the criterion of Theorem B.144, note that we can rewrite the dynamical system with variables u_t, v_t, where $u_t = q_t - \tilde{q}$ and $v_t = c_t^2 - \tilde{c}_t^2$ and parameter $\alpha = \eta - \eta_{c,m}$. This rewritten system then has a fixed point at $(0, 0)$ at the critical parameter value $\alpha_{c,m} = 0$. Also, note that the non-resonance (i.e., $\exp(ik\phi_0) \neq 1$ for $k = 1, 2, 3, 4$) is satisfied by lemma B.149. The crossing of the determinant at non-zero speed is satisfied by lemmas B.126 and B.127.

To proceed further, note the following assumption.

Assumption 1. *The First Lyapunov Coefficient (denoted $a(0)$ in Theorem B.144) is negative.*

We support the above assumption empirically with Figure 24, which shows the First Lyapunov Coefficient for the dynamical system over an evenly spaced subset of $m \in [0.15, 0.25]$.

Under the above assumption, the invariant curve generated from the fixed point following the bifurcation will be unique and stable. We know this from Theorem B.145 below.

Theorem B.145. *Consider the map*

$$z \to \tilde{z} = g(z, \bar{z}, \beta) = \exp(i\psi(\beta)) \left(1 + \beta + d(\beta)|z|^2\right) + G(z, \tilde{z}, \beta), \tag{147}$$

where $z = x_1 + ix_2$, $d(\beta) = a(\beta) + ib(\beta)$; $a(\beta)$, $b(\beta)$, and $\psi(\beta)$ are smooth real-valued functions; $a(0) < 0$, $0 < \psi(0) < \pi$, and G is a smooth complex-valued function of $(z, \bar{z})$ and β such that $G = O\left(|z|^4\right)$ for small z.

The real planar map $x \to \tilde{x} = f(x, \beta)$ corresponding to the map 147 has in a neighborhood of the origin a unique stable closed invariant curve for sufficiently small $\beta > 0$. Moreover, any nontrivial orbit starting in this neighborhood converges to the closed invariant curve under iteration of this map.

Within the proof for Theorem B.145, it is also shown that, for sufficiently small $\beta > 0$, the invariant curve for the normal form can be approximated as a circle. We explicitly show this argument in Lemma B.146 below.

Lemma B.146. *Consider the map*

$$z \to \tilde{z} = g(z, \bar{z}, \beta) = \exp(i\psi(\beta)) \left(1 + \beta + d(\beta)|z|^2\right) + G(z, \tilde{z}, \beta), \tag{148}$$

where $z = x_1 + ix_2$, $d(\beta) = a(\beta) + ib(\beta)$; $a(\beta)$, $b(\beta)$, and $\psi(\beta)$ are smooth real-valued functions; $a(0) < 0$, $0 < \psi(0) < \pi$, and G is a smooth complex-valued function of $(z, \bar{z})$ and β such that $G = O\left(|z|^4\right)$ for small z.

Consider sufficiently small $\beta > 0$ such that $a(\beta) < 0$ and $\beta^{5/4} > C\beta^{3/2}$ for any constant C. The real planar map $x \to \tilde{x} = f(x, \beta)$ corresponding to the map 147 has a unique stable closed invariant curve projecting from the origin that is contained within annulus of width $O(\beta^{3/4})$ centered around a circle of radius $\sqrt{-\frac{\beta}{a(\beta)}}$.

Proof of Lemma B.146. To prove the lemma, note that we adapt a part of the proof for Theorem B.145.

To begin the proof, we know that Equation 147 already has a unique stable closed invariant curve. So, we will show that the invariant curve is contained within annulus of width $O(\beta^{3/4})$ centered around a circle of radius $\sqrt{-\frac{\beta}{a(\beta)}}$.

To prove this, we rewrite Equation 147 into the polar form (ρ, ψ), where

$$\left\{\begin{array}{c} \rho \to \tilde{\rho} = \rho\left(1 + \beta + a(\beta)\right) + \rho^4 R(\rho, \theta, \beta) \\ \theta \to \tilde{\theta} = \theta + \psi(\beta) + \rho^2 Q(\rho, \theta, \beta), \end{array}\right\} \tag{149}$$

where R and Q are smooth functions of (ρ, θ, β) near $(\rho, \beta) = (0, 0)$ and

$$Q(\rho, \theta, \beta) = \frac{b(\beta)}{1 + \beta} + O\left(\rho^2\right).$$

Let $0 < \epsilon < 1$. Consider an annulus centered around a circle of radius $\sqrt{\frac{\beta}{a(\beta)}}$

$$A_{\beta,\epsilon} = \left\{(\rho, \theta) : \sqrt{\tfrac{\beta}{a(\beta)}}(1 - \epsilon) \le \rho \le \sqrt{\tfrac{\beta}{a(\beta)}}(1 + \epsilon),\ \theta \in [0, 2\pi]\right\}$$

Under the polar map (Equation 149), we can see that

$$\Delta\rho = \tilde{\rho} - \rho = \rho\left(\beta + a(\beta)\rho^2 + \rho^3 R(\rho, \theta, \beta)\right).$$

Since $a(\beta) < 0$ (under sufficiently small $\beta > 0$ assumption), then

$$\begin{aligned} &\Delta\rho \ge \rho\left(\beta\left(2\epsilon - \epsilon^2\right) + O\left(\beta^{3/2}\right)\right) && 0 < \rho \le \sqrt{-\frac{\beta}{a(\beta)}}(1 - \epsilon) \\ &\Delta\rho \le \rho\left(\beta\left(-2\epsilon - \epsilon^2\right) + O\left(\beta^{3/2}\right)\right) && \sqrt{-\frac{\beta}{a(\beta)}}(1 + \epsilon) \le \rho \le \epsilon_0, \end{aligned}$$

where the $O\left(\beta^{3/2}\right)$-terms are smooth functions of ϵ and $\epsilon_0 > 0$ is a constant independent of β. Take $\epsilon = \beta^{1/4}$. Then, the above inequalities imply

$$\begin{aligned} &\Delta\rho > 0 && 0 < \rho \le \sqrt{-\frac{\beta}{a(\beta)}}\left(1 - \beta^{1/4}\right) \\ &\Delta\rho < 0 && \sqrt{-\frac{\beta}{a(\beta)}}\left(1 + \beta^{1/4}\right) \le \rho \le \epsilon_0 \end{aligned}$$

by our assumption on β being sufficiently small, so the $\pm 2\beta^{5/4}$-terms dominate. Since any nontrivial orbit starting in a small neighborhood of the origin containing the annulus moves towards and enters the annulus $A_{\beta,\beta^{1/4}}$ after a finite number of iterations, we know the invariant curve is contained in it. Since $A_{\beta,\beta^{1/4}}$ has width of order $O\left(\beta^{3/4}\right)$, this completes the proof.

□

Remarks. *Note that the functions Q and R introduced in the lemma are 2π-periodic.*

We next consider the corresponding areas for sharpness/loss increase/decrease that we discussed earlier in the space of the original dynamical system. Before proceeding further, we introduce the Inverse Function Theorem below, which is used in the following argument.

Theorem B.147 (Inverse Function Theorem)**.** *Suppose $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^1 on E, E open in $\mathbb{R}^n$, and at $x_0 \in E$, $Df(x_0)$ is invertible. Then,*

1. *There exists open sets U and V, $x_0 \in U \subset E$, $y_0 = f(x_0) \in V$, such that f is one-to-one on U and $f(U) = V$*
2. *Let g be the inverse of f defined on V, then g is C^1 on V and $\forall y \in V. Dg(y) = Df(g(y))^{-1}$.*

Remarks. *If f is C^r on E, $r \geq 1$, then so is $g = f^{-1}$ on V.*

As noted in the remarks for lemmas B.142 and B.143, the coordinate transformations, denoted as L_1 and L_2 respectively, are diffeomorphic and near-identity for some neighborhood of the fixed point at the origin by Inverse Function Theorem. So, the composition of the transformations $L = L_2 \circ L_1$ is also near-identity and diffeomorphic for some neighborhood about the origin by Inverse Function Theorem. Inverse Function Theorem tells us that for L to diffeomorphic, we are restricted to the neighbhorhood where $\|I - DL(z)\|_2 < \frac{1}{2}$. To ensure that the transformation is near-identity, we can shrink to neighborhood to satisfy $\|I - DL(z)\|_2 < \delta_1 << \frac{1}{2}$. By Weyl's Inequality for singular value perturbation, we then see that $|1 - \sigma_k(DL(z))| < \|I - DL(z)\|_2 < \delta_1$ (for $k = 1, 2$), so the coordinate transformation L applies a negligible amount of warping to the space of the original dynamical system in the specified neighborhood. For convenience, we denote the neighborhood about the fixed point in the space of the original dynamical system satisfying $\|I - DL(z)\|_2 < \delta_1$ as U, and the corresponding open set in the space of the normal form system as V (i.e., $V = L(U)$). Note that we further restrict U such that it is contained within the rectangle defined by $q \in \left[\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944, \frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15\right]$, $c \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025]$.

So, we can control $\beta > 0$ to be sufficiently small to satisfy the following. First, we need $\beta > 0$ sufficiently small based on the constraints of the transformation between α (i.e., $\alpha = \eta - \eta_{c,m}$, where $0.2 \leq \eta \leq 0.3$) and β being diffeomorphic and Theorem B.144. Second, we need $\beta > 0$ sufficiently small as specified in Theorem B.145. The constraints on β considered in Theorem B.145 already take into account the constraints for lemma B.146. Third, we need $\beta > 0$ sufficiently small such that the annulus described in lemma B.146 is contained in the open set V. For convenience, we denote the corresponding annulus in U (using the smooth inverse mapping via the diffeomorphic coordinate transformation) as $U_A \subset U$.

By choice of $\beta > 0$ sufficiently small satisfying the above constraints, we know that segments of the invariant curve in the space of the original dynamical system (contained in U) corresponding to regions 1 and 3 are warped by a negligible amount under the coordinate transformation to the corresponding segments on the (approximately) invariant circle in V. So, if we denote $I_1 = [\Omega_{1,1}, \Omega_{1,2}]$ and $I_3 = [\Omega_{3,1}, \Omega_{3,2}]$ as the corresponding radial segments to regions 1 ($[0, 0.245\pi]$) and 3 ($[\pi, 1.245\pi]$), then we know that $||\Omega_{1,1} - \Omega_{1,2}| - 0.245\pi| = O(\delta_1)$, $||\Omega_{3,1} - \Omega_{3,2}| - 0.245\pi| = O(\delta_1)$, $||\Omega_{1,2} - \Omega_{3,1}| - 0.755\pi| = O(\delta_1)$, and $||\Omega_{3,2} - \Omega_{1,1}| - 0.755\pi| = O(\delta_1)$.

If we initialize our trajectory iterate in the annulus (corresponding to U_A in the original dynamical system space), we know we will converge to the invariant curve/circle by Theorem B.145. Once on the invariant circle, we then note the dynamics on the circle by the following equations in polar form (borrowed from lemma B.146).

$$\left\{\begin{array}{c} \rho \to \tilde{\rho} = \rho(1 + \beta + a(\beta)) + \rho^4 R(\rho, \theta, \beta) \\ \theta \to \tilde{\theta} = \theta + \psi(\beta) + \rho^2 Q(\rho, \theta, \beta), \end{array}\right\}$$

where R and Q are smooth 2π-periodic functions of (ρ, θ, β) near $(\rho, \beta) = (0, 0)$ and

$$Q(\rho, \theta, \beta) = \frac{b(\beta)}{1 + \beta} + O\left(\rho^2\right).$$

Note that we consider ρ to be fixed at $\sqrt{-\frac{\beta}{a(\beta)}}$ by our invariant circle approximation, leaving

$$\theta \to \tilde{\theta} = \theta + \psi(\beta) + \rho^2 Q(\rho, \theta, \beta),$$

which we rewrite as

$$\theta \to \tilde{\theta} = \theta + \psi(\beta) - \beta \frac{b(\beta)}{a(\beta)(1+\beta)} + \tilde{Q}(\theta, \beta),$$

where $\tilde{Q}(\theta, \beta_\delta) = O(\beta_\delta^2) = -\frac{\beta_\delta}{a(\beta_\delta)} \left(Q(\rho, \theta, \beta_\delta) - \frac{b(\beta_\delta)}{1+\beta_\delta} \right)$.

To proceed further, we make the following assumption based on how our choice of β is sufficiently small the meet the constraints for our diffeormorphic transformation with $\alpha = \eta - \eta_{c,m}$.

Assumption 2. *$\psi(\beta)$ is between 0.205π and 0.115π for choice of $\beta > 0$ sufficiently small.*

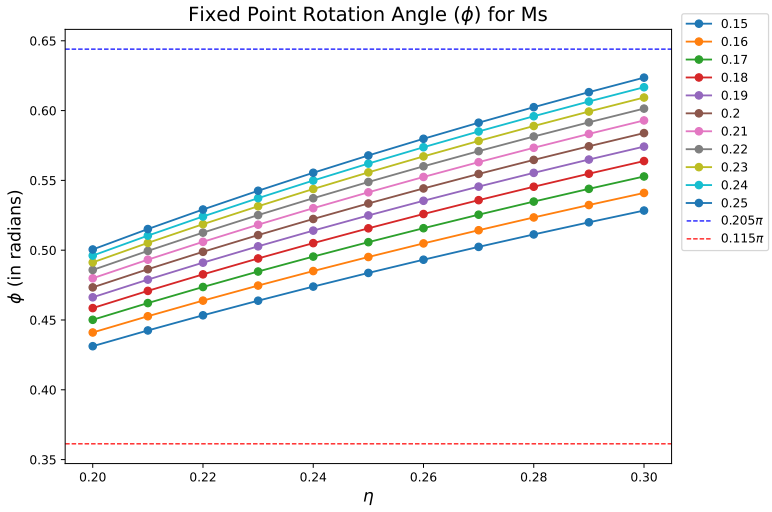


Figure 25: For an evenly-space set of $m \in [0.15, 0.25]$, we show that $\phi(\alpha)$ (where $\mu_{1,2}(\alpha) = r(\alpha)\exp(\pm i\phi(\alpha))$ are the eigenvalues of the system Jacobian at the fixed point), are between 0.205π (blue dashed line) and 0.115π (red dashed line).

We support the above assumption empirically with Figure 25, which shows that for $\eta \in [0.2, 0.3]$, $\phi(\alpha)$ (where $\mu_{1,2}(\alpha) = r(\alpha)\exp(\pm i\phi(\alpha))$ are the eigenvalues of the system Jacobian at the fixed point), are between 0.205π and 0.115π. Note that this point transfers to $\psi(\beta)$ based on our choice of sufficiently small $\beta > 0$.

By our assumption on $\psi(\beta)$, we further control $\beta > 0$ to be sufficiently small such that $\frac{\pi}{9} \leq \psi(\beta) - \beta\frac{b(0)}{a(0)} \leq \frac{2\pi}{9}$ and $\left| \frac{b(0)}{a(0)} - \frac{b(\beta)}{a(\beta)(1+\beta)} \right| < \delta_2$. Note for conciseness, we denote $\Omega = \psi(\beta) - \beta\frac{b(0)}{a(0)}$.

We then proceed with the following approximation for the dynamics on the circle and show the approximation error is negligible for the following cases.

$$\theta \to \tilde{\theta} \approx \theta + \Omega_\beta$$

Case 1: the trajectory iterate is not in I_1 or I_3. In this case, we will show how the trajectory iterate can enter I_1 and I_3 with negligible approximation error. WLOG, consider the iterate located at $\Omega_{1,2} < \theta_0 < \Omega_{3,1}$. To reach I_3 from θ_0, it would take $\left\lceil \frac{\Omega_{3,1} - \theta_0}{\Omega_\beta} \right\rceil$ steps to reach I_3 by our approximation. We can upper bound $\frac{\Omega_{3,1} - \theta_0}{\Omega_\beta}$ by $\frac{0.755\pi}{\frac{\pi}{9}} = 9(0.755) < 7$. Then, our approximation error is accumulated over 7 steps.

Case 2: the trajectory iterate is in I_1 or I_3. WLOG, assume that the trajectory iterate is in I_1 (denoted as $\Omega_{1,1} \leq \theta_0 \leq \theta_{1,2}$). We will show that it can enter I_3 with negligible approximation error. To

reach I_3 from θ_0, it would take $\left\lceil \frac{\Omega_{3,1}-\theta_0}{\Omega_\beta} \right\rceil$ steps to reach I_3 by our approximation. We can upper bound $\frac{\Omega_{3,1}-\theta_0}{\Omega_\beta}$ by $\frac{\pi}{\frac{\pi}{9}} = 9$. Then, our approximation error is accumulated over 9 steps.

So, our approximation accumulates the error from at most 9 steps to show perpetual oscillation in loss and sharpness.

We now formalize the above into the following theorem. Before stating the theorem, note the following assumptions made about the perturbation of parameter β for the normal form system and neighborhood about the fixed point $(\tilde{q}, \tilde{c})$ that we consider in our analysis.

Assumption 3. *Consider perturbation of parameter β, $\beta_\delta > 0$, sufficiently small such that it satisfies the following. For open neighborhood H_{diff} about $\eta_{c,m}$ such that $\forall \eta \in H_{diff}.0.2 \leq \eta \leq 0.3$ and the smooth parameter transformation $\beta(\alpha)$ ($\alpha = \eta - \eta_{c,m}$) is diffeomorphic, $\beta_\delta \in B_{diff}$, where B_{diff} is an open neighborhood about $\beta = 0$ corresponding to H_{diff} under the diffeomorphism $\beta(\alpha)$. Furthermore, any $\beta \in B_{diff}$ satisfies lemmas B.142, B.143, B.145, Assumption 1 and Assumption 2.*

Recall earlier that we introduce open neighborhood U about the fixed point $(\tilde{q}, \tilde{c})$ such that the smooth invertible coordinate transformations applied to transform the original dynamical system into the normal form are near-identity and diffeomorphic within the U. Also recall that V is the corresponding open neighborhood to U in the space of the normal form system under the coordinate transformation. In the assumption below, we formally define neighborhoods U and V.

Assumption 4. *Let U denote the neighborhood about the fixed point $(\tilde{q}, \tilde{c})$ such that $\|I - DL(z)\|_2 < \delta_1 << \frac{1}{2}$, where smooth coordinate transformation $L = L_2 \circ L_1$ for the smooth invertible coordinate transformations L_1 and L_2 given in lemmas B.142 and B.143, $\delta_1 > 0$, $z = (q - \tilde{q}) + i(c - \tilde{c})$, and $V = L(U)$. Furthermore, U is contained within the rectangle defined by $q \in \left[\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944, \frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15\right]$, $c \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025]$.*

Assumption 5. *Consider perturbation of parameter β, $\beta_\delta > 0$, sufficiently small such that $\frac{\pi}{9} \leq \psi(\beta_\delta) - \beta_\delta \frac{b(0)}{a(0)} \leq \frac{2\pi}{9}$, $\left|\frac{b(0)}{a(0)} - \frac{b(\beta_\delta)}{a(\beta_\delta)(1+\beta_\delta)}\right| < \delta_2$ for $\delta_2 > 0$, and the open ball of radius $\sqrt{-\frac{\beta_\delta}{a(\beta_\delta)}}\left(1 + \beta_\delta^{1/4}\right)$ is contained in V (where V and U are as defined in Assumption 4), where the corresponding open set about the fixed point $(\tilde{q}\tilde{c})$ for the open ball is U_A under the coordinate diffeomorphism, where $U_A \subset U$ (defined below).*

Theorem B.148. *For our dynamical system, let $m \in [0.15, 0.25]$ and the corresponding critical η value, $\eta_{c,m}$, where the complex eigenvalues of the system Jacobian at fixed point $(\tilde{q}, \tilde{c})$ (of the form $\mu_{1,2}(\eta) = r(\eta)\exp(\pm i\phi(\eta))$) cross the unit circle at nonzero speed, characteristic of a Neimark-Sacker Bifurcation. Consider the dynamical system transformed into the normal form by smooth coordinate transformation $L = L_2 \circ L_1$, where L_1 and L_2 are the smooth invertible coordinate transformations given in lemmas B.142 and B.143, and smooth parameter transformation $\beta(\alpha)$ such that the eigenvalues of the system Jacobian at fixed point $(\tilde{q}, \tilde{c})$ can be written as $\mu_{1,2}(\beta) = (1 + \beta)\exp(\pm i\psi(\beta))$ for smooth $\psi(\beta)$. Under Assumptions 3, 4, and 5, for sufficiently small perturbation $\eta_{c,m} + \epsilon \in H_{diff}$, where $\epsilon > 0$, and $\eta_{c,m} + \epsilon$ maps to β_δ under the parameter diffeomorphism, if we initialize our trajectory in U_A (excluding the fixed point $(\tilde{q}, \tilde{c})$), our trajectory will exhibit perpetual oscillation in loss and sharpness following convergence to the unique, stable invariant curve projected from the fixed point.*

Refer to Section B.5.4 for the proof of the theorem.

Rotation Angle Lemmas

Lemma B.149. *Consider some $m \in [0.15, 0.25]$. For the fixed point $(\tilde{q}, \tilde{c})$ at $\eta_{c,m}$ where the eigenvalues of the Jacobian are on the circle (i.e., of the form $\exp(i\theta_{c,m})$), $0 < \theta_{c,m} < \frac{\pi}{2}$.*

Lemmas for Loss and Sharpness Behavior

Lemma B.150. *Consider $q \in (3.74483, 4.84375)$, $c^2 \in (0.1175, 0.2375)$, $m \in [0.15, 0.25]$, $\eta \in (0.2, 0.3)$. For our sharpness approximation, $(\psi_1 + \psi_2)q^2$, where ψ_1, ψ_2 are shown below, it is*

sufficient for the trajectory iterate to enter region 3 *to see the approximation increase, and it is sufficient for the iterate to enter region* 1 *to see the approximation decrease.*

$$\psi_1 = \left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)$$
$$\psi_2 = \left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)$$

Lemma B.151. *Consider* $q \in [3.74483, 4.84375]$, $c^2 \in [0.1175, 0.2375]$, $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$. *Then,*

$$2+\frac{1}{2}q\left(m+\sqrt{c^2}\right)-q\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\left(m+\sqrt{c^2}\right) \tag{150}$$

is positive.

Lemma B.152. *Consider* $q \in [3.74483, 4.84375]$, $c^2 \in [0.1175, 0.2375]$, $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$. *Then,*

$$2+\frac{1}{2}q\left(m-\sqrt{c^2}\right)-q\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\left(m-\sqrt{c^2}\right) \tag{151}$$

is positive.

Lemma B.153. *Consider* $q \in (3.74483, 4.84375)$, $c^2 \in (0.1175, 0.2375)$, $m \in [0.15, 0.25]$, $\eta \in (0.2, 0.3)$. *Then,*

$$1+\frac{\eta}{4}\left(\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}+m\right)-\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}-m\right)\right)$$

is positive.

Lemma B.154. *Consider* $q \in [3.74483, 4.84375]$, $c^2 \in [0.1175, 0.2375]$, $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$. *Then,* $\tau_{2,p}+\tau_{2,m}$ *is negative, where*

$$\tau_{2,p} = \left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)$$
$$\cdot\left(1-2\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)$$

$$\tau_{2,m} = \left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)$$
$$\cdot\left(1-2\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)$$

Lemma B.155. *Consider* $q \in [3.74483, 4.84375]$, $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$. *Then,*

$$-0.2375\exp\left(q\sqrt{0.2375}\right)-6m^2\exp\left(qm\right)+\left(m+\sqrt{0.2375}\right)^2\exp\left(q\left(m+\sqrt{0.2375}\right)\right)$$
$$-\frac{1}{4}\left(3m+\sqrt{0.2375}\right)^2\exp\left(\frac{1}{2}q\left(3m+\sqrt{0.2375}\right)\right)-0.2375\exp\left(-q\sqrt{0.2375}\right)$$
$$-\frac{1}{4}\left(3m-\sqrt{0.2375}\right)^2\exp\left(\frac{1}{2}q\left(3m-\sqrt{0.2375}\right)\right) \tag{152}$$

is positive.

Lemma B.156. *Consider* $q \in [3.74483, 4.84375]$, $c^2 \in [0.1175, 0.2375]$, $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$. *Then,*

$$-\exp(qu) + \exp(q(m+u)) - \frac{1}{4}\exp\left(\frac{1}{2}q(3m+u)\right) \tag{153}$$

is positive.

Lemma B.157. *Consider* $q \in (3.74483, 4.84375)$, $u \in \left(\sqrt{0.1175}, \sqrt{0.2375}\right)$, $m \in [0.15, 0.25]$, $\eta \in (0.2, 0.3)$. *Then,*

$$-\exp(-qu) + \exp(q(m-u)) - \frac{1}{4}\exp\left(-\frac{1}{2}q(u-3m)\right) + \frac{1}{4}\exp\left(\frac{1}{2}q(u-m)\right)$$

is positive.

Lemma B.158. *Consider some* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$. *For trajectories of the asymptotic dynamical system sufficiently close to the fixed point* $(\tilde{q}, \tilde{c})$ *such that it is contained in the rectangle defined where* $q \in [3.74483, 4.84375], c \in [0.1175, 0.2375]$, *it is sufficient for the trajectory iterate on the invariant curve to enter region* 1 *to see the loss increase, and it is sufficient for the iterate to enter region* 3 *to see the loss decrease.*

Lemma B.159. *Consider* $q \in [3.74483, 4.84375]$, $c^2 \in [0.1175, 0.2375]$, $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$. *Then,*

$$\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right) - \exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)$$

can be lower bounded by

$$\left(\exp\left(\frac{1}{2}(3.74483)(0.15)\right)\left(\exp\left(\frac{1}{2}(3.74483)\sqrt{0.1175}\right) - \exp\left(-\frac{1}{2}(3.74483)\sqrt{0.1175}\right)\right)\right).$$

Lemma B.160. *Consider* $q \in [3.74483, 4.84375]$, $c^2 \in [0.1175, 0.2375]$, $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$. *Then,*

$$\left(1+\frac{1}{2}m\eta\right) + \left(1+\frac{1}{4}\eta\left(\sqrt{c^2}+m\right)\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right) + \left(1-\frac{1}{4}\eta\left(\sqrt{c^2}-m\right)\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right) + \exp(mq)$$

can upper bounded by

$$(1+0.1(0.25)) + \left(1+0.05\left(\sqrt{0.2375}+(0.25)\right)\right)\exp\left(2.421875\left((0.25)-\sqrt{0.2375}\right)\right) + \left(1-0.05\left(\sqrt{0.2375}-(0.25)\right)\right)\exp\left(2.421875\left((0.25)+\sqrt{0.2375}\right)\right) + \exp(4.84375(0.25)).$$

Lemmas Defining Regions 1 and 3

Lemma B.161. *Consider* $m \in [0.15, 0.25]$, $\eta \in [0.2, 0.3]$. *Under the linearization of Equations 7 and 8 (scaled by* $\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)$*) about the fixed point* $\left(\tilde{q}, \tilde{c}^2\right)$, *as long as the trajectory iterate remains inside the rectangle defined where* $q \in \left[\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944, \frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15\right]$ *and* $c^2 \in [2m\eta + 0.25m + 0.02, 2m\eta + 0.25m + 0.025]$, *it is sufficient for the trajectory iterate to be between* 0 *and* 0.245π *radians inclusive (relative to the fixed point* $\left(\tilde{q}, \tilde{c}^2\right)$ *) for the iterate to be in region* 1. *Furthermore, it is sufficient for the trajectory iterate to be between* π *and* 1.245π *radians inclusive (relative to the fixed point* $\left(\tilde{q}, \tilde{c}^2\right)$*) for the iterate to be in region* 3.

Lemma B.162. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right], u \in \left[\sqrt{2m\eta + 0.25m + 0.02}, \sqrt{2m\eta + 0.25m + 0.025}\right]$,

$$-\eta\left(\exp\left(\frac{1}{2}\tilde{q}\tilde{u}\right) - \exp\left(-\frac{1}{2}\tilde{q}\tilde{u}\right)\right) + m\tilde{u}\exp\left(\frac{1}{2}\tilde{q}m\right) - \frac{1}{2}\left((\eta\tilde{q}-\tilde{u})(\tilde{u}+m)\exp\left(\frac{1}{2}\tilde{q}\tilde{u}\right) + (\eta\tilde{q}+\tilde{u})(\tilde{u}-m)\exp\left(-\frac{1}{2}\tilde{q}\tilde{u}\right)\right) \tag{154}$$

is negative.

Lemma B.163. *For $m \in [0.15, 0.25]$,*

$$\begin{aligned}
&1 + \left(-6.03673m^5 - 0.18576m^4 - 7.98788m^3 + 2.05971m^2 - 2.55872m + 1\right) \exp\left(\left(4.5m^2 + 4.544\right) m\right) \\
&+ \left(\chi_{1,1} + \frac{\chi_{1,2}}{\sqrt{0.65m + 0.02}}\right) \exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right) \\
&+ \left(\chi_{2,1} + \frac{\chi_{2,2}}{\sqrt{0.65m + 0.02}}\right) \exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)
\end{aligned} \tag{155}$$

is negative, where

$$\begin{aligned}
\chi_{1,1} =& 4.55625m^7 + 2.29859m^6 + 8.13037m^5 + 2.64178m^4 + 3.18541m^3 + 0.04218m^2 - 0.54356m - 0.84235, \\
\chi_{1,2} =& 3.40857m^7 - 2.54649m^6 + 5.37261m^5 - 4.36439m^4 + 2.99594m^3 - 1.43286m^2 + 0.86916m + 0.03152, \\
\chi_{2,1} =& 4.55625m^7 + 2.29860m^6 + 8.13037m^5 + 2.64179m^4 + 3.18543m^3 + 0.04219m^2 - 0.54356m - 0.84235,
\end{aligned}$$

and

$$\chi_{2,2} = -3.40858m^7 + 2.54650m^6 - 5.37262m^5 + 4.36439m^4 - 2.99594m^3 + 1.43287m^2 - 0.86914m - 0.03151.$$

Lemma B.164. *For $x \in [0.15, 0.25]$,*

$$\exp\left(\frac{1}{2}\left(4.5x^2 + 4.544\right)\left(\sqrt{0.65x + 0.02} - x\right)\right) \tag{156}$$

is concave in x.

Lemma B.165. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\begin{aligned}
&\left(\sigma_{1,1} + \sigma_{1,2}\sqrt{2m\eta + 0.25m + 0.02}\right) \exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\
&+ \left(\sigma_{2,1} + \sigma_{2,2}\sqrt{2m\eta + 0.25m + 0.02}\right) \exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\
&+ \left(4.12768m^2\eta^2 + 0.51596m^2\eta + 0.04128m\eta + \eta\right) \exp\left(m\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\right) + \eta,
\end{aligned} \tag{157}$$

is concave in η, where

$$\begin{aligned}
\sigma_{1,1} =& -4.01210m\eta^3 + \left(-1.85745m^3 + 2.06384m^2 - 1.57470m - 1.92968\right)\eta^2 \\
&+ \left(-0.23218m^3 - 1.51019m^2 - 0.11351m - 0.02161\right)\eta + \left(-0.405m^4 - 0.468m^2 - 0.1352\right), \\
\sigma_{1,2} =& \left(0.05779m - 2.06383\right)\eta^2 + \left(-0.92872m^3 - 0.27861m + 0.99264\right)\eta + \left(0.45m^2 + 0.26\right), \\
\sigma_{2,1} =& -4.01210m\eta^3 + \left(-1.85745m^3 + 2.06384m^2 - 1.57470m - 1.92968\right)\eta^2 \\
&+ \left(-0.23218m^3 - 1.51019m^2 - 0.11351m - 0.02161\right)\eta + \left(-0.405m^4 - 0.468m^2 - 0.1352\right),
\end{aligned}$$

and

$$\sigma_{2,2} = \left(-0.05778m + 2.06384\right)\eta^2 + \left(0.92873m^3 + 0.27862m - 0.99263\right)\eta + \left(-0.45m^2 - 0.26\right).$$

Lemma B.166. *For $m \in [0.15, 0.25], \eta \in [0.15, 0.25]$,*

$$\begin{aligned}
&\left(0.56172m^3 - 68.20562m^2\right)\eta^4 + \left(15.98899m^5 + 1.85744m^4 + 19.05418m^3 - 3.16346m^2 - 8.36065m\right)\eta^3 \\
&+ \left(9.19437m^7 - 7.42982m^6 + 13.89926m^5 + 3.34381m^4 + 6.30440m^3 + 4.73651m^2 - 0.83898m - 0.07718\right)\eta^2 + \\
&\left(2.08963m^7 + 1.51836m^6 + 2.29023m^5 + 0.96504m^4 + 0.66078m^3 + 0.05220m^2 + 0.02029m + 0.00086\right)\eta \\
&+ \left(0.10449m^7 + 0.25109m^6 + 0.12100m^5 + 0.04015m^4 + 0.02191m^3 - 0.06097m^2 - 0.00758m - 0.00020\right)
\end{aligned} \tag{158}$$

is negative.

Lemma B.167. *For* $m \in [0.15, 0.25]$*,*

$$0.08783m^9 + 0.01406m^8 + 0.18114m^7 - 0.11006m^6 + 0.21808m^5 - 0.04141m^4 + 0.05439m^3 - 0.00438m^2 - 0.00004m + 0.00006 \tag{159}$$

is positive.

Lemma B.168. *For* $m \in [0.15, 0.25]$*,*

$$2.52924m^9 + 0.11242m^8 + 2.97441m^7 - 2.19152m^6 + 1.70855m^5 - 1.16531m^4 + 0.32618m^3 - 0.05019m^2 + 0.01047m + 0.00013 \tag{160}$$

is positive.

Lemma B.169. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$*,*

$$\begin{aligned}&\Big(8.25536m^2\eta^3 + \left(-7.42982m^5 - 4.29278m^3\right)\eta^2 \\ &+ \Big(3.34343m^8 + 3.86351m^6 + 1.11613m^4\Big)\eta \\ &+ \Big(0.41793m^8 + 0.03344m^7 + 1.29294m^6 + 0.03864m^5 + 1.07552m^4 + 0.01117m^3 + 0.2704m^2\Big)\Big)\end{aligned} \tag{161}$$

is positive.

Lemma B.170. *For* $m \in [0.15, 0.25]$*,*

$$-50.15112m^6 + 158.21496m^5 - 212.79648m^4 + 77.41236m^3 - 349.32075m^2 + 111.9462m - 13.93608 \tag{162}$$

is negative.

Lemma B.171. *For* $m \in [0.15, 0.25]$*,*

$$\begin{aligned}&\Big(\left(-0.405m^4 - 0.12073m^3 - 0.68748m^2 - 0.11778m - 0.21670\right) \\ &+ \left(-0.18574m^3 + 0.45m^2 - 0.05341m + 0.37598\right)\sqrt{0.65m + 0.02}\Big)\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right) \\ &+ \Big(\left(-0.405m^4 - 0.120734m^3 - 0.68748m^2 - 0.11778m - 0.21670\right) \\ &+ \left(0.18575m^3 - 0.45m^2 + 0.05342m - 0.37597\right)\sqrt{0.65m + 0.02}\Big)\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right) \\ &+ \left(0.26830m^2 + 0.00826m + 0.2\right)\exp\left(\left(4.5m^2 + 4.544\right)m\right) + 0.2\end{aligned} \tag{163}$$

is negative.

Lemma B.172. *For* $m \in [0.15, 0.25]$,

$$\begin{aligned}
&\Big(\left(-0.405m^4 - 0.12073m^3 - 0.68841m^2 - 0.11675m - 0.21764\right) \\
&+ \left(-0.18574m^3 + 0.45m^2 - 0.05341m + 0.37701\right)\sqrt{0.65m + 0.025}\Big) \\
&\cdot \exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.025}\right)\right) \\
&+ \Big(\left(-0.405m^4 - 0.12073m^3 - 0.68841m^2 - 0.11675m - 0.21764\right) \\
&+ \left(0.18575m^3 - 0.45m^2 + 0.05342m - 0.37700\right)\sqrt{0.65m + 0.025}\Big)\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.025}\right)\right) \\
&+ \left(0.26830m^2 + 0.01032m + 0.2\right)\exp\left(\left(4.5m^2 + 4.544\right)m\right) + 0.2
\end{aligned} \tag{164}$$

is negative.

Lemma B.173. *For* $x \in [0.15, 0.25]$,

$$\exp\left(-4.12516x^3 - 0.51084x^2 - 4.16549x - 0.51584\right) \tag{165}$$

is convex in x.

Lemma B.174. *For* $x \in [0.15, 0.25]$,

$$\exp\left(-0.38223x^3 + 0.50733x^2 - 0.38597x + 0.51229\right) \tag{166}$$

is convex in x.

Lemma B.175. *For* $m \in [0.15, 0.25]$,

$$\begin{aligned}
&1 + \left(\left(-6.03673\right)m^5 + \left(-0.2322\right)m^4 + \left(-7.98788\right)m^3 + \left(2.03288\right)m^2 + \left(-2.5484\right)m + 1\right)\exp\left(\left(4.5m^2 + 4.544\right)m\right) \\
&+ \left(\left(\nu_{1,1}\right) + \frac{\nu_{1,2}}{\sqrt{0.65m + 0.025}}\right)\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.025}\right)\right) \\
&+ \left(\left(\nu_{2,1}\right) + \frac{\nu_{2,2}}{\sqrt{0.65m + 0.025}}\right)\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.025}\right)\right),
\end{aligned} \tag{167}$$

is negative, where

$$\begin{aligned}
\nu_{1,1} &= \left(4.55625\right)m^7 + \left(2.29857\right)m^6 + \left(8.15126\right)m^5 + \left(2.60482\right)m^4 + \left(3.20583\right)m^3 \\
&\quad + \left(-0.00524\right)m^2 + \left(-0.53257\right)m + \left(-0.86143\right), \\
\nu_{1,2} &= \left(-3.40858\right)m^7 + \left(2.51328\right)m^6 + \left(-5.35880\right)m^5 + \left(4.31604\right)m^4 + \left(-2.97205\right)m^3 \\
&\quad + \left(1.40257\right)m^2 + \left(-0.86535\right)m + \left(-0.03967\right), \\
\nu_{2,1} &= \left(4.55625\right)m^7 + \left(2.29856\right)m^6 + \left(8.15126\right)m^5 + \left(2.60480\right)m^4 + \left(3.20582\right)m^3 \\
&\quad + \left(-0.00525\right)m^2 + \left(-0.53257\right)m + \left(-0.86143\right),
\end{aligned}$$

and

$$\begin{aligned}
\nu_{2,2} &= \left(3.40857\right)m^7 + \left(-2.51327\right)m^6 + \left(5.35878\right)m^5 + \left(-4.31604\right)m^4 + \left(2.97205\right)m^3 \\
&\quad + \left(-1.40255\right)m^2 + \left(0.86537\right)m + \left(0.03968\right).
\end{aligned}$$

Lemma B.176. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&\left(k_{1,1} + k_{1,2}\sqrt{2m\eta + 0.25m + 0.025}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right) \\
&+ \left(k_{2,1} + k_{2,2}\sqrt{2m\eta + 0.25m + 0.025}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m - \sqrt{2m\eta + 0.25m + 0.025}\right)\right) \\
&\quad + \left(4.12768m^2\eta^2 + \left(0.51596m^2 + 0.05160m + 1\right)\eta\right)\exp\left(m\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\right) + \eta
\end{aligned} \tag{168}$$

is concave in η, where

$$k_{1,1} = -4.01208m\eta^3 + \left(-1.85744m^3 + 2.06384m^2 - 1.57468m - 1.93970\right)\eta^2 + \left(-0.23217m^3 - 1.51483m^2 - 0.10834\right.$$
$$\left. - 0.405m^4 - 0.468m^2 - 0.1352\right.$$

$$k_{1,2} = \left(0.05779m - 2.06383\right)\eta^2 + \left(-0.92872m^3 - 0.27861m + 0.99780\right)\eta + 0.45m^2 + 0.26,$$

$$\begin{aligned} k_{2,1} = &-4.01208m\eta^3 + \left(-1.85744m^3 + 2.06384m^2 - 1.57468m - 1.93970\right)\eta^2 \\ &+ \left(-0.23217m^3 - 1.51483m^2 - 0.10834m - 0.02429\right)\eta \\ &- 0.405m^4 - 0.468m^2 - 0.1352, \end{aligned}$$

and

$$k_{2,2} = \left(-0.05778m + 2.06384\right)\eta^2 + \left(0.92873m^3 + 0.27862m - 0.99779\right)\eta - 0.45m^2 - 0.26.$$

Lemma B.177. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$*

$$\begin{aligned} &8.25536m^2\eta^3 + \left(-7.42982m^5 - 4.29278m^3\right)\eta^2 \\ &+ \left(3.34343m^8 + 3.86351m^6 + 1.11613m^4\right)\eta \\ &+ \left(0.41793m^8 + 0.04180m^7 + 1.29294m^6 + 0.04830m^5 + 1.07552m^4 + 0.01396m^3 + 0.2704m^2\right) \end{aligned} \tag{169}$$

is positive.

Lemma B.178. *For $m \in [0.15, 0.25]$,*

$$\begin{aligned} 0.07417m^9 - 0.07628m^8 + 0.22109m^7 - 0.10111m^6 + 0.18100m^5 - 0.04019m^4 \\ +0.04496m^3 - 0.00403m^2 - 0.00006m + 0.00009 \end{aligned} \tag{170}$$

is positive.

Lemma B.179. *For $m \in [0.15, 0.25]$,*

$$\begin{aligned} 2.13609m^9 - 0.6103m^8 + 3.03742m^7 - 2.06811m^6 + 1.28525m^5 - 1.12373m^4 \\ +0.31345m^3 - 0.07334m^2 + 0.01307m + 0.00020 \end{aligned} \tag{171}$$

is positive.

Lemma B.180. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\begin{aligned}
&\left((0.50546)\, m^3 + (-66.19943)\, m^2\right) \eta^5 \\
&+ \Big((9.74786)\, m^5 + (0.92866)\, m^4 + (15.19740)\, m^3 \\
&+ (-5.71182)\, m^2 + (-8.56121)\, m \Big) \eta^4 \\
&+ \Big((4.59718)\, m^7 + (-3.71491)\, m^6 + (7.16878)\, m^5 + (1.49440)\, m^4 + (3.85748)\, m^3 \\
&+ (2.45348)\, m^2 + (-0.89277)\, m + (-0.09698) \Big) \eta^3 \\
&+ \Big((0.62689)\, m^7 + (-0.19917)\, m^6 + (0.72951)\, m^5 + (0.37100)\, m^4 + (0.25660)\, m^3 \\
&+ (0.28201)\, m^2 + (0.02565)\, m + (0.00068) \Big) \eta^2 \\
&+ \Big((-0.729)\, m^8 + (-1.0611)\, m^6 + (0.02025)\, m^5 + (-0.49608)\, m^4 \\
&+ (0.0234)\, m^3 + (-0.07300)\, m^2 + (0.00676)\, m \Big) \eta \\
&+ \Big((-0.09112)\, m^8 + (-0.00911)\, m^7 + (-0.13263)\, m^6 + (-0.01073)\, m^5 + (-0.06175)\, m^4 \\
&+ (-0.00327)\, m^3 + (-0.00883)\, m^2 + (-0.00006)\, m + (0.00009) \Big)
\end{aligned} \tag{172}$$

is negative.

Lemma B.181. *For $m \in [0.15, 0.25]$,*

$$\begin{aligned}
(-0.60515)\, m^9 + (0.70963)\, m^8 + (-2.08336)\, m^7 + (1.67635)\, m^6 + (-1.31178)\, m^5 + (1.08053)\, m^4 \\
+ (-0.53880)\, m^3 + (-0.02702)\, m^2 + (-0.07495)\, m + (-0.00248)
\end{aligned} \tag{173}$$

is negative.

Lemma B.182. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&\left(-72.2176m^3 - 272.4548m^2 - 584.53918m + 596.2734\right) m^2\eta^3 \\
&+ \Bigg(\left(-4.17923\right) m^6 + \left(13.04756\right) m^5 + \left(-14.28125\right) m^4 + \left(13.37413\right) m^3 \\
&+ \left(-10.62202\right) m^2 + \left(-5.93916\right) m + \left(-1.12268\right)\Bigg) \left(12\right) m\eta^2 \\
&+ \Bigg(\left(2.14452\right) m^8 + \left(-3.40547\right) m^7 + \left(3.90331\right) m^6 + \left(0.70639\right) m^5 + \left(2.66559\right) m^4 \\
&+ \left(1.90678\right) m^3 + \left(-1.28116\right) m^2 + \left(-0.35723\right) m + \left(-0.01928\right)\Bigg) \left(6\right) \eta \\
&+ \Bigg(\left(0.85819\right) m^8 + \left(-0.95933\right) m^7 + \left(1.42341\right) m^6 + \left(-0.49957\right) m^5 \\
&+ \left(0.73852\right) m^4 + \left(0.01875\right) m^3 + \left(0.11625\right) m^2 + \left(-0.00742\right) m + \left(-0.00046\right)\Bigg) \left(2\right)
\end{aligned} \tag{174}$$

is negative.

Lemma B.183. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right], u \in \left[\sqrt{2m\eta + 0.25m + 0.02}, \sqrt{2m\eta + 0.25m + 0.025}\right]$,

$$\begin{aligned}
&\Bigg(\cot\left(0.245\pi\right) u^3 + \cot\left(0.245\pi\right) mu^2 - \cot\left(0.245\pi\right) nqu^2 - 0.5nq^2 \\
&+ 0.5qu - \cot\left(0.245\pi\right) nmqu - 2\cot\left(0.245\pi\right) nu + 1\Bigg) \exp\left(\frac{1}{2}q\left(m + u\right)\right) \\
&+ \Bigg(-\cot\left(0.245\pi\right) u^3 - \cot\left(0.245\pi\right) nqu^2 + \cot\left(0.245\pi\right) mu^2 + \cot\left(0.245\pi\right) nmqu \\
&+ 2\cot\left(0.245\pi\right) nu - 0.5qu - 0.5nq^2 + 1\Bigg) \exp\left(\frac{1}{2}q\left(m - u\right)\right) \\
&+ \left(2\cot\left(0.245\pi\right) mu^2 + 1\right) \exp\left(mq\right) + 1
\end{aligned} \tag{175}$$

is convex in u.

Lemma B.184. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&\left(\mu_{1,1}\sqrt{2m\eta + 0.25m + 0.02} + \mu_{1,2}\right) \exp\left(-\frac{1}{2}\left(\frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\
&+ \left(\mu_{2,1}\sqrt{2m\eta + 0.25m + 0.02} + \mu_{2,2}\right) \exp\left(-\frac{1}{2}\left(\frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\
&+ 4m\eta^3
\end{aligned} \tag{176}$$

is positive, where

$$\begin{aligned}
\mu_{1,1} = &\left(-0.17336m - 11.55625\right)\eta^4 + \left(-1.23787m^3 - 10.105m^2 + 3.95679m + 2.06694\right)\eta^3 \\
&+ \left(-0.98301m^5 - 2.209m^4 + 1.12870m^3 - 0.65683m^2 + 0.83267m + 0.24058\right)\eta^2 \\
&+ \left(-0.20765m^7 - 0.26288m^5 + 0.69473m^4 - 0.10604m^3 + 0.70951m^2 - 0.01325m + 0.18113\right)\eta \\
&+ \left(0.10059m^6 + 0.15411m^4 + 0.07869m^2 + 0.01339\right),
\end{aligned}$$

$$\begin{aligned}\mu_{1,2} =& -4.96919m\eta^5 + \left(-6.51773m^3 + 2.31125m^2 + 4.32812m - 11.23808\right)\eta^4 \\ &+ \left(-2.84961m^5 + 2.021m^4 - 2.12698m^3 - 7.03097m^2 + 3.29251m - 0.77622\right)\eta^3 \\ &+ \left(-0.41530m^7 + 0.4418m^6 - 1.87601m^5 - 2.29338m^4 - 0.86585m^3 + 0.17728m^2 - 0.00827m + 0.74731\right)\eta^2 \\ &+ \left(-0.05192m^7 - 0.81424m^6 - 0.07511m^5 - 0.63336m^4 - 0.03610m^3 - 0.00994m^2 - 0.00576m + 0.05164\right)\eta \\ &+ \left(-0.09459m^8 - 0.19319m^6 - 0.14799m^4 - 0.05040m^2 - 0.00644\right), \\ \mu_{2,1} =& \left(0.17333m + 11.55625\right)\eta^4 + \left(1.23785m^3 + 10.105m^2 - 3.95682m - 2.06701\right)\eta^3 \\ &+ \left(0.98300m^5 + 2.209m^4 - 1.12871m^3 + 0.65671m^2 - 0.83269m - 0.24069\right)\eta^2 \\ &+ \left(0.20764m^7 + 0.26287m^5 - 0.69482m^4 + 0.10603m^3 - 0.70962m^2 + 0.01324m - 0.18120\right)\eta \\ &+ \left(-0.10062m^6 - 0.15414m^4 - 0.07872m^2 - 0.01341\right),\end{aligned}$$

and

$$\begin{aligned}\mu_{2,2} =& -4.96919m\eta^5 + \left(-6.51773m^3 + 2.31125m^2 + 4.32812m - 11.23808\right)\eta^4 \\ &+ \left(-2.84961m^5 + 2.021m^4 - 2.12698m^3 - 7.03097m^2 + 3.29251m - 0.77622\right)\eta^3 \\ &+ \left(-0.41530m^7 + 0.4418m^6 - 1.87601m^5 - 2.29338m^4 - 0.86585m^3 + 0.17728m^2 - 0.00827m + 0.74731\right)\eta^2 \\ &+ \left(-0.05192m^7 - 0.81424m^6 - 0.07511m^5 - 0.63336m^4 - 0.03610m^3 - 0.00994m^2 - 0.00576m + 0.05164\right)\eta \\ &+ \left(-0.09459m^8 - 0.19319m^6 - 0.14799m^4 - 0.05040m^2 - 0.00644\right).\end{aligned}$$

Lemma B.185. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}&\left(-0.17420m^2 - 11.77129m - 10.45032\right)\eta^4 + \left(-1.24409m^4 - 11.27536m^3 - 5.16119m^2 + 5.65555m + 1.86909\right)\eta^3 \\ &+ \left(-0.98796m^6 - 3.10908m^5 - 0.86320m^4 + 0.36068m^3 + 0.24285m^2 + 0.99490m + 0.21749\right)\eta^2 \\ &+ \Big(-0.20869m^8 - 0.18779m^7 - 0.26420m^6 + 0.46061m^5 + 0.52163m^4 + 0.61732m^3 \\ &+ 0.62824m^2 + 0.17014m + 0.16375\Big)\eta \\ &+ \left(0.10113m^7 + 0.09094m^6 + 0.15492m^5 + 0.13934m^4 + 0.07912m^3 + 0.07114m^2 + 0.01348m + 0.01208\right)\end{aligned} \tag{177}$$

is positive.

Lemma B.186. *For* $m \in [0.15, 0.25]$,

$$\exp\left(\left(-1.56666m^2 - 1.875\right)\left(m + \sqrt{0.65m + 0.02}\right)\right) \tag{178}$$

is convex in m *and can be upper bounded by* $-1.27225m + 0.581$.

Lemma B.187. *For* $m \in [0.15, 0.25]$,

$$\exp\left(\left(1.56666m^2 + 1.875\right)\left(\sqrt{0.65m + 0.02} - m\right)\right) \tag{179}$$

is concave in m *and can be lower bounded by* $-0.26721m + 1.4853$.

Lemma B.188. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$

$$\mu_{2,1}\sqrt{2m\eta + 0.25m + 0.02} + \mu_{2,2} \tag{180}$$

is negative, where

$$\begin{aligned}\mu_{2,1} =& \left(0.17333m + 11.55625\right)\eta^4 + \left(1.23785m^3 + 10.105m^2 - 3.95682m - 2.06701\right)\eta^3 \\ &+ \left(0.98300m^5 + 2.209m^4 - 1.12871m^3 + 0.65671m^2 - 0.83269m - 0.24069\right)\eta^2 \\ &+ \left(0.20764m^7 + 0.26287m^5 - 0.69482m^4 + 0.10603m^3 - 0.70962m^2 + 0.01324m - 0.18120\right)\eta \\ &+ \left(-0.10062m^6 - 0.15414m^4 - 0.07872m^2 - 0.01341\right)\end{aligned}$$

and

$$\begin{aligned}\mu_{2,2} =& -4.96919m\eta^5 + \left(-6.51773m^3 + 2.31125m^2 + 4.32812m - 11.23808\right)\eta^4 \\ &+ \left(-2.84961m^5 + 2.021m^4 - 2.12698m^3 - 7.03097m^2 + 3.29251m - 0.77622\right)\eta^3 \\ &+ \left(-0.41530m^7 + 0.4418m^6 - 1.87601m^5 - 2.29338m^4 - 0.86585m^3 + 0.17728m^2 - 0.00827m + 0.74731\right)\eta^2 \\ &+ \left(-0.05192m^7 - 0.81424m^6 - 0.07511m^5 - 0.63336m^4 - 0.03610m^3 - 0.00994m^2 - 0.00576m + 0.05164\right)\eta \\ &+ \left(-0.09459m^8 - 0.19319m^6 - 0.14799m^4 - 0.05040m^2 - 0.00644\right).\end{aligned}$$

Lemma B.189. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$

$$\mu_{1,1}\sqrt{2m\eta + 0.25m + 0.02} + \mu_{1,2} \tag{181}$$

is positive, where

$$\begin{aligned}\mu_{1,1} =& \left(-0.17336m - 11.55625\right)\eta^4 + \left(-1.23787m^3 - 10.105m^2 + 3.95679m + 2.06694\right)\eta^3 \\ &+ \left(-0.98301m^5 - 2.209m^4 + 1.12870m^3 - 0.65683m^2 + 0.83267m + 0.24058\right)\eta^2 \\ &+ \left(-0.20765m^7 - 0.26288m^5 + 0.69473m^4 - 0.10604m^3 + 0.70951m^2 - 0.01325m + 0.18113\right)\eta \\ &+ \left(0.10059m^6 + 0.15411m^4 + 0.07869m^2 + 0.01339\right)\end{aligned}$$

and

$$\begin{aligned}\mu_{1,2} =& -4.96919m\eta^5 + \left(-6.51773m^3 + 2.31125m^2 + 4.32812m - 11.23808\right)\eta^4 \\ &+ \left(-2.84961m^5 + 2.021m^4 - 2.12698m^3 - 7.03097m^2 + 3.29251m - 0.77622\right)\eta^3 \\ &+ \left(-0.41530m^7 + 0.4418m^6 - 1.87601m^5 - 2.29338m^4 - 0.86585m^3 + 0.17728m^2 - 0.00827m + 0.74731\right)\eta^2 \\ &+ \left(-0.05192m^7 - 0.81424m^6 - 0.07511m^5 - 0.63336m^4 - 0.03610m^3 - 0.00994m^2 - 0.00576m + 0.05164\right)\eta \\ &+ \left(-0.09459m^8 - 0.19319m^6 - 0.14799m^4 - 0.05040m^2 - 0.00644\right).\end{aligned}$$

Lemma B.190. *For* $\eta \in [0.2, 0.3]$, $\sqrt{0.3\eta + 0.0575}$ *can be lower bounded by* $0.41274\eta + 0.2602$.

Lemma B.191. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}&4m^3\eta^3 \exp\left(m\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\right) \\ &+ \left(s_{1,1}\sqrt{2m\eta + 0.25m + 0.02} + s_{1,2}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\ &+ \left(s_{2,1}\sqrt{2m\eta + 0.25m + 0.02} + s_{2,2}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right)\end{aligned} \tag{182}$$

is negative, where

$$
\begin{aligned}
s_{1,1} =& 1.83668m\eta^5 + \left(2.55093m^3 - 0.97122m^2 + 3.37530m + 14.49146\right)\eta^4 \\
&+ \left(1.18098m^5 - 0.47413m^4 + 6.83158m^3 + 5.47345m^2 - 3.88288m + 1.75596\right)\eta^3 \\
&+ \left(0.18225m^7 + 0.18549m^6 + 2.48855m^5 + 0.22007m^4 + 1.60775m^3 - 1.86430m^2 + 0.13333m - 1.11485\right)\eta^2 \\
&+ \Big(0.09113m^8 + 0.02279m^7 + 0.45258m^6 - 0.25061m^5 + 0.04249m^4 - 0.31242m^3 \\
&- 0.33384m^2 - 0.09244m - 0.12993\Big)\eta \\
&+ \Big(0.03976m^8 - 0.04414m^7 + 0.09188m^6 - 0.07651m^5 + 0.07963m^4 - 0.04420m^3 \\
&+ 0.03068m^2 - 0.00851m + 0.00444\Big),
\end{aligned}
$$

$$
\begin{aligned}
s_{1,2} =& \left(-1.78373m^2\eta^5 - 15.11654m\eta^5\right) + \left(-3.35222m^4 - 13.05201m^3 + 7.85083m^2 - 5.72655m - 7.71080\right)\eta^4 \\
&+ \Big(-1.95696m^6 - 2.3652m^5 + 1.18890m^4 - 8.85181m^3 - 1.24679m^2 - 1.28704m + 0.01762\Big)\eta^3 \\
&+ \Big(-0.3645m^8 + 0.2025m^7 - 1.23061m^6 - 0.61722m^5 - 2.66722m^4 \\
&+ 0.50528m^3 - 1.22083m^2 + 0.53709m - 0.01169\Big)\eta^2 \\
&+ \Big(-0.04552m^8 - 0.23330m^7 - 0.42377m^6 - 0.05215m^5 - 0.64237m^4 \\
&+ 0.17342m^3 - 0.35299m^2 + 0.07261m - 0.06616\Big)\eta \\
&+ \Big(-0.03972m^9 - 0.08078m^7 + 0.00090m^6 - 0.06045m^5 + 0.00155m^4 \\
&- 0.01959m^3 + 0.00091m^2 - 0.00229m + 0.00018\Big),
\end{aligned}
$$

$$
\begin{aligned}
s_{2,1} =& \left(-1.83665\right) m\eta^5 + \left(-2.55091m^3 + 0.97125m^2 - 3.37526m - 14.49133\right)\eta^4 \\
&+ \Big(-1.18098m^5 + 0.47415m^4 - 6.83152m^3 - 5.47316m^2 + 3.88303m - 1.75570\Big)\eta^3 \\
&+ \Big(-0.18225m^7 - 0.18549m^6 - 2.48848m^5 - 0.21978m^4 - 1.60751m^3 + 1.86472m^2 - 0.13320m + 1.11503\Big)\eta^2 \\
&+ \Big(-0.09112m^8 - 0.02277m^7 - 0.45243m^6 + 0.25074m^5 - 0.04222m^4 \\
&+ 0.31256m^3 + 0.33404m^2 + 0.09249m + 0.12999\Big)\eta \\
&+ \Big(-0.03972m^8 + 0.04417m^7 - 0.09181m^6 + 0.07656m^5 - 0.07956m^4 \\
&+ 0.04424m^3 - 0.03064m^2 + 0.00852m - 0.00442\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
s_{2,2} =& \left(-1.78373m^2 - 15.11654m\right)\eta^5 + \left(-3.35222m^4 - 13.05201m^3 + 7.85083m^2 - 5.72655m - 7.7108\right)\eta^4 \\
&+ \Big(-1.95696m^6 - 2.3652m^5 + 1.18871m^4 - 8.85181m^3 - 1.24683m^2 - 1.28707m + 0.0198\Big)\eta^3 \\
&+ \Big(-0.3645m^8 + 0.2025m^7 - 1.23078m^6 - 0.61722m^5 - 2.66737m^4 \\
&+ 0.50525m^3 - 1.21884m^2 + 0.53706m - 0.01051\Big)\eta^2 \\
&+ \Big(-0.04555m^8 - 0.23330m^7 - 0.42383m^6 - 0.05216m^5 - 0.64193m^4 \\
&+ 0.17341m^3 - 0.35245m^2 + 0.07261m - 0.06600\Big)\eta \\
&+ \Big(-0.03972m^9 - 0.08078m^7 + 0.00090m^6 - 0.06045m^5 + 0.00155m^4 \\
&- 0.01959m^3 + 0.00091m^2 - 0.00229m + 0.00018\Big).
\end{aligned}
$$

Lemma B.192. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in []$,

$$
\begin{aligned}
&\left(\nu_{1,1}\sqrt{2mn + 0.25m + 0.02} + \nu_{1,2}\right)\exp\left(\frac{1}{2}q\left(m + \sqrt{2mn + 0.25m + 0.02}\right)\right) \\
&+ \left(\nu_{2,1}\sqrt{2mn + 0.25m + 0.02} + \nu_{2,2}\right)\exp\left(\frac{1}{2}q\left(m - \sqrt{2mn + 0.25m + 0.02}\right)\right) \\
&\qquad + 4m\exp\left(mq\right),
\end{aligned}
\tag{183}
$$

is concave in q, where

$$
\nu_{1,1} = 0.125\tan\left(0.245\pi\right)q^3 - 0.25nmq^3 + 0.5mnq^2 + 0.0625mq^2 + 0.005q^2 - 2.5nq^2 + 2mq + 6,
$$

$$\begin{aligned}\nu_{1,2} =& -0.125\tan(0.245\pi)\,nq^4 - 0.5mn^2q^3 - 0.0625nmq^3 - 0.005nq^3 + 0.5m^2nq^2 + 0.0625m^2q^2 + 0.005mq^2\\ &+ 0.75\tan(0.245\pi)\,q^2 - nmq^2 + 6mnq + 0.75mq + 0.06q - 4nq + 2m,\end{aligned}$$

$$\nu_{2,1} = 0.25nmq^3 - 0.125\tan(0.245\pi)\,q^3 - 0.5mnq^2 - 0.0625mq^2 - 0.005q^2 + 2.5nq^2 - 2mq - 6,$$

and

$$\begin{aligned}\nu_{2,2} =& -0.125\tan(0.245\pi)\,nq^4 - 0.5mn^2q^3 - 0.0625mnq^3 - 0.005nq^3 + 0.5m^2nq^2 - nmq^2\\ &+ 0.75\tan(0.245\pi)\,q^2 + 0.25\,(0.25)\,m^2q^2 + 0.005mq^2 + 6mnq + 0.75mq + 0.06q - 4nq + 2m.\end{aligned}$$

Lemma B.193. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in []$,*

$$4m\exp(mq) + \left(\nu_{1,1}\sqrt{2mn + 0.25m + 0.02} + \nu_{1,2}\right)\exp\left(\frac{1}{2}q\left(m + \sqrt{2mn + 0.25m + 0.02}\right)\right) \tag{184}$$

is concave in q, where

$$\nu_{1,1} = 0.125\tan(0.245\pi)\,q^3 - 0.25nmq^3 + 0.5mnq^2 + 0.0625mq^2 + 0.005q^2 - 2.5nq^2 + 2mq + 6$$

and

$$\begin{aligned}\nu_{1,2} =& -0.125\tan(0.245\pi)\,nq^4 - 0.5mn^2q^3 - 0.0625nmq^3 - 0.005nq^3 + 0.5m^2nq^2\\ &+ 0.0625m^2q^2 + 0.005mq^2 + 0.75\tan(0.245\pi)\,q^2 - nmq^2 + 6mnq + 0.75mq + 0.06q - 4nq + 2m.\end{aligned}$$

Lemma B.194. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in []$,*

$$\begin{aligned}&\left(-0.06056mn^2 - 0.03028nm^2 - 0.00757nm - 0.00060n\right)q^4\\ &+ \Big(-0.25m^2n^3 + \left(-0.375m^3 - 0.0625m^2 - 0.005m\right)n^2\\ &+ \left(-0.04687m^3 + 0.11348m^2 - 0.48515m - 0.00002\right)n + \left(0.01515m^2 + 0.00122m\right)\Big)q^3\\ &+ \Big(\left(0.75m^3 - 6m^2\right)n^2 + \left(0.125m^4 - 0.0625m^3 - 0.735m^2 + 1.03021m - 1.45360\right)n\\ &+ \left(0.01563m^4 + 0.01297m^3 + 0.18358m^2 + 0.13664m + 0.01091\right)\Big)q^2\\ &+ \Big(\left(5m^2 - 15m\right)n^2 + \left(4.5m^3 - 1.75m^2 - 1.775m - 0.15\right)n\\ &+ \left(0.5625m^3 + 0.12313m^2 + 1.46611m + 0.00165\right)\Big)q\\ &+ \left(\left(18m^2 - 6m\right)n + 3.86211m^3 + 2.25m^2 + 0.18m + 1.45361\right)\end{aligned} \tag{185}$$

is negative.

Lemma B.195. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in [],$

$$
\begin{aligned}
&-0.06056nmq^4 \\
&+\left(-0.375m^2n^2+\left(-0.0625m^3-0.04687m^2+0.05682m-0.48453\right)n+\left(0.03029m^2+0.00758m+0.00061\right)\right)q^3 \\
&+\Bigg(\left(0.25m^2-2.75m\right)n^2+\left(0.375m^3-1.8125m^2-0.33875m-0.0275\right)n \\
&+\left(0.04688m^3+0.00766m^2+0.72743m+0.00032\right)\Bigg)q^2 \\
&+\left(\left(m^2+5m^2-5m-5.5m\right)n+\left(0.5m^3+0.75m^2+0.06m+2.18041\right)\right)q \\
&+\left(10mn-9n+4.5m^2+1.25m+0.1\right)
\end{aligned}
\tag{186}
$$

is negative.

Lemma B.196. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3].$

$$
\begin{aligned}
&\left(-0.93117m^2-5.346m\right)\eta^4+\left(-1.0872m^4-1.98219m^3+1.67123m^2-12.81866m-1.88455\right)\eta^3 \\
&+\left(-0.30375m^6+0.1188m^5-2.14624m^4-0.06414m^3-1.97949m^2+1.08452m+1.18946\right)\eta^2 \\
&+\left(-0.05062m^7-0.03796m^6-0.25633m^5-0.32343m^4+0.41135m^3-0.39853m^2+0.31348m-0.12959\right)\eta \\
&+\left(-0.04414m^7+0.02455m^6-0.07037m^5+0.02886m^4-0.03710m^3+0.00877m^2-0.00646m+0.00017\right)
\end{aligned}
\tag{187}
$$

decreases in η*.*

Lemma B.197. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$

$$
10m\eta^4-9\eta^4+4.5m^2\eta^3+1.25m\eta^3+0.1x^3+0.00491\eta+0.00215m^2+0.00110 \tag{188}
$$

is decreasing in η*.*

Lemma B.198. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in [],$

$$
\left(\nu_{2,1}\sqrt{2mn+0.25m+0.02}+\nu_{2,2}\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{2mn+0.25m+0.02}\right)\right), \tag{189}
$$

is concave in η*, where*

$$
\nu_{2,1} = 0.25nmq^3-0.125\tan\left(0.245\pi\right)q^3-0.5mnq^2-0.0625mq^2-0.005q^2+2.5nq^2-2mq-6,
$$

and

$$
\begin{aligned}
\nu_{2,2} =& -0.125\tan\left(0.245\pi\right)nq^4-0.5mn^2q^3-0.0625mnq^3-0.005nq^3+0.5m^2nq^2-nmq^2 \\
&+0.75\tan\left(0.245\pi\right)q^2+0.25\left(0.25\right)m^2q^2+0.005mq^2+6mnq+0.75mq+0.06q-4nq+2m.
\end{aligned}
$$

Lemma B.199. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in []$,

$$
\begin{aligned}
&0.06057nmq^4 \\
&+ \left(0.375m^2n^2 + \left(0.0625m^3 + 0.04688m^2 - 0.05681m + 0.48454\right)n - 0.03028m^2 - 0.00757m - 0.00060\right)q^3 \\
&+ \Bigg(\left(2.75m - 0.25m^2\right)n^2 + \left(-0.375m^3 + 1.8125m^2 + 0.33875m + 0.0275\right)n \\
&+ \left(-0.04687m^3 - 0.00765m^2 - 0.72742m - 0.00002\right)\Bigg)q^2 \\
&+ \left(\left(-6m^2 + 10.5m\right)n + \left(-0.5m^3 - 0.75m^2 - 0.06m - 2.18040\right)\right)q \\
&+ \left(-10mn + 9n - 4.5m^2 - 1.25m - 0.1\right).
\end{aligned} \tag{190}
$$

is positive.

Lemma B.200. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$
\begin{aligned}
&\left(0.93117m^2 + 5.346m\right)n^4 \\
&+ \left(1.0872m^4 + 1.98219m^3 - 1.67119m^2 + 12.81877m + 1.88459\right)n^3 \\
&+ \left(0.30375m^6 - 0.1188m^5 + 2.14628m^4 + 0.06431m^3 + 1.97960m^2 - 1.08439m - 1.18880\right)n^2 \\
&+ \left(0.05062m^7 + 0.03797m^6 + 0.25639m^5 + 0.32349m^4 - 0.41123m^3 + 0.39887m^2 - 0.31345m + 0.12977\right)n \\
&+ \left(0.04414m^7 - 0.02454m^6 + 0.07038m^5 - 0.02884m^4 + 0.03711m^3 - 0.00876m^2 + 0.00646m - 0.00017\right).
\end{aligned} \tag{191}
$$

grows in η.

Lemma B.201. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$
-10mn^3 + 9n^3 - 4.5m^2n^2 - 1.25mn^2 - 0.1n^2 \tag{192}
$$

grows in η.

Lemma B.202. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right], u \in \left[\sqrt{2m\eta + 0.25m + 0.02}, \sqrt{2m\eta + 0.25m + 0.025}\right]$,

$$
\begin{aligned}
&0.125\cot\left(0.245\pi\right)q^3u^3 + \Bigg(0.125\cot\left(0.245\pi\right)mq^3u^2 + 0.75\cot\left(0.245\pi\right)q^2u^2 - 0.125\cot\left(0.245\pi\right)nq^4u^2 \\
&+ 1.5\cot\left(0.245\pi\right)q^2u^2\Bigg) + \Bigg(\cot\left(0.245\pi\right)mq^2u - 1.25\cot\left(0.245\pi\right)nq^3u + 3\cot\left(0.245\pi\right)qu + 0.0625q^4u \\
&- 0.125\cot\left(0.245\pi\right)nmq^4u + 0.5\cot\left(0.245\pi\right)mq^2u - 0.5\cot\left(0.245\pi\right)nq^3u + 6\cot\left(0.245\pi\right)qu\Bigg) \\
&+ \Bigg(-2\cot\left(0.245\pi\right)nq^2 - 0.0625nq^5 + \cot\left(0.245\pi\right)mq + 0.375q^3 - 0.5\cot\left(0.245\pi\right)nmq^3 \\
&+ 2\cot\left(0.245\pi\right)mq - 2.5\cot\left(0.245\pi\right)nq^2 + 6\cot\left(0.245\pi\right) + 0.125q^3 - 0.25\cot\left(0.245\pi\right)nmq^3\Bigg)
\end{aligned} \tag{193}
$$

is positive.

Lemma B.203. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,

$$
\left(0.06056 - 0.125nm\right)q^3 + \left(0.25mn + 0.03125m + 0.0025 - 1.75n\right)q^2 + 1.5mq + 9 \tag{194}
$$

is positive.

Lemma B.204. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,

$$0.25\cot(0.245\pi)\, mq^3 + 4.5\cot(0.245\pi)\, q^2 - 0.25\cot(0.245\pi)\, nq^4 \tag{195}$$

is positive.

Lemma B.205. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,

$$(0.0625 - 0.125\cot(0.245\pi)\, nm)\, q^4 - 1.75\cot(0.245\pi)\, nq^3 + 1.5\cot(0.245\pi)\, mq^2 + 9\cot(0.245\pi)\, q \tag{196}$$

is positive.

Lemma B.206. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right], u \in \left[\sqrt{2m\eta + 0.25m + 0.02}, \sqrt{2m\eta + 0.25m + 0.025}\right]$,

$$\begin{aligned}
&0.125\cot(0.245\pi)\, q^3u^3 + \left(0.125\cot(0.245\pi)\, nq^2 - 0.125\cot(0.245\pi)\, mq - 2.25\cot(0.245\pi)\right) q^2u^2 \\
&+ \left(-1.75\cot(0.245\pi)\, nq^2 + 1.5\cot(0.245\pi)\, mq + 9\cot(0.245\pi) - 0.125\cot(0.245\pi)\, nmq^3 + 0.0625q^3\right) qu \\
&+ \left(0.0625nq^5 + 4.5\cot(0.245\pi)\, nq^2 - 3\cot(0.245\pi)\, mq - 6\cot(0.245\pi) + 0.75\cot(0.245\pi)\, nmq^3 - 0.5q^3\right)
\end{aligned} \tag{197}$$

is positive.

Lemma B.207. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,

$$\begin{aligned}
&\Bigg((0.25797m\eta + 0.03224m + 0.00322 - 1.80586n)\, q^3 + 1.54787mq^2 \\
&+ (-0.12899\eta m + 0.0625)\, q^4 + 9.28727q\Bigg)\sqrt{2m\eta + 0.25m + 0.025} \\
&+ \Bigg(\left(0.25797m\eta^2 + 0.03224nm + 0.00322n\right) q^4 + \left(-0.25798m^2\eta - 0.03225m^2 - 0.00323m + 0.77393\eta m - 0.5\right) q^3 \\
&+ (-4.64364m\eta - 0.58046m - 0.05805 + 4.64363n)\, q^2 + 0.0625\eta q^5 - 3.09576mq - 6.19152\Bigg)
\end{aligned} \tag{198}$$

is positive.

Lemma B.208. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&(0.05293m - 13.26705)\, n^5 + \left(-0.77942m^3 - 18.42646m^2 + 5.63612m + 8.32429\right) n^4 \\
&+ \left(-1.15061m^5 - 8.53079m^4 + 4.41566m^3 + 0.18655m^2 + 2.93536m + 2.95557\right) n^3 \\
&+ \left(-0.54322m^7 - 1.31649m^6 + 0.46450m^5 - 1.11885m^4 + 1.08076m^3 + 0.02556m^2 + 0.36459m + 0.13433\right) n^2 \\
&+ \Bigg(-0.08465m^9 - 0.17212m^7 + 0.35659m^6 - 0.12880m^5 + 0.61814m^4 - 0.04178m^3 \\
&+ 0.35715m^2 - 0.00491m + 0.06876\Bigg) n \\
&+ \left(0.04099m^8 + 0.09475m^6 + 0.08213m^4 + 0.03162m^2 + 0.00456\right)
\end{aligned} \tag{199}$$

is positive.

Lemma B.209. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&\left(0.18225m^6 + 0.3159m^4 + 0.18252m^2 + 0.03515\right) \\
&+ \left(-0.37615m^7 - 0.57364m^5 + 1.18880m^4 - 0.28618m^3 + 1.37372m^2 - 0.04640m + 0.39685\right) n \\
&+ \left(-1.81054m^5 - 4.38825m^4 + 1.03245m^3 - 2.48615m^2 + 1.20091m + 0.02847\right) n^2 \\
&+ \left(-2.55667m^3 - 18.95722m^2 + 4.90645m + 0.20738\right) n^3 \\
&+ \left(-20.47379 - 0.86588m\right) n^4
\end{aligned} \tag{200}$$

is positive.

Lemma B.210. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&\left(0.20502m^8 + 0.47383m^6 + 0.41066m^4 + 0.15818m^2 + 0.02284\right) \\
&+ \left(0.09400m^7 + 1.70245m^6 + 0.15508m^5 + 1.78123m^4 + 0.08506m^3 + 0.35332m^2 + 0.01550m - 0.06212\right) n \\
&+ \left(0.75223m^7 - 0.62690m^6 + 3.79369m^5 + 4.73741m^4 + 2.55170m^3 + 0.94445m^2 + 0.35296m - 1.15671\right) n^2 \\
&+ \left(4.87450m^5 - 2.70818m^4 + 6.71443m^3 + 14.82452m^2 - 3.13704m + 3.78615\right) n^3 \\
&+ \left(10.52891m^3 - 2.92483m^2 - 2.24934m + 22.61210\right) n^4 + 7.58082mn^5.
\end{aligned} \tag{201}$$

is positive.

Lemma B.211. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&6.01792mn^3 + \left(2.78607m^3 - 1.54788m^2 + 7.00540m + 4.79903\right) n^2 \\
&+ \left(0.34819m^3 + 4.21527m^2 + 0.18179m - 0.45271\right) n + \left(1.0125m^4 + 1.17m^2 + 0.338\right)
\end{aligned} \tag{202}$$

is positive.

Lemma B.212. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,

$$\begin{aligned}
&\Big(-0.12899nmq^4 + 0.0625q^4 + 0.25797mnq^3 + 0.03224mq^3 + 0.00257q^3 \\
&- 1.80586nq^3 + 1.54788mq^2 + 9.28727q\Big)\sqrt{2mn + 0.25m + 0.02} \\
&+ \Big(0.0625nq^5 + 0.25797mn^2q^4 + 0.03224nmq^4 + 0.00257nq^4 \\
&+ 0.77393nmq^3 - 0.5q^3 - 0.25798m^2nq^3 - 0.03225m^2q^3 - 0.00258mq^3 - 4.64364mnq^2 \\
&- 0.58046mq^2 - 0.04644q^2 + 4.64363nq^2 - 3.09576mq - 6.19152\Big)
\end{aligned} \tag{203}$$

is positive.

Lemma B.213. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&(0.05293m - 13.26705)\, n^5 + \left(-0.77942m^3 - 18.42646m^2 + 5.63616m + 8.31953\right) n^4 \\
&+ \left(-1.15061m^5 - 8.53079m^4 + 4.41570m^3 + 0.17995m^2 + 2.93539m + 2.95175\right) n^3 \\
&+ \left(-0.54322m^7 - 1.31649m^6 + 0.46451m^5 - 1.12189m^4 + 1.08078m^3 + 0.02204m^2 + 0.36459m + 0.13331\right) n^2 \\
&+ \Big(-0.08465m^9 - 0.17212m^7 + 0.35612m^6 - 0.12880m^5 + 0.61733m^4 \\
&- 0.04178m^3 + 0.35668m^2 - 0.00491m + 0.06867\Big) n \\
&+ \left(0.04099m^8 + 0.09475m^6 + 0.08213m^4 + 0.03162m^2 + 0.00456\right)
\end{aligned} \tag{204}$$

is positive.

Lemma B.214. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&\left(-0.37615nm^7 - 0.57365nm^5 + 1.18721m^4n - 0.28619nm^3 + 1.37187m^2n - 0.04640nm + 0.39631n\right) \\
&+ \left(-1.81054n^2m^5 - 4.38825n^2m^4 + 1.03243n^2m^3 - 2.49301m^2n^2 + 1.20091n^2m + 0.02451n^2\right) \\
&+ \left(-2.55667m^3n^3 - 18.95723n^3m^2 + 4.90646mn^3 + 0.19999n^3\right) \\
&+ \left(-0.86588mn^4 - 20.47379n^4\right) + 0.18225m^6 + 0.3159m^4 + 0.18252m^2 + 0.03515
\end{aligned} \tag{205}$$

is positive.

Lemma B.215. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}
&0.20502m^8 + 0.47383m^6 + 0.41066m^4 + 0.15817m^2 + 0.02284 \\
&+ \Big(0.09400nm^7 + 1.70055m^6n + 0.15665nm^5 + 1.77793nm^4 + 0.08688nm^3 + 0.35141nm^2 + 0.01603nm \\
&- 0.0625n\Big) \\
&+ \Big(0.75223m^7n^2 - 0.62690m^6n^2 + 3.79367n^2m^5 + 4.72511m^4n^2 + 2.55849n^2m^3 + 0.95116n^2m^2 + 0.3569mn^2 \\
&- 1.14876n^2\Big) \\
&+ \left(4.87449m^5n^3 - 2.70819m^4n^3 + 6.71436m^3n^3 + 14.79798n^3m^2 - 3.1297mn^3 + 3.81597n^3\right) \\
&+ \left(10.52891m^3n^4 - 2.92483m^2n^4 - 2.24938mn^4 + 22.59301n^4\right) + 7.58082mn^5.
\end{aligned} \tag{206}$$

is positive.

Lemma B.216. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$
\begin{aligned}
& m\left(4\cot\left(0.245\pi\right)m^2n + 0.5\cot\left(0.245\pi\right)m^2 + 0.04\cot\left(0.245\pi\right)m + 1\right)\exp\left(m\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)\right) \\
& \quad + \left(k_{1,1}\sqrt{2mn + 0.25m + 0.02} + k_{1,2}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)\left(m - \sqrt{2mn + 0.25m + 0.02}\right)\right) \\
& \quad + \left(k_{2,1} + k_{2,2}\sqrt{2mn + 0.25m + 0.02}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)\left(m + \sqrt{2mn + 0.25m + 0.02}\right)\right)
\end{aligned} \tag{207}
$$

is negative, where

$$
\begin{aligned}
k_{1,1} =& 0.25n\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)^2 \\
& + \Bigg(\cot\left(0.245\pi\right)mn^2 + \left(0.125\right)\cot\left(0.245\pi\right)mn + \left(0.01\right)\cot\left(0.245\pi\right)n \\
& + \left(0.5\right)\cot\left(0.245\pi\right)m^2n + \left(-0.25\right)m\Bigg)\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right) \\
& + \Bigg(-\cot\left(0.245\pi\right)m^2n + \left(-0.125\right)\cot\left(0.245\pi\right)m^2 + \left(-0.01\right)\cot\left(0.245\pi\right)m + \left(-0.5\right) + \cot\left(0.245\pi\right)nm \\
& - 0.5 - \cot\left(0.245\pi\right)m^2n + \cot\left(0.245\pi\right)mn + \left(-0.125\right)\cot\left(0.245\pi\right)m^2 + \left(-0.01\right)\cot\left(0.245\pi\right)m\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
k_{1,2} =& \left(-0.25\right)mn\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)^2 \\
& + \Bigg(-\cot\left(0.245\pi\right)m^2n^2 + \left(-0.125\right)\cot\left(0.245\pi\right)m^2n + \left(-0.01\right)\cot\left(0.245\pi\right)mn \\
& + \left(\left(-0.5\right)\cot\left(0.245\pi\right)nm + 0.25\right)\left(\left(2\right)mn + \left(0.25\right)m + \left(0.02\right)\right) - n\Bigg)\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right) \\
& + \Bigg(\cot\left(0.245\pi\right)m^3n + \left(0.125\right)\cot\left(0.245\pi\right)m^3 + \left(0.01\right)\cot\left(0.245\pi\right)m^2 + \left(0.5\right)m + \Bigg(\cot\left(0.245\pi\right)mn \\
& - \cot\left(0.245\pi\right)n + \left(0.125\right)\cot\left(0.245\pi\right)m + \left(0.01\right)\cot\left(0.245\pi\right)\Bigg)\left(\left(2\right)mn + \left(0.25\right)m + \left(0.02\right)\right) \\
& - 2\cot\left(0.245\pi\right)mn^2 - 0.25\cot\left(0.245\pi\right)mn - 0.02\cot\left(0.245\pi\right)n\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
k_{2,1} =& -0.25nm\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)^2 \\
&+\Big(-\cot\left(0.245\pi\right)m^2n^2-0.125\cot\left(0.245\pi\right)m^2n-0.01\cot\left(0.245\pi\right)mn \\
&+\left(0.25+\left(-0.5\right)\cot\left(0.245\pi\right)nm\right)\left(\left(2\right)mn+\left(0.25\right)m+\left(0.02\right)\right)-n\Big)\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right) \\
&+\Big(\cot\left(0.245\pi\right)m^3n+0.125\cot\left(0.245\pi\right)m^3+0.01\cot\left(0.245\pi\right)m^2 \\
&+\left(0.5\right)m+\Big(\cot\left(0.245\pi\right)mn-\cot\left(0.245\pi\right)n+0.125\cot\left(0.245\pi\right)m \\
&+0.01\cot\left(0.245\pi\right)\Big)\left(\left(2\right)mn+\left(0.25\right)m+\left(0.02\right)\right) \\
&-2\cot\left(0.245\pi\right)mn^2-0.25\cot\left(0.245\pi\right)mn-0.02\cot\left(0.245\pi\right)n\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
k_{2,2} =& -0.25n\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)^2 \\
&+\Big(-\cot\left(0.245\pi\right)mn^2-0.125\cot\left(0.245\pi\right)mn-0.01\cot\left(0.245\pi\right)n+0.25m \\
&+\left(-0.5\right)\cot\left(0.245\pi\right)nm^2\Big)\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right) \\
&+\Big(\cot\left(0.245\pi\right)m^2n+0.125\cot\left(0.245\pi\right)m^2+0.01\cot\left(0.245\pi\right)m+\left(0.5\right)+\cot\left(0.245\pi\right)m^2n \\
&-\cot\left(0.245\pi\right)mn+0.125\cot\left(0.245\pi\right)m^2+0.01\cot\left(0.245\pi\right)m+0.5-\cot\left(0.245\pi\right)nm\Big).
\end{aligned}
$$

Lemma B.217. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$
\begin{aligned}
&\left(-0.18005m^2-3.60239m\right)\eta^3+\left(-1.03883m^4+0.34011m^3+1.17662m^2+1.10947m-1.73262\right)\eta^2 \\
&+\left(0.4093m^5-0.00823m^4+0.50440m^3-1.14477m^2-0.17576m-0.63816\right)\eta \\
&+\left(-0.02401m^5-0.14529m^4-0.03041m^3-0.04881m^2-0.00955m+0.0203\right)
\end{aligned} \tag{208}
$$

is negative.

Lemma B.218. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$
\begin{aligned}
&\left(-3.64165m^2-4.56116m\right)\eta^3+\left(-2.64141m^4+1.72676m^3-3.23630m^2-1.91812m-2.19375\right)\eta^2 \\
&+\left(-0.39199m^5-0.43049m^4-1.09446m^3-0.81376m^2+0.87846m-0.40435\right)\eta \\
&+\left(-0.37344m^5+0.14618m^4-0.32666m^3+0.06368m^2-0.06407m-0.01200\right)
\end{aligned} \tag{209}
$$

is negative.

Lemma B.219. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$
\begin{aligned}
&2.00606m\eta^3+\left(0.92873m^3-1.06079m^2+2.85120m+0.96485\right)\eta^2 \\
&+\left(0.46437m^4+0.11610m^3+0.89443m^2-0.43954m-0.48919\right)\eta \\
&+\left(0.2025m^4-0.225m^3+0.234m^2-0.13m+0.0676\right)
\end{aligned} \tag{210}
$$

is positive.

Lemma B.220. *For $m \in [0.15, 0.25], x \in [0.2, 0.3]$,*

$$\begin{aligned} &-2.00603mx^3 + \left(-0.92871m^3 + 1.06084m^2 - 2.85115m - 0.96482\right)x^2 \\ &+ \left(-0.46435m^4 - 0.11608m^3 - 0.89439m^2 + 0.43957m + 0.48920\right)x \\ &+ \left(-0.2025m^4 + 0.225m^3 - 0.234m^2 + 0.13m - 0.0676\right) \end{aligned} \tag{211}$$

is negative.

Lemma B.221. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,*

$$\begin{aligned} &\left(d_{1,1} + d_{1,2}\sqrt{2m\eta + 0.25m + 0.02}\right)\exp\left(\frac{1}{2}q\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\ &+ \left(d_{2,1} + d_{2,2}\sqrt{2m\eta + 0.25m + 0.02}\right)\exp\left(\frac{1}{2}q\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\ &+ \left(4\cot\left(0.245\pi\right)m^2\eta + 0.5\cot\left(0.245\pi\right)m^2 + 0.04\cot\left(0.245\pi\right)m + 1\right)\exp\left(mq\right) + 1 \end{aligned} \tag{212}$$

is concave in q, where

$$\begin{aligned} d_{1,1} =& -0.5\eta q^2 \\ &+ \left(-2\cot\left(0.245\pi\right)m\eta^2 - 0.25\cot\left(0.245\pi\right)m\eta - 0.02\cot\left(0.245\pi\right)\eta\right)q \\ &+ 2\cot\left(0.245\pi\right)m^2\eta + 0.25\cot\left(0.245\pi\right)m^2 + 0.02\cot\left(0.245\pi\right)m + 1, \end{aligned} \tag{213}$$

$$\begin{aligned} d_{1,2} =& \left(0.5 - \cot\left(0.245\pi\right)\eta m\right)q \\ &+ 2\cot\left(0.245\pi\right)m\eta - 2\cot\left(0.245\pi\right)\eta + 0.25\cot\left(0.245\pi\right)m + 0.02\cot\left(0.245\pi\right), \end{aligned} \tag{214}$$

$$\begin{aligned} d_{2,1} =& -0.5\eta q^2 \\ &+ \left(-2\cot\left(0.245\pi\right)m\eta^2 - 0.25\cot\left(0.245\pi\right)m\eta - 0.02\cot\left(0.245\pi\right)\eta\right)q \\ &+ 2\cot\left(0.245\pi\right)m^2\eta + 0.25\cot\left(0.245\pi\right)m^2 + 0.02\cot\left(0.245\pi\right)m + 1, \end{aligned} \tag{215}$$

and

$$\begin{aligned} d_{2,2} =& \left(\cot\left(0.245\pi\right)\eta m - 0.5\right)q \\ &+ \left(-2\right)\cot\left(0.245\pi\right)m\eta + 2\cot\left(0.245\pi\right)\eta + \left(-0.25\right)\cot\left(0.245\pi\right)m + \left(-0.02\right)\cot\left(0.245\pi\right). \end{aligned} \tag{216}$$

Lemma B.222. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\begin{aligned} &-2.21858m^2n^4 + \left(-0.96998m^3 + 1.79159m^2 - 3.91155m - 0.27861\right)mn^3 \\ &+ \left(-0.17253m^5 - 0.22098m^4 + 0.55343m^3 + 1.08950m^2 + 0.78915m - 0.43959\right)n^2 \\ &+ \left(0.21374m^6 - 0.16956m^5 + 0.65550m^4 - 0.51352m^3 - 0.04725m^2 - 0.16620m + 0.13409\right)n \\ &+ \left(0.08068m^6 - 0.07461m^5 + 0.08780m^4 - 0.02643m^3 + 0.02197m^2 + 0.00596m - 0.00093\right) \end{aligned} \tag{217}$$

is positive.

Lemma B.223. *For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,*

$$\begin{aligned} &-2.00601m^2n^4 + \left(-0.92871m^3 - 1.58222m^2 - 10.27007m - 1.65383\right)mn^3 \\ &+ \left(-2.56903m^5 + 0.86609m^4 - 3.16004m^3 - 1.66368m^2 - 2.99704m - 1.43642\right)n^2 \\ &+ \left(-0.19599m^6 - 0.57377m^5 - 0.49909m^4 - 2.03613m^3 + 0.77832m^2 - 0.74240m - 0.10139\right)n \\ &+ \left(-0.27218m^6 + 0.02592m^5 - 0.21224m^4 - 0.01245m^3 - 0.03245m^2 - 0.01584m - 0.00039\right). \end{aligned} \tag{218}$$

is negative.

Lemma B.224. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned} &-1.0319m^2n^3 + \left(-0.18355m^2 + 0.14734m - 0.50661\right)mn^2 \\ &+ \left(0.22738m^4 - 0.18039m^3 + 0.49773m^2 - 0.24605m + 0.25688\right)n \\ &+ \left(0.08583m^4 - 0.07938m^3 + 0.04957m^2 + 0.01241m - 0.00195\right) \end{aligned} \tag{219}$$

is positive.

Lemma B.225. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned} &-1.0319m^2n^3 + \left(-2.85448m^3 - 0.59245m^2 - 0.50661m\right)n^2 \\ &+ \left(-0.21777m^4 - 0.63753m^3 - 0.02864m^2 - 2.09181m - 0.22106\right)n \\ &+ \left(-0.30243m^4 + 0.02879m^3 - 0.06109m^2 - 0.03047m - 0.00076\right) \end{aligned} \tag{220}$$

is negative.

Lemma B.226. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,

$$\begin{aligned} &0.25nmq^2 + \left(0.25797m^3n + 1.54787m^2n^2 + 0.19349m^2n - 0.125m^2 - 0.23453mn - 0.03125m + n - 0.0025\right)q \\ &+ \Bigg(-1.54787m^3n - 1.03191m^2n^2 - 0.19349m^3 + 1.2899m^2n + 3.09576mn^2 - 0.03159m^2 \\ &+ 0.36635mn - 1.00256m + 0.03096n - 0.0001\Bigg) \end{aligned} \tag{221}$$

is positive.

Lemma B.227. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,

$$\begin{aligned} &-0.25nmq^2 + \Bigg(-0.25797m^3n - 1.54785m^2n^2 - 0.19347m^2n + 0.125m^2 \\ &+ 0.23455mn + 0.03125m - n + 0.0025\Bigg)q \\ &+ \Bigg(1.54788m^3n + 1.03192m^2n^2 + 0.1935m^3 - 1.28987m^2n - 3.09573mn^2 \\ &+ 0.03161m^2 - 0.3663mn + 1.00258m - 0.03093n + 0.00011\Bigg) \end{aligned} \tag{222}$$

is negative.

Lemma B.228. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned} &m\left(4\cot\left(0.245\pi\right)m^2n + 0.5\cot\left(0.245\pi\right)m^2 + 0.05\cot\left(0.245\pi\right)m + 1\right)\exp\left(m\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)\right) \\ &+ \left(c_{1,1}\sqrt{2mn + 0.25m + 0.025} + c_{1,2}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)\left(m - \sqrt{2mn + 0.25m + 0.025}\right)\right) \\ &+ \left(c_{2,1} + c_{2,2}\sqrt{2mn + 0.25m + 0.025}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)\left(m + \sqrt{2mn + 0.25m + 0.025}\right)\right) \end{aligned} \tag{223}$$

is negative, where

$$
\begin{aligned}
c_{1,1} =& 0.25n\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)^2 \\
&+\left(\cot\left(0.245\pi\right)mn^2+\left(0.125\right)\cot\left(0.245\pi\right)mn+\left(0.0125\right)\cot\left(0.245\pi\right)n+\left(0.5\right)\cot\left(0.245\pi\right)nm^2\right. \\
&\left.+\left(-0.25\right)m\right)\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right) \\
&-\cot\left(0.245\pi\right)m^2n+\left(-0.125\right)\cot\left(0.245\pi\right)m^2+\left(-0.0125\right)\cot\left(0.245\pi\right)m+\left(-0.5\right)+\cot\left(0.245\pi\right)mn \\
&-\cot\left(0.245\pi\right)m^2n+\left(-0.125\right)\cot\left(0.245\pi\right)m^2+\left(-0.0125\right)\cot\left(0.245\pi\right)m+\cot\left(0.245\pi\right)nm-0.5,
\end{aligned}
\tag{224}
$$

$$
\begin{aligned}
&-0.25nm\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)^2 \\
&+\left(-\cot\left(0.245\pi\right)m^2n^2+\left(-0.125\right)\cot\left(0.245\pi\right)m^2n+\left(-0.0125\right)\cot\left(0.245\pi\right)nm+\left(-1\right)\cot\left(0.245\pi\right)m^2n^2\right. \\
&+0.25\left(2\right)mn+\left(-0.125\right)\cot\left(0.245\pi\right)m^2n \\
&\left.+\left(0.0625\right)m+\left(-0.0125\right)\cot\left(0.245\pi\right)nm+\left(0.00625\right)-n\right)\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right) \\
&+\left(-2\right)\cot\left(0.245\pi\right)mn^2+\left(-0.25\right)\cot\left(0.245\pi\right)mn+\left(-0.025\right)\cot\left(0.245\pi\right)n+\cot\left(0.245\pi\right)m^3n \\
&+\left(0.125\right)\cot\left(0.245\pi\right)m^3 \\
&+\left(0.0125\right)\cot\left(0.245\pi\right)m^2+\left(0.5\right)m+\left(-2\right)\cot\left(0.245\pi\right)mn^2 \\
&+\left(-0.25\right)\cot\left(0.245\pi\right)mn+\left(-0.025\right)\cot\left(0.245\pi\right)n+\left(2\right)\cot\left(0.245\pi\right)m^2n^2 \\
&+\left(0.25\right)\cot\left(0.245\pi\right)m^2n \\
&+\left(0.025\right)\cot\left(0.245\pi\right)mn \\
&+\left(0.25\right)\cot\left(0.245\pi\right)m^2n+\left(0.03125\right)\cot\left(0.245\pi\right)m^2+\left(0.003125\right)\cot\left(0.245\pi\right)m+\left(0.025\right)\cot\left(0.245\pi\right)mn \\
&+\left(0.003125\right)\cot\left(0.245\pi\right)m+\left(\left(0.025\right)0.0125\right)\cot\left(0.245\pi\right),
\end{aligned}
\tag{225}
$$

$$
\begin{aligned}
c_{2,1} =& -0.25m\eta\left(\frac{0.9}{\eta}m^2+\frac{0.52}{\eta}+1.944\right)^2 \\
&+\Big((-1)\cot(0.245\pi)\,m^2\eta^2+(-0.125)\cot(0.245\pi)\,m^2\eta+(-0.0125)\cot(0.245\pi)\,m\eta \\
&+(2)\,0.25m\eta-0.5\,(2)\cot(0.245\pi)\,m^2\eta^2+(0.25)\,0.25m-0.5\,(0.25)\cot(0.245\pi)\,m^2\eta+(0.025)\,0.25 \\
&-0.5\,(0.025)\cot(0.245\pi)\,\eta m-\eta\Big)\left(\frac{0.9}{\eta}m^2+\frac{0.52}{\eta}+1.944\right) \\
&+\Big(\cot(0.245\pi)\,m^3\eta+(0.125)\cot(0.245\pi)\,m^3 \\
&+(0.0125)\cot(0.245\pi)\,m^2+0.5m-(2)\cot(0.245\pi)\,m\eta^2+(2)\cot(0.245\pi)\,m^2\eta^2 \\
&+(0.125)\,(2)\cot(0.245\pi)\,m^2\eta \\
&+(0.0125)\,(2)\cot(0.245\pi)\,m\eta-(0.25)\cot(0.245\pi)\,m\eta+(0.25)\cot(0.245\pi)\,m^2\eta \\
&+(0.125)\cot(0.245\pi)\,(0.25)\,m^2+(0.0125)\,(0.25)\cot(0.245\pi)\,m-(0.025)\cot(0.245\pi)\,\eta \\
&+(0.025)\cot(0.245\pi)\,m\eta+(0.125)\,(0.025)\cot(0.245\pi)\,m+(0.0125)\,(0.025)\cot(0.245\pi) \\
&+(-2)\cot(0.245\pi)\,m\eta^2+(-0.25)\cot(0.245\pi)\,m\eta+(-0.025)\cot(0.245\pi)\,\eta\Big),
\end{aligned}
\tag{226}
$$

and

$$
\begin{aligned}
c_{2,2} =& -0.25n\left(\frac{0.9}{\eta}m^2+\frac{0.52}{\eta}+1.944\right)^2 \\
&+\Big((-1)\cot(0.245\pi)\,m\eta^2+(-0.125)\cot(0.245\pi)\,m\eta+(-0.0125)\cot(0.245\pi)\,\eta+0.25m \\
&-0.5\cot(0.245\pi)\,m^2\eta\Big)\left(\frac{0.9}{\eta}m^2+\frac{0.52}{\eta}+1.944\right) \\
&+\Big(\cot(0.245\pi)\,m^2\eta+(0.125)\cot(0.245\pi)\,m^2+(0.0125)\cot(0.245\pi)\,m+0.5 \\
&-\cot(0.245\pi)\,m\eta+\cot(0.245\pi)\,m^2\eta+(0.125)\cot(0.245\pi)\,m^2+(0.0125)\cot(0.245\pi)\,m+0.5 \\
&-\cot(0.245\pi)\,\eta m\Big).
\end{aligned}
\tag{227}
$$

Lemma B.229. *For* $m\in[0.15,0.25],\eta\in[0.2,0.3]$,

$$
\begin{aligned}
&\left(-0.19875m^2-3.58738m\right)\eta^3+\left(-1.04749m^4+0.35695m^3+1.14212m^2+1.12646m-1.73437\right)\eta^2 \\
&+\left(0.40498m^5-0.00584m^4+0.49431m^3-1.13528m^2-0.17609m-0.63890\right)\eta \\
&+\left(-0.02590m^5-0.14168m^4-0.03427m^3-0.04472m^2-0.01115m+0.02146\right)
\end{aligned}
\tag{228}
$$

is negative.

Lemma B.230. *For* $m\in[0.15,0.25],\eta\in[0.2,0.3]$,

$$
\begin{aligned}
&\left(-3.61346m^2-4.57998m\right)\eta^3+\left(-2.62837m^4+1.70314m^3-3.18629m^2-1.93520m-2.21426\right)\eta^2 \\
&+\left(-0.38546m^5-0.43322m^4-1.08956m^3-0.82199m^2+0.87438m-0.39751\right)\eta \\
&+\left(-0.37059m^5+0.14112m^4-0.32126m^3+0.06079m^2-0.06190m-0.01199\right)
\end{aligned}
\tag{229}
$$

is negative.

Lemma B.231. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned} &2.00606m\eta^3 + \left(0.92873m^3 - 1.06079m^2 + 2.85120m + 0.96987\right)\eta^2 \\ &+ \left(0.46437m^4 + 0.11610m^3 + 0.89675m^2 - 0.44470m - 0.48785\right)\eta \\ &+ \left(0.2025m^4 - 0.225m^3 + 0.234m^2 - 0.13m + 0.0676\right) \end{aligned} \tag{230}$$

is positive.

Lemma B.232. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned} &-2.00603m\eta^3 + \left(-0.92871m^3 + 1.06084m^2 - 2.85115m - 0.96984\right)\eta^2 \\ &+ \left(-0.46435m^4 - 0.11608m^3 - 0.89671m^2 + 0.44473m + 0.48786\right)\eta \\ &+ \left(-0.2025m^4 + 0.225m^3 - 0.234m^2 + 0.13m - 0.0676\right) \end{aligned} \tag{231}$$

is negative.

Lemma B.233. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$,

$$\begin{aligned} &\left(f_{1,1} + f_{1,2}\sqrt{2m\eta + 0.25m + 0.025}\right)\exp\left(\frac{1}{2}q\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right) \\ &+ \left(f_{2,1} + f_{2,2}\sqrt{2m\eta + 0.25m + 0.025}\right)\exp\left(\frac{1}{2}q\left(m - \left(\sqrt{2m\eta + 0.25m + 0.025}\right)\right)\right) \\ &+ \left(4\cot\left(0.245\pi\right)m^2\eta + 0.5\cot\left(0.245\pi\right)m^2 + 0.05\cot\left(0.245\pi\right)m + 1\right)\exp\left(mx\right) + 1 \end{aligned} \tag{232}$$

is concave in q, where

$$\begin{aligned} f_{1,1} =& -0.5\eta q^2 + \left((-2)\cot\left(0.245\pi\right)m\eta^2 + (-0.25)\cot\left(0.245\pi\right)m\eta + (-0.025)\cot\left(0.245\pi\right)\eta\right)q \\ &+ (2)\cot\left(0.245\pi\right)m^2\eta + (0.25)\cot\left(0.245\pi\right)m^2 + (0.025)\cot\left(0.245\pi\right)m + 1, \end{aligned}$$

$$\begin{aligned} f_{1,2} =& \left(0.5 - \cot\left(0.245\pi\right)\eta m\right)q \\ &- 2\cot\left(0.245\pi\right)\eta + (2)\cot\left(0.245\pi\right)m\eta + (0.25)\cot\left(0.245\pi\right)m + (0.025)\cot\left(0.245\pi\right), \end{aligned}$$

$$\begin{aligned} f_{2,1} =& -0.5\eta q^2 + \left((-2)\cot\left(0.245\pi\right)m\eta^2 + (-0.25)\cot\left(0.245\pi\right)m\eta + (-0.025)\cot\left(0.245\pi\right)\eta\right)q \\ &+ (2)\cot\left(0.245\pi\right)m^2\eta + (0.25)\cot\left(0.245\pi\right)m^2 + (0.025)\cot\left(0.245\pi\right)m + 1, \end{aligned}$$

and

$$\begin{aligned} f_{2,2} =& \left(\cot\left(0.245\pi\right)\eta m - 0.5\right)q \\ &+ 2\cot\left(0.245\pi\right)\eta + (-2)\cot\left(0.245\pi\right)m\eta + (-0.25)\cot\left(0.245\pi\right)m + (-0.025)\cot\left(0.245\pi\right). \end{aligned}$$

Lemma B.234. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned} &-2.00601m^2n^4 + \left(-0.92871m^4 - 1.55450m^3 - 10.24515m^2 - 1.69290m\right)\eta^3 \\ &+ \left(-2.54948m^5 + 0.83833m^4 - 3.12021m^3 - 1.66893m^2 - 3.00375m - 1.45914\right)\eta^2 \\ &+ \left(-0.19272m^6 - 0.57351m^5 - 0.49864m^4 - 2.02848m^3 + 0.75940m^2 - 0.72574m - 0.10346\right)\eta \\ &+ \left(-0.26934m^6 + 0.02243m^5 - 0.20880m^4 - 0.01372m^3 - 0.03154m^2 - 0.01541m - 0.00047\right) \end{aligned} \tag{233}$$

is negative.

Lemma B.235. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}&-2.21858m^2n^4+\left(-0.96998m^4+2.26666m^3-3.35040m^2-0.39672m\right)\eta^3\\&+\left(0.12852m^5-0.47678m^4+1.08958m^3+1.20305m^2+1.19882m-0.51594\right)\eta^2\\&+\left(0.26392m^6-0.14032m^5+0.78411m^4-0.38715m^3-0.21298m^2-0.04355m+0.11984\right)\eta\\&+\left(0.12444m^6-0.10623m^5+0.13289m^4-0.04573m^3+0.03323m^2+0.00435m-0.00111\right)\end{aligned} \tag{234}$$

is positive.

Lemma B.236. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}&-1.0319m^2n^3+\left(-2.83276m^3-0.60697m^2-0.51177m\right)\eta^2\\&+\left(-0.21414m^4-0.63724m^3-0.03403m^2-2.08165m\right)\eta\\&+\left(-0.29927m^4+0.02492m^3-0.05909m^2-0.02965m-0.23170n-0.00091\right)\end{aligned} \tag{235}$$

is negative.

Lemma B.237. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\begin{aligned}&-1.0319m^2n^3+\left(0.13672m^3+0.09383m^2-0.51177m\right)\eta^2\\&+\left(0.28076m^4-0.14928m^3+0.54340m^2-0.04761m+0.22105\right)\eta\\&+\left(0.13238m^4-0.11302m^3+0.07377m^2+0.00906m-0.00232\right)\end{aligned} \tag{236}$$

is positive.

Lemma B.238. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta}+\frac{0.5}{\eta}+1.944, \frac{0.94m^2}{\eta}+\frac{0.48}{\eta}+2.15\right]$,

$$\begin{aligned}&-0.25\eta mq^2\\&+\left(-0.25797m^3\eta-1.54786m^2\eta^2-0.19347m^2\eta+0.125m^2+0.23067m\eta+0.03125m-\eta+0.00313\right)q\\&+\Bigg(1.54788m^3\eta+1.03192m^2\eta^2+0.19349m^3-1.28988m^2\eta-3.09574m\eta^2+0.03548m^2\\&\quad-0.36116m\eta+1.00323m-0.03868\eta+0.00017\Bigg),\end{aligned} \tag{237}$$

is negative.

Lemma B.239. *For* $m \in [0.15, 0.25], \eta \in [0.2, 0.3], q \in \left[\frac{0.9m^2}{\eta}+\frac{0.5}{\eta}+1.944, \frac{0.94m^2}{\eta}+\frac{0.48}{\eta}+2.15\right]$,

$$\begin{aligned}&0.25\eta mq^2\\&+\left(0.25797m^3\eta+1.54787m^2\eta^2+0.19348m^2\eta-0.125m^2-0.23066m\eta-0.03125m+\eta-0.00312\right)q\\&+\Bigg(-1.54787m^3\eta-1.03191m^2\eta^2-0.19348m^3+1.2899m^2\eta+3.09576mn^2-0.03547m^2\\&\quad+0.36118m\eta-1.00322m+0.0387\eta-0.00016\Bigg).\end{aligned} \tag{238}$$

is positive.

Proof of Theorem B.148 In this section, we introduce the formal proof of Theorem B.148 based on the intuition introduced in Section B.5.4.

Proof of Theorem B.148. To prove the theorem, we proceed by focusing on the dynamics in the space of the normal form for our dynamical system. Since we initialize our trajectory in U_A, under the coordinate diffeomorphism, this corresponds our iterate being contained within the circle of radius $\sqrt{-\frac{\beta_\delta}{a(\beta_\delta)}}\left(1+\beta_\delta^{1/4}\right)$. By Assumption 1 and Theorem B.145, we know that our iterate will converge to the invariant curve. By lemma B.146, we can approximate the invariant curve in the normal form space as a circle of radius $\sqrt{-\frac{\beta_\delta}{a(\beta_\delta)}}$. Now, we assume that the trajectory iterate has converged to the (approximately) invariant circle. From lemmas B.150, B.158,and B.161, we know that it is sufficient for our trajectory iterate to exhibit sharpness decrease and loss increase when it is between 0 and 0.245π radians relative to $(\tilde{q},\tilde{c})$ (region 3) and exhibit sharpness increase and loss decrease when it between π and 1.245π radians (region 1) relative to $(\tilde{q},\tilde{c})$ in the space of our original dynamical system. To proceed, we will show that under the coordinate diffeomorphism, these regions undergo negligible warping.

To see this, first note that for our smooth coordinate transformation L, $DL(0)=0$, where the origin corresponds to the fixed point $(\tilde{q},\tilde{c})$ of our original dynamical system under parameter-dependent shifting. Then, by the Inverse Function Theorem, we know that L is a diffeomorphism on the neighborhood about the origin, where $\|I-DL(z)\|_2 < \frac{1}{2}$, where $z=(q-\tilde{q})+i(c-\tilde{c})$. Since we restrict our neighborhood U to satisfy $\|I-DL(z)\|_2 < \delta_1 << \frac{1}{2}$, we also know by Weyl's Inequality for singular values that $|1-\sigma_k(DL(z))| < \|I-DL(z)\|_2 < \delta_1$ for $k\in\{1,2\}$. So, regions 1 and 3 restricted to our neighborhood U are warped negligibly by the coordinate diffeomorphism L. This means that in the space of the normal form for our dynamical system, the corresponding regions $I_1=[\Omega_{1,1},\Omega_{1,2}]$ and $I_3=[\Omega_{3,1},\Omega_{3,2}]$ satisfy $||\Omega_{1,1}-\Omega_{1,2}|-0.245\pi| = O(\delta_1)$, $||\Omega_{3,1}-\Omega_{3,2}|-0.245\pi| = O(\delta_1)$, $||\Omega_{1,2}-\Omega_{3,1}|-0.755\pi| = O(\delta_1)$, and $||\Omega_{3,2}-\Omega_{1,1}|-0.755\pi| = O(\delta_1)$.

With the regions for sharpness and loss oscillation established, we show how our dynamics on the invariant circle can enter these regions within one rotation. We introduce polar form dynamics of the normal form system, as presented in the proof of lemma B.146.

$$\left\{\begin{array}{c}\rho\to\tilde{\rho}=\rho(1+\beta+a(\beta))+\rho^4R(\rho,\theta,\beta)\\ \theta\to\tilde{\theta}=\theta+\psi(\beta)+\rho^2Q(\rho,\theta,\beta),\end{array}\right\}$$

where R and Q are 2π-periodic smooth functions of (ρ,θ,β) near $(\rho,\beta)=(0,0)$ and

$$Q(\rho,\theta,\beta)=\frac{b(\beta)}{1+\beta}+R(\rho,\theta,\beta),$$

where $R(\rho,\theta,\beta)=O\left(\rho^2\right)$.

Note that we consider ρ to be fixed at $\sqrt{-\frac{\beta_\delta}{a(\beta_\delta)}}$ by our invariant circle approximation, leaving

$$\theta\to\tilde{\theta}=\psi+\theta(\beta_\delta)+\rho^2Q(\rho,\theta,\beta_\delta),$$

which we rewrite as

$$\theta\to\tilde{\theta}=\theta+\psi(\beta_\delta)-\beta\frac{b(\beta_\delta)}{a(\beta_\delta)(1+\beta_\delta)}+\tilde{R}(\theta,\beta_\delta),$$

where $\tilde{R}(\theta,\beta_\delta)=O\left(\beta_\delta^2\right)=-\frac{\beta_\delta}{a(\beta_\delta)}R\left(\sqrt{-\frac{\beta_\delta}{a(\beta_\delta)}},\theta,\beta\right)$ and $O\left(\beta_\delta^2\right)$ is uniform in θ. To see that $O\left(\beta_\delta^2\right)$ is uniform in θ, note that $\tilde{R}(\theta,\beta_\delta)$ is a smooth 2π-periodic function in θ for our choice of $\beta_\delta>0$. This means that the extrema (in θ) for $\frac{1}{\beta_\delta^2}R(\tilde{\theta},\beta_\delta)$ are finite.

Since we know that $0.115\pi<\psi(\beta_\delta)<0.205\pi$ (note Assumption 2), we know it is possible for $\beta_\delta>0$ to be sufficiently small such that $\frac{\pi}{9}\le\psi(\beta_\delta)-\beta_\delta\frac{b(0)}{a(0)}\le\frac{2\pi}{9}$ and $\left|\frac{b(0)}{a(0)}-\frac{b(\beta_\delta)}{a(\beta_\delta)(1+\beta_\delta)}\right|<\delta_2$. Note for conciseness, we denote $\Omega_{\beta_\delta}=\psi(\beta_\delta)-\beta_\delta\frac{b(0)}{a(0)}$.

We then proceed with the following approximation for the dynamics on the circle and show the approximation error is negligible for the following two cases.

$$\theta\to\tilde{\theta}\approx\theta+\Omega_{\beta_\delta}$$

Case 1: the trajectory iterate is not in I_1 or I_3. In this case, we will show how the trajectory iterate can enter I_1 and I_3 with negligible approximation error. WLOG, consider the iterate located at $\Omega_{1,2} < \theta_0 < \Omega_{3,1}$. To reach I_3 from θ_0, it would take $\left\lceil \frac{\Omega_{3,1}-\theta_0}{\Omega_{\beta_\delta}} \right\rceil$ steps to reach I_3 by our approximation. We can upper bound $\frac{\Omega_{3,1}-\theta_0}{\Omega_{\beta_\delta}}$ by $\frac{0.755\pi}{\frac{\pi}{9}} = 9\,(0.755) < 7$. Then, our approximation error is $7\left(\beta_\delta \delta_2 + O\left(\beta_\delta^2\right)\right)$.

Case 2: the trajectory iterate is in I_1 or I_3. WLOG, assume that the trajectory iterate is in I_1 (denoted as $\Omega_{1,1} \leq \theta_0 \leq \Omega_{1,2}$). We will show that it can enter I_3 with negligible approximation error. To reach I_3 from θ_0, it would take $\left\lceil \frac{\Omega_{3,1}-\theta_0}{\Omega_{\beta_\delta}} \right\rceil$ steps to reach I_3 by our approximation. We can upper bound $\frac{\Omega_{3,1}-\theta_0}{\Omega_{\beta_\delta}}$ by $\frac{\pi}{\frac{\pi}{9}} = 9$. Then, our approximation error is $9\left(\beta_\delta \delta_2 + O\left(\beta_\delta^2\right)\right)$.

So, our approximation accumulates the error from at most 9 steps to show perpetual oscillation in loss and sharpness within at most one rotation of the invariant circle.

This completes the proof.

□

Proof of Lemma in Section B.5.4

Proof of Lemma B.149. To prove the lemma, we first note that we already showing the eigenvalues of the system Jacobian at fixed point $(\tilde{q}, \tilde{c})$ are complex at $\eta_{c,m}$. So, we know that $\theta_{c,m} > 0$. To show that $\theta_{c,m} < \frac{\pi}{2}$, we will show that the trace of the system Jacobian is positive at $(\tilde{q}, \tilde{c})$ for $\theta_{c,m}$. Consider the trace of the system Jacobian below at $(\tilde{q}, \tilde{c})$ for $0.2 \leq \eta \leq 0.3$ (with $\eta\tilde{q}$ factored out).

$$\begin{aligned}
&\left(-\frac{1}{2}\left(\sqrt{\tilde{c}}+m\right)^2+\tilde{q}\right)\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}}\right)\right)\\
&+\left(-\frac{1}{2}\left(\sqrt{\tilde{c}}-m\right)^2+\tilde{q}\right)\left(1+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m-\sqrt{\tilde{c}}\right)\right)
\end{aligned} \tag{239}$$

To show the trace is positive, we will show that $\left(-\frac{1}{2}\left(\sqrt{\tilde{c}}+m\right)^2+\tilde{q}\right)$ and $\left(-\frac{1}{2}\left(\sqrt{\tilde{c}}-m\right)^2+\tilde{q}\right)$ are positive.

We start with $\left(-\frac{1}{2}\left(\sqrt{\tilde{c}}+m\right)^2+\tilde{q}\right)$. Note that it can be lower bounded by $-\frac{1}{2}\left(\sqrt{0.2375}+0.25\right)^2+3.74483$, which is positive.

For $\left(-\frac{1}{2}\left(\sqrt{\tilde{c}}-m\right)^2+\tilde{q}\right)$, note that is can be lower bounded by $-0.5\,(0.2375)+(0.15)\sqrt{0.1175}-0.5\,(0.25)^2+3.74483$, which is positive.

This completes the proof.

□

Proofs of Lemmas in Section B.5.4

Proof of Lemma B.150. To prove the lemma, we will show that the directional derivative for our sharpness approximation is positive is region 3 and negative in region 1. Consider the directional derivative for the sharpness approximation shown below (after factoring out q^2, η, and $\frac{1}{\sqrt{c^2}}$).

$$
\begin{aligned}
&\sqrt{c^2}\left[\tau_{1,p}+\tau_{1,m}\right]\\
&\cdot\left[\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}+m\right)-\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}-m\right)\right]\\
&\cdot\left[1+\frac{\eta}{4}\left(\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}+m\right)-\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}-m\right)\right)\right]\\
&-q^2\left[\tau_{2,p}+\tau_{2,m}\right]\\
&\cdot\left[\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\right]\\
&\cdot\left[\sqrt{c^2}-\eta q\left(\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\right)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_{1,p}&=\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\\
&\quad\cdot\left(2+\frac{1}{2}q\left(m+\sqrt{c^2}\right)-q\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\left(m+\sqrt{c^2}\right)\right)\\
\tau_{1,m}&=\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\\
&\quad\cdot\left(2+\frac{1}{2}q\left(m-\sqrt{c^2}\right)-q\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\left(m-\sqrt{c^2}\right)\right)\\
\tau_{2,p}&=\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\\
&\quad\cdot\left(1-2\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)\\
\tau_{2,m}&=\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-2}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\\
&\quad\cdot\left(1-2\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)
\end{aligned}
$$

To prove the lemma, we note the following. First, since $\sqrt{c^2}>\sqrt{0.1175}>0.25\geq m$, then

$$
\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}
$$

is positive.

Second, from lemma B.151, we know that

$$2+\frac{1}{2}q\left(m+\sqrt{c^2}\right)-q\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\left(m+\sqrt{c^2}\right)$$

is positive.

Third, from lemma B.152, we know that

$$2+\frac{1}{2}q\left(m-\sqrt{c^2}\right)-q\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\left(m-\sqrt{c^2}\right)$$

is positive.

Fourth, from lemma B.153, we know that

$$1+\frac{\eta}{4}\left(\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}+m\right)-\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}-m\right)\right)$$

is positive.

And last, from lemma B.154, we know that $\tau_{2,p}+\tau_{2,m}$ is negative.

So, if the trajectory iterate satisfies

$$\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}+m\right)-\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}-m\right)>0$$

$$\sqrt{c^2}-\eta q\left(\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\right)>0$$

(corresponding to region 3), then the sharpness approximation will increase.

Similarly, if the trajectory iterate satisfies

$$\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}+m\right)-\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}-m\right)<0$$

$$\sqrt{c^2}-\eta q\left(\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\right)<0$$

(corresponding to region 1), then our sharpness approximation will decrease.

This completes the proof. □

Proof of Lemma B.151. To prove the lemma, we begin by rewriting Equation 150 as

$$2+\frac{1}{2}q\left(m+\sqrt{c^2}\right)\left(1-2\left(1+\exp\left(-\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\right).$$

Note that $1-2\left(1+\exp\left(-\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}$ is negative, which we can see with the following upper bound, $1-2\left(1+\exp\left(-\frac{1}{2}\left(3.74483\right)\left(0.15+\sqrt{0.1175}\right)\right)\right)^{-1}$ that is negative. So, we can lower bound

$$2+\frac{1}{2}q\left(m+\sqrt{c^2}\right)\left(1-2\left(1+\exp\left(-\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\right).$$

by

$$2+\frac{1}{2}\left(4.84375\right)\left(0.25+\sqrt{0.2375}\right)\left(1-2\left(1+\exp\left(-\frac{1}{2}\left(4.84375\right)\left(0.25+\sqrt{0.2375}\right)\right)\right)^{-1}\right),$$

which is positive.

This completes the proof. □

Proof of Lemma B.152. To begin the proof, first note that we rewrite Equation 151 as

$$2-\frac{1}{2}q\left(\sqrt{c^2}-m\right)\left(1-2\left(1+\exp\left(-\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\right).$$

Note that $\left(1-2\left(1+\exp\left(\frac{1}{2}q\left(\sqrt{c^2}-m\right)\right)\right)^{-1}\right)$ is positive. To see this, we can lower bound it with $1-2\left(1+\exp\left(\frac{1}{2}\left(3.74483\right)\left(\sqrt{0.1175}-0.25\right)\right)\right)^{-1}$, which is positive.

From this, we can lower bound

$$2-\frac{1}{2}q\left(\sqrt{c^2}-m\right)\left(1-2\left(1+\exp\left(-\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\right)$$

with

$$2-\frac{1}{2}\left(4.84375\right)\left(\sqrt{0.2375}-0.15\right)\left(1-2\left(1+\exp\left(\frac{1}{2}\left(4.84375\right)\left(\sqrt{0.2375}-0.15\right)\right)\right)^{-1}\right),$$

which is positive.

This completes the proof.

□

Proof of Lemma B.153. To prove the lemma, we first lower bound Equation B.153 by

$$1+\frac{\eta}{4}\Bigg(\left(1+\exp\left(\frac{1}{2}\left(4.84375\right)\left(0.25+\sqrt{0.2375}\right)\right)\right)^{-1}\left(\sqrt{c}+m\right)$$
$$-\left(1+\exp\left(\frac{1}{2}\left(4.84375\right)\left(0.15-\sqrt{0.2375}\right)\right)\right)^{-1}\left(\sqrt{c}-m\right)\Bigg),$$

which simplifies (by rounding down after five digits of precision) down to $1+\frac{n}{4}\left(-0.55001\sqrt{c^2}+0.83719m\right)$. We further lower bound the above by $1+\frac{0.3}{4}\left(-0.55001\sqrt{0.2375}+0.83719\left(0.15\right)\right)$, which is positive.

This completes the proof. □

Proof of Lemma B.154. To prove the lemma, we begin by introducing a change of variable, where $u=\sqrt{c^2}$ for $\tau_{2,p}+\tau_{2,m}$ and rescaling by $\left(1+\exp\left(\frac{1}{2}q\left(m+u\right)\right)\right)^3\left(1+\exp\left(\frac{1}{2}q\left(m-u\right)\right)\right)^3$.

This yields the following expression:

$$6-6\exp\left(qm\right)-\exp\left(\frac{1}{2}q\left(3m-u\right)\right)-\exp\left(qu\right)+\exp\left(q\left(m-u\right)\right)+\exp\left(\frac{1}{2}q\left(u-m\right)\right)-\exp\left(\frac{1}{2}q\left(3m+u\right)\right)$$
$$+\exp\left(q\left(m+u\right)\right)-\exp\left(-qu\right)+\exp\left(-\frac{1}{2}q\left(m+u\right)\right) \quad (240)$$

To show that the above is negative, we begin by showing that it is locally increasing in u (where we show convexity in u and that first partial u at $u=\sqrt{0.1175}$ is positive). After upper bounding the equation at $u=\sqrt{0.2375}$, we then show this upper bound is locally convex in q and that the upper bound evaluated at both q ends, $3.74483, 4.84375$, is negative.

We begin with the argument for convexity in u. After factoring out q, we show the second partial derivative with respect to u below.

$$\frac{1}{4}\exp\left(-\frac{1}{2}q\left(m+u\right)\right)+\Bigg(-\exp\left(-qu\right)+\exp\left(q\left(m-u\right)\right)-\frac{1}{4}\exp\left(-\frac{1}{2}q\left(u-3m\right)\right)+\frac{1}{4}\exp\left(\frac{1}{2}q\left(u-m\right)\right)\Bigg)$$
$$+\left(-\exp\left(qu\right)+\exp\left(q\left(m+u\right)\right)-\frac{1}{4}\exp\left(\frac{1}{2}q\left(3m+u\right)\right)\right)$$

To see that the second partial derivative is locally positive, we show that the following subexpressions of the second partial derivative are locally positive in lemmas B.156 and B.157.

$$-\exp\left(-qu\right)+\exp\left(q\left(m-u\right)\right)-\frac{1}{4}\exp\left(-\frac{1}{2}q\left(u-3m\right)\right)+\frac{1}{4}\exp\left(\frac{1}{2}q\left(u-m\right)\right)$$

$$-\exp\left(qu\right)+\exp\left(q\left(m+u\right)\right)-\frac{1}{4}\exp\left(\frac{1}{2}q\left(3m+u\right)\right)$$

Now that we've shown convexity in u, we next show that first partial u at $u=\sqrt{0.1175}$ is positive. Consider the partial derivative below (after factoring out q).

$$\begin{aligned}\frac{1}{2}\exp\left(\frac{1}{2}q\left(3m-\sqrt{0.1175}\right)\right)-\exp\left(q\sqrt{0.1175}\right)-\exp\left(q\left(m-\sqrt{0.1175}\right)\right)+\frac{1}{2}\exp\left(\frac{1}{2}q\left(\sqrt{0.1175}-m\right)\right)\\ -\frac{1}{2}\exp\left(\frac{1}{2}q\left(3m+\sqrt{0.1175}\right)\right)+\exp\left(q\left(m+\sqrt{0.1175}\right)\right)+\exp\left(-q\sqrt{0.1175}\right)\\ -\frac{1}{2}\exp\left(-\frac{1}{2}q\left(m+\sqrt{0.1175}\right)\right)\end{aligned}\tag{241}$$

To show that the above partial derivative is positive, we first show that it is convex in q. Consider the second partial derivative with respect to q below.

$$\begin{aligned}&\frac{1}{8}\left(\sqrt{0.1175}-m\right)^2\exp\left(\frac{1}{2}x\left(\sqrt{0.1175}-m\right)\right)+\frac{1}{8}\left(3m-\sqrt{0.1175}\right)^2\exp\left(\frac{1}{2}x\left(3m-\sqrt{0.1175}\right)\right)\\ &+\left(-\frac{1}{8}\left(3m+\sqrt{0.1175}\right)^2\exp\left(\frac{1}{2}x\left(3m+\sqrt{0.1175}\right)\right)+\left(m+\sqrt{0.1175}\right)^2\exp\left(x\left(m+\sqrt{0.1175}\right)\right)\right.\\ &\qquad\left.-0.1175\exp\left(x\sqrt{0.1175}\right)-\left(\sqrt{0.1175}-m\right)^2\exp\left(-x\left(\sqrt{0.1175}-m\right)\right)\right)\\ &\quad+\left(-\frac{1}{8}\left(m+\sqrt{0.1175}\right)^2\exp\left(-\frac{1}{2}x\left(m+\sqrt{0.1175}\right)\right)+0.1175\exp\left(-x\sqrt{0.1175}\right)\right)\end{aligned}$$

To see that this second partial derivative is positive, we will show that the following subexpressions are positive.

$$\begin{aligned}-\frac{1}{8}\left(3m+\sqrt{0.1175}\right)^2\exp\left(\frac{1}{2}q\left(3m+\sqrt{0.1175}\right)\right)-\left(\sqrt{0.1175}-m\right)^2\exp\left(-q\left(\sqrt{0.1175}-m\right)\right)\\ -0.1175\exp\left(q\sqrt{0.1175}\right)+\left(m+\sqrt{0.1175}\right)^2\exp\left(q\left(m+\sqrt{0.1175}\right)\right)\end{aligned}\tag{242}$$

$$-\frac{1}{8}\left(m+\sqrt{0.1175}\right)^2\exp\left(-\frac{1}{2}q\left(m+\sqrt{0.1175}\right)\right)+0.1175\exp\left(-q\sqrt{0.1175}\right)\tag{243}$$

For subexpression 1, note that we can lower bound it by

$$\begin{aligned}-\frac{1}{8}\left(3\left(0.25\right)+\sqrt{0.1175}\right)^2\exp\left(-\frac{1}{2}\left(3.74483\right)\left(\sqrt{0.1175}-0.25\right)\right)\\ -\left(\sqrt{0.1175}-0.15\right)^2\exp\left(-2\left(3.74483\right)\sqrt{0.1175}\right)\\ -0.1175\exp\left(-\left(3.74483\right)\left(0.15\right)\right)+\left(0.15+\sqrt{0.1175}\right)^2,\end{aligned}$$

which is positive.

For subexpression 2, note that we can lower bound it by

$$-\frac{1}{8}\left(0.25+\sqrt{0.1175}\right)^2\exp\left(\frac{1}{2}\left(4.84375\right)\left(\sqrt{0.1175}-0.15\right)\right)+0.1175,$$

which is positive.

After proving convexity in q, we will show the the first partial q of Equation 241 is positive at $q=3.74483$, shown below.

$$\begin{aligned}
&\frac{1}{4}\left(3m-\sqrt{0.1175}\right)\exp\left(\frac{1}{2}\left(3.74483\right)\left(3m-\sqrt{0.1175}\right)\right)-\sqrt{0.1175}\exp\left(\left(3.74483\right)\sqrt{0.1175}\right)\\
&+\left(\sqrt{0.1175}-m\right)\exp\left(-\left(3.74483\right)\left(\sqrt{0.1175}-m\right)\right)+\frac{1}{4}\left(\sqrt{0.1175}-m\right)\exp\left(\frac{1}{2}\left(3.74483\right)\left(\sqrt{0.1175}-m\right)\right)\\
&-\frac{1}{4}\left(3m+\sqrt{0.1175}\right)\exp\left(\frac{1}{2}\left(3.74483\right)\left(3m+\sqrt{0.1175}\right)\right)+\left(m+\sqrt{0.1175}\right)\exp\left(\left(3.74483\right)\left(m+\sqrt{0.1175}\right)\right)\\
&-\sqrt{0.1175}\exp\left(-\left(3.74483\right)\sqrt{0.1175}\right)+\frac{1}{4}\left(m+\sqrt{0.1175}\right)\exp\left(-\frac{1}{2}\left(3.74483\right)\left(m+\sqrt{0.1175}\right)\right)
\end{aligned}$$

We scale the first partial q by 4 and lower bound it with the following

$$\begin{aligned}
&3\exp\left(\frac{1}{2}\left(3.74483\right)\left(3\left(0.15\right)-\sqrt{0.1175}\right)\right)q-\sqrt{0.1175}\exp\left(\frac{1}{2}\left(3.74483\right)\left(3\left(0.15\right)-\sqrt{0.1175}\right)\right)\\
&+q\exp\left(-\frac{1}{2}\left(3.74483\right)\left(0.25+\sqrt{0.1175}\right)\right)+\sqrt{0.1175}\exp\left(-\frac{1}{2}\left(3.74483\right)\left(0.25+\sqrt{0.1175}\right)\right)\\
&-3\exp\left(\frac{1}{2}\left(3.74483\right)\left(3\left(0.25\right)+\sqrt{0.1175}\right)\right)q-\sqrt{0.1175}\exp\left(\frac{1}{2}\left(3.74483\right)\left(3\left(0.25\right)+\sqrt{0.1175}\right)\right)\\
&+\sqrt{0.1175}\exp\left(\frac{1}{2}\left(3.74483\right)\left(\sqrt{0.1175}-0.25\right)\right)-q\exp\left(\frac{1}{2}\left(3.74483\right)\left(\sqrt{0.1175}-0.25\right)\right)\\
&+4\sqrt{0.1175}\exp\left(-\left(3.74483\right)\left(\sqrt{0.1175}-0.15\right)\right)-4q\exp\left(-\left(3.74483\right)\left(\sqrt{0.1175}-0.15\right)\right)\\
&+4q\exp\left(\left(3.74483\right)\left(0.15+\sqrt{0.1175}\right)\right)+4\sqrt{0.1175}\exp\left(\left(3.74483\right)\left(0.15+\sqrt{0.1175}\right)\right)\\
&-4\sqrt{0.1175}\exp\left(\left(3.74483\right)\sqrt{0.1175}\right)-4\sqrt{0.1175}\exp\left(-\left(3.74483\right)\sqrt{0.1175}\right),
\end{aligned}$$

which we simplify (by rounding down after five digits of precision) to $2.97180m+1.46608$, which is positive.

So, we can lower bound Equation 241 at $q=3.74483$, we simplifies (by rounding down after five digits of precision) to

$$\left(-0.68682+3.33281\exp\left(-\frac{1}{2}\left(3.74483\right)m\right)+0.68681\exp\left(-2\left(3.74483\right)m\right)\right)\exp\left(\frac{3}{2}\left(3.74483\right)m\right)-3.33282,$$

that can be lower bounded by

$$\left(-0.68682+3.33281\exp\left(-\frac{1}{2}\left(3.74483\right)\left(0.25\right)\right)+0.68681\exp\left(-2\left(3.74483\right)\left(0.25\right)\right)\right)\exp\left(\frac{3}{2}\left(3.74483\right)\left(0.15\right)\right)-3.33282,$$

which is positive.

So, we can lower bound Equation 240 by $u=\sqrt{0.2375}$. We will show that this lower bounded equation is convex in q.

Consider the second partial derivative with respect to q below.

$$\left(\frac{1}{4}\left(m+\sqrt{0.2375}\right)^2\exp\left(-\frac{1}{2}q\left(m+\sqrt{0.2375}\right)\right)+\frac{1}{4}\left(\sqrt{0.2375}-m\right)^2\exp\left(\frac{1}{2}q\left(\sqrt{0.2375}-m\right)\right)\right.$$
$$\left.+\left(\sqrt{0.2375}-m\right)^2\exp\left(-q\left(\sqrt{0.2375}-m\right)\right)\right)$$

$$+\left(-0.2375\exp\left(q\sqrt{0.2375}\right)-6m^2\exp\left(qm\right)+\left(m+\sqrt{0.2375}\right)^2\exp\left(q\left(m+\sqrt{0.2375}\right)\right)\right.$$
$$-\frac{1}{4}\left(3m+\sqrt{0.2375}\right)^2\exp\left(\frac{1}{2}q\left(3m+\sqrt{0.2375}\right)\right)-0.2375\exp\left(-q\sqrt{0.2375}\right)$$
$$\left.-\frac{1}{4}\left(3m-\sqrt{0.2375}\right)^2\exp\left(\frac{1}{2}q\left(3m-\sqrt{0.2375}\right)\right)\right)$$

To see that the second partial derivative above is positive, note that the following subexpression is positive by lemma B.155.

$$-0.2375\exp\left(q\sqrt{0.2375}\right)-6m^2\exp\left(qm\right)+\left(m+\sqrt{0.2375}\right)^2\exp\left(q\left(m+\sqrt{0.2375}\right)\right)$$
$$-\frac{1}{4}\left(3m+\sqrt{0.2375}\right)^2\exp\left(\frac{1}{2}q\left(3m+\sqrt{0.2375}\right)\right)-0.2375\exp\left(-q\sqrt{0.2375}\right)$$
$$-\frac{1}{4}\left(3m-\sqrt{0.2375}\right)^2\exp\left(\frac{1}{2}q\left(3m-\sqrt{0.2375}\right)\right)$$

By local convexity in q, we evaluate Equation 240 (lower bounded at $u=\sqrt{0.2375}$) at both q ends, $3.74483, 4.84375$. For $q=3.74483$, we simplify (by rounding up after five digits of precision) the expression to

$$-0.36403+\left(0.36404\exp\left(-\frac{1}{2}\left(3.74483\right)m\right)-2.89206+2.89207\exp\left(-2\left(3.74483\right)m\right)\right)\exp\left(\frac{3}{2}\left(3.74483\right)m\right)$$

that we can upper bound by

$$\left(0.36404\exp\left(-\frac{1}{2}\left(3.74483\right)\left(0.15\right)\right)-2.89206+2.89207\exp\left(-2\left(3.74483\right)\left(0.15\right)\right)\right)\exp\left(\frac{3}{2}\left(3.74483\right)\left(0.15\right)\right)$$
$$-0.36403,$$

which is negative.

For $q=4.84375$, we simplify (by rounding up after five digits of precision) the expression to the following

$$-4.69116+\left(4.69117\exp\left(-\frac{1}{2}\left(4.84375\right)m\right)-3.56246+3.56247\exp\left(-2\left(4.84375\right)m\right)\right)\exp\left(\frac{3}{2}\left(4.84375\right)m\right)$$

that can be upper bounded by

$$\left(4.69117\exp\left(-\frac{1}{2}\left(4.84375\right)\left(0.15\right)\right)-3.56246+3.56247\exp\left(-2\left(4.84375\right)\left(0.15\right)\right)\right)\exp\left(\frac{3}{2}\left(4.84375\right)\left(0.25\right)\right)$$
$$-4.69116,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.155. To prove the lemma, we first factor out $\exp\left(q\left(m+\sqrt{0.2375}\right)\right)$ from Equation 152.

Note that we can lower bound the factored expression by

$$-0.2375\exp\left(-\left(3.74483\right)m\right)-6m^2\exp\left(-\left(3.74483\right)\sqrt{0.2375}\right)$$
$$-\frac{1}{4}\left(3m+\sqrt{0.2375}\right)^2\exp\left(-\frac{1}{2}\left(3.74483\right)\left(\sqrt{0.2375}-m\right)\right)-0.2375\exp\left(-\left(3.74483\right)\left(m+2\sqrt{0.2375}\right)\right)$$
$$-\frac{1}{4}\left(3m-\sqrt{0.2375}\right)^2\exp\left(-\frac{1}{2}\left(3.74483\right)\left(3\sqrt{0.2375}-m\right)\right)+\left(m+\sqrt{0.2375}\right)^2,$$

which we can simplify (by rounding down after five digits of precision) to

$$\left(-0.24368+\left(-1.04913m^2-0.50514\sqrt{0.2375}m-0.02769\right)\exp\left(\frac{3}{2}\left(3.74483\right)m\right)\right)\exp\left(-\left(3.74483\right)m\right)$$
$$+0.03269x^2+2\sqrt{0.2375}m+0.2375.$$

We can then lower bound the above by

$$\left(-0.24368+\left(-1.04913m^2-0.50514\sqrt{0.2375}m-0.02769\right)\exp\left(\frac{3}{2}\left(3.74483\right)\left(0.25\right)\right)\right)\exp\left(-\left(3.74483\right)m\right)$$
$$+0.03269m^2+2\sqrt{0.2375}m+0.2375$$

which simplifies (by rounding down after five digits of precision to

$$\left(-4.27282m^2-1.00261m-0.35646\right)\exp\left(-\left(3.74483\right)m\right)+0.03269m^2+0.97467m+0.2375.$$

To further lower bound the equation, note that $\exp\left(-\left(3.74483\right)m\right)$ is convex in m, so we can locally upper bound with a linear function $-1.78112m+0.838$. To see that this is a local upper bound, note that $-1.78112m+0.838-\exp\left(-\left(3.74483\right)m\right)$ is concave, so we can evaluate the difference at both m ends $(0.15, 0.25)$, which are positive.

Substituting in the linear function yields the following expression (simplified by rounding down after five digits of precision), $7.61040m^3-1.76217m^2+0.76938m-0.06122$, that can be lower bounded by $\left(\left(7.61040\left(0.15\right)-1.76217\right)\left(0.25\right)+0.76938\right)\left(0.15\right)-0.06122$, which is positive.

This completes the proof.

□

Proof of Lemma B.156. To prove the lemma, we first factor out $\exp\left(qu\right)$ from Equation 153. We first lower bound the factored expression in u with by setting $u=\sqrt{0.1175}$. Note that this lower bound is increasing in q. To see this, note that the partial derivative with respect to q (after factoring out $\exp\left(-qm\right)$), $m-\frac{1}{8}\left(3m-\sqrt{0.1175}\right)\exp\left(\frac{1}{2}x\left(m-\sqrt{0.1175}\right)\right)$, can be lower bounded by $0.15-\frac{1}{8}\left(3\left(0.25\right)-\sqrt{0.1175}\right)\exp\left(\frac{1}{2}\left(3.74483\right)\left(0.25-\sqrt{0.1175}\right)\right)$, which is positive. So, we further lower bound by evaluating at $q=3.74483$, which produces $-1+\exp\left(\left(3.74483\right)m\right)-\frac{1}{4}\exp\left(\frac{1}{2}\left(3.74483\right)\left(3m-\sqrt{0.1175}\right)\right)$. This can be further lower bounded by $-1+\exp\left(\left(3.74483\right)\left(0.15\right)\right)-\frac{1}{4}\exp\left(\frac{1}{2}\left(3.74483\right)\left(3\left(0.25\right)-\sqrt{0.1175}\right)\right)$, which is positive.

This completes the proof.

□

Proof of Lemma B.157. To prove the lemma, we first factor out $\exp\left(-qu\right)$ from Equation B.157.

To prove the factored expression is positive, we first show that it is increasing in u, so we can lower bound at $u=\sqrt{0.1175}$. Then, we show that this is lower bound is decreasing in m, so we can lower bound at $m=0.25$. Last, we show that this lower bound is decreasing in q, so we can lower bound at $q=4.84375$.

First, we show that the first partial u is positive. Consider the partial derivative (after factoring out $\frac{1}{8}q\exp\left(\frac{1}{2}qu\right)$), $-\exp\left(\frac{3}{2}qm\right)+3\exp\left(\frac{1}{2}q\left(2u-m\right)\right)$. We can lower bound this by $-\exp\left(\frac{3}{2}\left(4.84375\right)\left(0.25\right)\right)+3\exp\left(\frac{1}{2}\left(3.74483\right)\left(2\sqrt{0.1175}-0.25\right)\right)$, which is positive.

So, we can lower bound our factored version of Equation B.157 in u at $u = \sqrt{0.1175}$, and now we show that the lower bound is decreasing in m. Consider the partial derivative with respect to m (after factoring out $8q\exp(qm)$), $8-3\exp\left(\frac{1}{2}q\left(m+\sqrt{0.1175}\right)\right)-\exp\left(\frac{3}{2}q\left(\sqrt{0.1175}-m\right)\right)$, that we can upper bound by $8-3\exp\left(\frac{1}{2}\left(3.74483\right)\left(0.15+\sqrt{0.1175}\right)\right)-\exp\left(\frac{3}{2}\left(3.74483\right)\left(\sqrt{0.1175}-0.25\right)\right)$, which is negative.

Then, we can lower bound at $m = 0.25$ and show that partial q is negative. Consider the partial q (simplified by rounding up after five digits of precision and factoring out $\exp\left(0.54639q\right)$), $0.25\exp\left(-0.29639q\right) - 0.13659 + 0.09730\exp\left(-0.15721q\right)$, that we can upper bound by $0.25\exp\left(-0.29639\left(3.74483\right)\right)-0.13659+0.09730\exp\left(-0.15721\left(3.74483\right)\right)$, which is negative.

So, we can lower bound at $q = 4.84375$, resulting in the following

$$-1+\exp\left(0.25\left(4.84375\right)\right)-\frac{1}{4}\exp\left(\frac{1}{2}\left(4.84375\right)\left(0.75+\sqrt{0.1175}\right)\right)+\frac{1}{4}\exp\left(\frac{1}{2}\left(4.84375\right)\left(3\sqrt{0.1175}-0.25\right)\right)$$

that is positive.

This completes the proof.

□

Proof of Lemma B.158. To prove the theorem, we will show that the directional derivative of the loss is positive in region 1 and negative in region 3. Consider the directional derivative below (after factoring out $\eta q\exp(-qm)\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}$).

$$-\frac{1}{2}\mu_{1,1}^2\mu_{1,2} - \frac{q}{\sqrt{c^2}}\mu_{2,1}^2\mu_{2,2}, \tag{244}$$

where

$$\mu_{1,1} = \left(2m+\left(\sqrt{c^2}+m\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)-\left(\sqrt{c^2}-m\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)$$

$$\mu_{1,2} = \left(1+\frac{\eta}{4}\left(\left(1+\exp\left(\frac{1}{2}q\left(\sqrt{c^2}+m\right)\right)\right)^{-1}\left(\sqrt{c^2}+m\right)-\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}\left(\sqrt{c^2}-m\right)\right)\right)$$

$$\mu_{2,1} = \exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)-\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)$$

$$\mu_{2,2} = \sqrt{c^2}-\eta q\left(\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}-\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\right)$$

Using lemma B.153, we can see that a necessary condition for the directional derivative to be positive is $\mu_{2,2} < 0$. So, the trajectory iterate entering region 3 is a sufficient condition for the loss to decrease.

To see that the trajectory iterate entering region 1 is a sufficient condition for the loss to increase, we proceed as follows. First, we derive a simpler sufficient condition for the loss to increase using the directional derivative expression. Then, we linearize the sufficient condition about the fixed point $\left(\tilde{q},\tilde{c}^2\right)$ (since the invariant curve is assumed close to the fixed point) and use the linearization to show that the trajectory iterate entering region 1 causes the loss to increase.

We begin with deriving a simpler sufficient condition for the loss to increase, leveraging that $q \in [3.74483, 4.84375], c^2 \in [0.1175, 0.2375]$. Note that we rescale the directional derivative by $\sqrt{c^2}\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)^{-1}\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^{-1}$ to get Equation 245 > Equation 246 as the loss increase condition, for Equation 245, 246 defined below.

$$2q\left(\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)-\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)^2\left(\eta q\left(\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)-\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)\right.$$
$$\left.-\sqrt{c}\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)+\exp\left(qm\right)\right)\right) \quad (245)$$

$$\sqrt{c^2}\left(2m+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\left(\sqrt{c^2}+m\right)-\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\left(\sqrt{c^2}-m\right)\right)^2$$
$$\cdot\left(\left(1+\frac{1}{2}m\eta\right)+\left(1+\frac{1}{4}\eta\left(\sqrt{c^2}+m\right)\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right.$$
$$\left.+\left(1-\frac{1}{4}\eta\left(\sqrt{c^2}-m\right)\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)+\exp\left(mq\right)\right) \quad (246)$$

To simpler the condition, we lower bound

$$\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)-\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)$$

by

$$\left(\exp\left(\frac{1}{2}\left(3.74483\right)\left(0.15\right)\right)\left(\exp\left(\frac{1}{2}\left(3.74483\right)\sqrt{0.1175}\right)-\exp\left(-\frac{1}{2}\left(3.74483\right)\sqrt{0.1175}\right)\right)\right)$$

and upper bound

$$\left(1+\frac{1}{2}m\eta\right)+\left(1+\frac{1}{4}\eta\left(\sqrt{c^2}+m\right)\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)+\left(1-\frac{1}{4}\eta\left(\sqrt{c^2}-m\right)\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)$$
$$+\exp\left(mq\right)$$

by

$$\left(1+0.1\left(0.25\right)\right)+\left(1+0.05\left(\sqrt{0.2375}+\left(0.25\right)\right)\right)\exp\left(2.421875\left(0.25-\sqrt{0.2375}\right)\right)$$
$$+\left(1-0.05\left(\sqrt{0.2375}-\left(0.25\right)\right)\right)\exp\left(2.421875\left(0.25+\sqrt{0.2375}\right)\right)+\exp\left(4.84375\left(0.25\right)\right).$$

We prove these bounds in lemmas B.159 and B.160.

From these bounds, we have that the following a sufficient condition for the loss to increase.

$$\left(\sqrt{c^2}\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)+\exp\left(qm\right)\right)-\eta q\left(\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right.\right.$$
$$\left.\left.-\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)\right)$$

$$+0.21353\left(2m+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\left(\sqrt{c^2}+m\right)-\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\left(\sqrt{c^2}-m\right)\right)^2<0$$

By linearizing the above sufficient condition about the fixed point, we see that the loss will increase as long as the linearization of

$$\sqrt{c^2}\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)+\exp\left(qm\right)\right)-\eta q\left(\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right.$$
$$\left.-\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right),$$

about the fixed point is negative, which is satisfied in region 1 based on the proof of lemma B.161.

This completes the proof.

□

Proof of Lemma B.159. To prove the lemma, we begin by showing that

$$\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)-\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)$$

grow in c^2.

To see this, note that first partial c^2, shown below, which is positive.

$$\frac{q}{4\sqrt{c^2}}\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)+\frac{q}{4\sqrt{c^2}}\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)$$

So, we can lower bound in c^2 by $c^2=0.1175$.

Next, note that our lower bound grows in q. To see this, note the first partial q of the lower bound below, which is positive (since $\sqrt{0.1175}-0.25>0$).

$$\frac{1}{2}\left(m+\sqrt{0.1175}\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{0.1175}\right)\right)+\frac{1}{2}\left(\sqrt{0.1175}-m\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{0.1175}\right)\right)$$

So, we can further lower bound in q by $q=3.74483$, resulting the following expression.

$$\exp\left(\frac{1}{2}\left(3.74483\right)m\right)\left(\exp\left(\frac{1}{2}\left(3.74483\right)\sqrt{0.1175}\right)-\exp\left(-\frac{1}{2}\left(3.74483\right)\sqrt{0.1175}\right)\right)$$

We can lower bound the above in m by $m=0.15$.

This completes the proof.

□

Proof of Lemma B.160. To prove the lemma, we begin with a change of variable, where $u=\sqrt{c^2}$, resulting in the following expression.

$$\left(1+\frac{1}{2}m\eta\right)+\left(1+\frac{1}{4}\eta\left(u+m\right)\right)\exp\left(\frac{1}{2}q\left(m-u\right)\right)+\left(1-\frac{1}{4}\eta\left(u-m\right)\right)\exp\left(\frac{1}{2}q\left(m+u\right)\right)+\exp\left(mq\right) \tag{247}$$

To upper bound the above, we begin by showing that it is locally convex in u, and the first partial u is positive at $u=\sqrt{0.1175}$, so we can upper bound in u by $u=\sqrt{0.2375}$. Then, we show that the resulting upper bound is locally convex in q, and the first partial q is positive at $q=3.74483$, so we can upper bound in q by $q=4.84375$. Next, we show the upper bound is decreasing in η, so can upper bound in η by $\eta=0.2$. Last, we show that the resulting expression is locally convex in m, and the first partial m at $m=0.15$ is positive, so we can upper bound in m by $m=0.25$.

We begin with showing the Equation 247 is convex in u. To do so, we'll show that $\left(1+\frac{1}{4}\eta\left(u+m\right)\right)\exp\left(\frac{1}{2}q\left(m-u\right)\right)$ and $\left(1-\frac{1}{4}\eta\left(u-m\right)\right)\exp\left(\frac{1}{2}q\left(m+u\right)\right)$ are convex in u.

First, consider $\left(1+\frac{1}{4}\eta\left(u+m\right)\right)\exp\left(\frac{1}{2}q\left(m-u\right)\right)$. The second partial derivative with respect to u (after factoring out $q\exp\left(\frac{1}{2}q\left(m-u\right)\right)$ is $-\frac{1}{4}\eta+\frac{1}{4}q+\frac{1}{16}\eta q\left(u+m\right)$. The above can be lower bounded by $-\frac{1}{4}\left(0.3\right)+\frac{1}{4}\left(3.74483\right)+\frac{1}{16}\left(0.2\right)\left(3.74483\right)\left(\sqrt{0.1175}+0.15\right)$, which is positive.

Second, consider $\left(1-\frac{1}{4}\eta\left(u-m\right)\right)\exp\left(\frac{1}{2}q\left(m+u\right)\right)$. The second partial derivative with respect to u (after factoring out $q\exp\left(\frac{1}{2}q\left(m+u\right)\right)$) is $-\frac{1}{4}\eta+\frac{1}{4}q-\frac{1}{16}\eta q\left(u-m\right)$. The above can lower bounded by $-\frac{1}{4}\left(0.3\right)+\left(\frac{1}{4}-\frac{1}{16}\left(\sqrt{0.2375}-0.15\right)\left(0.3\right)\right)\left(3.74483\right)$, which is positive.

After showing convexity in u, we will show that the first partial u of Equation 247 is positive at $u=\sqrt{0.1175}$. Consider the first partial u below (after factoring out $\exp\left(\frac{1}{2}q\left(m-\sqrt{0.1175}\right)\right)$).

$$\frac{1}{4}\eta-\frac{1}{2}q-\frac{1}{8}\eta\left(\sqrt{0.1175}+m\right)q+\left(-\frac{1}{4}\eta+\frac{1}{2}q-\frac{1}{8}\eta\left(\sqrt{0.1175}-m\right)q\right)\exp\left(\sqrt{0.1175}q\right)$$

We can lower bound the above by

$$\frac{1}{4}(0.2)+\left(-\frac{1}{2}-\frac{1}{8}(0.3)\left(\sqrt{0.1175}+0.25\right)\right)(4.84375)$$
$$+\left(-\frac{1}{4}(0.3)+\left(\frac{1}{2}-\frac{1}{8}(0.3)\left(\sqrt{0.1175}-0.15\right)\right)(3.74483)\right)\exp\left(\sqrt{0.1175}(3.74483)\right),$$

which is positive.

After showing that Equation 247 is locally increasing in u, we can upper bound in u by $u = \sqrt{0.2375}$, shown below.

$$\left(1+\frac{1}{2}m\eta\right)+\left(1+\frac{1}{4}\eta\left(\sqrt{0.2375}+m\right)\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{0.2375}\right)\right)$$
$$+\left(1-\frac{1}{4}\eta\left(\sqrt{0.2375}-m\right)\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{0.2375}\right)\right)+\exp(mq) \quad (248)$$

Next, we will show that the upper bound is convex in q. To see this, note that the relevant terms are $\left(1+\frac{1}{4}\eta\left(\sqrt{0.2375}+m\right)\right)\exp\left(\frac{1}{2}q\left(m-\sqrt{0.2375}\right)\right)$, $\left(1-\frac{1}{4}\eta\left(\sqrt{0.2375}-m\right)\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{0.2375}\right)\right)$, and $\exp(mq)$, which are all convex. In the case of $\left(1-\frac{1}{4}\eta\left(\sqrt{0.2375}-m\right)\right)\exp\left(\frac{1}{2}q\left(m+\sqrt{0.2375}\right)\right)$, note that $1-\frac{1}{4}\eta\left(\sqrt{0.2375}-m\right)$ can be lower bounded by $1-\frac{1}{4}(0.3)\left(\sqrt{0.2375}-0.15\right)$, which is positive.

After showing convexity in q, we will show that the first partial q is positive at $q = 3.74483$. Consider the first partial q below (after factoring out $\exp\left(1.872415\left(m+\sqrt{0.2375}\right)\right)$ and simplifying by rounding down after five digits of precision).

$$0.20438+0.58060m-0.03448\eta+0.14515\eta m^2+m\exp\left(1.872415\left(m-\sqrt{0.2375}\right)\right)$$

Note that we can lower bound the above by the following

$$0.20438+0.58060(0.15)-0.03448(0.3)+0.14515(0.2)(0.15)^2+0.15\exp\left(1.872415\left(0.15-\sqrt{0.2375}\right)\right),$$

which is positive.

So, we can upper bound Equation 248 in q by $q = 4.84375$.

Next, we will show that the upper bound is decreasing in η. Note the first partial η (simplified by rounding up after five digits of precision)

$$\frac{1}{2}m+(-0.35917+0.89062m)\exp(2.421875m)$$

that can be upper bounded by

$$\frac{1}{2}(0.25)+(-0.35917+0.89062(0.25))\exp(2.421875(0.15)),$$

which is negative.

So, we can upper bound in η by $\eta = 0.2$, resulting in the following expression.

$$(1+0.1m)+\left(1+0.05\left(\sqrt{0.2375}+m\right)\right)\exp\left(2.421875\left(m-\sqrt{0.2375}\right)\right)$$
$$+\left(1-0.05\left(\sqrt{0.2375}-m\right)\right)\exp\left(2.421875\left(m+\sqrt{0.2375}\right)\right)+\exp(4.84375m) \quad (249)$$

Last, we will show that the above is convex in m and the first partial m at $m = 0.15$ is positive.

To show the above is convex in m, we will show that $\left(1+0.05\left(\sqrt{0.2375}+m\right)\right)\exp\left(2.421875\left(m-\sqrt{0.2375}\right)\right)$ and $\left(1-0.05\left(\sqrt{0.2375}-m\right)\right)\exp\left(2.421875\left(m+\sqrt{0.2375}\right)\right)$ are convex in m.

For $\left(1+0.05\left(\sqrt{0.2375}+m\right)\right)\exp\left(2.421875\left(m-\sqrt{0.2375}\right)\right)$, consider the second partial m

$$(2.421875)\left(2.521875+0.05\,(2.421875)\left(\sqrt{0.2375}+m\right)\right)\exp\left(2.421875\left(m-\sqrt{0.2375}\right)\right),$$

which is positive.

For $\left(1-0.05\left(\sqrt{0.2375}-m\right)\right)\exp\left(2.421875\left(m+\sqrt{0.2375}\right)\right)$, consider the second partial m (after factoring out $(2.421875)\exp\left(2.421875\left(m+\sqrt{0.2375}\right)\right)$)

$$2.521875-0.05\,(2.421875)\left(\sqrt{0.2375}-m\right)$$

that can be lower bounded by

$$2.521875+0.05\,(2.421875)\left(0.15-\sqrt{0.2375}\right),$$

which is positive.

After showing convexity in m, we evaluate the first partial m of Equation 249 at $m=0.15$, shown below, which is positive.

$$\begin{aligned}0.1+&\left(2.471875+0.05\,(2.421875)\left(\sqrt{0.2375}+(0.15)\right)\right)\exp\left(2.421875\left((0.15)-\sqrt{0.2375}\right)\right)\\+&\left(2.471875-0.05\,(2.421875)\left(\sqrt{0.2375}-(0.15)\right)\right)\exp\left(2.421875\left((0.15)+\sqrt{0.2375}\right)\right)\\&+4.84375\exp\left(4.84375\,(0.15)\right)\end{aligned}$$

So, we can upper bound Equation 249 in m by $m=0.25$.

This completes the proof.

□

Proofs of Lemmas in Section B.5.4

Proof of Lemma B.161. To prove the lemma, we begin by noting the linearization of Equations 7 and 8 (scaled by $\left(1+\exp\left(\frac{1}{2}q\left(m+\sqrt{c^2}\right)\right)\right)\left(1+\exp\left(\frac{1}{2}q\left(m-\sqrt{c^2}\right)\right)\right)$) about the fixed point $\left(\tilde{q},\tilde{c}^2\right)$, shown below.

$$\begin{aligned}&-\frac{1}{2}\left(\tilde{c}^2-m^2\right)\left[\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}^2}\right)\right)+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}^2}\right)\right)\right]\cdot\beta\cos\theta\\&+\frac{1}{2\sqrt{\tilde{c}^2}}\Bigg[-\frac{\tilde{q}}{2}\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}^2}\right)\right)\left(\sqrt{\tilde{c}^2}+m\right)-\frac{\tilde{q}}{2}\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}^2}\right)\right)\left(\sqrt{\tilde{c}^2}-m\right)\\&\qquad-\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}^2}\right)\right)+\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}^2}\right)\right)\Bigg]\cdot\beta\sin\theta\end{aligned}\tag{250}$$

$$\zeta_1\cdot\beta\cos\theta+\frac{1}{2\sqrt{\tilde{c}^2}}\zeta_2\cdot\beta\sin\theta,\tag{251}$$

where

$$\begin{aligned}\zeta_1=&-\eta\left(\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}^2}\right)\right)-\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}^2}\right)\right)\right)\\&-\frac{1}{2}\left(\left(\eta\tilde{q}-\sqrt{\tilde{c}^2}\right)\left(\sqrt{\tilde{c}^2}+m\right)\exp\left(\frac{1}{2}\tilde{q}\left(m+\sqrt{\tilde{c}^2}\right)\right)+\left(\eta\tilde{q}+\sqrt{\tilde{c}^2}\right)\left(\sqrt{\tilde{c}^2}-m\right)\exp\left(\frac{1}{2}\tilde{q}\left(m-\sqrt{\tilde{c}^2}\right)\right)\right)\\&+m\sqrt{\tilde{c}^2}\exp\left(\tilde{q}m\right)\end{aligned}$$

and

$$\zeta_2 = 1 + \exp\left(\frac{1}{2}\tilde{q}\left(m + \sqrt{\tilde{c}^2}\right)\right) + \exp\left(\frac{1}{2}\tilde{q}\left(m - \sqrt{\tilde{c}^2}\right)\right) - \frac{\tilde{q}}{2}\left(\eta\tilde{q} - \sqrt{\tilde{c}^2}\right)\exp\left(\frac{1}{2}\tilde{q}\left(m + \sqrt{\tilde{c}^2}\right)\right)$$
$$- \frac{\tilde{q}}{2}\left(\eta\tilde{q} + \sqrt{\tilde{c}^2}\right)\exp\left(\frac{1}{2}\tilde{q}\left(m - \sqrt{\tilde{c}^2}\right)\right) + \exp\left(\tilde{q}m\right)$$

For Equation 250 (the rescaled linearization of Equation 7), note that $\left(\tilde{c}^2 - m^2\right)$ and

$$\left[\exp\left(\frac{1}{2}\tilde{q}\left(m + \sqrt{\tilde{c}^2}\right)\right) + \exp\left(\frac{1}{2}\tilde{q}\left(m - \sqrt{\tilde{c}^2}\right)\right)\right]$$

are positive. Further note that

$$-\exp\left(\frac{1}{2}\tilde{q}\left(m + \sqrt{\tilde{c}^2}\right)\right) + \exp\left(\frac{1}{2}\tilde{q}\left(m - \sqrt{\tilde{c}^2}\right)\right)$$

is negative.

So, Equation 250 is negative for $\theta \in \left[0, \frac{\pi}{2}\right]$ and positive for $\theta \in \left[\pi, \frac{3\pi}{2}\right]$.

So, we will show that Equation 251 is negative for $\theta \in [0, 0.245\pi]$ and positive for $\theta \in [\pi, 1.245\pi]$. Since $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$, it is sufficient to prove that Equation 251 is negative for $\theta \in [0, 0.245\pi]$.

We split up the analysis into two cases. Case 1: $\theta = 0$ For this case, we will show that ζ_1 is negative in lemma B.162.

Case 2: $\theta \in \left(0, 0.245\pi\right]$ In this case, we will scale Equation 251 by $\frac{2\sqrt{\tilde{c}^2}}{\beta}\sin(\theta)$, apply the $\tilde{u} = \sqrt{\tilde{c}^2}$ change of variable, and show that the resulting expression (shown below) is negative locally.

$$2\tilde{u}\zeta_1 \cot\theta + \zeta_2 \tag{252}$$

Note that ζ_1 is negative by lemma B.162. So, we can upper bound Equation 252 in θ by $\theta = 0.245\pi$ (since $\cot\theta$ decreases in θ for $\theta \in \left(0, \frac{\pi}{5}\right]$).

To show that the resulting expression is negative, we begin by showing that it is convex in $\tilde{u}$ for $\tilde{u} \in \left[\sqrt{2m\eta + 0.25m + 0.02}, \sqrt{2m\eta + 0.25m + 0.025}\right]$ in lemma B.183. Then, we split the expression into two cases.

Case 2.1: $\tilde{u} = \sqrt{2m\eta + 0.25m + 0.02}$. For this case, we show that the resulting expression in decreasing in $\tilde{q}$ for $\tilde{q} \in \left[\frac{0.9m^2}{\eta} + \frac{0.5}{\eta} + 1.944, \frac{0.94m^2}{\eta} + \frac{0.48}{\eta} + 2.15\right]$ in lemmas B.221 (concavity in q) and B.216 (first partial q at $q = \frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944$ is negative). By upper bounding in q by $q = \left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)$, we next show that the resulting expression (scaled by η) is decreasing in η in lemmas B.165 (concavity in η) and B.163 (first partial η at $\eta = 0.2$ (simplified by rounding up after five digits of precision) is negative). Last, we upper bound in η by $\eta = 0.2$ and show the resulting expression (simplified by rounding up after five digits of precision) is negative in lemma B.171.

Case 2.2: $\tilde{u} = \sqrt{2m\eta + 0.25m + 0.025}$. For this case, we also show that the resulting expression in decreasing in $\tilde{q}$ for $\tilde{q} \in []$ in lemmas B.233 (concavity in q) and B.228 (first partial q at $q = \frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944$ is negative). By upper bounding in q by $q = \left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)$, we next show that the resulting expression (scaled by η) is decreasing in η in lemmas B.5.4 (concavity in η) and B.5.4 (first partial η at $\eta = 0.2$ (simplified by rounding up after five digits of precision) is negative). Last, we upper bound in η by $\eta = 0.2$ and show the resulting expression (simplified by rounding up after five digits of precision) is negative in lemma B.172.

□

Proof. To prove the lemma, note that we can upper bound Equation **??** in η by $\eta = 0.2$ Next, we upper bound the resulting expression in m with the following.

$$-0.2\left(\exp\left(\frac{1}{2}\tilde{q}\tilde{u}\right)-\exp\left(-\frac{1}{2}\tilde{q}\tilde{u}\right)\right)+0.25\tilde{u}\exp\left(\frac{1}{2}\left(0.25\right)\tilde{q}\right)$$
$$-\frac{1}{2}\left(\left(0.2\tilde{q}-\tilde{u}\right)\left(\tilde{u}+0.15\right)\exp\left(\frac{1}{2}\tilde{q}\tilde{u}\right)+\left(0.2\tilde{q}+\tilde{u}\right)\left(\tilde{u}-0.25\right)\exp\left(-\frac{1}{2}\tilde{q}\tilde{u}\right)\right)$$

We can further upper bound the above by (note that $-0.1\sqrt{0.1175}+0.025$ is negative)

$$0.25\sqrt{0.2375}\exp\left(0.125\left(4.84375\right)\right)+\left(-0.2+\left(-0.1\tilde{u}-0.015\right)\left(3.74483\right)+\frac{1}{2}\tilde{u}^2+0.075\tilde{u}\right)\exp\left(\frac{1}{2}\tilde{q}\tilde{u}\right)$$
$$+\left(0.2+\left(-0.1\tilde{u}+0.025\right)\left(3.74483\right)-\frac{1}{2}\tilde{u}^2+0.125\tilde{u}\right)\exp\left(-\frac{1}{2}\tilde{q}\tilde{u}\right),$$

which we can simplify (by rounding up after five digits of precision) to

$$\left(\frac{1}{2}\tilde{u}^2-0.29948\tilde{u}-0.25617\right)\exp\left(\frac{1}{2}\tilde{q}\tilde{u}\right)+\left(-\frac{1}{2}\tilde{u}^2-0.24948\tilde{u}+0.29363\right)\exp\left(-\frac{1}{2}\tilde{q}\tilde{u}\right)$$
$$+0.25\sqrt{0.2375}\exp\left(0.125\left(4.84375\right)\right) \quad (253)$$

From the above, note that $\frac{1}{2}\tilde{u}^2-0.29948\tilde{u}-0.25617$ is negative (i.e. can be upper bounded by $\frac{1}{2}\left(\sqrt{0.2375}\right)^2-0.29948\sqrt{0.1175}-0.25617$) and $-\frac{1}{2}\tilde{u}^2-0.24948\tilde{u}+0.29363$ is positive (i.e., can be lower bounded by $-\frac{1}{2}\left(\sqrt{0.2375}\right)^2-0.24948\sqrt{0.2375}+0.29363$).

So, our Equation 253 can be upper bounded by

$$\left(\frac{1}{2}\left(\sqrt{0.2375}\right)^2-0.29948\left(\sqrt{0.1175}\right)-0.25617\right)\exp\left(\frac{1}{2}\left(3.74483\right)\left(\sqrt{0.1175}\right)\right)$$
$$+\left(-\frac{1}{2}\left(\sqrt{0.1175}\right)^2-0.24948\left(\sqrt{0.1175}\right)+0.29363\right)\exp\left(-\frac{1}{2}\left(3.74483\right)\left(\sqrt{0.1175}\right)\right)$$
$$+0.25\sqrt{0.2375}\exp\left(0.125\left(4.84375\right)\right),$$

which is negative. This completes the proof. □

Proof of Lemma B.163. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we split Equation 155 into two parts (RU). Note that we factor out $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ in the first part. We also scale by $\left(0.65m+0.02\right)^{\frac{3}{2}}$, factor out $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$, and linearize the $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(\sqrt{0.65m+0.02}-m\right)\right)$ term at $m=0.15$ (note the corresponding coefficient is a positive constant and lemma B.164) in the second part.

Part 1

$$\left(4.55625m^7 + 2.29859m^6 + 8.13037m^5 + 2.64178m^4 + 3.18541m^3 + 0.04218m^2 - 0.54356m - 0.84235\right)$$

$$+ \frac{3.40857m^7 - 2.54649m^6 + 5.37261m^5 - 4.36439m^4 + 2.99594m^3 - 1.43286m^2 + 0.86916m + 0.03152}{\sqrt{0.65m + 0.02}}$$

$$+\Bigg(-6.03673m^5 - 0.18576m^4 - 7.98788m^3 + 2.05971m^2 - 2.55872m + 1\Bigg) \exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$$

$$+ 0.178 \exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right) \tag{254}$$

Part 2

$$\Bigg(4.55625m^7 + 2.29860m^6 + 8.13037m^5 + 2.64179m^4 + 3.18543m^3 + 0.04219m^2 - 0.53116m + 0.44207\Bigg)\sqrt{0.65m + 0.02}$$

$$+ \left(-3.40858m^7 + 2.54650m^6 - 5.37262m^5 + 4.36439m^4 - 2.99594m^3 + 1.43287m^2 - 0.86914m - 0.03151\right) \tag{255}$$

For Equation 254, note that

$$-6.03673m^5 - 0.18576m^4 - 7.98788m^3 + 2.05971m^2 - 2.55872m + 1 \tag{256}$$

can be lower bounded by

$$-6.03673\left(0.25\right)^5 - 0.18576\left(0.25\right)^4 - 7.98788\left(0.25\right)^3 + 2.05971\left(0.15\right)^2 - 2.55872\left(0.25\right) + 1,$$

which is positive.

So, we replace $\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(4.5\left(0.15\right)^2 + 4.544\right)\left(\left(0.15\right) + \sqrt{0.65\left(0.15\right) + 0.02}\right)\right)$ and $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(4.5\left(0.15\right)^2 + 4.544\right)\left(\sqrt{0.65\left(0.25\right) + 0.02} - \left(0.25\right)\right)\right)$ (note $\sqrt{0.65m + 0.02} - m$ is concave and is smaller at $m = 0.25$ than $m = 0.15$).

Furthermore, note that

$$3.40857m^7 - 2.54649m^6 + 5.37261m^5 - 4.36439m^4 + 2.99594m^3 - 1.43286m^2 + 0.86916m + 0.03152 \tag{257}$$

can be lower bounded by

$$3.40857\left(0.15\right)^7 - 2.54649\left(0.15\right)^6 + 5.37261\left(0.15\right)^5 - 4.36439\left(0.15\right)^4 + 2.99594\left(0.15\right)^3 - 1.43286\left(0.15\right)^2 + 0.86916\left(0.15\right) + 0.03152,$$

which is positive.

So, we can upper bound

$$\frac{3.40857m^7 - 2.54649m^6 + 5.37261m^5 - 4.36439m^4 + 2.99594m^3 - 1.43286m^2 + 0.86916m + 0.03152}{\sqrt{0.65m + 0.02}} \tag{258}$$

by

$$\frac{3.40857m^7 - 2.54649m^6 + 5.37261m^5 - 4.36439m^4 + 2.99594m^3 - 1.43286m^2 + 0.86916m + 0.03152}{\sqrt{0.65(0.15) + 0.02}}. \tag{259}$$

With the above, we can upper bound Equation 254 by

$$14.50008m^7 - 5.13028m^6 + 19.80390m^5 - 10.21353m^4 + 6.63263m^3 - 2.77311m^2 + 0.29662m - 0.03111. \tag{260}$$

We can then upper bound the above by

$$\Bigg(\Bigg(\Bigg(\Bigg(\Bigg(\Big(14.50008\,(0.25) - 5.13028\Big)\,(0.15) + 19.80390\Big)\,(0.25) - 10.21353\Big)\,(0.15) + 6.63263\Big)\,(0.25) - 2.77311\Big)\,(0.15) + 0.29662\Big)\,(0.25) - 0.03111,$$

which is negative.

For Equation 255, first note that

$$4.55625m^7 + 2.29860m^6 + 8.13037m^5 + 2.64179m^4 + 3.18543m^3 + 0.04219m^2 - 0.53116m + 0.44207, \tag{261}$$

can be lower bounded by

$$4.55625\,(0.15)^7 + 2.29860\,(0.15)^6 + 8.13037\,(0.15)^5 + 2.64179\,(0.15)^4 + 3.18543\,(0.15)^3 + 0.04219\,(0.15)^2 - 0.53116\,(0.25) + 0.44207,$$

which is positive.

So, we can linearize the $\sqrt{0.65m + 0.02}$ term at $m = 0.15$, resulting in the following (RU).

$$4.31992m^8 - 0.31536m^7 + 10.71618m^6 - 1.23715m^5 + 7.91446m^4 - 2.31703m^3 + 0.93773m^2 - 0.55653m + 0.05716 \tag{262}$$

We proceed by showing that the above is convex in m. Note that the second partial m can be lower bounded by

$$4.31992\,(56)\,(0.15)^6 - 0.31536\,(42)\,(0.25)^5 + 10.71618\,(30)\,(0.15)^4 - 1.23715\,(20)\,(0.25)^3 + 7.91446\,(12)\,(0.15)^2 - 2.31703\,(6)\,(0.25) + 0.93773\,(2)\,,$$

which is positive.

We evaluate Equation 262 at $m = 0.15, 0.25$, which are negative.

This completes the proof.

□

Proof of Lemma B.164. To prove the lemma, consider the second partial x of Equation 156, scaled by $(0.65m + 0.02)^{\frac{3}{2}}$ and rounded up after five digits of precision.

$$\left(29.61563x^5 + 14.27942x^4 + 20.59493x^3 - 2.75391x^2 + 3.21821x + 0.64848\right)\sqrt{0.65x + 0.02} + \left(-32.08359x^5 - 1.77693x^4 - 17.30286x^3 + 2.76737x^2 - 2.01361x - 0.30528\right) \tag{263}$$

First, note that

$$29.61563x^5 + 14.27942x^4 + 20.59493x^3 - 2.75391x^2 + 3.21821x + 0.64848 \tag{264}$$

can be lower bounded by

$$29.61563\,(0.15)^5 + 14.27942\,(0.15)^4 + 20.59493\,(0.15)^3 - 2.75391\,(0.25)^2 + 3.21821\,(0.15) + 0.64848,$$

which is positive.

So, we replace the $\sqrt{0.65x + 0.02}$ term in Equation 263 by $\sqrt{0.65\,(0.25) + 0.02}$.

Then, the resulting expression (rounded up after five digits of precision)

$$-19.43178x^5 + 4.32325x^4 - 8.50470x^3 + 1.59090x^2 - 0.63879x - 0.02824 \tag{265}$$

can be upper bounded by

$-19.43178\,(0.15)^5 + 4.32325\,(0.25)^4 - 8.50470\,(0.15)^3 + 1.59090\,(0.25)^2 - 0.63879\,(0.15) - 0.02824$,
which is negative.

This completes the proof.

□

Proof of Lemma B.165. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, note that we first split Equation 157 into two parts and prove that each part is concave in η.

Part 1

$$+\left(\sigma_{2,1} + \sigma_{2,2}\sqrt{2m\eta + 0.25m + 0.02}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \tag{266}$$

Part 2

$$\left(\sigma_{1,1} + \sigma_{1,2}\sqrt{2m\eta + 0.25m + 0.02}\right)\exp\left(\frac{1}{2}\left(\frac{0.9}{x}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right)$$
$$+\left(4.12768m^2\eta^2 + 0.51596m^2\eta + 0.04128m\eta + \eta\right)\exp\left(m\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\right) \tag{267}$$

where

$$\begin{aligned}
\sigma_{1,1} =& -4.01210m\eta^3 + \left(-1.85745m^3 + 2.06384m^2 - 1.57470m - 1.92968\right)\eta^2 \\
&+ \left(-0.23218m^3 - 1.51019m^2 - 0.11351m - 0.02161\right)\eta + \left(-0.405m^4 - 0.468m^2 - 0.1352\right), \\
\sigma_{1,2} =& \left(0.05779m - 2.06383\right)\eta^2 + \left(-0.92872m^3 - 0.27861m + 0.99264\right)\eta + \left(0.45m^2 + 0.26\right), \\
\sigma_{2,1} =& -4.01210m\eta^3 + \left(-1.85745m^3 + 2.06384m^2 - 1.57470m - 1.92968\right)\eta^2 \\
&+ \left(-0.23218m^3 - 1.51019m^2 - 0.11351m - 0.02161\right)\eta + \left(-0.405m^4 - 0.468m^2 - 0.1352\right),
\end{aligned}$$

and

$$\sigma_{2,2} = \left(-0.05778m + 2.06384\right)\eta^2 + \left(0.92873m^3 + 0.27862m - 0.99263\right)\eta + \left(-0.45m^2 - 0.26\right).$$

We start with Equation 266. We split the second partial η into the following parts. Note that for each part we factored out the exponential term and simplified the expressions (RU). Furthermore, in the first part, we also factored out $\left(\left(\frac{-0.9}{x^2}m^3 + \frac{-0.52}{x^2}m\right) + \frac{\frac{0.9}{x}m^3 + \frac{0.225}{x^2}m^3 + \frac{0.018}{x^2}m^2 + \frac{0.52}{x}m + \frac{0.13}{x^2}m + \frac{0.0104}{x^2} - 1.944m}{\sqrt{2mx + 0.25m + 0.02}}\right)^2$. Last, in the second part, we also scale by $\eta^3\left(2mn + 0.25m + 0.02\right)^{\frac{3}{2}}$.

Part 1

$$\begin{aligned}
&-1.00302m\eta^3 + \left(-0.46436m^3 + 0.51596m^2 - 0.39367m - 0.48242\right)\eta^2 \\
&+ \left(-0.058045m^3 - 0.37754m^2 - 0.02837m - 0.00540\right)\eta + \left(-0.10125m^4 - 0.117m^2 - 0.0338\right) \\
&+ \left(\left(-0.014445m + 0.51596\right)\eta^2 + \left(0.23219m^3 + 0.06966m - 0.24815\right)\eta + \left(-0.1125m^2 - 0.065\right)\right)\sqrt{2m\eta + 0.25m + 0}
\end{aligned} \tag{268}$$

Part 2

$$\delta_1 + \delta_2\sqrt{2m\eta + 0.25m + 0.02}, \tag{269}$$

where

$$\begin{aligned}
\delta_1 =& 42.89738m^3\eta^6 + \left(-3.6109m^5 - 14.04234m^4 + 6.30890m^3 + 44.55514m^2\right)\eta^5 \\
&+ \left(-4.17926m^7 + 4.95566m^6 - 17.25014m^5 + 0.63031m^4 - 9.25207m^3 + 2.50196m^2 + 0.64539m\right)\eta^4 \\
&+ \Big(-1.6717m^8 - 1.35824m^7 - 1.18282m^6 - 2.19135m^5 - 0.77791m^4 - 1.00729m^3 \\
&\qquad - 1.04120m^2 - 0.05949m + 0.00166\Big)\eta^3 \\
&+ \Big(0.33779m^8 - 0.93118m^7 + 0.86692m^6 - 0.97392m^5 + 0.44800m^4 - 0.27053m^3 \\
&\qquad + 0.03392m^2 - 0.01077m - 0.00039\Big)\eta^2 \\
&+ \Big(0.27338m^8 - 0.28187m^7 + 0.44956m^6 - 0.31308m^5 + 0.24571m^4 - 0.07949m^3 \\
&\qquad + 0.04464m^2 + 0.00423m + 0.00001\Big)\eta \\
&+ \Big(0.02279m^8 - 0.02166m^7 + 0.03559m^6 - 0.02309m^5 + 0.01839m^4 - 0.00497m^3 \\
&\qquad + 0.00320m^2 + 0.00066m + 0.00003\Big)
\end{aligned}$$

and

$$\begin{aligned}
\delta_2 =& \left(0.50546m^3 - 66.19967m^2\right)\eta^5 + \left(9.74793m^5 - 0.9287m^4 + 15.19745m^3 - 5.73813m^2 - 8.36065m\right)\eta^4 \\
&+ \left(4.59721m^7 - 3.7149m^6 + 7.16886m^5 + 1.45730m^4 + 3.83894m^3 + 2.40639m^2 - 0.90337m - 0.07718\right)\eta^3 \\
&+ \left(0.62690m^7 - 0.21170m^6 + 0.73881m^5 + 0.35446m^4 + 0.25302m^3 + 0.27622m^2 + 0.02047m + 0.00043\right)\eta^2 \\
&+ \left(-0.729m^8 - 1.0611m^6 + 0.0162m^5 - 0.49608m^4 + 0.01872m^3 - 0.073m^2 + 0.00541m\right)\eta \\
&+ \Big(-0.09112m^8 - 0.00729m^7 - 0.13263m^6 - 0.00858m^5 - 0.06184m^4 - 0.00262m^3 \\
&\qquad - 0.00893m^2 - 0.00004m + 0.00006\Big).
\end{aligned}$$

For Equation 268, note that

$$-0.46436m^3 + 0.51596m^2 - 0.39367m - 0.48242 \tag{270}$$

can be upper bounded by

$$-0.46436\left(0.15\right)^3 + 0.51596\left(0.25\right)^2 - 0.39367\left(0.15\right) - 0.48242,$$

which is negative.

Also, note that

$$\left(-0.014445m + 0.51596\right)\eta^2 + \left(0.23219m^3 + 0.06966m - 0.24815\right)\eta + \left(-0.1125m^2 - 0.065\right) \tag{271}$$

can be upper bounded by

$$\begin{aligned}
&\left(-0.01444\left(0.15\right) + 0.51596\right)\left(0.3\right)^2 + \Big(0.23219\left(0.25\right)^3 + 0.06966\left(0.25\right) - 0.24815\Big)\left(0.2\right) \\
&+ \Big(-0.1125\left(0.15\right)^2 - 0.065\Big).
\end{aligned}$$

For Equation 269, note that δ_2 is negative. We see this since δ_2 can be upper bounded by

$$
\begin{aligned}
&\left(0.50546\left(0.25\right)-66.19967\right)\left(0.15\right)^{2}\eta^{5}\\
&+\left(9.74793\left(0.25\right)^{4}-0.9287\left(0.15\right)^{3}+15.19745\left(0.25\right)^{2}-5.73813\left(0.15\right)-8.36065\right)\left(0.15\right)\eta^{4}\\
&+\Bigg(4.59721\left(0.25\right)^{7}-3.7149\left(0.15\right)^{6}+7.16886\left(0.25\right)^{5}+1.45730\left(0.25\right)^{4}\\
&\qquad+3.83894\left(0.25\right)^{3}+2.40639\left(0.25\right)^{2}-0.90337\left(0.15\right)-0.07718\Bigg)\eta^{3}\\
&+\Bigg(0.62690\left(0.25\right)^{7}-0.21170\left(0.15\right)^{6}+0.73881\left(0.25\right)^{5}+0.35446\left(0.25\right)^{4}\\
&\qquad+0.25302\left(0.25\right)^{3}+0.27622\left(0.25\right)^{2}+0.02047\left(0.25\right)+0.00043\Bigg)\eta^{2}\\
&+\Bigg(-0.729\left(0.15\right)^{7}-1.0611\left(0.15\right)^{5}+0.0162\left(0.25\right)^{4}-0.49608\left(0.15\right)^{3}\\
&\qquad+0.01872\left(0.25\right)^{2}-0.073\left(0.15\right)+0.00541\Bigg)\left(0.15\right)\eta\\
&+\Bigg(-0.09112\left(0.15\right)^{8}-0.00729\left(0.15\right)^{7}-0.13263\left(0.15\right)^{6}-0.00858\left(0.15\right)^{5}-0.06184\left(0.15\right)^{4}\\
&\qquad-0.00262\left(0.15\right)^{3}-0.00893\left(0.15\right)^{2}-0.00004\left(0.15\right)+0.00006\Bigg),
\end{aligned}
\tag{272}
$$

which simplifies (RU) to

$$
-1.48664\eta^{5}-1.23548\eta^{4}+0.01063\eta^{3}+0.02891\eta^{2}-0.00090\eta-0.00018. \tag{273}
$$

We can upper bound the above by

$$
((((-1.48664\left(0.2\right)-1.23548\right)\left(0.2\right)+0.01063\right)\left(0.2\right)+0.02891\right)\left(0.2\right)-0.00090\right)\left(0.2\right)-0.00018,
$$

which is negative.

So, we can replace $\delta_2\sqrt{2m\eta+0.25m+0.02}$ with $\delta_2\sqrt{2m\left(0.2\right)+0.25m+0.02}$, which we then replace with $\delta_2\left(0.84417m+0.2161\right)$ (see lemma B.241).

We then get the resulting upper bound for Equation 269.

$$42.89738m^3\eta^6 + \left(-3.6109m^5 - 13.61564m^4 - 49.46564m^3 + 30.24940m^2\right)\eta^5 + \left(-4.17926m^7 + 13.18458m^6 - 15.92759m^5 + 13.25885m^4 - 10.81185m^3 - 5.79585m^2 - 1.16134m\right)\eta^4 + \Big(2.20913m^8 - 3.50079m^7 + 4.06613m^6 + 0.58805m^5 + 2.77774m^4 + 1.85371m^3 - 1.28377m^2 - 0.31986m - 0.01501\Big)\eta^3 + \Big(0.86701m^8 - 0.97441m^7 + 1.44486m^6 - 0.51503m^5 + 0.73820m^4 + 0.01733m^3 + 0.11090m^2 - 0.00598m - 0.00029\Big)\eta^2 + \Big(-0.61539m^9 + 0.11585m^8 - 1.17761m^7 + 0.23394m^6 - 0.72835m^5 + 0.15431m^4 - 0.13706m^3 + 0.03344m^2 + 0.00540m + 0.00001\Big)\eta + \Big(-0.07692m^9 - 0.00305m^8 - 0.13519m^7 - 0.00031m^6 - 0.07714m^5 + 0.00282m^4 - 0.01307m^3 + 0.00124m^2 + 0.00071m + 0.00005\Big) \tag{274}$$

We proceed by showing that the above is concave in η. To see this, we iteratively upper bound the second partial η in η. By noting that

$$-72.218\left(0.25\right)^3 - 272.3128\left(0.25\right)^2 - 731.92852\left(0.25\right) + 604.988$$

is positive,

$$-50.15112m^6 + 158.21496m^5 - 212.79648m^4 + 77.41236m^3 - 349.32075m^2 + 111.9462m - 13.93608 \tag{275}$$

is negative (see lemma B.170), and

$$\Big(\Big(\Big(\Big(\Big(\Big(13.25478\left(0.25\right) - 31.03496\Big)\left(0.15\right) + 56.03978\Big)\left(0.25\right) - 39.0309\Big)\left(0.15\right) + 32.14892\Big)\left(0.25\right) - 58.74189\Big)\left(0.15\right) + 14.68662\Big)\left(0.25\right)^2 - 4.70637\left(0.15\right) - 0.09006$$

is negative, we get the following upper bound in m for the second partial η.

$$4.38498m^8 - 8.15581m^7 + 14.09768m^6 - 8.83624m^5 + 7.90619m^4 - 11.71371m^3 + 3.15913m^2 - 0.95323m - 0.01859 \tag{276}$$

Note that the above can be upper bounded by

$$\Big(\Big(\Big(\Big(\Big(\Big(4.38498\left(0.25\right) - 8.15581\Big)\left(0.15\right) + 14.09768\Big)\left(0.25\right) - 8.83624\Big)\left(0.15\right) + 7.90619\Big)\left(0.25\right) - 11.71371\Big)\left(0.15\right) + 3.15913\Big)\left(0.25\right)^2 - 0.95323\left(0.15\right) - 0.01859,$$

which is negative.

So, we evaluate the first partial η of Equation 274 at $\eta = 0.2$ that can be upper bounded by

$$-0.61539\left(0.15\right)^9 + 0.72775\left(0.25\right)^8 - 2.12120\left(0.15\right)^7 + 1.72173\left(0.25\right)^6 - 1.40236\left(0.15\right)^5 + 1.09828\left(0.25\right)^4 - 0.56702\left(0.15\right)^3 - 0.01972\left(0.15\right)^2 - 0.07253\left(0.15\right) - 0.00190,$$

which is negative.

We can then upper bound Equation 274 by $\eta = 0.2$ and upper bound the resulting expression by

$$-0.19999\,(0.15)^9 + 0.07248\,(0.25)^8 - 0.44438\,(0.15)^7 + 0.15790\,(0.25)^6 - 0.26534\,(0.15)^5 + 0.10229\,(0.25)^4 \\ -0.05534\,(0.15)^3 + 0.00251\,(0.25)^2 - 0.00286\,(0.15) - 0.00007,$$

which is negative.

Next, we consider the second partial η of Equation 267 (RU), which we split into the following parts. Note that for of the parts, we factor out $\exp\left(\frac{1}{2}\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)\left(m + \sqrt{2mn + 0.25m + 0.02}\right)\right)$ Note for the first part that we also factor out $\left(\left(\frac{-0.9}{\eta^2}m^3 + \frac{-0.52}{\eta^2}m\right) + \frac{\frac{-0.9}{\eta}m^3 + \frac{-0.52}{\eta}m + \frac{-0.225}{\eta^2}m^3 + \frac{-0.018}{\eta^2}m^2 + \frac{-0.13}{\eta^2}m + \frac{-0.0104}{\eta^2} + 1.944m}{\sqrt{2m\eta + 0.25m + 0.02}}\right)^2$. Note for the second part that we also scale by $\eta^2\left(2mn + 0.25m + 0.02\right)^{\frac{3}{2}}$. Last, for the third part, we also scale by $\eta^3\left(2mn + 0.25m + 0.02\right)^{\frac{3}{2}}$ and replace $\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right)$ by $\exp\left(-\frac{1}{2}\left(\frac{0.9}{0.3}\left(0.15\right)^2 + \frac{0.52}{0.3} + 1.944\right)\left(\sqrt{2\left(0.15\right)\left(0.2\right) + 0.25\left(0.15\right) + 0.02} - 0.25\right)\right)$, since the corresponding coefficient is positive (see lemma B.169).

$$\Bigg(-1.00302m\eta^3 + \left(-0.46436m^3 + 0.51596m^2 - 0.39367m - 0.48242\right)\eta^2 \\ + \left(-0.05804m^3 - 0.37754m^2 - 0.02837m - 0.00540\right)\eta + \left(-0.10125m^4 - 0.117m^2 - 0.0338\right)\Bigg) \\ + \Bigg(\left(0.01445m - 0.51595\right)\eta^2 + \left(-0.23218m^3 - 0.06965m + 0.24816\right)\eta \\ + \left(0.1125m^2 + 0.065\right)\Bigg)\sqrt{2m\eta + 0.25m + 0.02} \tag{277}$$

$$\kappa_1 + \kappa_2\sqrt{2m\eta + 0.25m + 0.02}, \tag{278}$$

where

$$\begin{aligned}\kappa_1 = &-46.79712m^3\eta^5 + \left(7.22182m^5 + 16.04842m^4 - 4.70989m^3 - 46.43060m^2\right)\eta^4 \\ &+ \left(6.68682m^7 - 7.94992m^6 + 30.73709m^5 - 1.00646m^4 + 16.22756m^3 - 0.89253m^2 - 0.64537m\right)\eta^3 \\ &+ \Bigg(5.0151m^8 + 2.92550m^7 + 4.41785m^6 + 2.70705m^5 + 2.90495m^4 + 0.88774m^3 \\ &\qquad + 1.83510m^2 + 0.12056m - 0.00165\Bigg)\eta^2 \\ &+ \Bigg(1.04483m^8 - 0.36072m^7 + 1.74379m^6 - 1.23740m^5 + 0.87117m^4 - 0.55599m^3 \\ &\qquad + 0.12414m^2 + 0.02227m + 0.00081\Bigg)\eta \\ &+ \Bigg(0.05225m^8 - 0.07983m^7 + 0.12513m^6 - 0.13789m^5 + 0.04413m^4 - 0.05314m^3 \\ &\qquad - 0.00619m^2 - 0.00006m + 0.00001\Bigg)\end{aligned}$$

and

$$\begin{aligned}\kappa_2 =& \left(0.56172m^3 - 68.20562m^2\right)\eta^4 + \left(15.98899m^5 + 1.85744m^4 + 19.05418m^3 - 3.16346m^2 - 8.36065m\right)\eta^3 \\ &+\Big(9.19437m^7 - 7.42982m^6 + 13.89926m^5 + 3.34381m^4 + 6.30440m^3 \\ &\qquad + 4.73651m^2 - 0.83898m - 0.07718\Big)\eta^2 \\ &+\Big(2.08963m^7 + 1.51836m^6 + 2.29023m^5 + 0.96504m^4 + 0.66078m^3 \\ &\qquad + 0.05220m^2 + 0.02029m + 0.00086\Big)\eta \\ &+\Big(0.10449m^7 + 0.25109m^6 + 0.12100m^5 + 0.04015m^4 + 0.02191m^3 \\ &\quad -0.06097m^2 - 0.00758m - 0.00020\Big).\end{aligned}$$

$$\alpha_1 + \alpha_2\sqrt{2m\eta + 0.25m + 0.02}, \tag{279}$$

where

$$\begin{aligned}\alpha_1 =& 3.89977m^3\eta^6 + \left(-3.61089m^5 - 2.00605m^4 - 1.59882m^3 + 1.87565m^2\right)\eta^5 \\ &+ \left(-2.50755m^7 + 2.99427m^6 - 13.48690m^5 + 0.37622m^4 - 6.97541m^3 - 1.60932m^2\right)\eta^4 \\ &+\Big(-3.34336m^8 - 1.56722m^7 - 3.23500m^6 - 0.51565m^5 - 2.12697m^4 + 0.11962m^3 \\ &\qquad - 0.79386m^2 - 0.06104m\Big)\eta^3 \\ &+\Big(-1.38259m^8 + 1.29193m^7 + -2.59813m^6 + 2.19883m^5 - 1.31912m^4 + 0.82656m^3 \\ &\qquad - 0.15801m^2 - 0.01146m - 0.00040\Big)\eta^2 \\ &+\Big(-0.32561m^8 + 0.36172m^7 - 0.57467m^6 + 0.45099m^5 - 0.28982m^4 + 0.13265m^3 \\ &\qquad - 0.03842m^2 - 0.00414m\Big)\eta \\ &+\Big(-0.02278m^8 + 0.02167m^7 - 0.03558m^6 + 0.02310m^5 - 0.01838m^4 + 0.00499m^3 \\ &\qquad -0.00318m^2 - 0.00064m - 0.00002\Big)\end{aligned}$$

and

$$\begin{aligned}\alpha_2 = &\left(-0.05617m + 2.00605\right) m^2\eta^5 + \left(-6.24104m^3 - 2.78617m^2 + 10.02107m - 2.57463\right) m^2\eta^4 \\ &+ \left(-4.59717m^6 - 8.77505m^5 - 6.73043m^4 - 9.10295m^3 - 0.73075m^2 - 2.19135m - 0.06439\right) m\eta^3 \\ &+ \Bigg(5.62051m^9 + 5.03208m^7 - 3.29129m^6 + 0.19998m^5 - 1.51261m^4 - 0.47991m^3 \\ &\qquad + 0.22404m^2 + 0.00020m - 0.00042\Bigg)\eta^2 \\ &+ \Bigg(1.40513m^9 + 0.11242m^8 + 2.15188m^7 - 1.18226m^6 + 1.93777m^5 - 0.49867m^4 \\ &\qquad + 0.45139m^3 - 0.01201m^2 + 0.01300m + 0.00021\Bigg)\eta \\ &+ \Bigg(0.08783m^9 + 0.01406m^8 + 0.18114m^7 - 0.11006m^6 + 0.21808m^5 - 0.04141m^4 + 0.05439m^3 \\ &\qquad - 0.00438m^2 - 0.00004m + 0.00006\Bigg).\end{aligned}$$

For Equation 277, we first note that the square root coefficient can lower bounded by

$$\begin{aligned}&\left(0.01445\left(0.15\right) - 0.51595\right)\left(0.3\right)^2 + \Big(-0.23218\left(0.25\right)^3 - 0.06965\left(0.25\right) + 0.24816\Big)\left(0.2\right) \\ &\qquad + \Big(0.1125\left(0.15\right)^2 + 0.065\Big),\end{aligned}$$

which is positive.

So, we $\sqrt{2m\eta + 0.25m + 0.02}$ in Equation 277 by $\sqrt{2m\left(0.3\right) + 0.25m + 0.02}$, which we then linearize at $m = 0.15$.

We then upper bound the resulting expression (RU) (shown below) iteratively in η.

$$-1.00302m\eta^3 + \left(-0.46436m^3 + 0.53196m^2 - 0.96147m - 0.59493\right)\eta^2 + \left(-0.25693m^4 - 0.10867m^3 - 0.45461m^2 + 0\right. \tag{280}$$

Noting that

$$-0.46436\left(0.15\right)^3 + 0.53196\left(0.25\right)^2 - 1.16207\left(0.15\right) - 0.59493$$

is negative, we can upper bound Equation 280 by

$$-0.15263m^4 + 0.08420m^3 - 0.16210m^2 + 0.07166m - 0.03367. \tag{281}$$

We then upper bound the above by

$$-0.15263\left(0.15\right)^4 + 0.08420\left(0.25\right)^3 - 0.16210\left(0.15\right)^2 + 0.07166\left(0.25\right) - 0.03367,$$

which is negative.

For Equation 278, we first show that κ_2 is negative in lemma B.166. We can then replace $\sqrt{2m\eta + 0.25m + 0.02}$ with $\sqrt{2m\left(0.2\right) + 0.25m + 0.02}$, which we then replace with $\left(0.84417m + 0.2161\right)$ (see lemma B.241).

This produces the following expression (RU).

$$-46.79712m^3\eta^5 + \left(7.22182m^5 + 16.52261m^4 - 62.16564m^3 - 61.16983m^2\right)\eta^4 + \left(6.68682m^7 + 5.54751m^6 + 35.7603\right. \tag{282}$$

We show that the above is concave in η. Consider the second partial η,

$$-46.79712\left(20\right)m^3\eta^3 + \left(7.22182m^5 + 16.52261m^4 - 62.16564m^3 - 61.16983m^2\right)\left(12\right)\eta^2 + \left(6.68682m^7 + 5.54751m^6 + \right. \tag{283}$$

that can be upper bounded by

$$-46.79712\left(20\right)\left(0.15\right)^3\left(0.2\right)^3 + \Big(7.22182\left(0.25\right)^5 + 16.52261\left(0.25\right)^4 - 62.16564\left(0.15\right)^3 - 61.16983\left(0.15\right)^2\Big)\left(12\right)\left(0.2\right)^2$$

which is negative.

So, we evaluate the first partial η of Equation 282 at $\eta = 0.2$ that can be upper bounded by

$$7.91953\,(0.25)^8 + 1.63119\,(0.25)^7 + 10.48921\,(0.25)^6 + 8.00789\,(0.25)^5 + 7.60366\,(0.25)^4 + 1.88759\,(0.25)^3 - 1.98079\,(0$$

which is negative.

We then evaluate Equation 282 at $\eta = 0.2$ that can be upper bounded by

$$(((((((1.21330\,(0.25) + 0.42836)\,(0.25) + 1.70880)\,(0.25) + 0.57558)\,(0.25) + 0.90708)\,(0.25) + 0.10321)\,(0.25) - 0.0761$$

which is negative.

For Equation 279, we begin by iteratively upper bounding α_2. We start by upper bound

$$(-0.05617m + 2.00605)\,(0.3) + \left(-6.24104m^3 - 2.78617m^2 + 10.02107m - 2.57463\right) \quad (284)$$

by fixing $\eta = 0.3$, which simplifies (RU) to

$$-6.24104m^3 - 2.78617m^2 + 10.00422m - 1.97281. \quad (285)$$

We then upper bound

$$\left(-6.24104m^3 - 2.78617m^2 + 10.00422m - 1.97281\right) m\eta + \left(-4.59717m^6 - 8.77505m^5 - 6.73043m^4 - 9.10295m^3 - 0.\right. \quad (286)$$

by

$$(((-6.24104\,(0.15) - 2.78617)\,(0.15) + 10.00422)\,(0.25) - 1.97281)\,m\,(0.3) + \left(-4.59717m^6 - 8.77505m^5 - 6.73043m^4\right. \quad (287)$$

which simplifies (RU) to

$$-4.59717m^6 - 8.77505m^5 - 6.73043m^4 - 9.10295m^3 - 0.73075m^2 - 2.07475m - 0.06439. \quad (288)$$

Since

$$-4.59717\,(0.15)^6 - 8.77505\,(0.15)^5 - 6.73043\,(0.15)^4 - 9.10295\,(0.15)^3 - 0.73075\,(0.15)^2 - 2.07475\,(0.15) - 0.06439$$

is negative, we upper bound

$$\left(-4.59717m^6 - 8.77505m^5 - 6.73043m^4 - 9.10295m^3 - 0.73075m^2 - 2.07475m - 0.06439\right) m\eta + \left(5.62051m^9 + 5.032\right. \quad (289)$$

by $\eta = 0.2$, which simplifies (RU) to

$$5.62051m^9 + 4.11265m^7 - 5.0463m^6 - 1.14610m^5 - 3.3332m^4 - 0.62606m^3 - 0.19091m^2 - 0.01267m - 0.00042. \quad (290)$$

Since

$$5.62051\,(0.25)^9 + 4.11265\,(0.25)^7 - 5.0463\,(0.15)^6 - 1.14610\,(0.15)^5 - 3.3332\,(0.15)^4 - 0.62606\,(0.15)^3 - 0.19091\,(0.15$$

is negative, we upper bound

$$\left(5.62051m^9 + 4.11265m^7 - 5.0463m^6 - 1.14610m^5 - 3.3332m^4 - 0.62606m^3 - 0.19091m^2 - 0.01267m - 0.00042\right) \eta - \quad (291)$$

by $\eta = 0.2$, which simplifies (RU) to

$$2.52924m^9 + 0.11242m^8 + 2.97441m^7 - 2.19152m^6 + 1.70855m^5 - 1.16531m^4 + 0.32618m^3 - 0.05019m^2 + 0.01047m \quad (292)$$

We then show that

$$\alpha_{2,ub} = \left(2.52924m^9 + 0.11242m^8 + 2.97441m^7 - 2.19152m^6 + 1.70855m^5 - 1.16531m^4 + 0.32618m^3 - 0.05019m^2 + \right. \quad (293)$$

is positive in lemmas B.168 and B.167.

For our upper bound of Equation 279, $\alpha_1 + \alpha_{2,ub}\sqrt{2m\eta + 0.25m + 0.02}$, we replace $\sqrt{2m\eta + 0.25m + 0.02}$ with $\sqrt{2m\,(0.3) + 0.25m + 0.02}$, which we then linearize at $m = 0.15$.

We then show the resulting expression is concave in η. Consider the second partial η below, which we iteratively upper bound in η.

$$3.89977\,(30)\,m^3\eta^4 + \left(-3.61089m^5 - 2.00605m^4 - 1.59882m^3 + 1.87565m^2\right)(20)\,\eta^3 + \left(-2.50755m^7 + 2.99427m^6 - 13\right. \tag{294}$$

Noting that

$$\left(-72.2178\,(0.25)^3 - 40.121\,(0.25)^2 + 3.12153\,(0.15) + 37.513\right)$$

is positive and

$$-30.0906\,(0.15)^5 + 35.93124\,(0.25)^4 - 183.50814\,(0.15)^3 - 7.52166\,(0.15)^2 - 82.76846\,(0.15) - 8.05794$$

is negative, we can upper bound the second partial η by the following expression in m.

$$-6.77721m^8 - 0.50042m^7 - 7.64101m^6 - 3.56144m^5 - 5.49147m^4 - 1.51407m^3 - 1.59096m^2 - 0.09616m - 0.00080 \tag{295}$$

We then show that the first partial η of our upper bound of $\alpha_1 + \alpha_{2,ub}\sqrt{2m\eta + 0.25m + 0.02}$ at $\eta = 0.2$ is negative. Note that the first partial can be upper bounded by

$$2.79889\,(0.25)^{10} + 0.67596\,(0.25)^9 + 2.03619\,(0.25)^8 - 1.16633\,(0.15)^7 - 0.49350\,(0.15)^6 - 0.10878\,(0.15)^5 - 0.96987\,(0$$

which is negative.

So, we evaluate our upper bound of $\alpha_1 + \alpha_{2,ub}\sqrt{2m\eta + 0.25m + 0.02}$ at $\eta = 0.2$, which can be upper bounded by

$$(((((((((0.65698\,(0.25) + 0.16992)\,(0.25) + 0.69678)\,(0.25) - 0.30845)\,(0.15) + 0.22437)\,(0.25) - 0.00726)\,(0.25) - 0.07$$

which is negative.

This completes the proof.

□

Proof of Lemma B.166. To prove the lemma, note that upper bound Equation 158 iteratively in η. Noting that

$$15.98899\,(0.25)^4 + 1.85744\,(0.25)^3 + 19.16653\,(0.25)^2 - 16.80458\,(0.15) - 8.36065$$

is negative,

$$9.19437\,(0.25)^7 - 7.42982\,(0.15)^6 + 17.09706\,(0.25)^5 + 3.71530\,(0.25)^4 + 10.13771\,(0.25)^3 + 1.37560\,(0.25)^2 - 2.51111$$

is negative, and

$$3.92851\,(0.25)^7 + 0.03240\,(0.25)^6 + 5.70965\,(0.25)^5 + 1.7081\,(0.25)^4 + 2.68833\,(0.25)^3 + 0.32732\,(0.25)^2 - 0.48193\,(0.1 \tag{296}$$

is negative, we can upper bound Equation 158 by

$$0.89020m^7 + 0.25757m^6 + 1.26293m^5 + 0.38177m^4 + 0.55958m^3 + 0.00450m^2 - 0.10396m - 0.00311. \tag{297}$$

We can then upper bound the above by

$$0.89020\,(0.25)^7 + 0.25757\,(0.25)^6 + 1.26293\,(0.25)^5 + 0.38177\,(0.25)^4 + 0.55958\,(0.25)^3 + 0.00450\,(0.25)^2 - 0.10396\,(0.$$

which is negative.

This completes the proof.

□

Proof of Lemma B.167. To prove the lemma, note that Equation 159 can be lower bounded by

$$(((((((((0.08783\,(0.15) + 0.01406)\,(0.15) + 0.18114)\,(0.15) - 0.11006)\,(0.25) + 0.21808)\,(0.15) - 0.04141)\,(0.25) + 0.05$$

which is positive.

This completes the proof. □

Proof of Lemma **??**. To prove the lemma, note that Equation 160 can be lower bounded by

$$(((((((((2.52924\,(0.25)+0.11242)\,(0.25)+2.97441)\,(0.25)-2.19152)\,(0.25)+1.70855)\,(0.15)-1.16531)\,(0.25)+0.32$$

which is positive.

This completes the proof. □

Proof of Lemma B.169. . To prove the lemma, we focus on prove that the following subexpression from Equation 161 is positive.

$$8.25536\eta+\left(-7.42982m^2-4.29278\right)m \tag{298}$$

Note that we can lower bound the above by

$$8.25536\,(0.2)+\left(-7.42982\,(0.25)^2-4.29278\right)(0.25)\,,$$

which is positive.

This completes the proof. □

Proof of Lemma B.170. To prove the lemma, we first upper bound Equation 162 by

$$\left(-50.15112\,(0.15)^4+158.21496\,(0.25)^3-212.79648\,(0.15)^2+77.41236\,(0.25)-349.32075\right)m^2+111.9462m-13.936 \tag{299}$$

which simplifies (by rounding up after five digits of precision) to

$$-332.30886m^2+111.9462m-13.93608. \tag{300}$$

We evaluate the above at the maximum $m=\frac{111.9462}{(2)332.30886}$,, which is negative.

This completes the proof. □

Proof of Lemma B.171. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we first note the following.

First,

$$-0.18574m^3+0.45m^2-0.05341m+0.37598 \tag{301}$$

can be lower bounded by

$$-0.18574\,(0.25)^3+0.45\,(0.15)^2-0.05341\,(0.25)+0.37598,$$

which is positive.

Second,

$$0.18575m^3-0.45m^2+0.05342m-0.37597 \tag{302}$$

can be upper bounded by

$$0.18575\,(0.25)^3-0.45\,(0.15)^2+0.05342\,(0.25)-0.37597,$$

which is negative.

So, we replace the square root in

$$\left(0.18575m^3-0.45m^2+0.05342m-0.37597\right)\sqrt{0.65m+0.02} \tag{303}$$

with $(0.84417m+0.2161)$ (see lemma B.241), where the $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$ coefficient then simplifies (RU) to

$$-0.24819m^4-0.46046m^3-0.73962m^2-0.42361m-0.29794. \tag{304}$$

Since the above is negative, we linearize the square root term in $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$ at $m=0.1986$.

We also linearize the square root in

$$\left(-0.18574m^3+0.45m^2-0.05341m+0.37598\right)\sqrt{0.65m+0.02} \tag{305}$$

at $m=0.1986$, where the $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ coefficient then simplifies (RU) to

$$-0.56133m^4+0.21737m^3-0.63390m^2+0.18700m-0.13437. \tag{306}$$

Since the above can be upper bounded by

$$-0.56133\left(0.15\right)^4+0.21737\left(0.25\right)^3-0.63390\left(0.15\right)^2+0.18700\left(0.25\right)-0.13437,$$

which is negative, we replace the square root term in $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m+\sqrt{0.65m+0.02}\right)\right)$ with $(0.84417m+0.2161)$ (see lemma B.241).

Note the resulting expression below.

$$\left(-0.56133m^4+0.21737m^3-0.63390m^2+0.18700m-0.13437\right)\exp\left(4.14938m^3+0.48622m^2+4.18995m+0.49097\right) \tag{307}$$

Note from the above that

$$0.35615m^3-0.49266m^2+0.35963m-0.49748 \tag{308}$$

is concave (i.e., second partial m can be upper bounded by $0.35615\left(6\right)\left(0.25\right)-0.49266\left(2\right)$). So, we can replace $\exp\left(0.35615m^3-0.49266m^2+0.35963m-0.49748\right)$ with $\exp\left(0.20619m-0.48435\right)$, where $0.20619m-0.48435$ is a linear lower bound for $0.35615m^3-0.49266m^2+0.35963m-0.49748$. To see this note that the difference between the linear bound and the polynomial is convex in m and negative at $m=0.15,0.25$.

We also replace $\exp\left(4.14938m^3+0.48622m^2+4.18995m+0.49097\right)$ by linearizing $4.14938m^3+0.48622m^2+4.18995m+0.49097$ at $m=0.1954$, since it is convex.

This gives us the following upper bound for Equation 307.

$$\left(-0.56133m^4+0.21737m^3-0.63390m^2+0.18700m-0.13437\right)\exp\left(4.85524m+0.41049\right)+\left(-0.24819m^4-0.46046\right. \tag{309}$$

We then show that the above is convex in m. Consider the second partial m (RD).

$$\left(48.89767m^6+1.50538m^5+69.36719m^4+22.74570m^3+30.52348m^2+10.44717m+4.74125\right)\exp\left(4.5m^3+4.544m\right) \tag{310}$$

Note that we replace $\exp\left(4.85524m+0.41049\right)$ with linear upper bound $19.51922m+0.19507$ and $\exp\left(0.20619m-0.48435\right)$ with linear upper bound $0.13239m+0.61560$. To see that these linear upper bounds are valid, note that the difference between the linear bounds and the respective exponential term is concave and positive for $m=0.15,0.25$. We also linearize $\exp\left(4.5m^3+4.544m\right)$ at $m=0.1765$

This yields the lower bound

$$554.91325m^7+30.91640m^6+529.34833m^5-50.45367m^4+49.27973m^3-7.24977m^2+2.03551m-0.19570 \tag{311}$$

that we further lower bound by

$$((((((554.91325\left(0.15\right)+30.91640\right)\left(0.15\right)+529.34833\right)\left(0.15\right)-50.45367\right)\left(0.15\right)+49.27973\right)\left(0.15\right)-7.24977\right)\left(0.15\right)+$$

which is positive.

We then evaluate Equation 309 at $m=0.15,0.25$, which are negative.

This completes the proof.

□

Proof of Lemma B.172. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we first note the following.

First,

$$-0.18574m^3 + 0.45m^2 - 0.05341m + 0.37701 \tag{312}$$

can be lower bounded by

$$-0.18574\,(0.25)^3 + 0.45\,(0.15)^2 - 0.05341\,(0.25) + 0.37701,$$

which is positive.

Second,

$$0.18575m^3 - 0.45m^2 + 0.05342m - 0.37700 \tag{313}$$

can be upper bounded by

$$0.18575\,(0.25)^3 - 0.45\,(0.15)^2 + 0.05342\,(0.25) - 0.37700,$$

which is negative.

So, we upper bound Equation B.172 by replacing the square root in

$$\left(0.18575m^3 - 0.45m^2 + 0.05342m - 0.37700\right)\sqrt{0.65m + 0.025} \tag{314}$$

with $0.83012m + 0.22548$ (see lemma **??**), resulting in the $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.025}\right)\right)$ coefficient simplifying (RU) to

$$-0.25080m^4 - 0.45240m^3 - 0.74553m^2 - 0.41766m - 0.30264. \tag{315}$$

Since the above is negative, we replace $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.025}\right)\right)$ by linearizing the square root at $m = 0.1955$.

We also linearize the square root in

$$\left(-0.18574m^3 + 0.45m^2 - 0.05341m + 0.37701\right)\sqrt{0.65m + 0.025} \tag{316}$$

at $m = 0.1955$, resulting in the $\exp\left(\frac{1}{2}\left(4.5x^2 + 4.544\right)\left(x + \sqrt{0.65x + 0.025}\right)\right)$ coefficient simplifying (RU) to

$$-0.55979m^4 + 0.21214m^3 - 0.63075m^2 + 0.18533m - 0.13204. \tag{317}$$

Since

$$-0.55979\,(0.15)^4 + 0.21214\,(0.25)^3 - 0.63075\,(0.15)^2 + 0.18533\,(0.25) - 0.13204$$

is negative, we replace the square root term in $\exp\left(\frac{1}{2}\left(4.5x^2 + 4.544\right)\left(x + \sqrt{0.65x + 0.025}\right)\right)$ with $0.83012m + 0.22548$ (see lemma **??**).

We then see that Equation 164 can be upper bounded by

$$\left(-0.55979x^4 + 0.21214x^3 - 0.63075x^2 + 0.18533x - 0.13204\right)\exp\left(4.11777x^3 + 0.50733x^2 + 4.15803x + 0.51229\right) + \left(\tag{318}$$

We then show that the above is concave in m. Consider the second partial m below with $\exp\left(4.5m^3 + 4.544m\right)$ factored out.

$$\left(48.89768x^6 + 1.88082x^5 + 69.36720x^4 + 22.99845x^3 + 30.63473x^2 + 10.48971x + 4.75998\right) + \left(-0.31714x^8 + 0.00421 \tag{319}$$

To proceed, note that

$$-0.31714x^8 + 0.00421x^7 - 0.37852x^6 - 2.29874x^5 - 2.28562x^4 - 2.13847x^3 - 2.46454x^2 - 3.06943x - 1.54139 \tag{320}$$

can be upper bounded by

$$-0.31714\,(0.15)^8 + 0.00421\,(0.25)^7 - 0.37852\,(0.15)^6 - 2.29874\,(0.15)^5 - 2.28562\,(0.15)^4 - 2.13847\,(0.15)^3 - 2.46454\,($$

which is negative, and

$$-85.42634x^8 + 18.34020x^7 - 149.02107x^6 - 39.39436x^5 - 72.98595x^4 - 49.60003x^3 - 18.50908x^2 - 9.82593x - 2.137 \tag{321}$$

can be upper bounded by

$$-85.42634\,(0.15)^8 + 18.34020\,(0.25)^7 - 149.02107\,(0.15)^6 - 39.39436\,(0.15)^5 - 72.98595\,(0.15)^4 - 49.60003\,(0.15)^3 - 1$$

which is also negative.

We can the upper bound Equation 319 by linearizing the exponential terms at $m = 0.15$ (see lemmas B.174 and B.173).

We can then upper bound the resulting expression (RU) by

$$(((((((35.73878\,(0.25) - 148.97504)\,(0.15) + 92.41611)\,(0.25) - 178.09899)\,(0.15) - 31.03597)\,(0.15) - 28.94640)\,(0.15$$

which is negative.

We then evaluate Equation 318 at $m = 0.15$ (note the first partial m at $m = 0.15$ is negative), which is negative,

This completes the proof.

□

Proof of Lemma B.173. To prove the lemma, note that we lower bound Equation 165 (with the exponential term factored out) by

$$(-24.75096\,(0.25) - 1.02168) + \left(12.37548\,(0.15)^2 + 1.02168\,(0.15) + 4.16549\right)^2,$$

which is positive.

This completes the proof. □

Proof of Lemma B.174. To prove the lemma, note that the second partial x of Equation **??** (with the exponential term factored out) can be lower bounded by

$$1.31489\,(0.15)^4 - 2.32701\,(0.25)^3 + 1.91471\,(0.15)^2 - 3.07664\,(0.25) + 1.16363,$$

which is positive.

This completes the proof. □

Proof. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, note that we split Equation 167 into two parts. For the first part, note that we factor out $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.025}\right)\right)$. For the second part, note that we factor out $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.025}\right)\right)$.

$$0.824\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(\sqrt{0.65m + 0.025} - m\right)\right) + \left((\nu_{1,1}) + \frac{\nu_{1,2}}{\sqrt{0.65m + 0.025}}\right) \tag{322}$$

$$\left((\nu_{2,1}) + \frac{\nu_{2,2}}{\sqrt{0.65m + 0.025}}\right) + \left((-6.03673)\,m^5 + (-0.2322)\,m^4 + (-7.98788)\,m^3 + (2.03288)\,m^2 + (-2.5484)\,m + 1\right) \tag{323}$$

where

$$\nu_{1,1} = (4.55625)\,m^7 + (2.29857)\,m^6 + (8.15126)\,m^5 + (2.60482)\,m^4 + (3.20583)\,m^3 + (-0.00524)\,m^2 + (-0.53257)\,m -$$

$$\nu_{1,2} = (-3.40858)\,m^7 + (2.51328)\,m^6 + (-5.35880)\,m^5 + (4.31604)\,m^4 + (-2.97205)\,m^3 + (1.40257)\,m^2 + (-0.86535)$$

$\nu_{2,1} = (4.55625)\, m^7 + (2.29856)\, m^6 + (8.15126)\, m^5 + (2.60480)\, m^4 + (3.20582)\, m^3 + (-0.00525)\, m^2 + (-0.53257)\, m -$
and
$\nu_{2,2} = (3.40857)\, m^7 + (-2.51327)\, m^6 + (5.35878)\, m^5 + (-4.31604)\, m^4 + (2.97205)\, m^3 + (-1.40255)\, m^2 + (0.86537)\, m$

For Equation 322, we note that $\sqrt{0.65x + 0.025} - x$ locally decreases in m, so we replace $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(\sqrt{0.65m + 0.025} - m\right)\right)$ with $\exp\left(\frac{1}{2}\left(4.5\,(0.25)^2 + 4.544\right)\left(\sqrt{0.65\,(0.15) + 0.025} - (0.15)\right)\right)$ and scale the resulting expression by $\sqrt{0.65m + 0.025}$. Further note that $\nu_{1,1}$ can be lower bounded by

$(4.55625)\,(0.15)^7 + (2.29857)\,(0.15)^6 + (8.15126)\,(0.15)^5 + (2.60482)\,(0.15)^4 + (3.20583)\,(0.15)^3 + (-0.00524)\,(0.25)^2 +$

which is positive. So we linearize $\sqrt{0.65m + 0.025}$ at $m = 0.15$. We then show that the resulting expression (RU) is convex in m. Consider the second partial m

$$4.23085\,(56)\, m^6 - 0.31408\,(42)\, m^5 + 10.56674\,(30)\, m^4 - 1.22238\,(20)\, m^3 + 7.84180\,(12)\, m^2 - 2.30138\,(6)\, m + 0.90694 \tag{324}$$

that we lower bound by

$4.23085\,(56)\,(0.15)^6 - 0.31408\,(42)\,(0.25)^5 + 10.56674\,(30)\,(0.15)^4 - 1.22238\,(20)\,(0.25)^3 + 7.84180\,(12)\,(0.15)^2 - 2.301$

which is positive.

So we evaluate our upper bound for Equation 322 at $m = 0.15, 0.25$, which are negative.

For Equation 323, note that

$$(-6.03673)\, m^5 + (-0.2322)\, m^4 + (-7.98788)\, m^3 + (2.03288)\, m^2 + (-2.5484)\, m + 1 \tag{325}$$

can be lower bounded by

$(-6.03673)\,(0.25)^5 + (-0.2322)\,(0.25)^4 + (-7.98788)\,(0.25)^3 + (2.03288)\,(0.15)^2 + (-2.5484)\,(0.25) + 1,$

which is positive.

So, we can replace $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.025}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(4.5\,(0.15)^2 + 4.544\right)\left(\sqrt{0.65\,(0.25) + 0.025} - (0.25)\right)\right)$ and $\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.025}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(4.5\,(0.15)^2 + 4.544\right)\left((0.15) + \sqrt{0.65\,(0.15) + 0.025}\right)\right.$ and scale by $\sqrt{0.65m + 0.025}$ yielding the following upper bound (RU).

$$\left((4.55625)\, x^7 + (2.29856)\, x^6 + (4.20487)\, x^5 + (2.45301)\, x^4 + (-2.01609)\, x^3 + (1.32371)\, x^2 + (-2.19853)\, x + (-0.1525\right. \tag{326}$$

Since

$$(4.55625)\, m^7 + (2.29856)\, m^6 + (4.20487)\, m^5 + (2.45301)\, m^4 + (-2.01609)\, m^3 + (1.32371)\, m^2 + (-2.19853)\, m + (-0.1 \tag{327}$$

can be upper bound by

$(4.55625)\,(0.25)^7 + (2.29856)\,(0.25)^6 + (4.20487)\,(0.25)^5 + (2.45301)\,(0.25)^4 + (-2.01609)\,(0.15)^3 + (1.32371)\,(0.25)^2 +$

which is negative, we can replace $\sqrt{0.65m + 0.025}$ in Equation 326 with $\sqrt{0.65\,(0.15) + 0.025}$, yielding the following.

$$(5.00326)\, m^7 + (-1.70877)\, m^6 + (6.83049)\, m^5 + (-3.45748)\, m^4 + (2.26642)\, m^3 + (-0.93925)\, m^2 + (0.09589)\, m + (-0 \tag{328}$$

We then upper bound the above with

$((((((5.00326)\,(0.25) + (-1.70877))\,(0.15) + (6.83049))\,(0.25) + (-3.45748))\,(0.15) + (2.26642))\,(0.25) + (-0.93925))$

which is negative.

This completes the proof.

□

Proof. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To proceed, we split up Equation 168 into two parts (shown below) and show both are concave in η.

$$\left(\left(-4.01208mx^3+\left(-1.85744m^3+2.06384m^2-1.57468m-1.93970\right)x^2+\left(-0.23217m^3-1.51483m^2-0.10834m-\right.\right. \tag{329}$$

$$\left(4.12768m^2\eta^2+\left(0.51596m^2+0.05160m+1\right)\eta\right)\exp\left(m\left(\frac{0.9}{\eta}m^2+\frac{0.52}{\eta}+1.944\right)\right)+\left(k_{1,1}+k_{1,2}\sqrt{2m\eta+0.25m+\right. \tag{330}$$

where

$$k_{1,1}=-4.01208m\eta^3+\left(-1.85744m^3+2.06384m^2-1.57468m-1.93970\right)\eta^2+\left(-0.23217m^3-1.51483m^2-0.10834\right.$$

$$k_{1,2}=\left(0.05779m-2.06383\right)\eta^2+\left(-0.92872m^3-0.27861m+0.99780\right)\eta+0.45m^2+0.26.$$

First, we show that Equation 329 concave in η. Consider the second partial η of Equation 329 below, split into two parts (RU with the exponential term factored out). Note that the second part is further scaled by $\eta^3\left(2m\eta+0.25m+0.025\right)^{\frac{3}{2}}$.

$$\left(-1.00302m\eta^3+\left(-0.46436m^3+0.51596m^2-0.39367m-0.484925\right)\eta^2+\left(-0.0580425m^3-0.3787075m^2-0.02708\right. \tag{331}$$

$$h_1\sqrt{2m\eta+0.25m+0.025}+h_2, \tag{332}$$

where

$$h_1=\left(\left(0.50546\right)m^3+\left(-66.19943\right)m^2\right)\eta^5+\left(\left(9.74786\right)m^5+\left(0.92866\right)m^4+\left(15.19740\right)m^3+\left(-5.71182\right)m^2+\left(-8.56\right.$$

and

$$h_2=\left(42.89718\right)m^3\eta^6+\left(\left(-3.61088\right)m^5+\left(-14.04234\right)m^4+\left(6.30880\right)m^3+\left(44.74031\right)m^2\right)\eta^5+\left(\left(-4.17923\right)m^7+\left(\right.$$

To show that Equation 331 is negative, we show that the following are negative.

$$-1.00302m\eta^3+\left(-0.46436m^3+0.51596m^2-0.39367m-0.484925\right)\eta^2+\left(-0.0580425m^3-0.3787075m^2-0.027085\right. \tag{333}$$

$$\left(-0.014445m+0.51596\right)\eta^2+\left(0.2321825m^3+0.069655m-0.2494475\right)\eta-0.1125m^2-0.065 \tag{334}$$

To show that Equation 333 is negative, we upper bound it by

$$-1.00302\left(0.15\right)\left(0.2\right)^3+\left(-0.46436\left(0.15\right)^3+0.51596\left(0.25\right)^2-0.39367\left(0.15\right)-0.484925\right)\left(0.2\right)^2+\left(-0.05804\left(0.15\right)^3\right.$$

which is negative.

To show that Equation 334 is negative, we upper bound it by

$$\left(-0.01444\left(0.15\right)+0.51596\right)\left(0.3\right)^2+\left(0.23219\left(0.25\right)^3+0.06966\left(0.25\right)-0.24944\right)\left(0.2\right)-0.1125\left(0.15\right)^2-0.065,$$

which is negative.

To show that Equation 332 is negative, we show that h_1 is negative in lemma B.180, so we upper bound $h_1\sqrt{2m\eta+0.25m+0.025}$ by $h_1\sqrt{0.65m+0.025}$. We then upper bound $h_1\sqrt{0.65m+0.025}$ by $h_1(0.83012m+0.22548)$ (see lemma **??**). Then, $h_1(0.83012m+0.22548)+h_2$ simplifies (RU) to

$$\left(42.89718\right)m^3\eta^6+\left(\left(-3.61088\right)m^5+\left(-13.62274\right)m^4+\left(-48.53069\right)m^3+\left(29.81367\right)m^2\right)\eta^5+\left(\left(-4.17923\right)m^7+\left(13\right.$$

We proceed then by showing that the above is concave in η in lemma B.182 and that the first partial η at $\eta = 0.2$ is negative in lemma B.181.

We then upper bound Equation 335 by $\eta = 0.2$, which simplifies (RU) to

$$(-0.19667)\, m^9 + (0.06798)\, m^8 + (-0.43665)\, m^7 + (0.15438)\, m^6 + (-0.26103)\, m^5 + (0.09789)\, m^4 + (-0.05187)\, m^3 + \tag{336}$$

We upper bound the above by

$$(-0.19667)\,(0.15)^9 + (0.06798)\,(0.25)^8 + (-0.43665)\,(0.15)^7 + (0.15438)\,(0.25)^6 + (-0.26103)\,(0.15)^5 + (0.09789)\,(0.2$$

which is negative.

Second, we show that Equation 330 is concave in η. We divide up the second partial η into the following three parts. Note that the first part has

$$\left(\left(\frac{(-0.9)}{\eta^2}m^3 + \frac{(-0.52)}{\eta^2}m\right) + \frac{\frac{(-0.9)}{\eta}m^3 + \frac{(-0.52)}{\eta}m + \frac{(-0.225)}{\eta^2}m^3 + \frac{(-0.0225)}{\eta^2}m^2 + \frac{(-0.13)}{\eta^2}m + \frac{(-0.013)}{\eta^2} + 1.944m}{\sqrt{2m\eta + 0.25m + 0.025}}\right)^2 \mathrm{e}$$

factored out.

The second part is scaled by $\eta^2\left(2m\eta + 0.25m + 0.025\right)^{\frac{3}{2}}$ and has $\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right)$ factored out.

The third part is scaled by $\eta^3\left(2mn + 0.25m + 0.025\right)^{\frac{3}{2}}$ with $\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right)$ factored out. Furthermore, we replace $\exp\left(\frac{1}{2}\left(\frac{0.9}{x}m^2 + \frac{0.52}{x} + 1.944\right)\left(m - \sqrt{2mx + 0.25m + 0.025}\right)\right)$ in the expression by $\exp\left(-\frac{1}{2}\left(\frac{0.9}{0.3}(0.15)^2 + \frac{0.52}{0.3} + 1.944\right)\left(\sqrt{2(0.25)\,(0.2) + 0.25(0.25) + 0.025} - (0.25)\right)\right)$, since the corresponding coefficient is positive (see lemma B.177).

$$\left(-1.00302m\eta^3 + \left(-0.46436m^3 + 0.51596m^2 - 0.39367m - 0.484925\right)\eta^2 + \left(-0.05804m^3 - 0.37870m^2 - 0.027085m - \right. \tag{337}$$

$$(\mu_1)\sqrt{2m\eta + 0.25m + 0.025} + (\mu_2)\,, \tag{338}$$

where

$$\mu_1 = \left((0.56172)\, m^3 + (-68.20538)\, m^2\right)\eta^4 + \left((15.98889)\, m^5 + (1.85748)\, m^4 + (19.05413)\, m^3 + (-3.13213)\, m^2 + (-8.5\right.$$

and

$$\mu_2 = (-46.7969)\, m^3\eta^5 + \left((7.22178)\, m^5 + (16.04842)\, m^4 + (-4.70978)\, m^3 + (-46.62551)\, m^2\right)\eta^4 + \left((6.6868)\, m^7 + (-\right.$$

$$\tau_1 + \tau_2\sqrt{2m\eta + 0.25m + 0.025}, \tag{339}$$

where

$$\tau_1 = (3.899745)\, m^3m^6 + \left(-3.61087m^5 - 2.00605m^4 - 1.59883m^3 + 1.88539m^2\right)m^5 + \left((-2.50754)\, m^7 + (2.99423)\, m^6\right.$$

and

$$\tau_2 = \left(-0.05617m^3 + 2.00605m^2\right)m^5 + \left(-6.24100m^5 - 2.78617m^4 + 7.86381m^3 - 2.57964m^2\right)m^4 + \left(-4.59715m^7 - \right.$$

To show that Equation 337 is negative, note that

$$(0.01445m - 0.51595)\,\eta^2 + \left(-0.23218m^3 - 0.06965m + 0.24945\right)\eta + 0.1125m^2 + 0.065$$

can be lower bounded by

$$(0.01445\,(0.15) - 0.51595)\,(0.3)^2 + \left(-0.23218\,(0.25)^3 - 0.06965\,(0.25) + 0.24945\right)(0.2) + 0.1125\,(0.15)^2 + 0.065,$$

which is positive.

So, we can upper bound Equation 337 by replacing $\sqrt{2m\eta + 0.25m + 0.025}$ by $\sqrt{2\left(0.25\right)\left(0.3\right) + 0.25\left(0.25\right) + 0.025}$, yielding the following expression (RU).

$$-1.00302m\eta^3 + \left(-0.46436m^3 + 0.51597m^2 - 0.38662m - 0.73636\right)\eta^2 + \left(-0.17119m^3 - 0.3787m^2 - 0.06102m + 0.1\right. \tag{340}$$

Note that the above can be upper bounded by

$$-1.00302\left(0.15\right)\left(0.2\right)^3 + \left(-0.46436\left(0.15\right)^3 + 0.51597\left(0.25\right)^2 - 0.38662\left(0.15\right) - 0.73636\right)\left(0.2\right)^2 + \left(-0.17119\left(0.15\right)^3 +$$

which is negative.

To show that Equation 338 is negative, we first show that the square root coefficient (μ_1) is negative in lemma B.5.4. So, we can replace $\sqrt{2m\eta + 0.25m + 0.025}$ in Equation 338 by $\sqrt{2m\left(0.2\right) + 0.25m + 0.025}$, which we replace with $\left(\left(0.83012\right)m + \left(0.22548\right)\right)$ (see lemma **??**).

We then show that the resulting expression (RU) decreases in η. First, we show concavity in η. Consider the second partial η below, which we iteratively upper bound in η.

$$\left(-46.7969\right)\left(20\right)m^3m^3 + \left(\left(7.22178\right)m^5 + \left(16.51472\right)m^4 + \left(-61.20177\right)m^3 + \left(-62.00445\right)m^2\right)\left(12\right)m^2 + \left(\left(6.6868\right)m^7\right. \tag{341}$$

Observing that expressions

$$86.66136\left(0.25\right)^3 + 198.17664\left(0.25\right)^2 - 921.60884\left(0.15\right) - 744.0534$$

and

$$40.1208\left(0.25\right)^6 + 31.93668\left(0.25\right)^5 + 232.63700\left(0.25\right)^4 + 131.87915\left(0.25\right)^3 - 76.41764\left(0.15\right)^2 - 200.54688\left(0.15\right) - 16$$

are negative, we get the resulting expression in m for the second partial η.

$$25.29504m^8 + 5.6861m^7 + 35.06548m^6 + 63.67914m^5 + 44.65013m^4 - 2.69800m^3 - 35.42903m^2 - 3.50812m - 0.0488 \tag{342}$$

We can then upper bound the above by

$$25.29504\left(0.25\right)^8 + 5.6861\left(0.25\right)^7 + 35.06548\left(0.25\right)^6 + 63.67914\left(0.25\right)^5 + 44.65013\left(0.25\right)^4 - 2.69800\left(0.15\right)^3 - 35.4290$$

which is negative.

We then evaluate the first partial η our upper bound for Equation 338 at $\eta = 0.2$ that we upper bound by

$$\left(7.83848\right)\left(0.25\right)^8 + \left(1.76127\right)\left(0.25\right)^7 + \left(10.36259\right)\left(0.25\right)^6 + \left(8.10193\right)\left(0.25\right)^5 + \left(7.67534\right)\left(0.25\right)^4 + \left(2.00508\right)\left(0.25\right)^3 +$$

which is negative.

So, we evaluate out upper bound for Equation 338 at $\eta = 0.2$ that we upper bound by

$$(((((((\left(1.20079\right)\left(0.25\right) + \left(0.44976\right))\left(0.25\right) + \left(1.69912\right))\left(0.25\right) + \left(0.60002\right))\left(0.25\right) + \left(0.91237\right))\left(0.25\right) + \left(0.11862\right))\left(0.2\right. \tag{343}$$

which is negative.

For Equation **??**, note that we first upper bound τ_2 iterative in η.

To do so, we note that

$$-6.241\left(0.15\right)^3 - 2.78617\left(0.15\right)^2 + 7.84696\left(0.25\right) - 1.97782$$

is negative,

$$-4.59715\left(0.15\right)^6 - 6.8335\left(0.15\right)^5 - 7.97859\left(0.15\right)^4 - 8.56673\left(0.15\right)^3 + 0.54115\left(0.15\right)^2 - 2.59556\left(0.15\right) - 0.08048$$

is negative, and

$$4.74682\left(0.25\right)^9 + 3.10305\left(0.25\right)^7 - 4.44454\left(0.15\right)^6 - 1.67575\left(0.15\right)^5 - 3.11696\left(0.15\right)^4 - 0.36474\left(0.15\right)^3 - 0.30378\left(0.\right.$$

is negative.

So, τ_2 can be upper bounded by

$$\left(2.13609m^9 - 0.6103m^8 + 3.03742m^7 - 2.06811m^6 + 1.28525m^5 - 1.12373m^4 + 0.31345m^3 - 0.07334m^2 + 0.01307m \right. \tag{344}$$

To show that the above is positive, we show that the constant (in η) and linear term in η are positive in lemmas B.179 and B.178.

We can then replace $\sqrt{2m\eta + 0.25m + 0.025}$ with $\sqrt{2m\,(0.3) + 0.25m + 0.025}$, which we linearize at $m = 0.15$.

For the resulting expression (RU), we show it is concave in η. Consider the second partial η below, which we iteratively upper bound in η.

$$(3.899745)\,(30)\,m^3\eta^4 + \left(-3.61087m^5 - 2.00605m^4 - 1.59883m^3 + 1.88539m^2\right)(20)\,\eta^3 + \left((-2.50754)\,m^7 + (2.99423)\right. \tag{345}$$

Noting that

$$-72.2174\,(0.25)^3 - 40.121\,(0.25)^2 + 3.12111\,(0.15) + 37.7078$$

is positive,

$$-30.09048\,(0.15)^7 + 35.93076\,(0.25)^6 - 183.5079\,(0.15)^5 - 8.28018\,(0.15)^4 - 82.82870\,(0.15)^3 - 8.4375\,(0.15)^2$$

is negative, and

$$-20.06034\,(0.15)^8 - 15.42129\,(0.15)^7 - 12.41200\,(0.15)^6 - 39.46872\,(0.15)^5 - 14.99361\,(0.15)^4 - 15.82462\,(0.15)^3 - 6.$$

is negative, we can upper bound the second partial η by

$$-6.77724m^8 - 0.53381m^7 - 7.72462m^6 - 3.51670m^5 - 5.63502m^4 - 1.53838m^3 - 1.65280m^2 - 0.1218m - 0.00126. \tag{346}$$

We then show that the first partial η for our upper bound of Equation **??** is negative. Note that the first partial η can be upper bounded by

$$2.32475\,(0.25)^{10} - 0.17873\,(0.15)^9 + 1.88714\,(0.25)^8 - 0.96450\,(0.15)^7 - 0.98338\,(0.15)^6 - 0.13451\,(0.15)^5 - 0.99373m$$

which is negative.

Evaluating our upper bound for Equation **??** at $\eta = 0.2$, we upper bound it by

$$0.54568\,(0.25)^{10} - 0.10190\,(0.15)^9 + 0.68674\,(0.25)^8 - 0.24582\,(0.15)^7 + 0.08542\,(0.25)^6 - 0.01869\,(0.15)^5 - 0.08693\,(0$$

which is negative.

This completes the proof.

□

Proof of Lemma B.177. To prove the lemma, note that we iteratively lower bound Equation 169 in η. , after factoring out m^2.

Noting that

$$-7.42982\,(0.25)^3 - 4.29278\,(0.25) + 1.65107$$

is positive and

$$3.34343\,(0.15)^6 + 3.86351\,(0.15)^4 - 1.48597\,(0.25)^3 + 1.11613\,(0.15)^2 - 0.85856\,(0.25) + 0.33021$$

is positive, we can lower bound Equation 169 by the following expression in m.

$$1.08661m^6 + 0.0418m^5 + 2.06564m^4 - 0.24890m^3 + 1.29874m^2 - 0.15776m + 0.33644 \tag{347}$$

We can then lower bound the above by

$$1.08661\,(0.15)^6 + 0.0418\,(0.15)^5 + 2.06564\,(0.15)^4 - 0.24890\,(0.25)^3 + 1.29874\,(0.15)^2 - 0.15776\,(0.25) + 0.33644,$$

which is positive.

This completes the proof. □

Proof of Lemma B.178. To prove the lemma, we show that Equation 170 is increasing in m. First, we show the equation is convex in m. Note that we can lower bound the second partial m by

$$0.07417\,(72)\,(0.15)^7 - 0.07628\,(56)\,(0.25)^6 + 0.22109\,(42)\,(0.15)^5 - 0.10111\,(30)\,(0.25)^4 + 0.18100\,(20)\,(0.15)^3 - 0.0401$$

which is positive.

Then, we evaluate Equation 170 at $m = 0.15$ (first partial m at $m = 0.15$ is positive), which is positive.

This concludes the proof.

□

Proof of Lemma B.179. To prove the lemma, we first show that Equation **??** is decreasing in m. We first show the second partial m (shown below) is negative.

$$2.13609\,(72)\,m^7 - 0.6103\,(56)\,m^6 + 3.03742\,(42)\,m^5 - 2.06811\,(30)\,m^4 + 1.28525\,(20)\,m^3 - 1.12373\,(12)\,m^2 + 0.31345 \tag{348}$$

The above can be upper bounded by

$$(((((2.13609\,(72)\,(0.25) - 0.6103\,(56))\,(0.25) + 3.03742\,(42))\,(0.25) - 2.06811\,(30))\,(0.15) + 1.28525\,(20))\,(0.25) - 1.12 \tag{349}$$

which simplifies (by rounding up after five digits of precision) to $-8.17913m^2 + 0.31345\,(6)\,m - 0.07334\,(2)$. We evaluate $-8.17913m^2 + 0.31345\,(6)\,m - 0.07334\,(2)$ the maximum $m = \frac{0.31345(6)}{(2)8.17913}$, which is negative.

We then evaluate the first partial m of Equation **??** at $m = 0.15$, which is negative.

So, we evaluate Equation **??** at $m = 0.25$, which is positive.

This concludes the proof.

□

Proof. For $m \in [0.15, 0.25], \eta \in [0.2, 0.3]$,

$$\left((0.56172)\,m^3 + (-68.20538)\,m^2\right)\eta^4 + \left((15.98889)\,m^5 + (1.85748)\,m^4 + (19.05413)\,m^3 + (-3.13213)\,m^2 + (-8.56121 \right. \tag{350}$$

is negative. □

Proof of Lemma B.5.4. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we iteratively upper bound Equation 350 in η.

We first upper bound

$$((0.56172)\,m + (-68.20538))\,m\eta + \left((15.98889)\,m^4 + (1.85748)\,m^3 + (19.05413)\,m^2 + (-3.13213)\,m + (-8.56121)\right) \tag{351}$$

by fixing $\eta = 0.2$, which simplifies (RU) to

$$(15.98889)\,m^4 + (1.85748)\,m^3 + (19.16648)\,m^2 + (-16.77320)\,m + (-8.56121)\,. \tag{352}$$

Since

$$(15.98889)\,(0.25)^4 + (1.85748)\,(0.25)^3 + (19.16648)\,(0.25)^2 + (-16.77320)\,(0.15) + (-8.56121)$$

is negative, we upper bound

$$\left((15.98889)\,m^4 + (1.85748)\,m^3 + (19.16648)\,m^2 + (-16.77320)\,m + (-8.56121)\right)m\eta + \left((9.19435)\,m^7 - 7.42982m^6 + \right. \tag{353}$$

by $\eta = 0.2$, which simplifies (RU) to

$$(9.19435)\, m^7 - 7.42982m^6 + (17.09696)\, m^5 + (3.78078)\, m^4 + (10.18411)\, m^3 + (1.44536)\, m^2 + (-2.52451)\, m + (-0.09 \tag{354}$$

Since

$$(9.19435)\,(0.25)^7 - 7.42982\,(0.15)^6 + (17.09696)\,(0.25)^5 + (3.78078)\,(0.25)^4 + (10.18411)\,(0.25)^3 + (1.44536)\,(0.25)^2 +$$

is negative, we upper bound

$$\left((9.19435)\, m^7 - 7.42982m^6 + (17.09696)\, m^5 + (3.78078)\, m^4 + (10.18411)\, m^3 + (1.44536)\, m^2 + (-2.52451)\, m + (-0.0 \right. \tag{355}$$

by $\eta = 0.2$, which simplifies (RU) to

$$3.92849m^7 + 0.07419m^6 + 5.68174m^5 + 1.76894m^4 + 2.69026m^3 + 0.35579m^2 - 0.47955m - 0.01804. \tag{356}$$

Since

$$3.92849\,(0.25)^7 + 0.07419\,(0.25)^6 + 5.68174\,(0.25)^5 + 1.76894\,(0.25)^4 + 2.69026\,(0.25)^3 + 0.35579\,(0.25)^2 - 0.47955\,(0.15) - 0$$

is negative, we upper bound

$$\left(3.92849m^7 + 0.07419m^6 + 5.68174m^5 + 1.76894m^4 + 2.69026m^3 + 0.35579m^2 - 0.47955m - 0.01804\right)\eta + \left((0.10449 \right. \tag{357}$$

by $\eta = 0.2$, which simplifies (RU) to

$$0.89019m^7 + 0.27011m^6 + 1.26126m^5 + 0.39769m^4 + 0.55898m^3 + 0.01076m^2 - 0.10536m - 0.00392. \tag{358}$$

The above can be upper bounded by

$$0.89019\,(0.25)^7 + 0.27011\,(0.25)^6 + 1.26126\,(0.25)^5 + 0.39769\,(0.25)^4 + 0.55898\,(0.25)^3 + 0.01076\,(0.25)^2 - 0.10536\,(0.$$

which is negative.

This completes the proof.

□

Proof of Lemma B.180. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we iteratively upper bound Equation 172 in η. First, we upper bound

$$((0.50546)\, m + (-66.19943))\, m\eta + \left((9.74786)\, m^4 + (0.92866)\, m^3 + (15.19740)\, m^2 + (-5.71182)\, m + (-8.56121)\right) \tag{359}$$

by fixing $\eta = 0.2$, which simplifies (RU) to $9.74786m^4 + 0.92866m^3 + 15.29850m^2 - 18.95170m - 8.56121$.

Since

$$(9.74786)\,(0.25)^4 + (0.92866)\,(0.25)^3 + 15.29850\,(0.25)^2 - 18.95170\,(0.15) - 8.56121$$

is negative, we upper bound

$$\left(9.74786m^4 + 0.92866m^3 + 15.29850m^2 - 18.95170m - 8.56121\right)m\eta + \left((4.59718)\, m^7 + (-3.71491)\, m^6 + (7.16878)\, m \right. \tag{360}$$

by fixing $\eta = 0.2$, which simplifies (RU) to

$$4.59718m^7 - 3.71491m^6 + 9.11836m^5 + 1.68014m^4 + 6.91718m^3 - 1.33686m^2 - 2.60501m - 0.09698. \tag{361}$$

Since

$$4.59718\,(0.25)^7 - 3.71491\,(0.15)^6 + 9.11836\,(0.25)^5 + 1.68014\,(0.25)^4 + 6.91718\,(0.25)^3 - 1.33686\,(0.15)^2 - 2.60501\,(0.$$

is negative, we upper bound

$$\left(4.59718m^7 - 3.71491m^6 + 9.11836m^5 + 1.68014m^4 + 6.91718m^3 - 1.33686m^2 - 2.60501m - 0.09698\right)\eta + \left((0.62689 \right. \tag{362}$$

by fixing $\eta = 0.2$, which simplifies (RU) to

$$1.54633m^7 - 0.94215m^6 + 2.55319m^5 + 0.70703m^4 + 1.64004m^3 + 0.01464m^2 - 0.49535m - 0.01871. \tag{363}$$

Since

$$1.54633\,(0.25)^7 - 0.94215\,(0.15)^6 + 2.55319\,(0.25)^5 + 0.70703\,(0.25)^4 + 1.64004\,(0.25)^3 + 0.01464\,(0.25)^2 - 0.49535\,(0.$$

is negative, we upper bound

$$\left(1.54633m^7 - 0.94215m^6 + 2.55319m^5 + 0.70703m^4 + 1.64004m^3 + 0.01464m^2 - 0.49535m - 0.01871\right)\eta + \left((-0.729\right) \tag{364}$$

by fixing $\eta = 0.2$, which simplifies (RU) to

$$-0.729m^8 + 0.30927m^7 - 1.24953m^6 + 0.53089m^5 - 0.35467m^4 + 0.35141m^3 - 0.07007m^2 - 0.09231m - 0.00374. \tag{365}$$

Since

$$(-0.729)\,(0.15)^8 + 0.30927\,(0.25)^7 + (-1.24953)\,(0.15)^6 + (0.53089)\,(0.25)^5 + (-0.35467)\,(0.15)^4 + \left(0.35141\,(0.25)^3\right)$$

is negative, we upper bound

$$\left(-0.729m^8 + 0.30927m^7 - 1.24953m^6 + 0.53089m^5 - 0.35467m^4 + 0.35141m^3 - 0.07007m^2 - 0.09231m - 0.00374\right) \tag{366}$$

by fixing $\eta = 0.2$, which simplifies (RU) to

$$-0.23692m^8 + 0.05275m^7 - 0.38253m^6 + 0.09545m^5 - 0.13268m^4 + 0.06702m^3 - 0.02284m^2 - 0.01852m - 0.00065. \tag{367}$$

We then upper bound the above by

$$-0.23692\,(0.15)^8 + 0.05275\,(0.25)^7 - 0.38253\,(0.15)^6 + 0.09545\,(0.25)^5 - 0.13268\,(0.15)^4 + 0.06702\,(0.25)^3 - 0.02284\,($$

which is negative.

This completes the proof. □

Proof of Lemma B.181. To prove the lemma, note that we upper bound Equation 173 by

$$(-0.60515)\,(0.15)^9 + (0.70963)\,(0.25)^8 + (-2.08336)\,(0.15)^7 + (1.67635)\,(0.25)^6 + (-1.31178)\,(0.15)^5 + (1.08053)\,(0.2$$

which is negative.

This completes the proof. □

Proof. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we iteratively upper bound Equation 174 in η. We first upper bound

$$(42.89718)\,(30)\,m\eta + \left((-3.61088)\,m^3 + (-13.62274)\,m^2 + (-48.53069)\,m + (29.81367)\right)(20) \tag{368}$$

by fixing $\eta = 0.3$, which simplifies (RU) to $-72.2176m^3 - 272.4548m^2 - 584.53918m + 596.2734$.

Since

$$-72.2176\,(0.25)^3 - 272.4548\,(0.25)^2 - 584.53918\,(0.25) + 596.2734$$

is positive, then we can upper bound

$$\left(-72.2176m^3 - 272.4548m^2 - 584.53918m + 596.2734\right)m\eta + \left((-4.17923)\,m^6 + (13.04756)\,m^5 + (-14.28125)\,m^4 + (\right. \tag{369}$$

by fixing $\eta = 0.3$, which simplifies (RU) to

$$-50.15076m^6 + 156.57072m^5 - 193.04028m^4 + 78.75312m^3 - 302.82599m^2 + 107.6121m - 13.47216. \tag{370}$$

To show that Equation 370 is negative, we upper bound it by

$$((((-50.15076\,(0.15)+156.57072)\,(0.25)-193.04028)\,(0.15)+78.75312)\,(0.25)-302.82599)\,m^2+107.6121m-13.47$$

which simplifies (RU) to $-288.97939m^2+107.6121m-13.47216$.

We evaluate $-288.97939m^2+107.6121m-13.47216$ at the global max $m=\frac{107.6121}{(2)288.97939}$, which is negative.

So, we can upper bound

$$\left(-50.15076m^6+156.57072m^5-193.04028m^4+78.75312m^3-302.82599m^2+107.6121m-13.47216\right)m\eta+\left((2.1445\right. \tag{371}$$

by fixing $\eta=0.2$, which simplifies (RU) to

$$12.86712m^8-30.46297m^7+54.73401m^6-34.36971m^5+31.74417m^4-49.12451m^3+13.83546m^2-4.83781m-0.1 \tag{372}$$

Since

$$12.86712\,(0.25)^8-30.46297\,(0.15)^7+54.73401\,(0.25)^6-34.36971\,(0.15)^5+31.74417\,(0.25)^4-49.12451\,(0.15)^3+13.8$$

is negative, we can upper bound

$$\left(12.86712m^8-30.46297m^7+54.73401m^6-34.36971m^5+31.74417m^4-49.12451m^3+13.83546m^2-4.83781m-0\right. \tag{373}$$

by fixing $\eta=0.2$, which simplifies (RU) to

$$4.28981m^8-8.01125m^7+13.79363m^6-7.87308m^5+7.82588m^4-9.78740m^3+2.99960m^2-0.98240m-0.02405. \tag{374}$$

The above can then be upper bounded by

$$(((((((4.28981\,(0.25)-8.01125)\,(0.15)+13.79363)\,(0.25)-7.87308)\,(0.15)+7.82588)\,(0.25)-9.78740)\,(0.15)+2.999$$

which is negative.

This completes the proof.

□

Proof of Lemma B.183. To prove the Equation 175 is convex in u, we first the second partial derivative with respect to u is growing in u.

To do so, we split the the third partial u (rounded down after five digits of precision) into the two parts.

$$\left(0.125\cot\left(0.245\pi\right)q^3u^3+\left(0.125\cot\left(0.245\pi\right)nq^4u^2-0.125\cot\left(0.245\pi\right)mq^3u^2-2.25\cot\left(0.245\pi\right)q^2u^2\right)+\left(-1.75\cot\right.\right. \tag{375}$$

$$\left(0.125\cot\left(0.245\pi\right)q^3u^3+\left(0.125\cot\left(0.245\pi\right)mq^3u^2+0.75\cot\left(0.245\pi\right)q^2u^2-0.125\cot\left(0.245\pi\right)nq^4u^2+1.5\cot\left(0.24\right.\right. \tag{376}$$

To proceed, we show that Equations 375 and 376 (with the exponential terms factored out) are positive in lemmas B.206 and B.202.

We can then lower bound the second partial u of Equation 175 in u by $u=\sqrt{2mn+0.25m+0.02}$. After scaling the second partial u by $\frac{1}{\cot(0.245\pi)}$, we show that the resulting expression is concave in q and that the first partial q of the expression is negative at $q=\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944$ (scaled by η^3 and simplified by rounding up after five digits of precision) in lemmas B.192 and B.191.

So, we can further lower bound the second partial u of Equation 175 in q by $q=\frac{0.94}{n}m^2+\frac{0.48}{n}+2.15$ (scaled by η^3 with $\exp\left(m\left(\frac{0.94}{n}m^2+\frac{0.48}{n}+2.15\right)\right)$ factored out and simplified by rounding down after five digits of precision) and show it is positive in lemma B.184.

This completes the proof.

□

Proof of Lemma B.184. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we first show that $\mu_{1,1}\sqrt{2m\eta+0.25m+0.02}+\mu_{1,2}$ is positive and $\mu_{2,1}\sqrt{2m\eta+0.25m+0.02}+\mu_{2,2}$ is negative, as defined for Equation 176, in lemma B.189 and B.188.

We then replace $\exp\left(-\frac{1}{2}\left(\frac{0.94}{\eta}m^2+\frac{0.48}{\eta}+2.15\right)\left(m-\sqrt{2m\eta+0.25m+0.02}\right)\right)$ with $\exp\left(\frac{1}{2}\left(\frac{0.94}{0.3}m^2+\frac{0.48}{0.3}+2.15\right)\left(\sqrt{2m\left(0.2\right)+0.25m+0.02}-m\right)\right)$, which we then replace with $(-0.26721m+1.4853)$ (see lemma B.5.4).

We also replace $\exp\left(-\frac{1}{2}\left(\frac{0.94}{\eta}m^2+\frac{0.48}{\eta}+2.15\right)\left(m+\sqrt{2m\eta+0.25m+0.02}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(\frac{0.94}{0.3}m^2+\frac{0.48}{0.3}+2.15\right)\left(m+\sqrt{2m\left(0.2\right)+0.25m+0.02}\right)\right)$, which we then replace with $(-1.27225m+0.581)$ (see lemma B.186).

The resulting expression simplifies (RD) to

$$\lambda_1\sqrt{2m\eta+0.25m+0.02}+\lambda_2, \tag{377}$$

where

$$\lambda_1=\left(\left(-0.17420m^2-11.77129m-10.45032\right)\eta^4+\left(-1.24409m^4-11.27536m^3-5.16119m^2+5.65555m+1.86909\right)$$

and

$$\lambda_2=\left(\left(7.64986m^2-10.26784m\right)\eta^5+\left(10.03378m^4-17.02567m^3-1.88724m^2+26.24376m-23.22125\right)\eta^4+\left(4.386\right.$$

We then show that λ_1 is positive in lemma B.185, so we can replace $\sqrt{2m\eta+0.25m+0.02}$ in Equation 377 with $\sqrt{2\left(0.15\right)\eta+0.25\left(0.15\right)+0.02}$, which we further replace with $0.41274\eta+0.2602$ (see lemma B.190).

With the above, we get the following lower bound for Equation 377.

$$\left(7.57796m^2-15.12633m-4.31327\right)\eta^5+\left(9.52029m^4-21.67947m^3-4.06280m^2+25.51514m-25.16898\right)\eta^4+\left(3.9\right. \tag{378}$$

We then show that the above is concave in η. Note the second partial η below.

$$\left(7.57796m^2-15.12633m-4.31327\right)\left(20\right)\eta^3+\left(9.52029m^4-21.67947m^3-4.06280m^2+25.51514m-25.16898\right)\left(12\right) \tag{379}$$

We can upper bound the above by

$$\left(\left(7.57796\left(0.25\right)-15.12633\right)\left(0.15\right)-4.31327\right)\left(20\right)\left(0.2\right)^3+\left(\left(\left(9.52029\left(0.25\right)-21.67947\right)\left(0.15\right)-4.06280\right)\left(0.15\right)+2\right.$$

which is negative.

So, we consider two cases, $\eta=0.2,0.3$. Case 1: $\eta=0.2$. In this case, we consider the following expression.

$$0.14561m^9-0.16822m^8+0.48689m^7-0.52592m^6+0.34800m^5-0.54875m^4+0.05088m^3-0.19215m^2+0.11858m \tag{380}$$

We can further lower bound the above by

$$\left(\left(\left(\left(\left(\left(0.14561\left(0.15\right)-0.16822\right)\left(0.25\right)+0.48689\right)\left(0.15\right)-0.52592\right)\left(0.25\right)+0.34800\right)\left(0.15\right)-0.54875\right)\left(0.25\right)+0.0508 \tag{381}$$

which simplifies (RD) to $-0.21154m^2+0.11858m+0.03754$. Since $-0.21154m^2+0.11858m+0.03754$ is concave in m, we evaluate at $m=0.15,0.25$, both of which are positive.

Case 2: $\eta=0.3$. In this case, we consider the following expression.

$$0.14561m^9-0.13799m^8+0.52001m^7-0.43838m^6+0.20475m^5-0.63498m^4-0.09307m^3-0.54532m^2+0.47016m \tag{382}$$

We lower bound the above by

$$(((((((0.14561\,(0.15) - 0.13799)\,(0.25) + 0.52001)\,(0.15) - 0.43838)\,(0.25) + 0.20475)\,(0.15) - 0.63498)\,(0.25) - 0.0930 \tag{383}$$

which simplifies (RD) to $-0.60721m^2 + 0.47016m - 0.05584$. Since $-0.60721m^2 + 0.47016m - 0.05584$ is concave in m, we evaluate at $m = 0.15, 0.25$, which are positive.

This completes the proof.

□

Proof of Lemma B.185. To prove the lemma, we first lower bound Equation 177 in m by

$$\left(-0.17420\,(0.25)^2 - 11.77129\,(0.25) - 10.45032\right)\eta^4 + \left(-1.24409\,(0.25)^4 - 11.27536\,(0.25)^3 - 5.16119\,(0.25)^2 + 5.655 \tag{384}$$

which simplifies (by rounding down after five digits of precision) to

$$-13.40403\eta^4 + 2.21381\eta^3 + 0.36675\eta^2 + 0.20570\eta + 0.01605. \tag{385}$$

We then lower bound the above by

$$(((-13.40403\,(0.3) + 2.21381)\,(0.3) + 0.36675)\,(0.3) + 0.20570)\,(0.2) + 0.01605,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.186. To prove the lemma, note that the second partial m of Equation 178 (simplified by rounding down after five digits of precision) can be lower bounded by

$$\left(-9.39996\,(0.25) + \frac{-2.48219\,(0.25)^2 - 0.12221\,(0.25) + 0.19678}{(0.65\,(0.15) + 0.02)^{\frac{3}{2}}}\right) + \left(4.69998\,(0.15)^2 + 1.875 + \frac{0.65\,(3.13332)\,(0.15)}{}\right.$$

which is positive.

For the linear upper bound, note the difference between the linear bound and the exponential function is concave in m, so we evaluate at $m = 0.15, 0.25$, which are positive.

This completes the proof.

□

Proof. To prove the lemma, note the second partial m for Equation 179 (with the exponential term factored out, scaled by $(0.65m + 0.02)^{\frac{3}{2}}$, and rounded up after five digits of precision).

$$-15.55490m^5 - 0.86150m^4 - 9.94048m^3 + 2.02395m^2 - 1.36783m - 0.24248 + \left(14.35839m^5 + 6.92305m^4 + 11.77530\right. \tag{386}$$

Since the square root coefficient is positive, we linearize $\sqrt{0.65m + 0.02}$ at $m = 0.25$.

The resulting expression (simplified by rounding up after five digits of precision)

$$-15.55490m^5 - 0.86150m^4 - 9.94048m^3 + 2.02395m^2 - 0.49260m + 0.03019 \tag{387}$$

can be upper bounded by

$$((((-15.55490\,(0.15) - 0.86150)\,(0.15) - 9.94048)\,(0.15) + 2.02395)\,(0.25) - 0.49260)\,(0.15) + 0.03019,$$

which is negative.

To verify the linear upper bound, note that the difference between the linear bound and the exponential function is convex in m, so we evaluate at $m = 0.15, 0.25$. which are negative.

This completes the proof.

□

Proof of Lemma B.188. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we first show that $\mu_{2,1}$ (as defined for Equation 180) is negative. Consider $\mu_{2,1}$ below.

$$\left(0.17333m + 11.55625\right)\eta^4 + \left(1.23785m^3 + 10.105m^2 - 3.95682m - 2.06701\right)\eta^3 + \left(0.98300m^5 + 2.209m^4 - 1.12871m \right. \tag{388}$$

Note that we can upper bond the above in m by

$$\left(0.17333\left(0.25\right) + 11.55625\right)\eta^4 + \left(1.23785\left(0.25\right)^3 + 10.105\left(0.25\right)^2 - 3.95682\left(0.15\right) - 2.06701\right)\eta^3 + \left(0.98300\left(0.25\right)^5 - \right. \tag{389}$$

which simplifies (RU) to

$$11.59959\eta^4 - 2.00962\eta^3 - 0.31876\eta^2 - 0.19228\eta - 0.01526. \tag{390}$$

We then upper bound the above by

$$\left(11.59959\left(0.3\right)^2 - 2.00962\left(0.2\right) - 0.31876\right)\eta^2 - 0.19228\eta - 0.01526, \tag{391}$$

which simplifies (RU) to $0.32328\eta^2 - 0.19228\eta - 0.01526$.

Since $0.32328\eta^2 - 0.19228\eta - 0.01526$ is convex in η, we evaluate at $\eta = 0.2, 0.3$, which are negative.

We can then replace $\sqrt{2m\eta + 0.25m + 0.02}$ in Equation 180 by $\sqrt{2\left(0.15\right)\left(0.2\right) + 0.25\left(0.15\right) + 0.02}$.

We then upper bound the resulting expression (RU) as follows. First, we upper bound

$$-4.96919m\eta^5 + \left(-6.51773m^3 + 2.31125m^2 + 4.38754m - 7.27679\right)\eta^4 + \left(-2.84961m^5 + 2.021m^4 - 1.70266m^3 - 3.5\right. \tag{392}$$

by

$$-4.96919\left(0.15\right)\eta^5 + \left(-6.51773\left(0.15\right)^3 + 2.31125\left(0.25\right)^2 + 4.38754\left(0.25\right) - 7.27679\right)\eta^4 + \left(-2.84961\left(0.15\right)^5 + 2.021\right. \tag{393}$$

which simplifies (RU) to

$$-0.74537\eta^5 - 6.05744\eta^4 - 1.07903\eta^3 + 0.64089\eta^2 - 0.01678\eta - 0.01287. \tag{394}$$

We then upper bound the above by

$$\left(\left(\left(-0.74537\left(0.2\right) - 6.05744\right)\left(0.2\right) - 1.07903\right)\left(0.2\right) + 0.64089\right)\eta^2 - 0.01678\eta - 0.01287, \tag{395}$$

which simplifies (RU) to $0.17683\eta^2 - 0.01678\eta - 0.01287$.

Since $0.17683\eta^2 - 0.01678\eta - 0.01287$ is convex in η, we evaluate at $\eta = 0.2, 0.3$, which are negative.

This completes the proof.

□

Proof of Lemma B.189. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To show that Equation 181 is positive, we will first show that $\mu_{1,1}$ (as defined for Equation 181) is positive.

Consider $\mu_{1,1}$ below.

$$\left(-0.17336m - 11.55625\right)\eta^4 + \left(-1.23787m^3 - 10.105m^2 + 3.95679m + 2.06694\right)\eta^3 + \left(-0.98301m^5 - 2.209m^4 + 1.12\right. \tag{396}$$

Note that we first lower bound the above in m by

$$(-0.17336\,(0.25) - 11.55625)\,\eta^4 + \left(-1.23787\,(0.25)^3 - 10.105\,(0.25)^2 + 3.95679\,(0.15) + 2.06694\right)\eta^3 + \left(-0.98301\,(0. \tag{397}$$

which simplifies (RD) to

$$-11.59959\eta^4 + 2.00955\eta^3 + 0.31864\eta^2 + 0.19220\eta + 0.01523. \tag{398}$$

We can then lower bound the above by

$$(((-11.59959\,(0.3) + 2.00955)\,(0.3) + 0.31864)\,(0.3) + 0.19220)\,(0.2) + 0.01523, \tag{399}$$

which is positive.

So, we can replace $\sqrt{2m\eta + 0.25m + 0.02}$ in Equation 181 by $\sqrt{2\,(0.15)\,\eta + 0.25\,(0.15) + 0.02}$, which we then replace by $0.41274\eta + 0.2602$ (see lemma B.190).

We then show that the resulting expression (RD) is negative. First, note that the resulting expression is concave in η. To see this, note the second partial η below.

$$(-5.04075m - 4.76973)\,(20)\,\eta^3 + \left(-7.02865m^3 - 1.85949m^2 + 5.91613m - 13.39191\right)(12)\,\eta^2 + \left(-3.25534m^5 + 1.109 \tag{400}$$

Note that the above can be upper bounded by

$$(-5.04075\,(0.15) - 4.76973)\,(20)\,(0.2)^3 + \left(-7.02865\,(0.15)^3 - 1.85949\,(0.15)^2 + 5.91613\,(0.25) - 13.39191\right)(12)\,(0.2)^2$$

which is negative.

So, we consider two cases for our lower bound of Equation 181, $\eta = 0.2, 0.3$.

Case 1: $\eta = 0.2$. In this case, we consider the expression below.

$$-0.09459m^8 - 0.04124m^7 - 0.30390m^6 - 0.14436m^5 - 0.28009m^4 - 0.06449m^3 - 0.05896m^2 + 0.05145m + 0.02921 \tag{401}$$

We can lower bound the above by

$$\left(-0.09459\,(0.3)^6 - 0.04124\,(0.3)^5 - 0.30390\,(0.3)^4 - 0.14436\,(0.3)^3 - 0.28009\,(0.3)^2 - 0.06449\,(0.3) - 0.05896\right)m^2 + \tag{402}$$

which simplifies (RD) to $-0.11005m^2 + 0.05145m + 0.02921$. Since

$$-0.11005m^2 + 0.05145m + 0.02921$$

is concave in m, we evaluate at $m = 0.15, 0.25$, which are positive.

Case 2: $\eta = 0.3$. In this case, we consider the following expression.

$$-0.09459m^8 - 0.07688m^7 - 0.35908m^6 - 0.33258m^5 - 0.42698m^4 - 0.18503m^3 - 0.22407m^2 + 0.17713m - 0.01588 \tag{403}$$

Since the above is concave in m, we evaluate at $m = 0.15, 0.25$, which are positive.

This completes the proof.

□

Proof of Lemma B.190. To prove the lemma, note that $0.41274\eta + 0.2602 - \sqrt{0.3\eta + 0.0575}$ is convex in η, so we evaluate at $\eta = 0.2, 0.3$, which are negative.

This completes the proof. □

Proof of Lemma B.191. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we split Equation 182 into two parts, shown below, and show that both are negative. Note, for the first part, that we factor out

$\exp\left(\frac{1}{2}\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)\left(m-\sqrt{2mn+0.25m+0.02}\right)\right)$. Also note that, for the second part, we factor out $\exp\left(\frac{1}{2}\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)\left(m+\sqrt{2mn+0.25m+0.02}\right)\right)$.

$$4m^3\eta^3\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2+\frac{0.52}{\eta}+1.944\right)\left(m+\sqrt{2m\eta+0.25m+0.02}\right)\right)+s_{1,1}\sqrt{2m\eta+0.25m+0.02}+s_{1,2} \tag{404}$$

$$s_{2,1}\sqrt{2m\eta+0.25m+0.02}+s_{2,2}, \tag{405}$$

where

$s_{1,1}=1.83668m\eta^5+\left(2.55093m^3-0.97122m^2+3.37530m+14.49146\right)\eta^4+\left(1.18098m^5-0.47413m^4+6.83158m^3+\right.$

$s_{1,2}=\left(-1.78373m^2\eta^5-15.11654m\eta^5\right)+\left(-3.35222m^4-13.05201m^3+7.85083m^2-5.72655m-7.71080\right)\eta^4+\left(-1\right.$

$s_{2,1}=\left(-1.83665\right)m\eta^5+\left(-2.55091m^3+0.97125m^2-3.37526m-14.49133\right)\eta^4+\left(-1.18098m^5+0.47415m^4-6.831\right.$

and

$s_{2,2}=\left(-1.78373m^2-15.11654m\right)\eta^5+\left(-3.35222m^4-13.05201m^3+7.85083m^2-5.72655m-7.7108\right)\eta^4+\left(-1.95\right.$

For Equation 404, note that we first upper bound it by replacing

$$4m^3\eta^3\exp\left(\frac{1}{2}\left(\frac{0.9}{\eta}m^2+\frac{0.52}{\eta}+1.944\right)\left(m+\sqrt{2m\eta+0.25m+0.02}\right)\right) \tag{406}$$

with

$$4\left(0.25\right)^3\left(0.3\right)^3\exp\left(\frac{1}{2}\left(\frac{0.9}{0.2}\left(0.25\right)^2+\frac{0.52}{0.2}+1.944\right)\left(\left(0.25\right)+\sqrt{2\left(0.25\right)\left(0.3\right)+0.25\left(0.25\right)+0.02}\right)\right),$$

which simplifies (RU) to 0.00988.

We then upper bound $s_{1,1}$ by

$$1.83668\left(0.25\right)\eta^5+\left(\left(\left(2.55093\left(0.25\right)-0.97122\right)\left(0.15\right)+3.37530\right)\left(0.25\right)+14.49146\right)\eta^4+\left(\left(\left(\left(1.18098\left(0.25\right)-0.4741\right.\right.\right.\right. \tag{407}$$

which simplifies (RU) to

$$0.45917\eta^5+15.32278\eta^4+1.44258\eta^3-1.12654\eta^2-0.15234\eta+0.00417. \tag{408}$$

We further upper bound the above by

$$\left(\left(\left(\left(0.45917\left(0.3\right)+15.32278\right)\left(0.3\right)+1.44258\right)\left(0.3\right)-1.12654\right)\eta^2-0.15234\eta+0.00417\right), \tag{409}$$

which simplifies (RU) to

$$0.69769\eta^2-0.15234\eta+0.00417. \tag{410}$$

Since $\left(0.69769\left(0.2\right)-0.15234\right)\left(0.3\right)+0.00417$ is positive, we replace $\sqrt{2m\eta+0.25m+0.02}$ with $\sqrt{2\left(0.25\right)\left(0.3\right)+0.25\left(0.25\right)+0.02}$ in Equation 404.

Next, we upper bound $s_{1,2}$ by

$$\left(-1.78373\left(0.15\right)^2-15.11654\left(0.15\right)\right)\eta^5+\left(\left(\left(\left(-3.35222\left(0.15\right)-13.05201\right)\left(0.15\right)+7.85083\right)\left(0.25\right)-5.72655\right)\left(0.15\right)+ \tag{411}$$

which simplifies (RU) to

$$-2.30761\eta^5-8.35162\eta^4-0.23269\eta^3+0.07762\eta^2-0.06054\eta+0.00967 \tag{412}$$

With the above upper bounds on $s_{1,1}$ and $s_{1,2}$, Equation 404 can be upper bounded by

$$-2.30761\eta^5-8.35162\eta^4-0.23269\eta^3+0.41404\eta^2-0.13399\eta+0.01169. \tag{413}$$

We further upper bound the above by $0.41404\eta^2 - 0.13399\eta - 0.00427$, which is convex in η. So, we evaluate at $\eta = 0.2, 0.3$, both of which are negative.

For Equation 405, note that $s_{2,1}$ by

$$\left(-1.83665\left(0.15\right)\eta^5 + \left(\left(\left(-2.55091\left(0.15\right) + 0.97125\right)\left(0.25\right) - 3.37526\right)\left(0.15\right) - 14.49133\right)\eta^4 + \left(\left(\left(\left(\left(-1.18098\left(0.15\right) + \right.\right.\right.\right.\right. \tag{414}$$

which simplifies (RU) to

$$-0.27549\eta^5 - 14.97554\eta^4 - 1.02819\eta^3 + 1.18237\eta^2 + 0.17889\eta - 0.00311 \tag{415}$$

Since

$$\left(\left(\left(\left(-0.27549\left(0.3\right) - 14.97554\right)\left(0.3\right) - 1.02819\right)\left(0.3\right) + 1.18237\right)\left(0.3\right) + 0.17889\right)\left(0.2\right) - 0.00311$$

is positive, we can replace $\sqrt{2m\eta + 0.25m + 0.02}$ in Equation **??** by $\sqrt{2\left(0.25\right)\left(0.3\right) + 0.25\left(0.25\right) + 0.02}$.

We can then upper bound the resulting expression in m by

$$\left(-1.78373\left(0.15\right)^2 - 15.11654\left(0.15\right) - 0.13283\right)\eta^5 + \left(\left(\left(\left(-3.35222\left(0.15\right) - 13.05201\right)\left(0.15\right) + 7.85083\right)\left(0.25\right) - 5.726\right.\right. \tag{416}$$

which simplifies (RU) to

$$-2.44044\eta^5 - 15.57256\eta^4 - 0.72629\eta^3 + 0.64899\eta^2 + 0.02590\eta - 0.00170. \tag{417}$$

We can then upper bound the above by

$$\left(\left(\left(\left(-2.44044\left(0.2\right) - 15.57256\right)\left(0.2\right) - 0.72629\right)\left(0.2\right) + 0.64899\right)\left(0.2\right) + 0.02590\right)\left(0.2\right) - 0.00170,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.192. To proceed, we split Equation 183 into two parts and show each are concave in q.

$$\left(\nu_{2,1}\sqrt{2mn + 0.25m + 0.02} + \nu_{2,2}\right)\exp\left(\frac{1}{2}q\left(m - \sqrt{2mn + 0.25m + 0.02}\right)\right), \tag{418}$$

$$4m\exp\left(mq\right) + \left(\nu_{1,1}\sqrt{2mn + 0.25m + 0.02} + \nu_{1,2}\right)\exp\left(\frac{1}{2}q\left(m + \sqrt{2mn + 0.25m + 0.02}\right)\right), \tag{419}$$

where

$$\nu_{1,1} = 0.125\tan\left(0.245\pi\right)q^3 - 0.25nmq^3 + 0.5mnq^2 + 0.0625mq^2 + 0.005q^2 - 2.5nq^2 + 2mq + 6,$$

$$\nu_{1,2} = -0.125\tan\left(0.245\pi\right)nq^4 - 0.5mn^2q^3 - 0.0625nmq^3 - 0.005nq^3 + 0.5m^2nq^2 + 0.0625m^2q^2 + 0.005mq^2 + 0.75\,t$$

$$\nu_{2,1} = 0.25nmq^3 - 0.125\tan\left(0.245\pi\right)q^3 - 0.5mnq^2 - 0.0625mq^2 - 0.005q^2 + 2.5nq^2 - 2mq - 6,$$

and

$$\nu_{2,2} = -0.125\tan\left(0.245\pi\right)nq^4 - 0.5mn^2q^3 - 0.0625mnq^3 - 0.005nq^3 + 0.5m^2nq^2 - nmq^2 + 0.75\tan\left(0.245\pi\right)q^2 + 0.$$

We show that Equation 418 and 418 are concave in q in lemmas B.198 and B.193.

This completes the proof.

□

Proof of Lemma B.193. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, consider the second partial η of Equation 184 below (RU), with $\exp\left(\frac{1}{2}q\left(m+\sqrt{2mn+0.25m+0.02}\right)\right)$ factored out.

$$4m^3\exp\left(\frac{1}{2}q\left(m-\sqrt{2mn+0.25m+0.02}\right)\right)+t_1\sqrt{2mn+0.25m+0.02}+t_2, \tag{420}$$

where

$$t_1 = -0.06056nmq^4+\left(-0.375m^2n^2+\left(-0.0625m^3-0.04687m^2+0.05682m-0.48453\right)n+\left(0.03029m^2+0.00758n\right.\right.$$

and

$$t_2 = \left(-0.06056mn^2-0.03028nm^2-0.00757nm-0.00060n\right)q^4+\left(-0.25m^2n^3+\left(-0.375m^3-0.0625m^2-0.005m\right)\right.$$

To proceed, we first replace $\exp\left(\frac{1}{2}q\left(m-\sqrt{2mn+0.25m+0.02}\right)\right)$ by $\exp\left(\frac{1}{2}\left(\frac{0.9}{0.3}(0.15)^2+\frac{0.52}{0.3}+1.944\right)\left(0.25-\sqrt{2(0.15)(0.2)+0.25(0.15)+0.02}\right)\right)$.

So, Equation 420 can be upper bounded by

$$t_1\sqrt{2mn+0.25m+0.02}+(t_2+3.36211m^3). \tag{421}$$

We show that t_1 and $(t_2+3.36211m^3)$ are negative in lemmas B.195 and B.194.

This completes the proof. □

Proof of Lemma B.194. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we iteratively upper bound Equation 185 in q. First, we upper bound

$$\left(-0.06056mn^2-0.03028nm^2-0.00757nm-0.00060n\right)q+\left(-0.25m^2n^3+\left(-0.375m^3-0.0625m^2-0.005m\right)n^2+\right. \tag{422}$$

by fixing $q=\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944$, which simplifies (RU) to

$$-0.25m^2n^3+\left(-0.375m^3-0.0625m^2-0.12272m\right)n^2+\left(-0.10137m^3+0.05462m^2-0.53135m-0.00118\right)n+\left(-0.\right. \tag{423}$$

Since

$$-0.10137(0.15)^3+0.05462(0.25)^2-0.53135(0.15)-0.00118$$

is negative, we upper bound

$$\left(-0.25m^2n^3+\left(-0.375m^3-0.0625m^2-0.12272m\right)n^2+\left(-0.10137m^3+0.05462m^2-0.53135m-0.00118\right)n+\left(-0\right. \tag{424}$$

by $q=\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944$, which simplifies (RU and scaling by η) to

$$-0.486m^2n^4+\left(-0.225m^4+0.021m^3-6.2515m^2-0.23856m\right)n^3+\left(-0.3375m^5+0.06875m^4-0.56501m^3-0.661\right. \tag{425}$$

To proceed, note the following expressions are negative.

$$-0.225(0.15)^3+0.021(0.25)^2-6.2515(0.15)-0.23856$$

$$-0.09123(0.15)^5+0.01182(0.25)^4-0.53118(0.15)^3+0.20873(0.25)^2-0.14493(0.15)+0.00970$$

$$-0.3375(0.15)^5+0.06875(0.25)^4-0.56501(0.15)^3-0.66131(0.15)^2-0.06654(0.15)-1.45589$$

So, we can upper bound

$$\left(-0.486m^2n^4 + \left(-0.225m^4 + 0.021m^3 - 6.2515m^2 - 0.23856m\right)n^3 + \left(-0.3375m^5 + 0.06875m^4 - 0.56501m^3 - 0.661\right. \tag{426}$$

by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$, which simplifies (RU and scaling by η) to

$$-0.94478m^2n^5 + \left(-0.8748m^4 + 0.04083m^3 - 7.40563m^2 - 15.46376m\right)n^4 + \left(-0.2025m^6 - 0.6372m^5 - 5.6097m^4 + \right. \tag{427}$$

To show that the above is negative, note that the following expressions are negative.

$$-0.02206\,(0.15)^8 - 0.00550\,(0.15)^7 - 0.02641\,(0.15)^6 - 0.00856\,(0.15)^5 - 0.00866\,(0.15)^4 - 0.00436\,(0.15)^3 - 0.00017$$

$$((((((-0.08210\,(0.15) - 0.03702)\,(0.15) - 0.53739)\,(0.15) + 0.16450)\,(0.25) - 0.41827)\,(0.15) + 0.11560)\,(0.25) - 0.078$$

$$((((((-0.30375\,(0.15) + 0.06188)\,(0.25) - 0.86136)\,(0.15) - 0.53645)\,(0.15) - 0.82380)\,(0.15) - 1.12528)\,(0.15) + 1.149$$

$$(((((-0.2025\,(0.15) - 0.6372)\,(0.15) - 5.6097)\,(0.15) + 3.19784)\,(0.25) - 6.28636)\,(0.15) - 2.02840)\,(0.15) - 2.98025$$

$$(((-0.8748\,(0.15) + 0.04083)\,(0.15) - 7.40563)\,(0.15) - 15.46376)\,(0.15)$$

So, we can upper bound

$$\left(-0.94478m^2n^5 + \left(-0.8748m^4 + 0.04083m^3 - 7.40563m^2 - 15.46376m\right)n^4 + \left(-0.2025m^6 - 0.6372m^5 - 5.6097m^4 + \right. \tag{428}$$

by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$, which simplifies (RU and scaling by η) to

$$-1.83665m^2n^6 + \left(-2.55091m^4 + 0.07938m^3 - 14.88783m^2 - 30.06154m\right)n^5 + \left(-1.18098m^6 - 1.20196m^5 - 18.0252\right. \tag{429}$$

To see that the above is negative, note the following negative expression.

$$(((-2.55091\,(0.15) + 0.07938)\,(0.15) - 14.88783)\,(0.15) - 30.06154)\,(0.15)$$

$$(((((-1.18098\,(0.15) - 1.20196)\,(0.15) - 18.02521)\,(0.15) - 7.67955)\,(0.15) + 1.92839)\,(0.25) - 17.98436)\,(0.15) - 5.79$$

$$(((((((-0.18225\,(0.15) - 1.16397)\,(0.15) - 5.03373)\,(0.15) + 0.87223)\,(0.25) - 9.61762)\,(0.15) + 2.09796)\,(0.25) - 5.88$$

$$(((((((-0.27337\,(0.15) + 0.05570)\,(0.25) - 1.09277)\,(0.15) - 0.52259)\,(0.15) - 2.23401)\,(0.15) - 0.97191)\,(0.15) - 0.2$$

$$(((((((-0.07389\,(0.15) - 0.07620)\,(0.15) - 0.53703)\,(0.15) + 0.07746)\,(0.15) - 0.67252)\,(0.15) + 0.17275)\,(0.25) - 0.2$$

$$-0.01985\,(0.15)^{10} - 0.00495\,(0.15)^9 - 0.03524\,(0.15)^8 - 0.01056\,(0.15)^7 - 0.02152\,(0.15)^6 - 0.00837\,(0.15)^5 - 0.00465$$

This completes the proof. □

Proof of Lemma B.195. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we iteratively upper bound Equation 186 in q.

First, we upper bound

$$-0.06056nmq + \left(-0.375m^2n^2 + \left(-0.0625m^3 - 0.04687m^2 + 0.05682m - 0.48453\right)n + \left(0.03029m^2 + 0.00758m + 0.0\right. \tag{430}$$

by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$, which simplifies (RU) to

$$-0.375m^2n^2 + \left(-0.0625m^3 - 0.04687m^2 - 0.06090m - 0.48453\right)n + \left(-0.05450m^3 + 0.03029m^2 - 0.02391m + 0.000\right. \tag{431}$$

Note that we can upper bound the above in η by $\eta = 0.2$, which simplifies (RU) to

$$-0.067m^3 + 0.00592m^2 - 0.03609m - 0.09629. \tag{432}$$

The above can be further upper bounded by

$$-0.067\,(0.15)^3 + 0.00592\,(0.25)^2 - 0.03609\,(0.15) - 0.09629,$$

which is negative.

So, we can upper bound

$$\left(-0.375m^2n^2 + \left(-0.0625m^3 - 0.04687m^2 - 0.06090m - 0.48453\right)n + \left(-0.05450m^3 + 0.03029m^2 - 0.02391m + 0.00\right. \tag{433}$$

by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$, which simplifies (RU and scaling by η) to

$$\left(-0.479m^2 - 2.75m\right)n^3 + \left(-0.3375m^4 + 0.2535m^3 - 2.09861m^2 - 0.45713m - 0.96942\right)n^2 + \left(-0.05625m^5 - 0.0421\right. \tag{434}$$

Since

$$\left(-0.479\,(0.15)^2 - 2.75\,(0.15)\right)(0.2)^3 + \left(-0.3375\,(0.15)^4 + 0.2535\,(0.25)^3 - 2.09861\,(0.15)^2 - 0.45713\,(0.15) - 0.9694\right.$$

is negative, we can upper bound

$$\left(\left(-0.479m^2 - 2.75m\right)n^3 + \left(-0.3375m^4 + 0.2535m^3 - 2.09861m^2 - 0.45713m - 0.96942\right)n^2 + \left(-0.05625m^5 - 0.042\right. \tag{435}$$

by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$, which simplifies (RU and scaling by η) to

$$\left(\left(-0.93117m^2 - 5.346m\right)n^4 + \left(-1.0872m^4 - 1.98219m^3 + 1.67123m^2 - 12.81866m - 1.88455\right)n^3 + \left(-0.30375m^6 + \right. \tag{436}$$

We then show that, from the above, the term linear in q decreases in η in lemma B.196, so we can upper bound it by fixing $\eta = 0.2$, yielding the following.

$$\left(-0.05426m^7 + 0.00481m^6 - 0.11688m^5 - 0.13037m^4 + 0.02675m^3 - 0.13823m^2 - 0.01148m + 0.00676\right)q + \left(10mn^3\right. \tag{437}$$

We further lower bound the term linear in q naively in m by

$$\Big(-0.05426\,(0.15)^7 + 0.00481\,(0.25)^6 - 0.11688\,(0.15)^5 - 0.13037\,(0.15)^4 + 0.02675\,(0.25)^3 - 0.13823\,(0.15)^2 - 0.01148 \tag{438}$$

which simplifies (RU) to

$$0.00228q + \left(10mn^3 - 9n^3 + 4.5m^2n^2 + 1.25mn^2 + 0.1n^2\right) \tag{439}$$

From the above, we upper bound by $q = \frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15$, scale by η, and show the resulting expression decreases in η in lemma B.197. We then upper bound the resulting expression in m by $m = 0.25$, which is negative.

This completes the proof.

□

Proof of Lemma B.196. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, we first show that the second partial η of Equation 187 is negative. Note that the second partial η can be upper bounded by

$$\Big(-0.93117\,(0.15)^2 - 5.346\,(0.15)\Big)\,(12)\,(0.2)^2 + \Big(-1.0872\,(0.15)^4 - 1.98219\,(0.15)^3 + 1.67123\,(0.25)^2 - 12.81866\,(0.15)$$

which is negative.

So, we can upper bound the first partial η of Equation 187 at $\eta = 0.2$, which simplifies (RU) to

$$-0.05062m^7 - 0.15946m^6 - 0.20881m^5 - 1.31239m^4 + 0.14784m^3 - 1.01957m^2 - 0.96202m + 0.12005. \tag{440}$$

We can then upper bound the above by

$$-0.05062\left(0.15\right)^7 - 0.15946\left(0.15\right)^6 - 0.20881\left(0.15\right)^5 - 1.31239\left(0.15\right)^4 + 0.14784\left(0.25\right)^3 - 1.01957\left(0.15\right)^2 - 0.96202($$

which is negative.

This completes the proof. □

Proof of Lemma **??**. To prove the lemma, consider the first partial η of Equation 188 below.

$$40m\eta^3 - 36\eta^3 + 13.5m^2\eta^2 + 3.75m\eta^2 + 0.3\eta^2 + 0.00491 \tag{441}$$

We upper bound the above by

$$\left(\left(40\left(0.25\right) - 36\right)\left(0.2\right) + \left(13.5\left(0.25\right)^2 + 3.75\left(0.25\right) + 0.3\right)\right)\left(0.2\right)^2 + 0.00491,$$

which is negative.

This completes the proof. □

Proof of Lemma B.198. Before we start the proof of the lemma, note that we round terms up/down after five digits of precision frequently throughout for simplicity, which we will accompany with RU/RD.

To prove the lemma, consider the second partial q of Equation 189 below with $\exp\left(\frac{1}{2}q\left(m - \sqrt{2mn + 0.25m + 0.02}\right)\right)$ factored out (RU).

$$\gamma_1 + \gamma_2\sqrt{2mn + 0.25m + 0.02}, \tag{442}$$

where

$$\gamma_1 = \left(-0.06056n^2m - 0.03028nm^2 - 0.00757nm - 0.00060n\right)q^4 + \left(-0.25m^2n^3 + \left(-0.375m^3 - 0.0625m^2 - 0.005m\right)\right.$$

and

$$\gamma_2 = 0.06057nmq^4 + \left(0.375m^2n^2 + \left(0.0625m^3 + 0.04688m^2 - 0.05681m + 0.48454\right)n - 0.03028m^2 - 0.00757m - 0.0$$

We proceed by showing that γ_2 is positive in lemma B.199. So, we can replace $\sqrt{2mn + 0.25m + 0.02}$ in Equation 442 by $\sqrt{2\left(0.25\right)\left(0.3\right) + 0.25\left(0.25\right) + 0.02}$, resulting in the following expression (RU).

$$\left(-0.06056n^2m - 0.03028nm^2 + 0.02164nm - 0.00060n\right)q^4 + \left(-0.25m^2n^3 + \left(-0.375m^3 + 0.11832m^2 - 0.005m\right)n^2 + \right. \tag{443}$$

We then upper bound the above iteratively in q. We first upper bound

$$\left(-0.06056nm - 0.03028m^2 + 0.02164m - 0.00060\right)nq + \left(-0.25m^2n^3 + \left(-0.375m^3 + 0.11832m^2 - 0.005m\right)n^2 + \left(-0\right. \tag{444}$$

by

$$\left(\left(-0.03028\left(0.15\right) + \left(0.02164 - 0.06056\left(0.2\right)\right)\right)\left(0.25\right) - 0.00060\right)n\left(\frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15\right) + \left(-0.25m^2n^3 + \left(-0.375\right.\right. \tag{445}$$

which simplifies (RU) to

$$-0.25m^2n^3 + \left(-0.375m^3 + 0.11832m^2 - 0.005m\right)n^2 + \left(-0.01673m^3 + 0.13609m^2 - 0.51254m + 0.23502\right)n + \left(0.001\right. \tag{446}$$

Since

$$-0.25\left(0.25\right)^{2}\left(0.3\right)^{3}+(((-0.375\left(0.25\right)+0.11832)\left(0.15\right)-0.005)\left(0.25\right))\left(0.3\right)^{2}+(((-0.01673\left(0.25\right)+0.13609)\left(0.15\right)$$

is positive, we can upper bound

$$\left(-0.25m^{2}n^{3}+\left(-0.375m^{3}+0.11832m^{2}-0.005m\right)n^{2}+\left(-0.01673m^{3}+0.13609m^{2}-0.51254m+0.23502\right)n+\left(0.00\right.\right.\tag{447}$$

by fixing $q=\frac{0.94}{n}m^{2}+\frac{0.48}{n}+2.15$, which simplifies (RU and scaling by η) to

$$-0.5375m^{2}n^{4}+\left(-0.235m^{4}-0.05625m^{3}-5.98615m^{2}+1.31526m\right)n^{3}+\left(-0.3525m^{5}+0.23623m^{4}-0.46397m^{3}+0\right.\tag{448}$$

Since

$$-0.5375\left(0.15\right)^{2}\left(0.2\right)^{4}+((((-0.235\left(0.15\right)-0.05625)\left(0.15\right)-5.98615)\left(0.15\right)+1.31526)\left(0.25\right))\left(0.3\right)^{3}+(((((-0.3525$$

is negative, we can upper bound

$$\left(\left(-0.5375m^{2}n^{4}+\left(-0.235m^{4}-0.05625m^{3}-5.98615m^{2}+1.31526m\right)n^{3}+\left(-0.3525m^{5}+0.23623m^{4}-0.46397m^{3}+\right.\right.\right.\tag{449}$$

by $q=\frac{0.9}{n}m^{2}+\frac{0.52}{n}+1.944$, which simplifies (RU and scaling by η) to

$$-1.0449m^{2}n^{5}+\left(-0.94059m^{4}-0.10935m^{3}-6.91657m^{2}-12.44313m\right)n^{4}+\left(-0.2115m^{6}-0.73588m^{5}-5.05050m^{4}\right.\tag{450}$$

Since

$$-1.0449\left(0.15\right)^{2}\left(0.2\right)^{5}+\left(-0.94059\left(0.15\right)^{4}-0.10935\left(0.15\right)^{3}-6.91657\left(0.15\right)^{2}-12.44313\left(0.15\right)\right)\left(0.2\right)^{4}+\left(-0.2115\right.$$

is negative, we upper bound

$$\left(-1.0449m^{2}n^{5}+\left(-0.94059m^{4}-0.10935m^{3}-6.91657m^{2}-12.44313m\right)n^{4}+\left(-0.2115m^{6}-0.73588m^{5}-5.05050m^{4}\right.\right.\tag{451}$$

by $q=\frac{0.9}{n}m^{2}+\frac{0.52}{n}+1.944$, which simplifies (RU and scaling by η) to

$$-2.03128m^{2}n^{6}+\left(-2.76891m^{4}-0.21257m^{3}-13.98916m^{2}-24.18944m\right)n^{5}+\left(-1.25768m^{6}-1.52896m^{5}-16.5321\right.\tag{452}$$

We can then upper bound the above by

$$-2.03128\left(0.15\right)^{2}\left(0.2\right)^{6}+\left(-2.76891\left(0.15\right)^{4}-0.21257\left(0.15\right)^{3}-13.98916\left(0.15\right)^{2}-24.18944\left(0.15\right)\right)\left(0.2\right)^{5}+\left(-1.257\right.$$

which is negative.

This completes the proof.

□

Proof of Lemma B.199. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first iteratively lower bounding Equation 190 in q.

We first lower bound

$$0.06057nmq+\left(0.375m^{2}n^{2}+\left(0.0625m^{3}+0.04688m^{2}-0.05681m+0.48454\right)n-0.03028m^{2}-0.00757m-0.00060\right)\tag{453}$$

by fixing $q=\left(\frac{0.9}{n}m^{2}+\frac{0.52}{n}+1.944\right)$, which simplifies (RD) to

$$0.375m^{2}n^{2}+\left(0.0625m^{3}+0.04688m^{2}+0.06093m+0.48454\right)n+\left(0.05451m^{3}-0.03028m^{2}+0.02392m-0.00060\right).\tag{454}$$

Since

$$0.375\left(0.14\right)^{2}\left(0.2\right)^{2}+\left(0.0625\left(0.15\right)^{3}+0.04688\left(0.15\right)^{2}+0.06093\left(0.15\right)+0.48454\right)\left(0.2\right)+\left(0.05451\left(0.15\right)^{3}-0.0302\right.$$

is positive, we can lower bound

$$\left(0.375m^2n^2 + \left(0.0625m^3 + 0.04688m^2 + 0.06093m + 0.48454\right)n + \left(0.05451m^3 - 0.03028m^2 + 0.02392m - 0.00060\right)\right) \tag{455}$$

by $q = \left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right)$, which simplifies (RD and scaling by η) to

$$\left(0.479m^2 + 2.75m\right)n^3 + \left(0.3375m^4 - 0.2535m^3 + 2.09863m^2 + 0.45719m + 0.96944\right)n^2 + \left(0.05625m^5 + 0.04219m^4 + \right. \tag{456}$$

Since the following expressions,

$$\left(0.3375\left(0.15\right)^4 - 0.2535\left(0.25\right)^3 + 2.09863\left(0.15\right)^2 + 0.45719\left(0.15\right) + 0.96944\right)$$

$$\left(0.05625\left(0.15\right)^5 + 0.04219\left(0.15\right)^4 + 0.14643\left(0.15\right)^3 + 0.39394\left(0.15\right)^2 - 0.64924\left(0.25\right) + 0.25077\right)$$

$$\left(0.04905\left(0.15\right)^5 - 0.02726\left(0.25\right)^4 + 0.04987\left(0.15\right)^3 - 0.01629\left(0.25\right)^2 + 0.01243\left(0.15\right) - 0.00032\right)$$

are positive, we can lower bound

$$\left(\left(0.479m^2 + 2.75m\right)n^3 + \left(0.3375m^4 - 0.2535m^3 + 2.09863m^2 + 0.45719m + 0.96944\right)n^2 + \left(0.05625m^5 + 0.04219m^4\right.\right. \tag{457}$$

by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$, which simplifies (RD and scaling by η) to

$$\left(0.93117m^2 + 5.346m\right)n^4 + \left(1.0872m^4 + 1.98219m^3 - 1.67119m^2 + 12.81877m + 1.88459\right)n^3 + \left(0.30375m^6 - 0.1188\right. \tag{458}$$

We then show that the above is growing in η in lemma B.200, so we can lower bound the above by $\eta = 0.2$.

We lower bound the resulting expression by

$$(((((((0.05426\left(0.15\right) - 0.00480)\left(0.15\right) + 0.11690)\left(0.15\right) + 0.13040)\left(0.15\right) - 0.02671)\left(0.25\right) + 0.13831)\left(0.15\right) + 0.0114$$

which simplifies (RD) to -0.00189.

We also show that

$$-10mn^3 + 9n^3 - 4.5m^2n^2 - 1.25mn^2 - 0.1n^2 \tag{459}$$

is growing in η by lemma B.5.4, so we lower bound it by $\eta = 0.2$.

We then lower bound the resulting expression by $\left(-0.18\left(0.25\right)^2 - 0.13\left(0.25\right) + 0.068\right)$, which simplifies (RD) to 0.02425.

This results in the following lower bound for Equation 190 (scaled by η^2), $-0.00189q + 0.02425$ that can be lower bounded by $-0.00189\left(\frac{0.94}{0.2}\left(0.25\right)^2 + \frac{0.48}{0.2} + 2.15\right) + 0.02425$, which is positive.

This completes the proof.

□

Proof of Lemma B.200. To prove the lemma, we first show that the second partial η of Equation 191 is positive.

To see this, note that the second partial η can be lower bounded by

$$\left(0.93117\left(0.15\right)^2 + 5.346\left(0.15\right)\right)\left(12\right)\left(0.2\right)^2 + \left(1.0872\left(0.15\right)^4 + 1.98219\left(0.15\right)^3 - 1.67119\left(0.25\right)^2 + 12.81877\left(0.15\right) + \right.$$

which is positive.

So, we evaluate the first partial η of Equation 191 at $\eta = 0.2$, which can be lower bounded by

$$0.05062\left(0.15\right)^7 + 0.15947\left(0.15\right)^6 + 0.20887\left(0.15\right)^5 + 1.31246\left(0.15\right)^4 - 0.14765\left(0.25\right)^3 + 1.01996\left(0.15\right)^2 + 0.96211\left(0.\right.$$

which is positive.

This completes the proof. □

Proof. To prove the lemma, consider the first partial η of Equation **??** below, with η factored out.

$$\left(-30m\eta + 27\eta - 9m^2 - 2.5m - 0.2\right) \tag{460}$$

Note that we can lower bound the above by

$$\left(-30\left(0.25\right) + 27\right)\left(0.2\right) - 9\left(0.25\right)^2 - 2.5\left(0.25\right) - 0.2,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.202. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first note that Equation 193 grows in u. To see this, we show the coefficient linear in u and the 0th order term in u of the above are positive in lemmas B.204 and B.205.

So, we can lower bound Equation 193 in u by $u = \sqrt{2mn + 0.25m + 0.02}$. We divide the resulting expression by $\cot\left(0.245\pi\right)$, which simplifies (RD) to

$$Equation\ \textbf{??} v_1\sqrt{2mn + 0.25m + 0.02} + v_2, \tag{461}$$

where

$$v_1 = 0.06056q^4 - 0.125nmq^4 + 0.25mnq^3 + 0.03125mq^3 + 0.0025q^3 - 1.75nq^3 + 1.5mq^2 + 9q$$

and

$$-0.06057nq^5 - 0.25mn^2q^4 - 0.03125nmq^4 - 0.0025nq^4 + 0.25m^2nq^3 + 0.03125m^2q^3 + 0.0025mq^3 + 0.48453q^3 - 0.75$$

Since v_1 is positive (see lemma B.203), we can lower bound $v_1\sqrt{2mn + 0.25m + 0.02}$ by $v_1\sqrt{2m\left(0.2\right) + 0.25m + 0.02}$. We then further lower bound $v_1\sqrt{0.65m + 0.02}$ by $v_1(0.84417m + 0.2161)$.

Next, we iteratively lower bound the resulting expression, shown below (RD), in q.

$$-0.06057nq^5 + \left(-0.25mn^2 - 0.10553nm^2 - 0.05827nm - 0.0025n + 0.05112m + 0.01308\right)q^4 + \left(0.46104m^2n - 2.173\right. \tag{462}$$

First, we lower bound

$$-0.06057nq + \left(-0.25mn^2 - 0.10553nm^2 - 0.05827nm - 0.0025n + 0.05112m + 0.01308\right) \tag{463}$$

by fixing $q = \frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15$, which simplifies (RD) to

$$-0.25mn^2 - 0.10553nm^2 - 0.05827nm - 0.13273n - 0.05694m^2 + 0.05112m - 0.01600.$$

Since

$$-0.25\left(0.15\right)\left(0.2\right)^2 - 0.10553\left(0.2\right)\left(0.15\right)^2 - 0.05827\left(0.2\right)\left(0.15\right) - 0.13273\left(0.2\right) - 0.05694\left(0.15\right)^2 + 0.05112\left(0.25\right) - 0$$

is negative, we can lower bound

$$\left(-0.25mn^2 - 0.10553nm^2 - 0.05827nm - 0.13273n - 0.05694m^2 + 0.05112m - 0.01600\right)q + \left(0.46104m^2n - 2.17328\right. \tag{464}$$

by fixing $q = \frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15$, which simplifies (after scaling by η and RD) to

$$-0.5375mn^3 - 0.235m^3n^2 + 0.23415m^2n^2 - 2.41857mn^2 - 0.66355n^2 - 0.09920nm^4 - 0.05478nm^3 - 0.24022nm^2 +$$

We lower bound the above in m by

$$-0.5375\left(0.25\right)n^3 + \left(\left(\left(-0.235\left(0.25\right) + 0.23415\right)\left(0.15\right) - 2.41857\right)\left(0.25\right) - 0.66355\right)n^2 + \left(\left(\left(-0.09920\left(0.25\right) - 0.0547\right.\right.\right.$$

which simplifies (RD) to

$$-0.13438n^3 - 1.26162n^2 + 0.39118n - 0.00540.$$

We further lower bound the above in η by

$$((-0.13438\,(0.3)-1.26162)\,(0.3)+0.39118)\,(0.2)-0.00540,$$

which is negative, so we can then lower bound

$$(((-0.13438\,(0.3)-1.26162)\,(0.3)+0.39118)\,(0.2)-0.00540)\,q+1.26625m^2n+0.88665mn+0.045n+4.5mn^2-4.5 \tag{465}$$

in q by $q=\frac{0.94}{n}m^2+\frac{0.48}{n}+2.15$, which simplifies (RD and scale by η) to

$$4.5mn^3-4.5n^3+1.26625m^2+0.88665m+0.045n^2-0.01138n-0.00498m^2-0.00254.$$

Since

$$\left(\left((4.5\,(0.25)-4.5)\,(0.2)+\left(1.26625\,(0.25)^2+0.88665\,(0.25)+0.045\right)\right)(0.2)-0.01138\right)(0.2)-0.00498\,(0.15)^2-0.0$$

is negative, we can lower bound

$$\left(4.5mn^3-4.5n^3+1.26625m^2+0.88665m+0.045n^2-0.01138n-0.00498m^2-0.00254\right)q+10.59753mn^2+1.9449 \tag{466}$$

by $q=\frac{0.94}{n}m^2+\frac{0.48}{n}+2.15$, which simplifies (RD and scale by η) to

$$(9.675m-9.675)\,n^4+\left(4.23m^3-4.23m^2+12.75753m-0.11835\right)n^3+\left(0.0423m^2-0.00287\right)n^2+\left(2.70103m^2+1.9\right.$$

We then further lower bound the above by

$$\left(\left(\left(((9.675\,(0.15)-9.675)\,(0.3)+(((4.23\,(0.15)-4.23)\,(0.25)+12.75753)\,(0.15)-0.11835))\,(0.3)+\left(0.0423\,(0.15)^2-\right.\right.\right.$$

which simplifies (RD) to 0.13212.

With the above, we get that Equation **??** can be lower bounded by $0.13212q+6n^3$.

This completes the proof.

□

Proof of Lemma B.203. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first lower bound $0.06056-0.125nm$ by $0.06056-0.125\,(0.3)\,(0.25)$ (simplifies RD to 0.05118), lower bound $0.25mn+0.03125m+0.0025-1.75n$ by $0.25\,(0.15)\,(0.2)+0.03125\,(0.15)+0.0025-1.75\,(0.3)$ (simplifies RD to -0.51032), and lower bound $1.5m$ by $1.5\,(0.15)$ (simplifies to 0.225).

We then iteratively lower bound the resulting expression (shown below) in q and scale by η.

$$0.05118q^3-0.51032q^2+0.225q+9 \tag{467}$$

We first lower bound $0.05118q-0.51032$ in q by $q=\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)$, which (after scaling by η) simplifies (RD) to $0.04606m^2+0.02661-0.41083n$. Since $0.04606\,(0.25)^2+0.02661-0.41083\,(0.2)$ is negative, we lower bound $\left(0.04606m^2+0.02661-0.41083n\right)q+0.225n$ in q by $q=\frac{0.94}{n}m^2+\frac{0.48}{n}+2.15$, which (after scaling by η) simplifies (RD) to

$$-0.65829n^2-0.28716nm^2-0.13999n+0.04329m^4+0.04712m^2+0.01277.$$

Since

$$-0.65829\,(0.2)^2-0.28716\,(0.2)\,(0.15)^2-0.13999\,(0.2)+0.04329\,(0.25)^4+0.04712\,(0.25)^2+0.01277$$

is negative, we lower bound

$$\left(-0.65829n^2-0.28716nm^2-0.13999n+0.04329m^4+0.04712m^2+0.01277\right)q+9n^2 \tag{468}$$

in q by $q = \frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15$, which (after scaling by η) simplifies (RD) to

$$7.58467n^3 + \left(-1.23619m^2 - 0.61696\right)n^2 + \left(-0.17686m^4 - 0.16812m^2 - 0.03974\right)n + \left(0.04069m^6 + 0.06507m^4 + 0.0\right. \tag{469}$$

We lower bound the above in m by

$$7.58467\eta^3 + \left(-1.23619\left(0.25\right)^2 - 0.61696\right)\eta^2 + \left(-0.17686\left(0.25\right)^4 - 0.16812\left(0.25\right)^2 - 0.03974\right)\eta + \left(0.04069\left(0.15\right)^6\right.$$

which simplifies (RD) to

$$7.58467\eta^3 - 0.69423\eta^2 - 0.05094\eta + 0.00693. \tag{470}$$

The above can then be lower bounded by

$$\left(\left(7.58467\left(0.2\right) - 0.69423\right)\left(0.2\right) - 0.05094\right)\left(0.2\right) + 0.00693,$$

which is positive.

This completes the proof.

□

Proof. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first factor out $q^2\cot\left(0.245\pi\right)$ from Equation 195.

We then show that the resulting expression

$$\left(-0.25nq + 0.25m\right)q + 4.5 \tag{471}$$

is positive.

To do so, we lower bound the above in q iteratively. We first lower bound $-0.25nq + 0.25m$ by $-0.25n\left(\frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15\right) + 0.25m$, which simplifies (RD) to $-0.235m^2 + 0.25m - 0.5375n - 0.12$.

Since

$$-0.235\left(0.15\right)^2 + 0.25\left(0.25\right) - 0.5375\left(0.2\right) - 0.12$$

is negative, we lower bound

$$\left(-0.235m^2 + 0.25m - 0.5375n - 0.12\right)q + 4.5 \tag{472}$$

by

$$\left(-0.235m^2 + 0.25m - 0.5375n - 0.12\right)\left(\frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15\right) + 4.5.$$

Scaling the above by η, we simplify (RD) the resulting expression to

$$\left(-1.15563n + \left(-1.0105m^2 + 0.5375m + 3.984\right)\right)n + \left(-0.2209m^4 + 0.235m^3 - 0.2256m^2 + 0.12m - 0.0576\right). \tag{473}$$

We then iteratively lower bound the above in η. First, we lower bound $-1.15563n + \left(-1.0105m^2 + 0.5375m + 3.984\right)$ by $-1.15563\left(0.3\right) + \left(-1.0105m^2 + 0.5375m + 3.984\right)$, which simplifies (RD) to $-1.0105m^2 + 0.5375m + 3.63731$.

Since

$$-1.0105\left(0.25\right)^2 + 0.5375\left(0.15\right) + 3.63731$$

is positive, we lower bound

$$\left(-1.0105m^2 + 0.5375m + 3.63731\right)n + \left(-0.2209m^4 + 0.235m^3 - 0.2256m^2 + 0.12m - 0.0576\right) \tag{474}$$

by fixing $\eta = 0.2$, which simplifies (RD) to

$$-0.2209m^4 + 0.235m^3 - 0.4277m^2 + 0.2275m + 0.66986. \tag{475}$$

The above can be lower bounded by

$$-0.2209\left(0.25\right)^4+0.235\left(0.15\right)^3-0.4277\left(0.25\right)^2+0.2275\left(0.15\right)+0.66986,$$

which is positive.

This concludes the proof.

□

Proof of Lemma B.205. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first divide out a factor of q from Equation 196.

We then note that $\left(0.0625-0.125\cot\left(0.245\pi\right)nm\right)$ from Equation 196 can be lower bounded by $0.0625-0.125\cot\left(0.245\pi\right)\left(0.3\right)\left(0.25\right)$, which is positive.

We then show that

$$\left(-1.75nq+1.5m\right)q+9 \tag{476}$$

is positive.

To proceed, we first lower bound $-1.75nq+1.5m$ by $-1.75\left(0.94m^2+0.48+2.15n\right)+1.5m$, which simplifies (RD) to $-1.645m^2+1.5m-3.7625n-0.84$.

Since

$$-1.645\left(0.15\right)^2+1.5\left(0.25\right)-3.7625\left(0.2\right)-0.84$$

is negative, we lower bound

$$\left(-1.645m^2+1.5m-3.7625n-0.84\right)q+9 \tag{477}$$

by

$$\left(-1.645m^2+1.5m-3.7625n-0.84\right)\left(\frac{0.94}{n}m^2+\frac{0.48}{n}+2.15\right)+9,$$

which simplifies (RD and scaling by η) to

$$-8.08938n^2+\left(-7.0735m^2+3.225m+5.388\right)n+\left(-1.5463m^4+1.41m^3-1.5792m^2+0.72m-0.4032\right). \tag{478}$$

We then lower bound the above in η as follows. First, we lower bound $-8.08938n^2$ by $-8.08938(0.3)n$, where

$$-8.08938\left(0.3\right)+\left(-7.0735m^2+3.225m+5.388\right)$$

simplifies (RD) to

$$-7.0735m^2+3.225m+2.96118.$$

Since

$$-7.0735\left(0.25\right)^2+3.225\left(0.15\right)+2.96118$$

is positive, we can lower bound

$$\left(-7.0735m^2+3.225m+2.96118\right)n+\left(-1.5463m^4+1.41m^3-1.5792m^2+0.72m-0.4032\right) \tag{479}$$

by fixing $\eta=0.2$, which simplifies (RD) to

$$-1.5463m^4+1.41m^3-2.9939m^2+1.365m+0.18903. \tag{480}$$

We then lower bound the above with

$$-1.5463\left(0.25\right)^4+1.41\left(0.15\right)^3-2.9939\left(0.25\right)^2+1.365\left(0.15\right)+0.18903,$$

which is positive.

This completes the proof. □

Proof. To prove the lemma, we first show that Equation 197 is concave in u. To do so, consider the second partial u of Equation 197 below (with $\cot(0.245\pi)\, q^2$ factored out).

$$0.75qu + 0.25nq^2 - 0.25mq - 4.5 \tag{481}$$

We first upper bound the above in u by $u = \sqrt{2mn + 0.25m + 0.025}$ and upper bound $0.25nq^2$ by $0.25n\left(\frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15\right)q$. Since

$$\left(0.75\sqrt{2\,(0.15)\,(0.2) + 0.25\,(0.15) + 0.025} - 0.25\,(0.25)\right)$$

is positive, we can upper bound

$$\left(0.25n\left(\frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15\right) + \left(0.75\sqrt{2mn + 0.25m + 0.025} - 0.25m\right)\right)q \tag{482}$$

by fixing $q = \left(\frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15\right)$.

We then scale the resulting expression by η and note that it can be upper bounded by

$$\left(0.235\,(0.25)^2 - 0.25\,(0.15) + 0.12 + 0.5375\,(0.3) + 0.75\sqrt{2\,(0.25)\,(0.3) + 0.25\,(0.25) + 0.025}\right)\left(0.94\,(0.25)^2 + 0.48 + 2\right.$$

which is negative.

After showing concavity in u, we show that at both $u = \sqrt{2m\eta + 0.25m + 0.02}, \sqrt{2m\eta + 0.25m + 0.025}$, Equation 197 (rounded down after five digits of precision) is positive in lemmas B.212 and B.207.

This completes the proof. □

Proof of Lemma B.207. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first that Equation 198 is convex in q and the first partial q evaluated at $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$ is positive.

We start with showing convexity in q. Consider the second partial q below.

$$(k_1)\sqrt{2mn + 0.25m + 0.025} + k_2, \tag{483}$$

where

$$k_1 = \left(\left(-1.54788nm + 0.75\right)q + 1.54782mn + 0.19344m + 0.01932 - 10.83516n\right)q + 3.09574m$$

and

$$k_2 = \left(\left(1.25nq + 3.09564mn^2 + 0.38688nm + 0.03864n\right)q - 1.54788m^2n - 0.1935m^2 - 0.01938m + 4.64358nm - 3\right)q$$

To proceed, we iteratively lower bound k_1 and k_2 in q. Since $-1.54788\,(0.3)\,(0.25) + 0.75$ is positive, we first lower bound

$$\left(-1.54788nm + 0.75\right)q + 1.54782mn + 0.19344m + 0.01932 - 10.83516n \tag{484}$$

and $1.25nq$ by fixing $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$ (and scale by η), yielding the following lower bounds (RD) for k_1 and k_2.

$$\left(\left(-1.39310m^3 - 0.61146m + 1.47732\right)n + \left(0.675m^2 + 0.39\right) + \left(-1.46126m - 10.83516\right)n^2\right)q + 3.09574nm \tag{485}$$

$$\left(\left(6.01792mn^3 + \left(2.78607m^3 - 1.54788m^2 + 7.00540m + 4.79903\right)n^2 + \left(0.34819m^3 + 4.21527m^2 + 0.18179m - 0.452\right.\right.\right.$$
(486)

We further lower bound

$$\left(-1.39310m^3 - 0.61146m + 1.47732\right) n + \left(0.675m^2 + 0.39\right) + \left(-1.46126m - 10.83516\right) n^2 \tag{487}$$

in k_1 by

$$\left(-1.39310\left(0.25\right)^3 - 0.61146\left(0.25\right) + 1.47732\right)\left(0.2\right) + \left(0.675\left(0.15\right)^2 + 0.39\right) + \left(-1.46126\left(0.25\right) - 10.83516\right)\left(0.3\right)^2,$$

which simplifies (RD) to -0.34232. We then lower bound $-0.34232q + 3.09574nm$ by fixing $q = \frac{0.94}{n}m^2 + \frac{0.48}{n} + 2.15$.

And, since Equation 486 is positive by lemma B.211, we lower bound Equation 486 in q by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$.

After scaling both lower bounds for k_1 and k_2 by η, we can lower bound Equation 483 by the following.

$$\left(-0.32179m^2 - 0.16432 - 0.73599n + 3.09574n^2m\right)\sqrt{2mn + 0.25m + 0.025} + \left(11.69883mn^4 + \left(10.83224m^3 - 3.0090\right.\right. \tag{488}$$

We then lower bound the above naively in η, m as follows, which is positive.

$$\left(-0.32179\left(0.25\right)^2 - 0.16432 - 0.73599\left(0.3\right) + 3.09574\left(0.2\right)^2\left(0.15\right)\right)\sqrt{2\left(0.25\right)\left(0.3\right) + 0.25\left(0.25\right) + 0.025} + \left(11.6988\right.$$

We next move to showing that first partial q of Equation 198 at $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$ is positive.

Consider the first partial q (scaled by η^3 and RD) below.

$$\phi_1\sqrt{2mn + 0.25m + 0.025} + \phi_2, \tag{489}$$

where

$$\phi_1 = \left(0.18225m^6 + 0.3159m^4 + 0.18252m^2 + 0.03515\right) + \left(-0.37615m^7 - 0.57364m^5 + 1.18880m^4 - 0.28618m^3 + 1.3\right.$$

and

$$\phi_2 = \left(0.20502m^8 + 0.47383m^6 + 0.41066m^4 + 0.15818m^2 + 0.02284\right) + \left(0.09400m^7 + 1.70245m^6 + 0.15508m^5 + 1.78\right.$$

We show that ϕ_1 and ϕ_2 are positive in lemmas B.209 and B.210.

So, we can lower bound Equation 198 in q by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$, shown below (scaled by η^4 and RD).

$$b_1\sqrt{2mn + 0.25m + 0.025} + b_2, \tag{490}$$

where

$$b_1 = \left(0.05293m - 13.26705\right)n^5 + \left(-0.77942m^3 - 18.42646m^2 + 5.63612m + 8.32429\right)n^4 + \left(-1.15061m^5 - 8.53079m\right.$$

and

$$b_2 = 3.68422mn^6 + \left(6.82263m^3 - 1.89533m^2 - 7.46080m + 19.33008\right)n^5 + \left(4.73792m^5 - 2.63241m^4 - 2.02464m^3 + \right.$$

To proceed, we first show the b_1 is positive in lemma B.208, so we lower bound $b_1\sqrt{2mn + 0.25m + 0.025}$ by $b_1\sqrt{2m\left(0.2\right) + 0.25m + 0.025}$. We then lower bound $b_1\sqrt{0.65m + 0.025}$ by $b_1(0.83012m + 0.22548)$ (see lemma **??**).

So, we can lower bound Equation 490 by the following (RD).

$$3.68422mx^6 + \left(6.82263m^3 - 1.85140m^2 - 18.46212m + 16.33968\right)\eta^5 + \left(4.73792m^5 - 3.27943m^4 - 17.49650m^3 + 19.\right. \tag{491}$$

To proceed, we first show that the above is growing in η. To see this, consider the first partial η below.

$$3.68422\left(6\right)mx^5 + \left(6.82263m^3 - 1.85140m^2 - 18.46212m + 16.33968\right)\left(5\right)x^4 + \left(4.73792m^5 - 3.27943m^4 - 17.49650m\right. \tag{492}$$

We then lower bound the above in m as follows. The 5th order η term can be lower bounded by $3.68422\,(6)\,(0.15)$. The quartic term in η can be lower bounded by

$$\left(6.82263\,(0.15)^3 - 1.85140\,(0.25)^2 - 18.46212\,(0.25) + 16.33968\right)(5)\,.$$

The cubic term in η can be lower bounded by

$$\left(4.73792\,(0.15)^5 - 3.27943\,(0.25)^4 - 17.49650\,(0.25)^3 + 19.11825\,(0.15)^2 - 2.80645\,(0.25) + 3.55027\right)(4)\,.$$

The quadratic term in η can be lower bounded by

$$\left(1.46230\,(0.15)^7 - 2.17389\,(0.25)^6 - 4.31964\,(0.25)^5 + 7.54488\,(0.15)^4 - 1.67336\,(0.25)^3 + 5.28919\,(0.15)^2 + 0.75716\,(0\right.$$

The linear term in η can be lower bounded by

$$\left(0.16924\,(0.15)^9 - 0.63903\,(0.25)^8 - 0.07726\,(0.25)^7 + 1.35048\,(0.15)^6 + 0.32407\,(0.15)^5 + 0.88764\,(0.15)^4 + 0.58242\,(0\right.$$

Last, the 0th order η term can be lower bounded by

$$\left(-0.07027\,(0.25)^{10} + 0.00205\,(0.15)^9 + 0.23421\,(0.15)^8 + 0.30371\,(0.15)^7 + 0.49414\,(0.15)^6 + 0.52235\,(0.15)^5 + 0.2518\right.$$

With the above, Equation 492 can be lower bounded by

$$3.31579\eta^5 + 58.15731\eta^4 + 11.97193\eta^3 + 0.97037\eta^2 - 0.91597\eta + 0.00002. \tag{493}$$

The above can be further lower bounded by

$$((((3.31579\,(0.2) + 58.15731)\,(0.2) + 11.97193)\,(0.2) + 0.97037)\,(0.2) - 0.91597)\,(0.2) + 0.00002,$$

which is positive.

So, we can lower bound Equation 491 in η by $\eta = 0.2$, shown below (RD).

$$0.02283m^{10} + 0.04119m^9 + 0.13709m^8 + 0.148m^7 + 0.27998m^6 + 0.15862m^5 + 0.23065m^4 + 0.07053m^3 + 0.07035m^2 \tag{494}$$

We then lower bound the above by fixing $m = 0.15$, which is positive.

This completes the proof.

□

Proof of Lemma B.208. To prove the lemma, we first lower bound Equation 199 in m as follows.

The 5th order term in η is lower bounded by $0.05293\,(0.15) - 13.26705$.

The quartic term in η is lower bounded by

$$((-0.77942\,(0.25) - 18.42646)\,(0.25) + 5.63612)\,(0.15) + 8.32429.$$

The cubic term in η is lower bounded by

$$(((((-1.15061\,(0.25) - 8.53079)\,(0.25) + 4.41566)\,(0.15) + 0.18655)\,(0.15) + 2.93536)\,(0.15) + 2.95557)\,.$$

The quadratic term in η is lower bounded by

$$(((((((-0.54322\,(0.25) - 1.31649)\,(0.25) + 0.46450)\,(0.15) - 1.11885)\,(0.25) + 1.08076)\,(0.15) + 0.02556)\,(0.15) + 0.364$$

The linear term in η is lower bounded by

$$\left(\left(\left(\left(\left(\left(\left(-0.08465\,(0.25)^2 - 0.17212\right)(0.25) + 0.35659\right)(0.15) - 0.12880\right)(0.25) + 0.61814\right)(0.15) - 0.04178\right)(0.15)\right.$$

The 0th order term in η is lower bounded by fixing $m = 0.15$.

With the above, Equation 199 can be lower bounded by

$$-13.25912\eta^5 + 8.47140\eta^4 + 3.40753\eta^3 + 0.19230\eta^2 + 0.07622\eta + 0.00531. \tag{495}$$

The above can be lower bounded by

$$-13.25912\,(0.3)^5 + 8.47140\,(0.2)^4 + 3.40753\,(0.2)^3 + 0.19230\,(0.2)^2 + 0.07622\,(0.2) + 0.00531,$$

which is positive.

This completes the proof.

□

Proof. To prove the lemma, we first show that Equation 200 is decreasing in η. Consider the first partial η below.

$$\left(-0.37615m^7 - 0.57364m^5 + 1.18880m^4 - 0.28618m^3 + 1.37372m^2 - 0.04640m + 0.39685\right) + \left(-1.81054m^5 - 4.3882 \tag{496}$$

We first upper bound the above in m as follows. We upper bound the 0th order η term by

$$\Big(\Big(\Big(\Big(\Big(\Big(-0.37615\,(0.15)^2 - 0.57364\Big)\,(0.15) + 1.18880\Big)\,(0.25) - 0.28618\Big)\,(0.15) + 1.37372\Big)\,(0.25) - 0.04640\Big)\,(0.25) +$$

We upper bound the linear term in η by

$$(((((-1.81054\,(0.15) - 4.38825)\,(0.15) + 1.03245)\,(0.25) - 2.48615)\,(0.15) + 1.20091)\,(0.25) + 0.02847)\,(2)\,.$$

We upper bound the quadratic term in η by

$$(((-2.55667\,(0.15) - 18.95722)\,(0.15) + 4.90645)\,(0.25) + 0.20738)\,(3)\,.$$

Last, we upper bound the cubic term in η by $(-20.47379 - 0.86588\,(0.15))\,(4)$.

With the above, we upper bound Equation 496 by the following.

$$0.47101 + 0.47719\eta + 2.12615\eta^2 - 82.41468\eta^3 \tag{497}$$

We then upper bound the above by

$$0.47101 + (0.47719 + (2.12615 - 82.41468\,(0.2))\,(0.2))\,(0.2)\,,$$

which is negative.

So, we lower bound Equation 200 in η by $\eta = 0.3$, resulting in the following expression (rounded down after five digits of precision).

$$-0.11285m^7 + 0.18225m^6 - 0.33505m^5 + 0.27759m^4 - 0.06197m^3 - 0.14097m^2 + 0.21962m - 0.00348 \tag{498}$$

Note that the above can be lower bounded by

$$-0.11285\,(0.25)^7 + 0.18225\,(0.15)^6 - 0.33505\,(0.25)^5 + 0.27759\,(0.15)^4 - 0.06197\,(0.25)^3 - 0.14097\,(0.25)^2 + 0.21962\,($$

which is positive.

This completes the proof.

□

Proof of Lemma B.210. To prove the lemma, we start by showing that Equation 201 is growing in η. To see this, note the first partial η below.

$$\left(0.09400m^7 + 1.70245m^6 + 0.15508m^5 + 1.78123m^4 + 0.08506m^3 + 0.35332m^2 + 0.01550m - 0.06212\right) + \left(0.75223m^7 \right. \tag{499}$$

We first lower bound the above in m as follows. We lower bound the 0th order term in η by fixing $m = 0.15$. We lower bound the linear term in η by

$$(((((((0.75223\,(0.15) - 0.62690)\,(0.25) + 3.79369)\,(0.15) + 4.73741)\,(0.15) + 2.55170)\,(0.15) + 0.94445)\,(0.15) + 0.3529$$

We lower bound the quadratic term in η by

$$(((((4.87450\,(0.15) - 2.70818)\,(0.25) + 6.71443)\,(0.15) + 14.82452)\,(0.15) - 3.13704)\,(0.25) + 3.78615)\,(3)\,.$$

We lower bound the cubic term in η by

$$(((10.52891\,(0.15) - 2.92483)\,(0.25) - 2.24934)\,(0.25) + 22.61210)\,(4)\,.$$

Last, we lower bound the quartic term in η by $7.58082\,(5)\,(0.15)$.

With the above, we can then lower bound Equation 499 by the following.

$$-0.05063 - 2.14246\eta + 10.77839\eta^2 + 87.86268\eta^3 + 5.68561\eta^4 \tag{500}$$

We further lower bound the above by

$$-0.05063 + (-2.14246 + (10.77839 + (87.86268 + 5.68561\,(0.2))\,(0.2))\,(0.2))\,(0.2)\,,$$

which is positive.

Then, we can lower bound Equation 201 in η by $\eta = 0.2$, shown below (rounded down after five digits of precision).

$$0.20502m^8 + 0.04888m^7 + 0.78924m^6 + 0.22175m^5 + 0.93473m^4 + 0.18964m^3 + 0.38053m^2 - 0.00906m + 0.03061 \tag{501}$$

We can lower bound the above by

$$0.20502\,(0.15)^8 + 0.04888\,(0.15)^7 + 0.78924\,(0.15)^6 + 0.22175\,(0.15)^5 + 0.93473\,(0.15)^4 + 0.18964\,(0.15)^3 + 0.38053\,(0.$$

which is positive.

This completes the proof.

□

Proof of Lemma B.211. To prove the lemma, note that we can lower bound Equation 202 naively in η, m as follows, which is positive.

$$6.01792\,(0.15)\,(0.2)^3 + \left(2.78607\,(0.15)^3 - 1.54788\,(0.25)^2 + 7.00540\,(0.15) + 4.79903\right)(0.2)^2 + \left(0.34819\,(0.15)^3 + 4.21\right.$$

This completes the proof. □

Proof of Lemma B.212. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first show that Equation 203 is convex in q and the first partial q at $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$ is positive.

First, we show convexity in q. Consider the second partial q below.

$$1.25nq^3 + l_1q^2 + l_2q + l_3, \tag{502}$$

where

$$l_1 = 3.09564mn^2 + 0.03084n + \left(0.38688 - 1.54788\sqrt{2mn + 0.25m + 0.02}\right)nm + 0.75\sqrt{2mn + 0.25m + 0.02},$$

$$l_2 = \left(4.64358 + 1.54782\sqrt{2mn + 0.25m + 0.02}\right)nm - 3 - 1.54788m^2n - 0.1935m^2 + \left(0.19344\sqrt{2mn + 0.25m + 0.02}\right.$$

and

$$l_3 = -9.28728mn - 1.16092m - 0.09288 + 9.28726n + 3.09576m\sqrt{2mn + 0.25m + 0.02}.$$

To proceed, we lower bound l_1, l_2, l_3 as follows. We lower bound l_1 by

$$3.09564\,(0.15)\,(0.2)^2 + 0.03084\,(0.2) + \left(0.38688 - 1.54788\sqrt{2(0.25)\,(0.3) + 0.25(0.25) + 0.02}\right)(0.3)\,(0.25) + 0.75\sqrt{2(0}$$

which simplifies (RD) to 0.25486.

We lower bound l_2 by

$$\left(\left(4.64358 + 1.54782\sqrt{2(0.15)\,(0.2) + 0.25(0.15) + 0.02}\right)(0.2)(0.15) - 3 - 1.54788\,(0.25)^2\,(0.3) - 0.1935\,(0.25)^2 + \left(0.\right.\right.$$

which simplifies (RD) to -4.44034.

Last, we lower bound l_3 by

$$\left(-9.28728(0.25)(0.3) - 1.16092(0.25) - 0.09288 + 9.28726(0.2) + 3.09576(0.15)\sqrt{2(0.15)\,(0.2) + 0.25(0.15) + 0.02}\right),$$

which simplifies (RD) to 0.93697.

Then, we can lower bound Equation 502 by $((1.25nq + 0.25486)\,q - 4.44034)\,q + 0.93697$, which we iteratively lower bound.

We first lower bound $(1.25nq + 0.25486)\,q - 4.44034$ by

$$\left(1.25n\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right) + 0.25486\right)\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right) - 4.44034, \quad (503)$$

which simplifies (RD) after scaling by η to $1.0125m^4 + 1.39937m^2 + 4.374nm^2 + 4.72392n^2 + 0.47052 - 1.41770n$.

Since

$$1.0125\,(0.15)^4 + 1.39937\,(0.15)^2 + \left(4.374\,(0.15)^2 + 4.72392\,(0.2) - 1.41770\right)(0.3) + 0.47052$$

is positive, this shows convexity in q.

We next show that the first partial q of Equation 203 at $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$ is positive.

Consider the first partial q (scaled by η^3 and RD) below.

$$\gamma_1\sqrt{2mn + 0.25m + 0.02} + \gamma_2, \quad (504)$$

where

$$\gamma_1 = \left(-0.37615nm^7 - 0.57365nm^5 + 1.18721m^4n - 0.28619nm^3 + 1.37187m^2n - 0.04640nm + 0.39631n\right) + \left(-1.810\right.$$

and

$$\gamma_2 = 0.20502m^8 + 0.47383m^6 + 0.41066m^4 + 0.15817m^2 + 0.02284 + \left(0.09400nm^7 + 1.70055m^6n + 0.15665nm^5 + 1.7\right.$$

We show that γ_1 and γ_2 are positive in lemmas B.5.4 and B.215.

So, we lower bound Equation 203 in q by $q = \frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944$ and scale the resulting expression by η^4, shown below (RD).

$$c_1\sqrt{2mn + 0.25m + 0.02} + c_2, \quad (505)$$

where

$$c_1 = (0.05293m - 13.26705)\,n^5 + \left(-0.77942m^3 - 18.42646m^2 + 5.63616m + 8.31953\right)n^4 + \left(-1.15061m^5 - 8.53079m\right.$$

and

$$c_2 = 3.68422mn^6 + \left(6.82263m^3 - 1.89533m^2 - 7.46080m + 19.32083\right)n^5 + \left(4.73792m^5 - 2.63241m^4 - 2.02464m^3 + \right.$$

We show c_1 is positive in lemma B.213 and lower bound $c_1\sqrt{2m(0.2) + 0.25m + 0.02}$, which we subsequently lower bound by $c_1(0.84417m + 0.2161)$ (see lemma **??**).

We simplify $c_1(0.84417m + 0.2161) + c_2$ to the following (RD).

$$
\begin{aligned}
&3.68422m\eta^6 + \left(6.82263m^3 - 1.85065m^2 - 18.64901m + 16.45382\right)\eta^5 \\
&+ \left(4.73792m^5 - 3.29038m^4 - 17.74815m^3 + 19.35172m^2 - 2.74111m + 3.50577\right)\eta^4 \\
&+ \left(1.46230m^7 - 2.19006m^6 + -4.42880m^5 + 7.67440m^4 - 1.71072m^3 + 5.35417m^2 + 0.77204m + 0.10914\right)\eta^3 \\
&+ \Big(0.16924m^9 - 0.64667m^8 - 0.09071m^7 + 1.36559m^6 + 0.30453m^5 + 0.91565m^4 + 0.57331m^3 \\
&- 0.80852m^2 + 0.19320m - 0.43740\Big)\eta^2 \\
&+ \Big(-0.07146m^{10} + 0.00284m^9 + 0.23138m^8 + 0.31040m^7 + 0.48794m^6 + 0.53236m^5 \\
&+ 0.24446m^4 + 0.30647m^3 + 0.01210m^2 + 0.05889m - 0.01091\Big)\eta \\
&+ \Big(0.03689m^{10} + 0.03460m^9 + 0.11543m^8 + 0.07998m^7 + 0.14365m^6 + 0.06933m^5 \\
&+ 0.08890m^4 + 0.02669m^3 + 0.02737m^2 + 0.00384m + 0.00335\Big)
\end{aligned}
\tag{506}
$$

We first show that the above is increasing in η. We lower bound the first partial η in m as follows. First, we lower bound the 5th order η term by $22.10532\,(0.15)$.

We lower bound the quartic η term by

$$((34.11315\,(0.15) - 9.25325)\,(0.25) - 93.24505)\,(0.25) + 82.2691.$$

We lower bound the cubic η term by

$$((((18.95168\,(0.15) - 13.16152)\,(0.25) - 70.9926)\,(0.25) + 77.40688)\,(0.15) - 10.96444)\,(0.25) + 14.02308.$$

We lower bound the quadratic η term by

$$\Big(((((4.3869\,(0.15) - 6.57018)\,(0.25) - 13.2864)\,(0.25) + 23.0232)\,(0.15) - 5.13216)\,(0.25) + 16.06251)\,(0.15) + 2.31612\Big)\,(0.15) + 0.32742.$$

We lower bound the linear η term by

$$\Big(\Big(\Big(\Big(\Big(((((0.33848\,(0.15) - 1.29334)\,(0.25) - 0.18142)\,(0.25) + 2.73118)\,(0.15) + 0.60906)\,(0.15) + 1.8313\Big)\,(0.15) + 1.14662\Big)\,(0.15) - 1.61704\Big)\,(0.25) + 0.3864\Big)\,(0.15) - 0.8748\Big).$$

We lower bound the 0th η term by

$$\Big(\Big(\Big(\Big(\Big(\Big(((((-0.07146\,(0.25) + 0.00284)\,(0.25) + 0.23138)\,(0.15) + 0.31040)\,(0.15) + 0.48794)\,(0.15) + 0.53236\Big)\,(0.15) + 0.24446\Big)\,(0.15) + 0.30647\Big)\,(0.15) + 0.01210\Big)\,(0.15) + 0.05889\Big)\,(0.15) - 0.01091\Big).$$

With the above, the first partial η of Equation 506 simplifies (RD) to

$$3.31579\eta^5 + 58.69932\eta^4 + 13.49498\eta^3 + 1.02368\eta^2 - 0.86936\eta - 0.00060,$$

which can be lower bounded by

$$3.31579\,(0.2)^5 + 58.69932\,(0.2)^4 + 13.49498\,(0.2)^3 + (1.02368\,(0.2) - 0.86936)\,(0.3) - 0.00060.$$

We can then lower bound Equation 506 in η by $\eta = 0.2$. We lower bound the resulting expression in m by $m = 0.15$, which is positive.

This completes the proof.

□

Proof of Lemma B.213. To prove the lemma, we first lower bound Equation 204 in m as follows.

We first lower bound the 5th order η term by $0.05293\,(0.15) - 13.26705$.

We lower bound the quartic η term by

$$((-0.77942\,(0.25) - 18.42646)\,(0.25) + 5.63616)\,(0.15) + 8.31953.$$

We lower bound the cubic η term by

$$((((-1.15061\,(0.25) - 8.53079)\,(0.25) + 4.41570)\,(0.15) + 0.17995)\,(0.15) + 2.93539)\,(0.15) + 2.95175.$$

We lower bound the quadratic η term by

$$\Bigg(\Bigg(\Bigg(((((-0.54322\,(0.25) - 1.31649)\,(0.25) + 0.46451)\,(0.15) - 1.12189)\,(0.25) + 1.08078)\,(0.15) + 0.02204\Bigg)\,(0.15) + 0.36459\Bigg)\,(0.15) + 0.13331\Bigg).$$

We lower bound the linear η term by

$$\Bigg(\Bigg(\Bigg(\Bigg(\Bigg(\Bigg(\Bigg(\Big(-0.08465\,(0.25)^2 - 0.17212\Big)\,(0.25) + 0.35612\Bigg)\,(0.15) - 0.12880\Bigg)\,(0.25) + 0.61733\Bigg)\,(0.15) - 0.04178\Bigg)\,(0.15) + 0.35668\Bigg)\,(0.15) - 0.00491\Bigg)\,(0.15) + 0.06867\Bigg).$$

We lower bound the 0th order η term by fixing 0.15.

With the above, Equation 204 can then be lower bounded by

$$-13.25912\eta^5 + 8.46665\eta^4 + 3.40356\eta^3 + 0.19120\eta^2 + 0.07611\eta + 0.00531. \tag{507}$$

The above can be lower bounded by

$$-13.25912\,(0.3)^5 + 8.46665\,(0.2)^4 + 3.40356\,(0.2)^3 + 0.19120\,(0.2)^2 + 0.07611\,(0.2) + 0.00531,$$

which is positive.

This completes the proof.

□

Proof. To prove the lemma, note that we lower bound Equation 205 in m as follows.

We lower bound the 0th order η term by

$$0.20502\,(0.15)^8 + 0.47383\,(0.15)^6 + 0.41066\,(0.15)^4 + 0.15817\,(0.15)^2 + 0.02284.$$

We lower bound the linear η term by

$$0.09400\,(0.15)^7 + 1.70055\,(0.15)^6 + 0.15665\,(0.15)^5 + 1.77793\,(0.15)^4 + 0.08688\,(0.15)^3 + 0.35141\,(0.15)^2 + 0.01603\,(0.15) - 0.0625.$$

We lower bound quadratic η term by

$$\Big(\Big((((0.75223\,(0.15) - 0.62690)\,(0.25) + 3.79367)\,(0.15) + 4.72511)\,(0.15) + 2.55849)\,(0.15) + 0.95116\Big)\,(0.15) + 0.3569\Big)\,(0.15) - 1.14876.$$

We lower bound the cubic η term by

$$((((4.87449\,(0.15) - 2.70819)\,(0.25) + 6.71436)\,(0.15) + 14.79798)\,(0.15) - 3.1297)\,(0.25) + 3.81597.$$

We lower bound the quartic η term by

$$((10.52891\,(0.15) - 2.92483)\,(0.25) - 2.24938)\,(0.25) + 22.59301.$$

We lower bound the 5th order η term by $7.58082\,(0.15)$.

Equation 205 then simplifies (by rounding down after five digits of precision) to

$$1.13712\eta^5 + 21.94657\eta^4 + 3.62345\eta^3 - 1.06252\eta^2 - 0.05097\eta + 0.02661. \tag{508}$$

The above can be lower bounded by

$$((((1.13712\,(0.2) + 21.94657)\,(0.2) + 3.62345)\,(0.2) - 1.06252)\,(0.2) - 0.05097)\,(0.2) + 0.02661,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.215. To prove the lemma, note that we first lower bound Equation 206 in m as follows.

The 0th order η term can be lower bounded by

$$\Big(0.18225\,(0.15)^6 + 0.3159\,(0.15)^4 + 0.18252\,(0.15)^2 + 0.03515\Big).$$

The linear η term can be lower bounded by

$$\Big(\Big(\Big(\Big(\Big(\Big(-0.37615\,(0.25)^2 - 0.57365\Big)\,(0.25) + 1.18721\Big)\,(0.15) - 0.28619\Big)\,(0.25) + 1.37187\Big)\,(0.15) - 0.04640\Big)\,(0.15) + 0.39631\Big).$$

The quadratic η term can be lower bounded by

$$(((((-1.81054\,(0.25) - 4.38825)\,(0.25) + 1.03243)\,(0.25) - 2.49301)\,(0.25) + 1.20091)\,(0.15) + 0.02451).$$

The cubic η term can be lower bounded by

$$(((-2.55667\,(0.25) - 18.95723)\,(0.25) + 4.90646)\,(0.15) + 0.19999).$$

The quartic η term can be lower bounded by $(-0.86588\,(0.25) - 20.47379)$.

Equation 206 then simplifies (by rounding down after five digits of precision) to

$$0.41948\eta + 0.10949\eta^2 + 0.20109\eta^3 - 20.69026\eta^4 + 0.03941. \tag{509}$$

The above can be upper bounded by

$$(((-20.69026\,(0.3) + 0.20109)\,(0.3) + 0.10949)\,(0.3) + 0.41948)\,(0.3) + 0.03941,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.216. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we begin by scaling Equation 207 by $\eta \exp\left(-m\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\right)$.

The resulting expression (RU) is shown below.

$$\begin{aligned}
&\left(4.12768m^3\eta^2 + 0.51596m^3\eta + 0.04128m^2\eta + m\eta\right) \\
&+ \left(t_{1,1}\sqrt{2m\eta + 0.25m + 0.02} + t_{1,2}\right)\exp\left(-\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\
&+ \left(t_{2,1} + t_{2,2}\sqrt{2m\eta + 0.25m + 0.02}\right)\exp\left(-\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right),
\end{aligned} \tag{510}$$

where

$$\begin{aligned}
t_{1,1} =& 2.00606m\eta^3 + \left(0.92873m^3 - 1.06079m^2 + 2.85120m + 0.96485\right)\eta^2 \\
&+ \left(0.46437m^4 + 0.11610m^3 + 0.89443m^2 - 0.43954m - 0.48919\right)\eta \\
&+ \left(0.2025m^4 - 0.225m^3 + 0.234m^2 - 0.13m + 0.0676\right), \\
t_{1,2} =& \left(-1.94822m^2 - 4.12766m\right)\eta^3 + \left(-1.85743m^4 + 1.03192m^3 - 1.05870m^2 - 0.48752m - 1.98526\right)\eta^2 + \\
&\left(-0.23216m^4 - 0.31436m^3 - 0.99156m^2 + 0.37050m - 0.51007\right)\eta \\
&+ \left(-0.2025m^5 - 0.17775m^3 + 0.0045m^2 - 0.0351m + 0.0026\right), \\
t_{2,1} =& \left(-1.94822m^2 - 4.12766m\right)\eta^3 + \left(-1.85743m^4 + 1.03192m^3 - 1.05870m^2 - 0.48752m - 1.98526\right)\eta^2 \\
&+ \left(-0.23216m^4 - 0.31436m^3 - 0.99156m^2 + 0.37050m - 0.51007\right)\eta \\
&+ \left(-0.2025m^5 - 0.17775m^3 + 0.0045m^2 - 0.0351m + 0.0026\right),
\end{aligned}$$

and

$$\begin{aligned}
t_{2,2} =& -2.00603m\eta^3 + \left(-0.92871m^3 + 1.06084m^2 - 2.85115m - 0.96482\right)\eta^2 \\
&+ \left(-0.46435m^4 - 0.11608m^3 - 0.89439m^2 + 0.43957m + 0.48920\right)\eta \\
&+ \left(-0.2025m^4 + 0.225m^3 - 0.234m^2 + 0.13m - 0.0676\right).
\end{aligned}$$

To proceed, we first show that $t_{1,1}$ is positive in lemma B.219, and $t_{2,2}$ is negative in lemma B.220.

Then, we upper bound $t_{1,1}\sqrt{2m\eta + 0.25m + 0.02}$ by $t_{1,1}\sqrt{2m(0.3) + 0.25m + 0.02}$, and upper bound $t_{2,2}\sqrt{2m\eta + 0.25m + 0.02}$ by $t_{2,2}\sqrt{2m(0.2) + 0.25m + 0.02}$.

We further upper bound $t_{1,1}\sqrt{0.85m + 0.02} + t_{1,2}$ by linearizing $\sqrt{0.85m + 0.02}$ at $m = 0.25$. We also upper bound $t_{2,1} + t_{2,2}\sqrt{0.65m + 0.02}$ by replacing $\sqrt{0.65m + 0.02}$ with $0.84417m + 0.2161$ (refer to lemma B.241).

With the resulting upper bounds, we upper bound Equation 510 with the following.

$$\begin{aligned}
&\left(4.12768m^3\eta^2 + 0.51596m^3\eta + 0.04128m^2\eta + m\eta\right) \\
&+ t_1\exp\left(-\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\
&+ t_2\exp\left(-\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right),
\end{aligned} \tag{511}$$

where

$$t_1 = \left(-0.18005m^2 - 3.60239m\right)\eta^3 + \left(-1.03883m^4 + 0.34011m^3 + 1.17662m^2 + 1.10947m - 1.73262\right)\eta^2 + \left(0.4093m^5 - 0.00823m^4 + 0.50440m^3 - 1.14477m^2 - 0.17576m - 0.63816\right)\eta + \left(-0.02401m^5 - 0.14529m^4 - 0.03041m^3 - 0.04881m^2 - 0.00955m + 0.0203\right)$$

and

$$t_2 = \left(-3.64165m^2 - 4.56116m\right)\eta^3 + \left(-2.64141m^4 + 1.72676m^3 - 3.23630m^2 - 1.91812m - 2.19375\right)\eta^2 + \left(-0.39199m^5 - 0.43049m^4 - 1.09446m^3 - 0.81376m^2 + 0.87846m - 0.40435\right)\eta + \left(-0.37344m^5 + 0.14618m^4 - 0.32666m^3 + 0.06368m^2 - 0.06407m - 0.01200\right).$$

We then show that t_1 and t_2 are negative in lemmas B.217 and B.218. Based on the lemmas, we then replace $\exp\left(-\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right)$ and $\exp\left(-\frac{1}{2}\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right)$ from Equation 511 with $\exp\left(-\frac{1}{2}\left(\frac{0.9}{0.2}(0.25)^2 + \frac{0.52}{0.2} + 1.944\right)\left((0.25) + \sqrt{2(0.25)(0.3) + 0.25(0.25) + 0.02}\right)\right)$ and $\exp\left(\frac{1}{2}\left(\frac{0.9}{0.3}(0.15)^2 + \frac{0.52}{0.3} + 1.944\right)\left(\sqrt{2(0.25)(0.2) + 0.25(0.25) + 0.02} - (0.25)\right)\right)$ (note that $\sqrt{0.65m + 0.02} - m$ is locally decreasing in m).

We show the resulting expression below (RU).

$$\Bigg(\left(\left(-5.10527m - 6.97155\right)m\eta + \left(-3.85826m^4 + 6.59199m^3 - 4.30853m^2 - 2.48318m - 3.35306\right)\right)\eta + \left(-0.47626m^5 - 0.60127m^4 - 0.92290m^3 - 1.28833m^2 + 2.19406m - 0.67252\right)\Bigg)\eta + \left(-0.52447m^5 + 0.17887m^4 - 0.46038m^3 + 0.08040m^2 - 0.09091m - 0.01325\right) \tag{512}$$

We now iteratively upper bound the above in η. We first upper bound

$$\left(-5.10527m - 6.97155\right)m\eta + \left(-3.85826m^4 + 6.59199m^3 - 4.30853m^2 - 2.48318m - 3.35306\right) \tag{513}$$

by fixing $\eta = 0.2$, which simplifies (RU) to $-3.85826m^4 + 6.59199m^3 - 5.32958m^2 - 3.87749m - 3.35306$.

Since

$$-3.85826\left(0.15\right)^4 + 6.59199\left(0.25\right)^3 - 5.32958\left(0.15\right)^2 - 3.87749\left(0.15\right) - 3.35306$$

is negative, we upper bound

$$\left(-3.85826m^4 + 6.59199m^3 - 5.32958m^2 - 3.87749m - 3.35306\right)\eta + \left(-0.47626m^5 - 0.60127m^4 - 0.92290m^3 - 1.28833m^2 + 2.19406m - 0.67252\right) \tag{514}$$

by fixing $\eta = 0.2$, which simplifies (RU) to $-0.47626m^5 - 1.37292m^4 + 0.39550m^3 - 2.35424m^2 + 1.41857m - 1.34313$.

Since

$$-0.47626\left(0.15\right)^5 - 1.37292\left(0.15\right)^4 + 0.39550\left(0.25\right)^3 - 2.35424\left(0.15\right)^2 + 1.41857\left(0.25\right) - 1.34313$$

is negative, we upper bound

$$\left(-0.47626m^5 - 1.37292m^4 + 0.39550m^3 - 2.35424m^2 + 1.41857m - 1.34313\right)\eta + \left(-0.52447m^5 + 0.17887m^4 - 0.46038m^3 + 0.08040m^2 - 0.09091m - 0.01325\right) \tag{515}$$

by fixing $\eta = 0.2$, which simplifies (RU) to

$$-0.61972m^5 - 0.09571m^4 - 0.38128m^3 - 0.39044m^2 + 0.19281m - 0.28187 \tag{516}$$

The above can be upper bounded by

$$-0.61972\,(0.15)^5-0.09571\,(0.15)^4-0.38128\,(0.15)^3-0.39044\,(0.15)^2+0.19281\,(0.25)-0.28187,$$

which is negative.

This completes the proof.

□

Proof. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we rewrite Equation 208 as

$$\begin{aligned}&\Big(\left(\left(-0.18005m-3.60239\right)mx+\left(-1.03883m^4+0.34011m^3+1.17662m^2+1.10947m-1.73262\right)\right)x\\&+\left(0.4093m^5-0.00823m^4+0.50440m^3-1.14477m^2-0.17576m-0.63816\right)\Big)x\\&+\left(-0.02401m^5-0.14529m^4-0.03041m^3-0.04881m^2-0.00955m+0.0203\right)\end{aligned}\tag{517}$$

and iteratively upper bound the above in η.

We first upper bound

$$\left(-0.18005m-3.60239\right)m\eta+\left(-1.03883m^4+0.34011m^3+1.17662m^2+1.10947m-1.73262\right)\tag{518}$$

by fixing $\eta=0.2$, which simplifies (RU) to $-1.03883m^4+0.34011m^3+1.14061m^2+0.38900m-1.73262$.

Since

$$-1.03883\,(0.15)^4+0.34011\,(0.25)^3+1.14061\,(0.25)^2+0.38900\,(0.25)-1.73262$$

is negative, we can upper bound

$$\begin{aligned}&\left(-1.03883m^4+0.34011m^3+1.14061m^2+0.38900m-1.73262\right)\eta\\&+\left(0.4093m^5-0.00823m^4+0.50440m^3-1.14477m^2-0.17576m-0.63816\right)\end{aligned}\tag{519}$$

by fixing $\eta=0.2$, which simplifies (RU) to $0.4093m^5-0.21599m^4+0.57243m^3-0.91664m^2-0.09796m-0.98468$.

Since

$$0.4093\,(0.25)^5-0.21599\,(0.15)^4+0.57243\,(0.25)^3-0.91664\,(0.15)^2-0.09796\,(0.15)-0.98468$$

is negative, we can upper bound

$$\begin{aligned}&\left(0.4093m^5-0.21599m^4+0.57243m^3-0.91664m^2-0.09796m-0.98468\right)\eta\\&+\left(-0.02401m^5-0.14529m^4-0.03041m^3-0.04881m^2-0.00955m+0.0203\right)\end{aligned}\tag{520}$$

by fixing $\eta=0.2$, which simplifies (RU) to

$$0.05785m^5-0.18848m^4+0.08408m^3-0.23213m^2-0.02914m-0.17663.\tag{521}$$

We then upper bound the above by

$$0.05785\,(0.25)^5-0.18848\,(0.15)^4+0.08408\,(0.25)^3-0.23213\,(0.15)^2-0.02914\,(0.15)-0.17663,$$

which is negative.

This completes the proof.

□

Proof. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we start by rewriting Equation 209 as

$$\begin{aligned}&\Big(\left(\left(-3.64165m-4.56116\right)m\eta+\left(-2.64141m^4+1.72676m^3-3.23630m^2-1.91812m-2.19375\right)\right)\eta\\&+\left(-0.39199m^5-0.43049m^4-1.09446m^3-0.81376m^2+0.87846m-0.40435\right)\Big)\eta\\&+\left(-0.37344m^5+0.14618m^4-0.32666m^3+0.06368m^2-0.06407m-0.01200\right)\end{aligned}\tag{522}$$

and iteratively upper bound the above in η.

We first upper bound

$$\left(-3.64165m-4.56116\right)m\eta+\left(-2.64141m^4+1.72676m^3-3.23630m^2-1.91812m-2.19375\right)\tag{523}$$

by fixing $\eta=0.2$, which simplifies (RU) to $-2.64141m^4+1.72676m^3-3.96463m^2-2.83035m-2.19375$.

Since

$$-2.64141\left(0.15\right)^4+1.72676\left(0.25\right)^3-3.96463\left(0.15\right)^2-2.83035\left(0.15\right)-2.19375$$

is negative, we can upper bound

$$\begin{aligned}&\left(-2.64141m^4+1.72676m^3-3.96463m^2-2.83035m-2.19375\right)\eta\\&+\left(-0.39199m^5-0.43049m^4-1.09446m^3-0.81376m^2+0.87846m-0.40435\right)\end{aligned}\tag{524}$$

by fixing $\eta=0.2$, which simplifies (RU) to $-0.39199m^5-0.95877m^4-0.74910m^3-1.60668m^2+0.31239m-0.8431$.

Since

$$-0.39199\left(0.15\right)^5-0.95877\left(0.15\right)^4-0.74910\left(0.15\right)^3-1.60668\left(0.15\right)^2+0.31239\left(0.25\right)-0.8431$$

is negative, we can upper bound

$$\begin{aligned}&\left(-0.39199m^5-0.95877m^4-0.74910m^3-1.60668m^2+0.31239m-0.8431\right)\eta\\&+\left(-0.37344m^5+0.14618m^4-0.32666m^3+0.06368m^2-0.06407m-0.01200\right)\end{aligned}\tag{525}$$

by fixing $\eta=0.2$, which simplifies (RU) to

$$-0.45183m^5-0.04557m^4-0.47648m^3-0.25765m^2-0.00159m-0.18062.\tag{526}$$

This completes the proof.

□

Proof of Lemma B.219. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first note that Equation 210 is convex in η. To see this, note the second partial η below.

$$2.00606\left(6\right)m\eta+\left(0.92873m^3-1.06079m^2+2.85120m+0.96485\right)\left(2\right)\tag{527}$$

Note that we lower bound in η by $\eta=0.2$. The resulting expression (RD) $1.85746m^3-2.12158m^2+8.10967m+1.9297$ can be lower bounded by

$$1.85746\left(0.15\right)^3-2.12158\left(0.25\right)^2+8.10967\left(0.15\right)+1.9297,$$

which is positive.

We then evaluate the first partial η of Equation 210 at $\eta = 0.2$ is positive. Note that the first partial η (RD),

$$0.46437m^4 + 0.48759m^3 + 0.47011m^2 + 0.94166m - 0.10325, \tag{528}$$

can be lower bounded by fixing $m = 0.15$, which is positive.

We then lower bound Equation 210 in η by $\eta = 0.2$. The resulting expression (RD),

$$0.29537m^4 - 0.16464m^3 + 0.37045m^2 - 0.08782m + 0.00835, \tag{529}$$

grows in m. To see this, note that the first partial m,

$$0.29537\,(4)\,m^3 - 0.16464\,(3)\,m^2 + 0.37045\,(2)\,m - 0.08782, \tag{530}$$

can be lower bounded by

$$((0.29537\,(4)\,(0.15) - 0.16464\,(3))\,(0.25) + 0.37045\,(2))\,(0.15) - 0.08782,$$

which is positive.

Then, we evaluate Equation 529 at $m = 0.15$, which is positive.

This completes the proof.

□

Proof of Lemma B.220. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we begin by noting that Equation 211 is concave in η. To see this, first note the second partial η,

$$-2.00603\,(6)\,m\eta + \left(-0.92871m^3 + 1.06084m^2 - 2.85115m - 0.96482\right)(2)\,, \tag{531}$$

that can be upper bounded by

$$-2.00603\,(6)\,(0.15)\,(0.2) + \left(-0.92871(0.15)^3 + 1.06084(0.25)^2 - 2.85115(0.15) - 0.96482\right)(2)\,,$$

which is negative.

We then show that the first partial η of Equation 211 at $\eta = 0.2$ is negative. Note that the first partial η (RU),

$$-0.46435m^4 - 0.48756m^3 - 0.47005m^2 - 0.94161m + 0.10328, \tag{532}$$

can be upper bounded by $m = 0.15$, which is negative.

Then, we upper bound Equation 211 in η by $\eta = 0.2$. Note that the resulting expression (RU) is decreasing in m. To see this, note that the first partial m,

$$-0.29537\,(4)\,m^3 + 0.16464\,(3)\,m^2 - 0.37044\,(2)\,m + 0.08782, \tag{533}$$

can be upper bounded by

$$((-0.29537\,(4)\,(0.15) + 0.16464\,(3))\,(0.25) - 0.37044\,(2))\,(0.15) + 0.08782,$$

which is negative.

So, we further upper bound Equation 211 in m by $m = 0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.221. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, consider the second partial derivative of Equation 212 with respect to q below (RU and scaled by $\exp(-mq)$).

$$
\begin{aligned}
&\left(4.12768m^4\eta + 0.51596m^4 + 0.04128m^3 + m^2\right) \\
&+ \left(z_{1,1} + z_{1,2}\sqrt{2m\eta + 0.25m + 0.02}\right) \exp\left(-\frac{1}{2}q\left(m + \sqrt{2m\eta + 0.25m + 0.02}\right)\right) \\
&+ \left(z_{2,1} + z_{2,2}\sqrt{2m\eta + 0.25m + 0.02}\right) \exp\left(-\frac{1}{2}q\left(m - \sqrt{2m\eta + 0.25m + 0.02}\right)\right),
\end{aligned} \tag{534}
$$

where

$$
\begin{aligned}
z_{1,1} = &\left(-0.125m^2\eta - 0.25m\eta^2 - 0.03125m\eta - 0.0025\eta\right) q^2 \\
&+ \Big(-1.54786m^3\eta^2 - 1.0319m^2\eta^3 - 0.19349m^3\eta - 0.25796m^2\eta^2 + 0.46840m^2\eta - 0.02061m\eta^2 \\
&+ 0.0625m^2 - 1.00257m\eta + 0.005m - 0.00010\eta\Big) q \\
&+ \Big(0.51595m^4\eta + 3.09575m^3\eta^2 + 0.06449m^4 + 0.77394m^3\eta - 6.19146m^2\eta^2 + 0.05352m^3 \\
&- 0.71201m^2\eta + 0.25772m^2 + 1.43811m\eta + 0.18781m - \eta + 0.015\Big),
\end{aligned}
$$

$$
\begin{aligned}
z_{1,2} = &0.25\eta m q^2 \\
&+ \left(0.25797m^3\eta + 1.54787m^2\eta^2 + 0.19349m^2\eta - 0.125m^2 - 0.23453m\eta - 0.03125m + \eta - 0.0025\right) q \\
&+ \Big(-1.54787m^3\eta - 1.03191m^2\eta^2 - 0.19349m^3 + 1.2899m^2\eta + 3.09576m\eta^2 - 0.03159m^2 \\
&+ 0.36635m\eta - 1.00256m + 0.03096\eta - 0.0001\Big),
\end{aligned}
$$

$$
\begin{aligned}
z_{2,1} = &\left(-0.125m^2\eta - 0.25m\eta^2 - 0.03125m\eta - 0.0025\eta\right) q^2 \\
&+ \Big(-1.54784m^3\eta^2 - 1.0319m^2\eta^3 - 0.19347m^3\eta - 0.25796m^2\eta^2 + 0.46842m^2\eta - 0.02061m\eta^2 \\
&+ 0.0625m^2 - 1.00257m\eta + 0.005m - 0.00010\eta\Big) q \\
&+ \Big(0.51596m^4\eta + 3.09576m^3\eta^2 + 0.0645m^4 + 0.77396m^3\eta - 6.19144m^2\eta^2 + 0.05354m^3 - 0.71197m^2\eta \\
&+ 0.25774m^2 + 1.43813m\eta + 0.18782m - \eta + 0.015\Big),
\end{aligned}
$$

and

$$z_{2,2} = -0.25\eta m q^2$$
$$+ \left(-0.25797m^3\eta - 1.54785m^2\eta^2 - 0.19347m^2\eta + 0.125m^2 + 0.23455m\eta + 0.03125m - \eta + 0.0025\right) q$$
$$+ \Bigg(1.54788m^3\eta + 1.03192m^2\eta^2 + 0.1935m^3 - 1.28987m^2\eta - 3.09573m\eta^2 + 0.03161m^2$$
$$- 0.3663m\eta + 1.00258m - 0.03093\eta + 0.00011\Bigg).$$

To proceed, we first show that $z_{1,2}$ is positive in lemma B.226 and $z_{2,2}$ is negative in lemma B.227.

We then upper bound $z_{1,2}\sqrt{2m\eta + 0.25m + 0.02}$ by $z_{1,2}\sqrt{2m(0.3) + 0.25m + 0.02}$ and $z_{2,2}\sqrt{2m\eta + 0.25m + 0.02}$ by $z_{2,2}\sqrt{2m(0.2) + 0.25m + 0.02}$.

Then, $z_{1,2}\sqrt{0.85m + 0.02}$ can be further upper bounded by linearizing $\sqrt{0.85m + 0.02}$ at $m = 0.25$. We can also upper bound $z_{2,2}\sqrt{0.65m + 0.02}$ by $z_{2,2}(0.84417m + 0.2161)$ (note lemma B.241).

We show the resulting upper bounds (RU) for $z_{1,1} + z_{1,2}\sqrt{2m\eta + 0.25m + 0.02}$ and $z_{2,1} + z_{2,2}\sqrt{2m\eta + 0.25m + 0.02}$ below.

$$\left(0.09536m^2 - 0.25m\eta + 0.03421m - 0.0025\right)\eta q^2$$
$$+ \Bigg(0.22738m^4\eta - 0.18355m^3\eta^2 - 1.0319m^2\eta^3 + 0.04461m^3\eta + 0.14734m^2\eta^2 - 0.11017m^3 + 0.31235m^2\eta$$
$$- 0.02061m\eta^2 + 0.00223m^2 - 0.18256m\eta - 0.00538m + 0.26174\eta - 0.00065\Bigg) q$$
$$+ \Bigg(-0.84835m^4\eta + 2.18622m^3\eta^2 - 0.10605m^4 + 1.50558m^3\eta - 3.73302m^2\eta^2 - 0.02498m^3$$
$$- 0.05135m^2\eta + 0.81060m\eta^2 - 0.63421m^2 + 1.56133m\eta - 0.07478m - 0.99189\eta + 0.01498\Bigg) \tag{535}$$

$$\left(-0.33604m^2 - 0.25m\eta - 0.08527m - 0.0025\right)\eta q^2$$
$$+ \Bigg(-0.21777m^4\eta - 2.85448m^3\eta^2 - 1.0319m^2\eta^3 - 0.41253m^3\eta - 0.59245m^2\eta^2 + 0.10553m^3$$
$$+ 0.62462m^2\eta - 0.02061m\eta^2 + 0.11590m^2 - 1.79605m\eta + 0.01387m - 0.2162\eta + 0.00054\Bigg) q$$
$$+ \Bigg(1.82264m^4\eta + 3.96688m^3\eta^2 + 0.22785m^4 + 0.01959m^3\eta - 8.58176m^2\eta^2 + 0.12204m^3$$
$$- 1.29993m^2\eta - 0.66898m\eta^2 + 1.11092m^2 + 1.33287m\eta + 0.40457m - 1.00668\eta + 0.01503\Bigg) \tag{536}$$

We then begin to upper bound Equation 535 by replacing $\left(0.09536m^2 - 0.25m\eta + 0.03421m - 0.0025\right)\eta q^2$ with $\left(0.09536m^2 - 0.25mn + 0.03421m - 0.0025\right)\left(0.9m^2 + 0.52 + 1.\right.$

(note that $(0.09536\,(0.25) - 0.25\,(0.2) + 0.03421)\,(0.25) - 0.0025$ is negative). Similarly, we also replace $\left(-0.33604m^2 - 0.25m\eta - 0.08527m - 0.0025\right)\eta q^2$ with $\left(-0.33604m^2 - 0.25mn - 0.08527m - 0.0025\right)\left(0.9m^2 + 0.52 + 1.944n\right)q$ to upper bound Equation 536.

We then show that the coefficients linear in q (RU) for our upper bounds of Equation 535 and 536 are positive and negative, respectively, in lemmas B.224 and B.225.

Then, the term linear in q for our upper bound of Equation 535 can be further upper bounded by fixing $q = \left(\frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15\right)$. Similarly, the term linear in q for our upper bound of Equation 536 can be further upper bounded by fixing $q = \left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)$.

We scale both terms by η and show the resulting upper bound for Equation 534 below (RU).

$$\left(4.12768m^4n^2 + 0.51596m^4n + 0.04128m^3n + m^2n\right) + w_1 \exp\left(-\frac{1}{2}q\left(m + \sqrt{2mn + 0.25m + 0.02}\right)\right) + w_2 \exp\left(-\frac{1}{2}q\right. \tag{537}$$

where

$$\begin{aligned} w_1 = &-2.21858m^2n^4 + \left(-0.96998m^3 + 1.79159m^2 - 3.91155m - 0.27861\right)mn^3 \\ &+ \left(-0.17253m^5 - 0.22098m^4 + 0.55343m^3 + 1.08950m^2 + 0.78915m - 0.43959\right)n^2 \\ &+ \left(0.21374m^6 - 0.16956m^5 + 0.65550m^4 - 0.51352m^3 - 0.04725m^2 - 0.16620m + 0.13409\right)n \\ &+ \left(0.08068m^6 - 0.07461m^5 + 0.08780m^4 - 0.02643m^3 + 0.02197m^2 + 0.00596m - 0.00093\right) \end{aligned} \tag{538}$$

and

$$\begin{aligned} w_2 = &-2.00601m^2n^4 + \left(-0.92871m^3 - 1.58222m^2 - 10.27007m - 1.65383\right)mn^3 \\ &+ \left(-2.56903m^5 + 0.86609m^4 - 3.16004m^3 - 1.66368m^2 - 2.99704m - 1.43642\right)n^2 \\ &+ \left(-0.19599m^6 - 0.57377m^5 - 0.49909m^4 - 2.03613m^3 + 0.77832m^2 - 0.74240m - 0.10139\right)n \\ &+ \left(-0.27218m^6 + 0.02592m^5 - 0.21224m^4 - 0.01245m^3 - 0.03245m^2 - 0.01584m - 0.00039\right). \end{aligned} \tag{539}$$

After showing that w_1 is positive in lemma B.222 and w_2 is negative in lemma B.223, we upper bound Equation 537 by replacing $\exp\left(-\frac{1}{2}q\left(m + \sqrt{2mn + 0.25m + 0.02}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(\frac{0.9}{0.3}\right)(0.15)^2 + \frac{0.52}{0.3} + 1.944\right)\left((0.15) + \sqrt{2(0.15)(0.2) + 0.25(0.15) + 0.02}\right)\Big)$ and $\exp\left(\frac{1}{2}q\left(\sqrt{2mn + 0.25m + 0.02} - m\right)\right)$ with $\exp\left(\frac{1}{2}\left(\frac{0.9}{0.3}m^2 + \frac{0.52}{0.3} + 1.944\right)\left(\sqrt{2(0.25)(0.2) + 0.25(0.25) + 0.02} - (0.25\right.\right.$ (note that $\sqrt{0.66m + 0.02} - m$ is locally decreasing in m).

Then, our upper bound for Equation 537 simplifies (RU) to

$$\begin{aligned} &\Bigg(\Bigg(\left(-3.67706m\eta + \left(-1.67963m^3 - 1.49269m^2 - 15.86557m - 2.41527\right)\right)m\eta \\ &+ \left(-3.64841m^5 + 5.24672m^4 - 4.18342m^3 - 1.88525m^2 - 3.86260m - 2.17630\right)\Bigg)\eta \\ &+ \left(-0.18815m^6 - 0.86691m^5 + 0.08103m^4 - 3.00008m^3 + 2.06578m^2 - 1.10056m - 0.08798\right)\Bigg)\eta \\ &+ \left(-0.34720m^6 + 0.00647m^5 - 0.26085m^4 - 0.02785m^3 - 0.03648m^2 - 0.01970m - 0.00091\right). \end{aligned} \tag{540}$$

We iteratively upper the above as follows. First, we upper bound

$$\left(-3.67706m\eta + \left(-1.67963m^3 - 1.49269m^2 - 15.86557m - 2.41527\right)\right)m\eta \tag{541}$$

by

$$\left(-3.67706m\,(0.2) + \left(-1.67963m^3 - 1.49269m^2 - 15.86557m - 2.41527\right)\right)m\,(0.2)\,, \tag{542}$$

which simplifies (RU) to $-3.64841m^5 + 4.91080m^4 - 4.48195m^3 - 5.20544m^2 - 4.34565m - 2.1763$.

Since

$$-3.64841\,(0.15)^5+4.91080\,(0.25)^4-4.48195\,(0.15)^3-5.20544\,(0.15)^2-4.34565\,(0.15)-2.1763$$

is negative, we can upper bound

$$\begin{aligned}&\left(-3.64841m^5 + 4.91080m^4 - 4.48195m^3 - 5.20544m^2 - 4.34565m - 2.1763\right)\eta\\&+\left(-0.18815m^6 - 0.86691m^5 + 0.08103m^4 - 3.00008m^3 + 2.06578m^2 - 1.10056m - 0.08798\right)\end{aligned} \tag{543}$$

by

$$\begin{aligned}&\left(-3.64841m^5 + 4.91080m^4 - 4.48195m^3 - 5.20544m^2 - 4.34565m - 2.1763\right)(0.2)\\&+\left(-0.18815m^6 - 0.86691m^5 + 0.08103m^4 - 3.00008m^3 + 2.06578m^2 - 1.10056m - 0.08798\right),\end{aligned} \tag{544}$$

which simplifies (RU) to

$$-0.18815m^6-1.59659m^5+1.06319m^4-3.89647m^3+1.02470m^2-1.96969m-0.52324. \tag{545}$$

Since

$$-0.18815\,(0.15)^6-1.59659\,(0.15)^5+1.06319\,(0.25)^4-3.89647\,(0.15)^3+1.02470\,(0.25)^2-1.96969\,(0.15)-0.52324$$

is negative, we can upper bound

$$\begin{aligned}&\left(-0.18815m^6 - 1.59659m^5 + 1.06319m^4 - 3.89647m^3 + 1.02470m^2 - 1.96969m - 0.52324\right)\eta\\&+\left(-0.34720m^6 + 0.00647m^5 - 0.26085m^4 - 0.02785m^3 - 0.03648m^2 - 0.01970m - 0.00091\right)\end{aligned} \tag{546}$$

by

$$\begin{aligned}&\left(-0.18815m^6 - 1.59659m^5 + 1.06319m^4 - 3.89647m^3 + 1.02470m^2 - 1.96969m - 0.52324\right)(0.2)\\&+\left(-0.34720m^6 + 0.00647m^5 - 0.26085m^4 - 0.02785m^3 - 0.03648m^2 - 0.01970m - 0.00091\right),\end{aligned} \tag{547}$$

which simplifies (RU) to

$$-0.38483m^6-0.31284m^5-0.04821m^4-0.80714m^3+0.16846m^2-0.41363m-0.10555. \tag{548}$$

Last, we upper bound the above by

$$-0.38483\,(0.15)^6-0.31284\,(0.15)^5-0.04821\,(0.15)^4-0.80714\,(0.15)^3+0.16846\,(0.25)^2-0.41363\,(0.15)-0.10555,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.222. Before we start the proof of the lemma, note that we round terms down/up after five digits of precision for simplicity, which we will accompany with RD/RU.

To prove the lemma, we start by first noting that Equation 217 is concave in η. To see this, consider the second partial η below.

$$\begin{aligned}&-2.21858\,(12)\,m^2\eta^2+\left(-0.96998m^3+1.79159m^2-3.91155m-0.27861\right)(6)\,m\eta\\&+\left(-0.17253m^5-0.22098m^4+0.55343m^3+1.08950m^2+0.78915m-0.43959\right)(2)\end{aligned} \tag{549}$$

Since

$$-0.96998\,(0.15)^3+1.79159\,(0.25)^2-3.91155\,(0.15)-0.27861$$

is negative, we can upper bound the second partial in η by fixing $\eta = 0.2$, which simplifies (RU) to

$$-0.34506m^5-1.60593m^4+3.25677m^3-3.57977m^2+1.24397m-0.87918. \tag{550}$$

The above can be upper bounded by

$$-0.34506\,(0.15)^5-1.60593\,(0.15)^4+3.25677\,(0.25)^3-3.57977\,(0.15)^2+1.24397\,(0.25)-0.87918,$$

which is negative.

So, we consider the following two cases.

Case 1: $\eta = 0.2$. In this case, the resulting expression simplifies (RD) to

$$0.12342m^6-0.11543m^5+0.20230m^4-0.09267m^3+0.02125m^2+0.00205m+0.00830. \quad (551)$$

The above can be lower bounded by

$$0.12342\,(0.15)^6 - 0.11543\,(0.25)^5 + 0.20230\,(0.15)^4 - 0.09267\,(0.25)^3 + 0.02125\,(0.15)^2 + 0.00205\,(0.15) + 0.00830,$$

which is positive.

Case 2: $\eta = 0.3$. The resulting expression can be simplified (RD) to

$$0.14480m^6-0.14101m^5+0.23837m^4-0.08231m^3-0.01774m^2+0.01960m-0.00027. \quad (552)$$

The above can be lower bounded by

$$0.14480\,(0.15)^6-0.14101\,(0.25)^5+0.23837\,(0.15)^4-0.08231\,(0.25)^3-0.01774\,(0.25)^2+0.01960\,(0.15)-0.00027,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.223. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we start by rewriting Equation 218 as follows.

$$\Big(\Big(\big(-2.00601m\eta + \big(-0.92871m^3 - 1.58222m^2 - 10.27007m - 1.65383\big)\big)\,mx + \big(-2.56903m^5 + 0.86609m^4 - 3.16004m^3 - 1.66368m^2 - 2.99704m - 1.43642\big)\Big)x + \big(-0.19599m^6 - 0.57377m^5 - 0.49909m^4 - 2.03613m^3 + 0.77832m^2 - 0.74240m - 0.10139\big)\Big)x + \big(-0.27218m^6 + 0.02592m^5 - 0.21224m^4 - 0.01245m^3 - 0.03245m^2 - 0.01584m - 0.00039\big) \quad (553)$$

First, we upper bound

$$-2.00601m\eta + \big(-0.92871m^3 - 1.58222m^2 - 10.27007m - 1.65383\big) \quad (554)$$

by fixing $\eta = 0.2$, which simplifies (RU) to $-0.92871m^3 - 1.58222m^2 - 10.67127m - 1.65383$.

We can the upper bound

$$\big(-0.92871m^3 - 1.58222m^2 - 10.67127m - 1.65383\big)\,m\eta + \big(-2.56903m^5 + 0.86609m^4 - 3.16004m^3 - 1.66368m^2 - 2.99704m - 1.43642\big) \quad (555)$$

by fixing $\eta = 0.2$, which simplifies (RU) to

$$-2.56903m^5 + 0.68035m^4 - 3.47648m^3 - 3.79793m^2 - 3.32780m - 1.43642. \quad (556)$$

Since

$$-2.56903\,(0.15)^5+0.68035\,(0.25)^4-3.47648\,(0.15)^3-3.79793\,(0.15)^2-3.32780\,(0.15)-1.43642$$

is negative, we can upper bound

$$\begin{aligned}&\left(-2.56903m^5+0.68035m^4-3.47648m^3-3.79793m^2-3.32780m-1.43642\right)\eta\\&+\left(-0.19599m^6-0.57377m^5-0.49909m^4-2.03613m^3+0.77832m^2-0.74240m-0.10139\right)\end{aligned}\tag{557}$$

in η by fixing $\eta = 0.2$, which simplifies (RU) to

$$-0.19599m^6-1.08757m^5-0.36302m^4-2.73142m^3+0.01874m^2-1.40796m-0.38867.\tag{558}$$

Since

$$-0.19599\left(0.15\right)^6-1.08757\left(0.15\right)^5-0.36302\left(0.15\right)^4-2.73142\left(0.15\right)^3+0.01874\left(0.25\right)^2-1.40796\left(0.15\right)-0.38867$$

is negative, we can upper bound

$$\begin{aligned}&\left(-0.19599m^6-1.08757m^5-0.36302m^4-2.73142m^3+0.01874m^2-1.40796m-0.38867\right)\eta\\&+\left(-0.27218m^6+0.02592m^5-0.21224m^4-0.01245m^3-0.03245m^2-0.01584m-0.00039\right)\end{aligned}\tag{559}$$

by fixing $\eta = 0.2$, which simplifies (RU) to

$$-0.31137m^6-0.19159m^5-0.28484m^4-0.55873m^3-0.02870m^2-0.29743m-0.07812.\tag{560}$$

This completes the proof.

□

Proof of Lemma B.224. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, first note that

$$-1.0319m^2\eta^3+\left(-0.18355m^2+0.14734m-0.50661\right)m\eta^2\tag{561}$$

can be lower bounded by

$$\left(-1.0319m\left(0.3\right)+\left(-0.18355m^2+0.14734m-0.50661\right)\right)m\eta^2,\tag{562}$$

which simplifies (RD) to $\left(-0.18355m^2-0.16223m-0.50661\right)m\eta^2$.

Then,

$$\begin{aligned}&\left(-0.18355m^2-0.16223m-0.50661\right)m\eta^2\\&+\left(0.22738m^4-0.18039m^3+0.49773m^2-0.24605m+0.25688\right)\eta\end{aligned}\tag{563}$$

can be lower bounded by

$$\begin{aligned}&\Big(\left(-0.18355m^2-0.16223m-0.50661\right)m\left(0.3\right)\\&+\left(0.22738m^4-0.18039m^3+0.49773m^2-0.24605m+0.25688\right)\Big)\eta,\end{aligned}\tag{564}$$

which simplifies (RD) to

$$\left(0.22738m^4-0.23546m^3+0.44906m^2-0.39804m+0.25688\right)\eta.\tag{565}$$

Since

$$0.22738\left(0.15\right)^4-0.23546\left(0.25\right)^3+0.44906\left(0.15\right)^2-0.39804\left(0.25\right)+0.25688$$

is positive, we can lower bound

$$\begin{aligned}&\left(0.22738m^4-0.23546m^3+0.44906m^2-0.39804m+0.25688\right)\eta\\&+\left(0.08583m^4-0.07938m^3+0.04957m^2+0.01241m-0.00195\right)\end{aligned}\tag{566}$$

by fixing $\eta = 0.2$, which simplifies (RD) to

$$0.13130m^4 - 0.12648m^3 + 0.13938m^2 - 0.06720m + 0.04942. \tag{567}$$

Note that the above can be lower bounded by

$$0.13130\,(0.15)^4 - 0.12648\,(0.25)^3 + 0.13938\,(0.15)^2 - 0.06720\,(0.25) + 0.04942,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.225. To prove the lemma, first note that we can upper bound Equation 220 in η by $\eta = 0.2$. This yields the following expression (RU)

$$-0.34598m^4 - 0.21289m^3 - 0.09877m^2 - 0.46909m - 0.04497. \tag{568}$$

This completes the proof. □

Proof of Lemma B.226. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we first lower bound

$$\left(0.25nmq + \left(0.25797m^3n + 1.54787m^2n^2 + 0.19349m^2n - 0.125m^2 - 0.23453mn - 0.03125m + n - 0.0025\right)\right) q \tag{569}$$

by

$$\Bigg(0.25m\left(0.9m^2 + 0.52 + 1.944n\right) + \left(0.25797m^3n + 1.54787m^2n^2 + 0.19349m^2n - 0.125m^2 - 0.23453mn - 0.03125m + n - 0.0025\right)\Bigg)q, \tag{570}$$

which simplifies (RD) to

$$\left(0.25797m^3n + 1.54787m^2n^2 + 0.225m^3 + 0.19349m^2n - 0.125m^2 + 0.25147mn + 0.09875m + n - 0.0025\right)x. \tag{571}$$

Since

$$1.54787\,(0.15)^2\,(0.2)^2 + \left(0.25797\,(0.15)^3 + 0.19349\,(0.15)^2 + 0.25147\,(0.15) + 1\right)(0.2) + \left(0.225\,(0.15)^3 - 0.125\,(0.25)^2 + 0.09875\,(0.15) - 0.0025\right)$$

is negative, we can lower bound

$$\left(1.54787m^2n^2 + \left(0.25797m^3 + 0.19349m^2 + 0.25147m + 1\right)n + \left(0.225m^3 - 0.125m^2 + 0.09875m - 0.0025\right)\right)q + \Bigg(-1.54787m^3n - 1.03191m^2n^2 - 0.19349m^3 + 1.2899m^2n + 3.09576mn^2 - 0.03159m^2 + 0.36635mn - 1.00256m + 0.03096n - 0.0001\Bigg) \tag{572}$$

by

$$\Big(1.54787m^2n^2+\left(0.25797m^3+0.19349m^2+0.25147m+1\right)n \\ +\left(0.225m^3-0.125m^2+0.09875m-0.0025\right)\Big)\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right) \\ +\Big(-1.54787m^3n-1.03191m^2n^2-0.19349m^3+1.2899m^2n+3.09576mn^2-0.03159m^2 \\ +0.36635mn-1.00256m+0.03096n-0.0001\Big), \tag{573}$$

which simplifies (RD) (after scaling by η) to

$$\Big(\Big(\Big(1.97714m+3.09576\Big)m\eta \\ +\left(1.39308m^4-1.04638m^3+2.47093m^2+0.85520m+1.97496\right)\Big)\eta \\ +\left(0.23217m^5+0.17414m^4+0.60437m^3+0.72602m^2-0.67983m+0.51504\right)\Big)\eta \\ +\left(0.2025m^5-0.1125m^4+0.20587m^3-0.06725m^2+0.05135m-0.0013\right). \tag{574}$$

We then iteratively lower bound the above as follows. First, we lower bound

$$\left(1.97714m+3.09576\right)m\eta+\left(1.39308m^4-1.04638m^3+2.47093m^2+0.85520m+1.97496\right) \tag{575}$$

in η by fixing $\eta = 0.2$, which simplifies (RD) to $1.39308m^4-1.04638m^3+2.86635m^2+1.47435m+1.97496$.

Since

$$1.39308\left(0.15\right)^4-1.04638\left(0.25\right)^3+2.86635\left(0.15\right)^2+1.47435\left(0.15\right)+1.97496$$

is positive, we can lower bound

$$\left(1.39308m^4-1.04638m^3+2.86635m^2+1.47435m+1.97496\right)\eta \\ +\left(0.23217m^5+0.17414m^4+0.60437m^3+0.72602m^2-0.67983m+0.51504\right) \tag{576}$$

in η by fixing $\eta = 0.2$, which simplifies (RD) to

$$0.23217m^5+0.45275m^4+0.39509m^3+1.29929m^2-0.38496m+0.91003. \tag{577}$$

Since

$$0.23217\left(0.15\right)^5+0.45275\left(0.15\right)^4+0.39509\left(0.15\right)^3+1.29929\left(0.15\right)^2-0.38496\left(0.25\right)+0.91003$$

is positive, we can lower bound

$$\left(0.23217m^5+0.45275m^4+0.39509m^3+1.29929m^2-0.38496m+0.91003\right)\eta \\ +\left(0.2025m^5-0.1125m^4+0.20587m^3-0.06725m^2+0.05135m-0.0013\right) \tag{578}$$

in η by fixing $\eta = 0.2$, which simplifies (RD) to

$$0.24893m^5-0.02195m^4+0.28488m^3+0.19260m^2-0.02565m+0.18070. \tag{579}$$

The above can be lower bounded by

$$0.24893\left(0.15\right)^5-0.02195\left(0.15\right)^4+0.28488\left(0.15\right)^3+0.19260\left(0.15\right)^2-0.02565\left(0.25\right)+0.18070,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.227. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To proceed, we first upper bound

$$\Big(-0.25nmq + \left(-0.25797m^3n - 1.54785m^2n^2 - 0.19347m^2n + 0.125m^2 + 0.23455mn + 0.03125m - n + 0.0025\right)\Big)q \tag{580}$$

by

$$\Big(-0.25m\left(0.9m^2 + 0.52 + 1.944n\right) + \left(-0.25797m^3n - 1.54785m^2n^2 - 0.19347m^2n + 0.125m^2 + 0.23455mn + 0.03125m - n + 0.0025\right)\Big)q, \tag{581}$$

which simplifies (RU) to

$$\left(-0.25797m^3n - 1.54785m^2n^2 - 0.225m^3 - 0.19347m^2n + 0.125m^2 - 0.25145mn - 0.09875m - n + 0.0025\right)q. \tag{582}$$

Since

$$-1.54785\left(0.15\right)^2\left(0.2\right)^2 + \Big(-0.25797\left(0.15\right)^3 - 0.19347\left(0.15\right)^2 - 0.25145\left(0.15\right) - 1\Big)\left(0.2\right) + \Big(-0.225\left(0.15\right)^3 + 0.125\left(0.25\right)^2 - 0.09875\left(0.15\right) + 0.0025\Big)$$

is negative, we upper bound Equation 582 by

$$\Big(-1.54785m^2n^2 + \left(-0.25797m^3 - 0.19347m^2 - 0.25145m - 1\right)n + \left(-0.225m^3 + 0.125m^2 - 0.09875m + 0.0025\right)\Big)\left(\frac{0.9}{n}m^2 + \frac{0.52}{n} + 1.944\right). \tag{583}$$

Then, Equation 222 can be upper bounded by (after scaling by η) by

$$\Big(\left(\left(-1.9771m - 3.09573\right)m\eta + \left(-1.39306m^4 + 1.04639m^3 - 2.47085m^2 - 0.85512m - 1.97493\right)\right)\eta + \left(-0.23217m^5 - 0.17412m^4 - 0.60434m^3 - 0.72599m^2 + 0.67986m - 0.51503\right)\Big)\eta + \left(-0.2025m^5 + 0.1125m^4 - 0.20587m^3 + 0.06725m^2 - 0.05135m + 0.0013\right). \tag{584}$$

We can further upper bound the above by first upper bounding

$$\left(-1.9771m - 3.09573\right)m\eta + \left(-1.39306m^4 + 1.04639m^3 - 2.47085m^2 - 0.85512m - 1.97493\right) \tag{585}$$

by

$$\left(-1.9771m - 3.09573\right)m\left(0.2\right) + \left(-1.39306m^4 + 1.04639m^3 - 2.47085m^2 - 0.85512m - 1.97493\right), \tag{586}$$

which simplifies (RU) to $-1.39306m^4 + 1.04639m^3 - 2.86627m^2 - 1.47426m - 1.97493$.

Since

$$-1.39306\left(0.15\right)^4 + 1.04639\left(0.25\right)^3 - 2.86627\left(0.15\right)^2 - 1.47426\left(0.15\right) - 1.97493$$

is negative, we can upper bound

$$\left(-1.39306m^4 + 1.04639m^3 - 2.86627m^2 - 1.47426m - 1.97493\right)\eta + \left(-0.23217m^5 - 0.17412m^4 - 0.60434m^3 - 0.72599m^2 + 0.67986m - 0.51503\right) \tag{587}$$

by

$$\begin{aligned}&\left(-1.39306m^4+1.04639m^3-2.86627m^2-1.47426m-1.97493\right)(0.2)\\&+\left(-0.23217m^5-0.17412m^4-0.60434m^3-0.72599m^2+0.67986m-0.51503\right),\end{aligned}\tag{588}$$

which simplifies (RU) to $-0.23217m^5-0.45273m^4-0.39506m^3-1.29924m^2+0.38501m-0.91001$.

Since

$$-0.23217\,(0.15)^5-0.45273\,(0.15)^4-0.39506\,(0.15)^3-1.29924\,(0.15)^2+0.38501\,(0.2)-0.91001$$

is negative, then

$$\begin{aligned}&\left(-0.23217m^5-0.45273m^4-0.39506m^3-1.29924m^2+0.38501m-0.91001\right)\eta\\&+\left(-0.2025m^5+0.1125m^4-0.20587m^3+0.06725m^2-0.05135m+0.0013\right)\end{aligned}\tag{589}$$

can be upper bounded by

$$\begin{aligned}&\left(-0.23217m^5-0.45273m^4-0.39506m^3-1.29924m^2+0.38501m-0.91001\right)(0.2)\\&+\left(-0.2025m^5+0.1125m^4-0.20587m^3+0.06725m^2-0.05135m+0.0013\right),\end{aligned}\tag{590}$$

which simplifies (RU) to

$$-0.24893m^5+0.02196m^4-0.28488m^3-0.19259m^2+0.02566m-0.18070.\tag{591}$$

The above can be upper bounded by

$$-0.24893\,(0.15)^5+0.02196\,(0.25)^4-0.28488\,(0.15)^3-0.19259\,(0.15)^2+0.02566\,(0.25)-0.18070,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.228. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU. To proceed, we first scale Equation 223 by η and factor out $\exp\left(m\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)\right)$.

The resulting expression simplifies (RU) to the following.

$$\begin{aligned}&\left(4.12768m^3n^2+0.51596m^3n+0.05160m^2n+mn\right)\\&+\left(h_{1,1}\sqrt{2mn+0.25m+0.025}+h_{1,2}\right)\exp\left(-\frac{1}{2}\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)\left(m+\sqrt{2mn+0.25m+0.025}\right)\right)\\&+\left(h_{2,1}+h_{2,2}\sqrt{2mn+0.25m+0.025}\right)\exp\left(-\frac{1}{2}\left(\frac{0.9}{n}m^2+\frac{0.52}{n}+1.944\right)\left(m-\sqrt{2mn+0.25m+0.025}\right)\right),\end{aligned}\tag{592}$$

where

$$\begin{aligned}h_{1,1}=&0.46437m^4n+0.92873m^3n^2+0.2025m^4+0.11610m^3n\\&-1.06079m^2n^2+2.00606mn^3-0.225m^3+0.89675m^2n\\&+2.85120mn^2+0.234m^2-0.44470mn+0.96987n^2-0.13m-0.48785n+0.0676,\end{aligned}\tag{593}$$

$$\begin{aligned}h_{1,2}=&-1.85743m^4n^2-0.2025m^5-0.23216m^4n+1.03192m^3n^2-1.94822m^2n^3-0.31901m^3n-1.05870m^2n^2\\&-4.12766mn^3-0.17775m^3-0.98898m^2n-0.48724mn^2+0.00563m^2+0.36912mn\\&-1.99558n^2-0.0351m-0.50752n+0.00325,\end{aligned}\tag{594}$$

$$h_{2,1} = -1.85743m^4n^2 - 0.2025m^5 - 0.23216m^4n + 1.03192m^3n^2 - 1.94822m^2n^3 - 0.31901m^3n - 1.05870m^2n^2 - 4.12766mn^3 - 0.17775m^3 - 0.98898m^2n - 0.48724mn^2 + 0.00563m^2 + 0.36912mn - 1.99558n^2 - 0.0351m - 0.50752n + 0.00325, \tag{595}$$

and

$$h_{2,2} = -0.46435m^4n - 0.92871m^3n^2 - 0.2025m^4 - 0.11608m^3n + 1.06084m^2n^2 - 2.00603mn^3 + 0.225m^3 - 0.89671m^2n - 2.85115mn^2 - 0.234m^2 + 0.44473mn - 0.96984n^2 + 0.13m + 0.48786n - 0.0676. \tag{596}$$

To proceed, we begin by showing that $h_{1,1}$ is positive and $h_{2,2}$ is negative in lemmas B.231 and B.232.

Then, we can upper bound $h_{1,1}\sqrt{2mn + 0.25m + 0.025}$ by $h_{1,1}\sqrt{2m(0.3) + 0.25m + 0.025}$ and upper bound $h_{2,2}\sqrt{2mn + 0.25m + 0.025}$ by $h_{2,2}\sqrt{2m(0.2) + 0.25m + 0.025}$.

We further upper bound $h_{1,1}\sqrt{0.85m + 0.025}$ by linearizing $\sqrt{0.85m + 0.025}$ at $m = 0.25$ and upper bound $h_{2,2}\sqrt{0.65m + 0.025}$ by $h_{2,2}(0.83012m + 0.22548)$ (note lemma **??**).

Then, our upper bound for $h_{1,1}\sqrt{2mn + 0.25m + 0.025} + h_{1,2}$ simplifies (RU) to

$$\left(-0.19875m^2 - 3.58738m\right)x^3 + \left(-1.04749m^4 + 0.35695m^3 + 1.14212m^2 + 1.12646m - 1.73437\right)\eta^2 + \left(0.40498m^5 - 0.00584m^4 + 0.49431m^3 - 1.13528m^2 - 0.17609m - 0.63890\right)\eta + \left(-0.02590m^5 - 0.14168m^4 - 0.03427m^3 - 0.04472m^2 - 0.01115m + 0.02146\right). \tag{597}$$

Furthermore, our upper bound for $h_{2,1} + h_{2,2}\sqrt{2mn + 0.25m + 0.025}$ simplifies (RU) to

$$\left(-3.61346m^2 - 4.57998m\right)x^3 + \left(-2.62837m^4 + 1.70314m^3 - 3.18629m^2 - 1.93520m - 2.21426\right)\eta^2 + \left(-0.38546m^5 - 0.43322m^4 - 1.08956m^3 - 0.82199m^2 + 0.87438m - 0.39751\right)\eta + \left(-0.37059m^5 + 0.14112m^4 - 0.32126m^3 + 0.06079m^2 - 0.06190m - 0.01199\right). \tag{598}$$

We then show that Equations 597 and 598 are negative in lemmas B.229 and B.230.

To further upper bound Equation 592, we replace $\exp\left(-\frac{1}{2}\left(\frac{0.9}{x}m^2 + \frac{0.52}{x} + 1.944\right)\left(m + \sqrt{2mx + 0.25m + 0.025}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(\frac{0.9}{0.2}(0.25)^2 + \frac{0.52}{0.2} + 1.944\right)\left((0.25) + \sqrt{2(0.25)(0.3) + 0.25(0.25) + 0.025}\right)\right)$ and replace $\exp\left(-\frac{1}{2}\left(\frac{0.9}{x}m^2 + \frac{0.52}{x} + 1.944\right)\left(m - \sqrt{2mx + 0.25m + 0.025}\right)\right)$ with $\exp\left(\frac{1}{2}\left(\frac{0.9}{0.3}(0.15)^2 + \frac{0.52}{0.3} + 1.944\right)\left(\sqrt{2(0.25)(0.2) + 0.25(0.25) + 0.025} - (0.25)\right)\right)$ (note that $\sqrt{0.65m + 0.025} - m$ is locally decreasing in m).

The resulting expression simplifies (RU) to the following.

$$\left(-5.12386m^2 - 7.05744m\right)\eta^3 + \left(-3.87943m^4 + 6.58717m^3 - 4.29575m^2 - 2.53597m - 3.41202\right)\eta^2 + \left(-0.474637m^5 - 0.611267m^4 - 0.93547m^3 - 1.297992m^2 + 2.202022m - 0.667829\right)\eta + \left(-0.52642m^5 + 0.17488m^4 - 0.45834m^3 + 0.07809m^2 - 0.08908m - 0.01326\right) \tag{599}$$

Note that we rewrite the above as follows and iteratively upper bound the expression in η.

$$\Big(\left(\left(-5.12386m^2 - 7.05744m\right)\eta + \left(-3.87943m^4 + 6.58717m^3 - 4.29575m^2 - 2.53597m - 3.41202\right)\right)\eta + \left(-0.474637m^5 - 0.611267m^4 - 0.93547m^3 - 1.297992m^2 + 2.202022m - 0.667829\right)\Big)\eta + \left(-0.52642m^5 + 0.17488m^4 - 0.45834m^3 + 0.07809m^2 - 0.08908m - 0.01326\right) \tag{600}$$

First, we upper bound

$$\left(-5.12386m^2 - 7.05744m\right)\eta + \left(-3.87943m^4 + 6.58717m^3 - 4.29575m^2 - 2.53597m - 3.41202\right) \tag{601}$$

by

$$\left(-5.12386m^2 - 7.05744m\right)(0.2) + \left(-3.87943m^4 + 6.58717m^3 - 4.29575m^2 - 2.53597m - 3.41202\right), \tag{602}$$

which simplifies (RU) to $-3.87943m^4 + 6.58717m^3 - 5.32052m^2 - 3.94745m - 3.41202$.

Since

$$-3.87943\,(0.15)^4 + 6.58717\,(0.25)^3 - 5.32052\,(0.15)^2 - 3.94745\,(0.15) - 3.41202$$

is negative, we upper bound

$$\begin{aligned}&\left(-3.87943m^4 + 6.58717m^3 - 5.32052m^2 - 3.94745m - 3.41202\right)(0.2)\\ &+ \left(-0.474637m^5 - 0.611267m^4 - 0.93547m^3 - 1.297992m^2 + 2.202022m - 0.667829\right),\end{aligned} \tag{603}$$

which simplifies (RU) to

$$-0.47463m^5 - 1.38715m^4 + 0.38197m^3 - 2.36209m^2 + 1.41254m - 1.35023. \tag{604}$$

Since

$$-0.47463\,(0.15)^5 - 1.38715\,(0.15)^4 + 0.38197\,(0.25)^3 - 2.36209\,(0.15)^2 + 1.41254\,(0.25) - 1.35023$$

is negative, we can upper bound

$$\begin{aligned}&\left(-0.47463m^5 - 1.38715m^4 + 0.38197m^3 - 2.36209m^2 + 1.41254m - 1.35023\right)\eta\\ &+ \left(-0.52642m^5 + 0.17488m^4 - 0.45834m^3 + 0.07809m^2 - 0.08908m - 0.01326\right)\end{aligned} \tag{605}$$

by

$$\begin{aligned}&\left(-0.47463m^5 - 1.38715m^4 + 0.38197m^3 - 2.36209m^2 + 1.41254m - 1.35023\right)(0.2)\\ &+ \left(-0.52642m^5 + 0.17488m^4 - 0.45834m^3 + 0.07809m^2 - 0.08908m - 0.01326\right),\end{aligned} \tag{606}$$

which simplifies (RU) to

$$-0.62134m^5 - 0.10255m^4 - 0.38194m^3 - 0.39432m^2 + 0.19343m - 0.28330. \tag{607}$$

Last, we upper bound the above by

$$-0.62134\,(0.15)^5 - 0.10255\,(0.15)^4 - 0.38194\,(0.15)^3 - 0.39432\,(0.15)^2 + 0.19343\,(0.25) - 0.28330,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.229. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To begin the proof, note that we rewrite Equation 228 as follows and iteratively upper bound it.

$$\begin{aligned}&\Big(\left(\left(-0.19875m^2 - 3.58738m\right)\eta + \left(-1.04749m^4 + 0.35695m^3 + 1.14212m^2 + 1.12646m - 1.73437\right)\right)\eta\\ &+ \left(0.40498m^5 - 0.00584m^4 + 0.49431m^3 - 1.13528m^2 - 0.17609m - 0.63890\right)\Big)\eta\\ &+ \left(-0.02590m^5 - 0.14168m^4 - 0.03427m^3 - 0.04472m^2 - 0.01115m + 0.02146\right)\end{aligned} \tag{608}$$

First, we upper bound

$$\left(-0.19875m^2 - 3.58738m\right)\eta + \left(-1.04749m^4 + 0.35695m^3 + 1.14212m^2 + 1.12646m - 1.73437\right) \tag{609}$$

by

$$\left(-0.19875m^2 - 3.58738m\right)(0.2) + \left(-1.04749m^4 + 0.35695m^3 + 1.14212m^2 + 1.12646m - 1.73437\right), \tag{610}$$

which simplifies (RU) to $-1.04749m^4 + 0.35695m^3 + 1.10237m^2 + 0.40899m - 1.73437$.

Since

$$-1.04749\,(0.15)^4 + 0.35695\,(0.25)^3 + 1.10237\,(0.25)^2 + 0.40899\,(0.25) - 1.73437$$

is negative, we upper bound

$$\begin{aligned}&\left(-1.04749m^4 + 0.35695m^3 + 1.10237m^2 + 0.40899m - 1.73437\right)\eta \\ &+ \left(0.40498m^5 - 0.00584m^4 + 0.49431m^3 - 1.13528m^2 - 0.17609m - 0.63890\right)\end{aligned} \tag{611}$$

by

$$\begin{aligned}&\left(-1.04749m^4 + 0.35695m^3 + 1.10237m^2 + 0.40899m - 1.73437\right)(0.2) \\ &+ \left(0.40498m^5 - 0.00584m^4 + 0.49431m^3 - 1.13528m^2 - 0.17609m - 0.63890\right),\end{aligned} \tag{612}$$

which simplifies (RU) to $0.40498m^5 - 0.21533m^4 + 0.5657m^3 - 0.91480m^2 - 0.09429m - 0.98577$.

Since

$$0.40498\,(0.25)^5 - 0.21533\,(0.15)^4 + 0.5657\,(0.25)^3 - 0.91480\,(0.15)^2 - 0.09429\,(0.15) - 0.98577$$

is negative, we upper bound

$$\begin{aligned}&\left(0.40498m^5 - 0.21533m^4 + 0.5657m^3 - 0.91480m^2 - 0.09429m - 0.98577\right)\eta \\ &+ \left(-0.02590m^5 - 0.14168m^4 - 0.03427m^3 - 0.04472m^2 - 0.01115m + 0.02146\right)\end{aligned} \tag{613}$$

by

$$\begin{aligned}&\left(0.40498m^5 - 0.21533m^4 + 0.5657m^3 - 0.91480m^2 - 0.09429m - 0.98577\right)(0.2) \\ &+ \left(-0.02590m^5 - 0.14168m^4 - 0.03427m^3 - 0.04472m^2 - 0.01115m + 0.02146\right),\end{aligned} \tag{614}$$

which simplifies (RU) to

$$0.05510m^5 - 0.18474m^4 + 0.07887m^3 - 0.22768m^2 - 0.03000m - 0.17569. \tag{615}$$

Note that the above can be upper bounded by

$$0.05510\,(0.25)^5 - 0.18474\,(0.15)^4 + 0.07887\,(0.25)^3 - 0.22768\,(0.15)^2 - 0.03000\,(0.15) - 0.17569,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.230. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To proceed, we first rewrite Equation 229 as follows and iteratively upper bound it in η.

$$\begin{aligned}&\Big(\left(\left(-3.61346m^2 - 4.57998m\right)\eta + \left(-2.62837m^4 + 1.70314m^3 - 3.18629m^2 - 1.93520m - 2.21426\right)\right)\eta \\ &+ \left(-0.38546m^5 - 0.43322m^4 - 1.08956m^3 - 0.82199m^2 + 0.87438m - 0.39751\right)\Big)\eta \\ &+ \left(-0.37059m^5 + 0.14112m^4 - 0.32126m^3 + 0.06079m^2 - 0.06190m - 0.01199\right)\end{aligned} \tag{616}$$

We first upper bound

$$\left(-3.61346m^2 - 4.57998m\right)\eta + \left(-2.62837m^4 + 1.70314m^3 - 3.18629m^2 - 1.93520m - 2.21426\right) \tag{617}$$

by

$$\left(-3.61346m^2 - 4.57998m\right)(0.2) + \left(-2.62837m^4 + 1.70314m^3 - 3.18629m^2 - 1.93520m - 2.21426\right), \tag{618}$$

which simplifies (RU) to $-2.62837m^4 + 1.70314m^3 - 3.90898m^2 - 2.85119m - 2.21426$.

Since

$$-2.62837\left(0.15\right)^4 + 1.70314\left(0.25\right)^3 - 3.90898\left(0.15\right)^2 - 2.85119\left(0.15\right) - 2.21426$$

is negative, we upper bound

$$\begin{aligned}&\left(-2.62837m^4 + 1.70314m^3 - 3.90898m^2 - 2.85119m - 2.21426\right)\eta \\ &+ \left(-0.38546m^5 - 0.43322m^4 - 1.08956m^3 - 0.82199m^2 + 0.87438m - 0.39751\right)\end{aligned} \tag{619}$$

by

$$\begin{aligned}&\left(-2.62837m^4 + 1.70314m^3 - 3.90898m^2 - 2.85119m - 2.21426\right)(0.2) \\ &+ \left(-0.38546m^5 - 0.43322m^4 - 1.08956m^3 - 0.82199m^2 + 0.87438m - 0.39751\right),\end{aligned} \tag{620}$$

which simplifies (RU) to $-0.38546m^5 - 0.95889m^4 - 0.74893m^3 - 1.60378m^2 + 0.30415m - 0.84036$.

Since

$$-0.38546\left(0.15\right)^5 - 0.95889\left(0.15\right)^4 - 0.74893\left(0.15\right)^3 - 1.60378\left(0.15\right)^2 + 0.30415\left(0.25\right) - 0.84036$$

is negative, we can upper bound

$$\begin{aligned}&\left(-0.38546m^5 - 0.95889m^4 - 0.74893m^3 - 1.60378m^2 + 0.30415m - 0.84036\right)\eta \\ &+ \left(-0.37059m^5 + 0.14112m^4 - 0.32126m^3 + 0.06079m^2 - 0.06190m - 0.01199\right)\end{aligned} \tag{621}$$

by

$$\begin{aligned}&\left(-0.38546m^5 - 0.95889m^4 - 0.74893m^3 - 1.60378m^2 + 0.30415m - 0.84036\right)(0.2) \\ &+ \left(-0.37059m^5 + 0.14112m^4 - 0.32126m^3 + 0.06079m^2 - 0.06190m - 0.01199\right),\end{aligned} \tag{622}$$

which simplifies (RU) to

$$-0.44768m^5 - 0.05065m^4 - 0.47104m^3 - 0.25996m^2 - 0.00107m - 0.18006. \tag{623}$$

This completes the proof.

□

Proof of Lemma B.231. To proceed, we first note that Equation 230 is convex in η and the first partial η evaluated at $\eta = 0.2$ is positive.

Consider the second partial η below.

$$2.00606\left(6\right)m\eta + \left(0.92873m^3 - 1.06079m^2 + 2.85120m + 0.96987\right)(2) \tag{624}$$

We lower bound the above in η by $\eta = 0.2$ and lower bound the resulting expression (RD) by the following.

$$1.85746\left(0.15\right)^3 - 2.12158\left(0.25\right)^2 + 8.10967\left(0.15\right) + 1.93974$$

Next, we evaluate the first partial η of Equation 230 at $\eta = 0.2$, shown below (RD).

$$0.46437m^4 + 0.48759m^3 + 0.47243m^2 + 0.93650m - 0.09991 \tag{625}$$

Note that we lower bound the above by $m = 0.15$, which is positive.

So, we lower bound Equation 230 in η by $\eta = 0.2$. Then, we show the resulting expression (RD) grows in m. Consider the first partial m below.

$$0.29537\,(4)\,m^3 - 0.16464\,(3)\,m^2 + 0.37091\,(2)\,m - 0.08885 \tag{626}$$

Note that we lower bound the above by

$$((0.29537\,(4)\,(0.15) - 0.16464\,(3))\,(0.25) + 0.37091\,(2))\,(0.15) - 0.08885,$$

which is positive.

So, we further lower bound Equation 230 in m by $m = 0.15$, which is positive.

This completes the proof.

□

Proof of Lemma B.232. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove Equation 231 is negative, we first show that it is concave in η and the first partial η at $\eta = 0.2$ is negative.

First note the second partial η below,

$$-2.00603\,(6)\,m\eta + \left(-0.92871m^3 + 1.06084m^2 - 2.85115m - 0.96984\right)(2)\,, \tag{627}$$

which we upper bound in η by $\eta = 0.2$.

The resulting expression (RU) $-1.85742m^3 + 2.12168m^2 - 8.10953m - 1.93968$ can be upper bounded by

$$-1.85742\,(0.15)^3 + 2.12168\,(0.25)^2 - 8.10953\,(0.15) - 1.93968,$$

which is negative.

Next, note the first partial η of Equation 231 at $\eta = 0.2$ (RU), $-0.46435m^4 - 0.48756m^3 - 0.47237m^2 - 0.93645m + 0.09993$. We upper bound the expression by

$$-0.46435\,(0.15)^4 - 0.48756\,(0.15)^3 - 0.47237\,(0.15)^2 - 0.93645\,(0.15) + 0.09993,$$

which is negative.

So, we upper bound Equation 231 at $\eta = 0.2$. Last, we show that the resulting expression (RU) is decreasing in m. Note that we upper bound the first partial m by

$$((-0.29537\,(4)\,(0.15) + 0.16464\,(3))\,(0.25) - 0.37090\,(2))\,(0.15) + 0.08886,$$

which is negative. So, we further upper bound Equation 231 in m by $m = 0.15$, which is negative.

This completes the proof.

□

Proof of Lemma B.233. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To begin the lemma, consider the second partial q of Equation 232 below (simplified by rounding up after five digits of precision and scaled by $\exp(-mq)$).

$$\begin{aligned}
&\left(p_{1,1}\sqrt{2m\eta + 0.25m + 0.025} + p_{1,2}\right)\exp\left(-\frac{1}{2}q\left(m - \left(\sqrt{2m\eta + 0.25m + 0.025}\right)\right)\right) \\
&+ \left(p_{2,1} + p_{2,2}\sqrt{2m\eta + 0.25m + 0.025}\right)\exp\left(-\frac{1}{2}q\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right) \\
&+ \left(4.12768m^4\eta + 0.51596m^4 + 0.05160m^3 + m^2\right),
\end{aligned} \tag{628}$$

where

$$
\begin{aligned}
p_{1,1} =& -0.25\eta m q^2 \\
&+ \left(-0.25797m^3\eta - 1.54786m^2\eta^2 - 0.19347m^2\eta + 0.125m^2 + 0.23067m\eta + 0.03125m - \eta + 0.00313\right) q \\
&+ \Bigg(1.54788m^3\eta + 1.03192m^2\eta^2 + 0.19349m^3 - 1.28988m^2\eta - 3.09574m\eta^2 + 0.03548m^2 \\
&\quad - 0.36116m\eta + 1.00323m - 0.03868\eta + 0.00017\Bigg),
\end{aligned} \tag{629}
$$

$$
\begin{aligned}
p_{1,2} =& \left(-0.125m^2\eta - 0.25m\eta^2 - 0.03125m\eta - 0.003125\eta\right) q^2 \\
&+ \Bigg(-1.54785m^3\eta^2 - 1.0319m^2\eta^3 - 0.19347m^3\eta - 0.25796m^2\eta^2 + 0.46454m^2\eta - 0.02577m\eta^2 + 0.0625m^2 \\
&- 1.00322m\eta + 0.00626m - 0.00016\eta\Bigg) q \\
&+ \Bigg(0.51596m^4\eta + 3.09576m^3\eta^2 + 0.0645m^4 + 0.77394m^3\eta - 6.19147m^2\eta^2 + 0.05483m^3 - 0.69652m^2\eta \\
&+ 0.25968m^2 + 1.42263m\eta + 0.18800m - \eta + 0.01875\Bigg),
\end{aligned} \tag{630}
$$

$$
\begin{aligned}
p_{2,1} =& \left(-0.125m^2\eta - 0.25m\eta^2 - 0.03125m\eta - 0.003125\eta\right) q^2 \\
&+ \Bigg(-1.54786m^3\eta^2 - 1.0319m^2\eta^3 - 0.19348m^3\eta - 0.25796m^2\eta^2 + 0.46453m^2\eta - 0.02577m\eta^2 \\
&+ 0.0625m^2 - 1.00322m\eta + 0.00625m - 0.00016\eta\Bigg) q \\
&+ \Bigg(0.51595m^4\eta + 3.09575m^3\eta^2 + 0.06449m^4 + 0.77393m^3\eta - 6.19149m^2\eta^2 + 0.05482m^3 - 0.69654m^2\eta \\
&+ 0.25967m^2 + 1.42262m\eta + 0.18799m - \eta + 0.01875\Bigg),
\end{aligned} \tag{631}
$$

and

$$
\begin{aligned}
p_{2,2} =& 0.25\eta m q^2 \\
&+ \left(0.25797m^3\eta + 1.54787m^2\eta^2 + 0.19348m^2\eta - 0.125m^2 - 0.23066m\eta - 0.03125m + \eta - 0.00312\right) q \\
&+ \Bigg(-1.54787m^3\eta - 1.03191m^2\eta^2 - 0.19348m^3 + 1.2899m^2\eta + 3.09576m\eta^2 - 0.03547m^2 \\
&\quad + 0.36118m\eta - 1.00322m + 0.0387\eta - 0.00016\Bigg).
\end{aligned} \tag{632}
$$

To proceed, we first show that $p_{1,1}$ is negative and $p_{2,2}$ is positive in lemmas B.238 and B.239.

Then, we can upper bound Equation 628 by replacing $p_{1,1}\sqrt{2m\eta + 0.25m + 0.025}$ with $p_{1,1}\sqrt{2m(0.2) + 0.25m + 0.025}$ and $p_{2,2}\sqrt{2m\eta + 0.25m + 0.025}$ with $p_{2,2}\sqrt{2m(0.3) + 0.25m + 0.025}$.

We can further upper bound $p_{2,2}\sqrt{2m\eta + 0.25m + 0.025}$ with $p_{2,2}\sqrt{0.85m + 0.025}$ by linearizing the $\sqrt{0.85m + 0.025}$ at $m = 0.15$ and upper bound $p_{1,1}\sqrt{0.65m + 0.025}$ with $p_{1,1}\left(0.83012m + 0.22548\right)$ (note lemma **??**).

Note the resulting expression below.

$$
\begin{aligned}
& b_1 \exp\left(-\frac{1}{2}q\left(m - \sqrt{2m\eta + 0.25m + 0.025}\right)\right) \\
& + b_2 \exp\left(-\frac{1}{2}q\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right) \\
& + \left(4.12768m^4\eta + 0.51596m^4 + 0.05160m^3 + m^2\right),
\end{aligned} \tag{633}
$$

where

$$
\begin{aligned}
b_1 = & \left(-0.33253m^2\eta - 0.25m\eta^2 - 0.08762m\eta - 0.00312\eta\right)q^2 \\
& + \Big(- 0.21414m^4\eta - 2.83276m^3\eta^2 - 1.0319m^2\eta^3 - 0.41224m^3\eta - 0.60697m^2\eta^2 + 0.10377m^3 + 0.6124m^2\eta \\
& \quad - 0.02577m\eta^2 + 0.11663m^2 - 1.78132m\eta + 0.01591m - 0.22564\eta + 0.00071\Big)q \\
& + \Big(1.80089m^4\eta + 3.95238m^3\eta^2 + 0.22512m^4 + 0.05221m^3\eta - 8.52862m^2\eta^2 + 0.12792m^3 - 1.28716m^2\eta \\
& \quad - 0.69802m\eta^2 + 1.10049m^2 + 1.30909m\eta + 0.41435m - 1.00872\eta + 0.01879\Big)
\end{aligned} \tag{634}
$$

and

$$
\begin{aligned}
b_2 = & \left(0.14708m^2\eta - 0.25m\eta^2 + 0.02557m\eta - 0.00312\eta\right)q^2 \\
& + \Big(0.28076m^4\eta + 0.13672m^3\eta^2 - 1.0319m^2\eta^3 + 0.07572m^3\eta + 0.09383m^2\eta^2 - 0.13604m^3 + 0.25747m^2\eta \\
& - 0.02577m\eta^2 + 0.00009m^2 + 0.03268m\eta - 0.00424m + 0.22711\eta - 0.00070\Big)q \\
& + \Big(- 1.16862m^4\eta + 1.97271m^3\eta^2 - 0.14607m^4 + 1.82597m^3\eta - 3.05683m^2\eta^2 - 0.02775m^3 \\
& - 0.01030m^2\eta + 0.70358m\eta^2 - 0.84021m^2 + 1.54683m\eta - 0.04018m - 0.99120\eta + 0.01872\Big)
\end{aligned} \tag{635}
$$

To proceed, we will upper bound b_1 and b_2 in q. First, note that

$$
\left(0.14708\left(0.25\right) - 0.25\left(0.2\right) + 0.02557\right)\left(0.25\right) - 0.00312
$$

is negative. So, we can upper bound $\left(0.14708m^2\eta - 0.25m\eta^2 + 0.02557m\eta - 0.00312\eta\right)q^2$ by $\left(0.14708m^2\eta - 0.25m\eta^2 + 0.02557m\eta - 0.00312\eta\right)\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)q$

and upper bound $\left(-0.33253m^2\eta - 0.25m\eta^2 - 0.08762m\eta - 0.00312\eta\right)q^2$ by $\left(-0.33253m^2\eta - 0.25m\eta^2 - 0.08762m\eta - 0.00312\eta\right)\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)q$.

We can then upper bound b_1 and b_2 with the following (RU), respectively.

$$\left(-0.21414m^4\eta - 2.83276m^3\eta^2 - 1.0319m^2\eta^3 - 0.29927m^4 - 0.63724m^3\eta - 0.60697m^2\eta^2 + 0.02492m^3 - 0.03403m^2\eta - 0.51177m\eta^2 - 0.05909m^2 - 2.08165m\eta - 0.02965m - 0.23170\eta - 0.00091\right)q + \left(1.80089m^4\eta + 3.95238m^3\eta^2 + 0.22512m^4 + 0.05221m^3\eta - 8.52862m^2\eta^2 + 0.12792m^3 - 1.28716m^2\eta - 0.69802m\eta^2 + 1.10049m^2 + 1.30909m\eta + 0.41435m - 1.00872\eta + 0.01879\right) \tag{636}$$

$$\left(0.28076m^4\eta + 0.13672m^3\eta^2 - 1.0319m^2\eta^3 + 0.13238m^4 - 0.14928m^3\eta + 0.09383m^2\eta^2 - 0.11302m^3 + 0.54340m^2\eta - 0.51177m\eta^2 + 0.07377m^2 - 0.04761m\eta + 0.00906m + 0.22105\eta - 0.00232\right)q + \left(-1.16862m^4\eta + 1.97271m^3\eta^2 - 0.14607m^4 + 1.82597m^3\eta - 3.05683m^2\eta^2 - 0.02775m^3 - 0.01030m^2\eta + 0.70358m\eta^2 - 0.84021m^2 + 1.54683m\eta - 0.04018m - 0.99120\eta + 0.01872\right) \tag{637}$$

We then show that Equation 636 is negative in lemma B.236 and 637 is positive in lemma B.237.

So, we can upper bound Equation 636 by fixing $q = \frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944$ and upper bound Equation 637 by fixing $q = \frac{0.94}{\eta}m^2 + \frac{0.48}{\eta} + 2.15$ and scale both by η.

We show that the upper bound for Equation 636 is negative in lemma B.234 and show that the upper bound for Equation]637 is positive in lemma B.235.

With the above modifications applied to b_1 and b_2, we can further upper bound Equation 628 by replacing $\exp\left(-\frac{1}{2}q\left(m - \left(\sqrt{2m\eta + 0.25m + 0.025}\right)\right)\right)$ with $\exp\left(\frac{1}{2}\left(\frac{0.9}{0.3}\left(0.15\right)^2 + \frac{0.52}{0.3} + 1.944\right)\left(\sqrt{2\left(0.25\right)\left(0.2\right) + 0.25\left(0.25\right) + 0.025} - \left(0.25\right)\right)\right)$ (note that $\sqrt{0.65m + 0.025} - m$ is locally decreasing in m) and $\exp\left(-\frac{1}{2}q\left(m + \sqrt{2m\eta + 0.25m + 0.025}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(\frac{0.9}{0.3}\left(0.15\right)^2 + \frac{0.52}{0.3} + 1.944\right)\left(\left(0.15\right) + \sqrt{2\left(0.15\right)\left(0.2\right) + 0.25\left(0\right.\right.\right.$

This yields the following upper bound for Equation 628.

$$\begin{aligned}&\left(-3.69583\right)m^2n^4 + \left(-1.68863m^4 - 1.30103m^3 - 15.74620m^2 - 2.54036m\right)n^3\\ &+ \left(-3.54108m^5 + 5.12169m^4 - 3.96822m^3 - 1.87929m^2 - 3.76133m - 2.25781\right)n^2\\ &+ \left(-0.16799m^6 - 0.86293m^5 + 0.12099m^4 - 2.95774m^3 + 1.98627m^2 - 1.03943m - 0.09875\right)n\\ &+ \left(-0.33062m^6 - 0.01005m^5 - 0.24203m^4 - 0.03725m^3 - 0.03140m^2 - 0.02000m - 0.00109\right)\end{aligned} \tag{638}$$

From the above, note the following expressions are negative.

$$\Big(-3.54108\,(0.15)^5 + 5.12169\,(0.25)^4 - 3.96822\,(0.15)^3 - 1.87929\,(0.15)^2 - 3.76133\,(0.15) - 2.25781\Big)$$

$$\Big(-0.16799\,(0.15)^6 - 0.86293\,(0.15)^5 + 0.12099\,(0.25)^4 - 2.95774\,(0.15)^3 + 1.98627\,(0.25)^2 - 1.03943\,(0.15) - 0.09875\Big)$$

This completes the proof.

□

Proof of Lemma B.234. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove that Equation 233 is negative, we rewrite it as follows.

$$\Big(\Big(\big(-2.00601mn + \big(-0.92871m^3 - 1.55450m^2 - 10.24515m - 1.69290\big)\big)\,mn + \big(-2.54948m^5 + 0.83833m^4 - 3.12021m^3 - 1.66893m^2 - 3.00375m - 1.45914\big)\Big)n + \big(-0.19272m^6 - 0.57351m^5 - 0.49864m^4 - 2.02848m^3 + 0.75940m^2 - 0.72574m - 0.10346\big)\Big)n + \big(-0.26934m^6 + 0.02243m^5 - 0.20880m^4 - 0.01372m^3 - 0.03154m^2 - 0.01541m - 0.00047\big) \tag{639}$$

We begin by upper bounding $-2.00601mn + \big(-0.92871m^3 - 1.55450m^2 - 10.24515m - 1.69290\big)$ by $-2.00601m\,(0.2) + \big(-0.92871m^3 - 1.55450m^2 - 10.24515m - 1.69290\big)$, which simplifies (RU) to $-0.92871m^3 - 1.5545m^2 - 10.64635m - 1.6929$.

Then, we upper bound

$$\begin{aligned}&\big(-0.92871m^3 - 1.5545m^2 - 10.64635m - 1.6929\big)\,m\eta \\ &+ \big(-2.54948m^5 + 0.83833m^4 - 3.12021m^3 - 1.66893m^2 - 3.00375m - 1.45914\big)\end{aligned} \tag{640}$$

by

$$\begin{aligned}&\big(-0.92871m^3 - 1.5545m^2 - 10.64635m - 1.6929\big)\,m\,(0.2) \\ &+ \big(-2.54948m^5 + 0.83833m^4 - 3.12021m^3 - 1.66893m^2 - 3.00375m - 1.45914\big)\end{aligned} \tag{641}$$

, which simplifies (RU) to $-2.54948m^5 + 0.65259m^4 - 3.43111m^3 - 3.7982m^2 - 3.34233m - 1.45914$.

Since

$$-2.54948\,(0.15)^5 + 0.65259\,(0.25)^4 - 3.43111\,(0.15)^3 - 3.7982\,(0.15)^2 - 3.34233\,(0.15) - 1.45914$$

is negative, we can upper bound

$$\begin{aligned}&\big(-2.54948m^5 + 0.65259m^4 - 3.43111m^3 - 3.7982m^2 - 3.34233m - 1.45914\big)\,\eta \\ &+ \big(-0.19272m^6 - 0.57351m^5 - 0.49864m^4 - 2.02848m^3 + 0.75940m^2 - 0.72574m - 0.10346\big)\end{aligned} \tag{642}$$

by

$$\begin{aligned}&\big(-2.54948m^5 + 0.65259m^4 - 3.43111m^3 - 3.7982m^2 - 3.34233m - 1.45914\big)\,(0.2) \\ &+ \big(-0.19272m^6 - 0.57351m^5 - 0.49864m^4 - 2.02848m^3 + 0.75940m^2 - 0.72574m - 0.10346\big)\,,\end{aligned} \tag{643}$$

which simplifies (RU) to

$$-0.19272m^6 - 1.08340m^5 - 0.36812m^4 - 2.71470m^3 - 0.00024m^2 - 1.39420m - 0.39528. \tag{644}$$

Lastly, we upper bound

$$\begin{aligned}&\left(-0.19272m^6 - 1.08340m^5 - 0.36812m^4 - 2.71470m^3 - 0.00024m^2 - 1.39420m - 0.39528\right) n \\ &+ \left(-0.26934m^6 + 0.02243m^5 - 0.20880m^4 - 0.01372m^3 - 0.03154m^2 - 0.01541m - 0.00047\right)\end{aligned} \tag{645}$$

by

$$\begin{aligned}&\left(-0.19272m^6 - 1.08340m^5 - 0.36812m^4 - 2.71470m^3 - 0.00024m^2 - 1.39420m - 0.39528\right)(0.2) \\ &+ \left(-0.26934m^6 + 0.02243m^5 - 0.20880m^4 - 0.01372m^3 - 0.03154m^2 - 0.01541m - 0.00047\right),\end{aligned} \tag{646}$$

which simplifies (RU) to

$$-0.30788m^6 - 0.19425m^5 - 0.28242m^4 - 0.55666m^3 - 0.03158m^2 - 0.29425m - 0.079526. \tag{647}$$

This completes the proof.

□

Proof of Lemma B.235. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove that Equation 234 is positive, we rewrite it as follows.

$$\begin{aligned}&\Bigg(\Bigg(\left(-2.21858mn + \left(-0.96998m^3 + 2.26666m^2 - 3.35040m - 0.39672\right)\right) mn \\ &+ \left(0.12852m^5 - 0.47678m^4 + 1.08958m^3 + 1.20305m^2 + 1.19882m - 0.51594\right)\Bigg) n \\ &+ \left(0.26392m^6 - 0.14032m^5 + 0.78411m^4 - 0.38715m^3 - 0.21298m^2 - 0.04355m + 0.11984\right)\Bigg) n \\ &+ \left(0.12444m^6 - 0.10623m^5 + 0.13289m^4 - 0.04573m^3 + 0.03323m^2 + 0.00435m - 0.00111\right)\end{aligned} \tag{648}$$

We proceed by lower bounding $-2.21858mn + \left(-0.96998m^3 + 2.26666m^2 - 3.35040m - 0.39672\right)$ by $-2.21858m\,(0.3) + \left(-0.96998m^3 + 2.26666m^2 - 3.35040m - 0.39672\right)$, which simplifies (RD) to $-0.96998m^3 + 2.26666m^2 - 4.01598m - 0.39672$.

Since $-0.96998\,(0.15)^3 + 2.26666\,(0.25)^2 - 4.01598\,(0.15) - 0.39672$ is negative, we can lower bound

$$\begin{aligned}&\left(-0.96998m^3 + 2.26666m^2 - 4.01598m - 0.39672\right) mn \\ &+ \left(0.12852m^5 - 0.47678m^4 + 1.08958m^3 + 1.20305m^2 + 1.19882m - 0.51594\right)\end{aligned} \tag{649}$$

by

$$\begin{aligned}&\left(-0.96998m^3 + 2.26666m^2 - 4.01598m - 0.39672\right) m\,(0.3) \\ &+ \left(0.12852m^5 - 0.47678m^4 + 1.08958m^3 + 1.20305m^2 + 1.19882m - 0.51594\right),\end{aligned} \tag{650}$$

which simplifies (RD) to

$$0.12852m^5 - 0.76777m^4 + 1.76957m^3 - 0.00175m^2 + 1.07980m - 0.51594. \tag{651}$$

Since

$$\Big(0.12852\,(0.25)^5 - 0.76777\,(0.15)^4 + 1.76957\,(0.25)^3 - 0.00175\,(0.15)^2 + 1.07980\,(0.25) - 0.51594\Big)$$

is negative, we can lower bound

$$\begin{aligned}&\left(0.12852m^5 - 0.76777m^4 + 1.76957m^3 - 0.00175m^2 + 1.07980m - 0.51594\right)\eta \\ &+ \left(0.26392m^6 - 0.14032m^5 + 0.78411m^4 - 0.38715m^3 - 0.21298m^2 - 0.04355m + 0.11984\right)\end{aligned} \tag{652}$$

by

$$\begin{aligned}&\left(0.12852m^5 - 0.76777m^4 + 1.76957m^3 - 0.00175m^2 + 1.07980m - 0.51594\right)(0.3) \\ &+ \left(0.26392m^6 - 0.14032m^5 + 0.78411m^4 - 0.38715m^3 - 0.21298m^2 - 0.04355m + 0.11984\right),\end{aligned} \tag{653}$$

which simplifies (RD) to

$$0.26392m^6 - 0.10177m^5 + 0.55377m^4 + 0.14372m^3 - 0.21351m^2 + 0.28039m - 0.03495. \tag{654}$$

Note the following expressions are positive.

$$(((((0.26392\,(0.15) - 0.10177)\,(0.25) + 0.55377)\,(0.15) + 0.14372)\,(0.15) - 0.21351)\,(0.25) + 0.28039)\,(0.15) - 0.03495$$

$$(((((0.12444\,(0.15) - 0.10623)\,(0.25) + 0.13289)\,(0.15) - 0.04573)\,(0.25) + 0.03323)\,(0.15) + 0.00435)\,(0.15) - 0.00111$$

This completes the proof.

□

Proof of Lemma B.236. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To show that Equation 235 is negative, we rewrite the expression as follows.

$$\begin{aligned}&\Big(\left(-1.0319mn + \left(-2.83276m^2 - 0.60697m - 0.51177\right)\right)mn \\ &+ \left(-0.21414m^4 - 0.63724m^3 - 0.03403m^2 - 2.08165m\right)\Big)n \\ &+ \left(-0.29927m^4 + 0.02492m^3 - 0.05909m^2 - 0.02965m - 0.23170n - 0.00091\right)\end{aligned} \tag{655}$$

We begin by upper bounding $-1.0319mn + \left(-2.83276m^2 - 0.60697m - 0.51177\right)$ by $-1.0319m\,(0.2) + \left(-2.83276m^2 - 0.60697m - 0.51177\right)$, which simplifies (RU) to $-2.83276m^2 - 0.81335m - 0.51177$.

Then, we upper bound $\left(-2.83276m^2 - 0.81335m - 0.51177\right)mn + \left(-0.21414m^4 - 0.63724m^3 - 0.03403m^2 - 2.08165m - 0.23170\right)$ by $\left(-2.83276m^2 - 0.81335m - 0.51177\right)m\,(0.2) + \left(-0.21414m^4 - 0.63724m^3 - 0.03403m^2 - 2.08165m - 0.23170\right)$, which simplifies (RU) to $-0.21414m^4 - 1.20379m^3 - 0.1967m^2 - 2.18400m - 0.2317$.

Lastly, we upper bound

$$\begin{aligned}&\left(-0.21414m^4 - 1.20379m^3 - 0.1967m^2 - 2.18400m - 0.2317\right)x \\ &+ \left(-0.29927m^4 + 0.02492m^3 - 0.05909m^2 - 0.02965m - 0.00091\right)\end{aligned} \tag{656}$$

by

$$\left(-0.21414m^4 - 1.20379m^3 - 0.1967m^2 - 2.18400m - 0.2317\right)(0.2) + \left(-0.29927m^4 + 0.02492m^3 - 0.05909m^2 - 0.02 \tag{657}$$

which simplifies (RU) to $-0.34209m^4 - 0.21583m^3 - 0.09843m^2 - 0.46645m - 0.04725$.

This completes the proof.

□

Proof of Lemma B.237. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To show that Equation 236 is positive, we first rewrite it as follows.

$$\begin{aligned}&\Big(\left(-1.0319m\eta+\left(0.13672m^2+0.09383m-0.51177\right)\right)m\eta\\&+\left(0.28076m^4-0.14928m^3+0.54340m^2-0.04761m+0.22105\right)\Big)\eta\\&+\left(0.13238m^4-0.11302m^3+0.07377m^2+0.00906m-0.00232\right)\end{aligned}\tag{658}$$

We begin by lower bounding $-1.0319m\eta+\left(0.13672m^2+0.09383m-0.51177\right)$ by $-1.0319m\left(0.3\right)+\left(0.13672m^2+0.09383m-0.51177\right)$, which simplifies (RD) to $0.13672m^2-0.21574m-0.51177$. Since $0.13672m^2-0.21574m-0.51177$ is negative (i.e., $0.13672\left(0.25\right)^2-0.21574\left(0.15\right)-0.51177$ is negative), then we can lower bound

$$\left(0.13672m^2-0.21574m-0.51177\right)m\eta+\left(0.28076m^4-0.14928m^3+0.54340m^2-0.04761m+0.22105\right)\tag{659}$$

by

$$\left(0.13672m^2-0.21574m-0.51177\right)m\left(0.3\right)+\left(0.28076m^4-0.14928m^3+0.54340m^2-0.04761m+0.22105\right),\tag{660}$$

which simplifies (RD) to $0.28076m^4-0.10827m^3+0.47867m^2-0.20115m+0.22105$.

Note the following expression are positive.

$$\left(0.28076\left(0.15\right)^4-0.10827\left(0.25\right)^3+0.47867\left(0.15\right)^2-0.20115\left(0.25\right)+0.22105\right)$$

$$((((0.13238\left(0.15\right)-0.11302)\left(0.25\right)+0.07377)\left(0.15\right)+0.00906)\left(0.15\right)-0.00232)$$

This completes the proof.

□

Proof of Lemma B.238. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we first start by writing

$$\begin{aligned}&-0.25\eta mq^2\\&+\left(-0.25797m^3\eta-1.54786m^2\eta^2-0.19347m^2\eta+0.125m^2+0.23067m\eta+0.03125m-\eta+0.00313\right)q\end{aligned}\tag{661}$$

as

$$\begin{aligned}&\Big(-0.25\eta mq\\&+\left(-0.25797m^3\eta-1.54786m^2\eta^2-0.19347m^2\eta+0.125m^2+0.23067m\eta+0.03125m-\eta+0.00313\right)\Big)q.\end{aligned}\tag{662}$$

Note that we upper bound $-0.25\eta mq$ by $-0.25m\left(0.9m^2+0.52+1.944\eta\right)$, where

$$\begin{aligned}&\Big(-0.25m\left(0.9m^2+0.52+1.944\eta\right)\\&+\left(-0.25797m^3\eta-1.54786m^2\eta^2-0.19347m^2\eta+0.125m^2+0.23067m\eta+0.03125m-\eta+0.00313\right)\Big)\end{aligned}\tag{663}$$

simplifies (RU) to

$$-0.25797m^3n-1.54786m^2n^2-0.225m^3-0.19347m^2n+0.125m^2-0.25533mn-0.09875m-n+0.00313\tag{664}$$

.

Since

$$\left(-1.54786\left(0.15\right)^{2}\left(0.2\right)+\left(-0.25797\left(0.15\right)^{3}-0.19347\left(0.15\right)^{2}-0.25533\left(0.15\right)-1\right)\right)\left(0.2\right)$$
$$+\left(-0.225\left(0.15\right)^{3}+0.125\left(0.25\right)^{2}-0.09875\left(0.15\right)+0.00313\right)$$

is negative, then we upper bound

$$\left(-0.25797m^{3}\eta-1.54786m^{2}\eta^{2}-0.225m^{3}-0.19347m^{2}\eta+0.125m^{2}-0.25533m\eta-0.09875m-\eta+0.00313\right)q$$
$$+\Bigg(1.54788m^{3}\eta+1.03192m^{2}\eta^{2}+0.19349m^{3}-1.28988m^{2}\eta-3.09574m\eta^{2}+0.03548m^{2}$$
$$-0.36116m\eta+1.00323m-0.03868\eta+0.00017\Bigg),\tag{665}$$

by

$$\Bigg(-0.25797m^{3}\eta-1.54786m^{2}\eta^{2}-0.225m^{3}-0.19347m^{2}\eta+0.125m^{2}$$
$$-0.25533m\eta-0.09875m-\eta+0.00313\Bigg)\left(\frac{0.9}{n}m^{2}+\frac{0.52}{n}+1.944\right)$$

$$+\Bigg(1.54788m^{3}\eta+1.03192m^{2}\eta^{2}+0.19349m^{3}-1.28988m^{2}\eta-3.09574m\eta^{2}+0.03548m^{2}$$
$$-0.36116m\eta+1.00323m-0.03868\eta+0.00017\Bigg),\tag{666}$$

which we scale by η and simplify (RU) to

$$\left(-1.97712m^{2}-3.09574m\right)\eta^{3}+\left(-1.39307m^{4}+1.04639m^{3}-2.47087m^{2}-0.85752m-1.98268\right)\eta^{2}$$
$$+\left(-0.23217m^{5}-0.17412m^{4}-0.60785m^{3}-0.72212m^{2}+0.67849m-0.51374\right)\eta$$
$$+\left(-0.2025m^{5}+0.1125m^{4}-0.20587m^{3}+0.06782m^{2}-0.05135m+0.00163\right).\tag{667}$$

From the above, note that the following expressions are negative.

$$\left(-0.23217\left(0.15\right)^{5}-0.17412\left(0.15\right)^{4}-0.60785\left(0.15\right)^{3}-0.72212\left(0.15\right)^{2}+0.67849\left(0.25\right)-0.51374\right)$$

$$\left(-0.2025\left(0.15\right)^{5}+0.1125\left(0.25\right)^{4}-0.20587\left(0.15\right)^{3}+0.06782\left(0.25\right)^{2}-0.05135\left(0.15\right)+0.00163\right)$$

$$\left(-1.39307\left(0.15\right)^{4}+1.04639\left(0.25\right)^{3}-2.47087\left(0.15\right)^{2}-0.85752\left(0.15\right)-1.98268\right)$$

This completes the proof.

□

Proof. Before we start the proof of the lemma, note that we round terms down after five digits of precision frequently throughout for simplicity, which we will accompany with RD.

To prove the lemma, we begin by writing

$$0.25\eta mq^{2}+\left(0.25797m^{3}\eta+1.54787m^{2}\eta^{2}+0.19348m^{2}\eta-0.125m^{2}-0.23066m\eta-0.03125m+\eta-0.00312\right)q\tag{668}$$

as

$$\left(0.25\eta mq + \left(0.25797m^3\eta + 1.54787m^2\eta^2 + 0.19348m^2\eta - 0.125m^2 - 0.23066m\eta - 0.03125m + \eta - 0.00312\right)\right) q. \tag{669}$$

We can then lower bound $0.25\eta mq$ by $0.25m\eta\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right)$, resulting in the following lower bound for Equation 669 (RD).

$$\left(0.25797m^3\eta + 1.54787m^2\eta^2 + 0.225m^3 + 0.19348m^2\eta - 0.125m^2 + 0.25534m\eta + 0.09875m + \eta - 0.00312\right) q \tag{670}$$

Note that

$$\left(1.54787\left(0.15\right)^2\left(0.2\right) + \left(0.25797\left(0.15\right)^3 + 0.19348\left(0.15\right)^2 + 0.25534\left(0.15\right) + 1\right)\right)\left(0.2\right) + \left(0.225\left(0.15\right)^3 - 0.125\left(0.25\right)^2 + 0.09875\left(0.15\right) - 0.00312\right)$$

is positive, so we can lower bound

$$\left(0.25797m^3\eta + 1.54787m^2\eta^2 + 0.225m^3 + 0.19348m^2\eta - 0.125m^2 + 0.25534m\eta + 0.09875m + \eta - 0.00312\right) q \tag{671}$$

by

$$\left(0.25797m^3\eta + 1.54787m^2\eta^2 + 0.225m^3 + 0.19348m^2\eta - 0.125m^2 + 0.25534m\eta + 0.09875m + \eta - 0.00312\right)\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right). \tag{672}$$

By scaling

$$\left(0.25797m^3\eta + 1.54787m^2\eta^2 + 0.225m^3 + 0.19348m^2\eta - 0.125m^2 + 0.25534m\eta + 0.09875m + \eta - 0.00312\right)\left(\frac{0.9}{\eta}m^2 + \frac{0.52}{\eta} + 1.944\right) + \left(-1.54787m^3\eta - 1.03191m^2\eta^2 - 0.19348m^3 + 1.2899m^2\eta + 3.09576m\eta^2 - 0.03547m^2 + 0.36118m\eta - 1.00322m + 0.0387\eta - 0.00016\right) \tag{673}$$

by η, the above simplifies (RD) to

$$\left(1.97714m^2 + 3.09576m\right)\eta^3 + \left(1.39308m^4 - 1.04638m^3 + 2.47091m^2 + 0.85756m + 1.9827\right)\eta^2 + \left(0.23217m^5 + 0.17413m^4 + 0.60787m^3 + 0.72214m^2 - 0.67848m + 0.51377\right)\eta + \left(0.2025m^5 - 0.1125m^4 + 0.20587m^3 - 0.06781m^2 + 0.05135m - 0.00163\right). \tag{674}$$

To show the above is positive, we rewrite the above as

$$\left(\left(\left(1.97714m^2 + 3.09576m\right)\eta + \left(1.39308m^4 - 1.04638m^3 + 2.47091m^2 + 0.85756m + 1.9827\right)\right)\eta + \left(0.23217m^5 + 0.17413m^4 + 0.60787m^3 + 0.72214m^2 - 0.67848m + 0.51377\right)\right)\eta + \left(0.2025m^5 - 0.1125m^4 + 0.20587m^3 - 0.06781m^2 + 0.05135m - 0.00163\right). \tag{675}$$

We first lower bound $\left(1.97714m^2 + 3.09576m\right)\eta$ by $\left(1.97714m^2 + 3.09576m\right)(0.2)$. Then,

$$\left(1.97714m^2 + 3.09576m\right)\eta + \left(1.39308m^4 - 1.04638m^3 + 2.47091m^2 + 0.85756m + 1.9827\right) \tag{676}$$

simplifies (RD) to

$$\left(1.39308m^4 - 1.04638m^3 + 2.86633m^2 + 1.47671m + 1.9827\right). \tag{677}$$

Since

$$\left(1.39308\left(0.15\right)^4 - 1.04638\left(0.25\right)^3 + 2.86633\left(0.15\right)^2 + 1.47671\left(0.15\right) + 1.9827\right)$$

is positive, we can lower bound

$$\begin{aligned}&\left(1.39308m^4 - 1.04638m^3 + 2.86633m^2 + 1.47671m + 1.9827\right)\eta \\ &\quad + \left(0.23217m^5 + 0.17413m^4 + 0.60787m^3 + 0.72214m^2 - 0.67848m + 0.51377\right)\end{aligned} \tag{678}$$

by

$$\begin{aligned}&\left(1.39308m^4 - 1.04638m^3 + 2.86633m^2 + 1.47671m + 1.9827\right)(0.2) \\ &\quad + \left(0.23217m^5 + 0.17413m^4 + 0.60787m^3 + 0.72214m^2 - 0.67848m + 0.51377\right),\end{aligned} \tag{679}$$

which simplifies (RD) to

$$\left(0.23217m^5 + 0.45274m^4 + 0.39859m^3 + 1.29540m^2 - 0.38314m + 0.91031\right). \tag{680}$$

Since

$$\left(0.23217\left(0.15\right)^5 + 0.45274\left(0.15\right)^4 + 0.39859\left(0.15\right)^3 + 1.29540\left(0.15\right)^2 - 0.38314\left(0.25\right) + 0.91031\right)$$

is positive, then

$$\begin{aligned}&\left(0.23217m^5 + 0.45274m^4 + 0.39859m^3 + 1.29540m^2 - 0.38314m + 0.91031\right)\eta \\ &\quad + \left(0.2025m^5 - 0.1125m^4 + 0.20587m^3 - 0.06781m^2 + 0.05135m - 0.00163\right)\end{aligned} \tag{681}$$

can be lower bounded by

$$\begin{aligned}&\left(0.23217m^5 + 0.45274m^4 + 0.39859m^3 + 1.29540m^2 - 0.38314m + 0.91031\right)(0.2) \\ &\quad + \left(0.2025m^5 - 0.1125m^4 + 0.20587m^3 - 0.06781m^2 + 0.05135m - 0.00163\right),\end{aligned} \tag{682}$$

which simplifies (RD) to

$$0.24893m^5 - 0.02196m^4 + 0.28558m^3 + 0.19127m^2 - 0.02528m + 0.18043. \tag{683}$$

The above can be lower bounded by

$0.24893\left(0.15\right)^5 - 0.02196\left(0.25\right)^4 + 0.28558\left(0.15\right)^3 + 0.19127\left(0.15\right)^2 - 0.02528\left(0.25\right) + 0.18043$,

which is positive.

This completes the proof.

□

B.5.5 Miscellaneous Lemmas

Lemmas For Square Root Terms In this section, we include the lemmas regarding local lower bounds for the square root terms encountered during our analysis in lemmas B.123, B.122, B.125, and B.124. Refer to Section B.5.5 for the proofs of the below lemmas.

Lemma B.240. *For* $m \in [0.15, 0.25]$, $\sqrt{0.65m + 0.02}$ *can be lower bounded by* $0.84417m + 0.2161$.

Lemma B.241. *For* $m \in [0.15, 0.25]$, $\sqrt{0.65m + 0.025}$ *can be lower bounded by* $0.83012m + 0.22548$.

Lemma B.242. *For* $m \in [0.15, 0.25]$, $\sqrt{0.85m + 0.025}$ *can be lower bounded by* $0.96827m + 0.2452$.

Lemma B.243. *For* $m \in [0.15, 0.25]$, $\sqrt{0.85m + 0.02}$ *can be lower bounded by* $0.98125m + 0.2368$.

Lemmas for Exponential Terms ($\eta = 0.2$) In this section, we include the lemmas regarding the convexity/concavity of the exponential terms encountered during our analysis in lemmas B.123 and B.122. Refer to Section B.5.5 for the proofs of the below lemmas.

Lemma B.244. *For $m \in [0.15, 0.25]$, $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ is convex in m*

Lemma B.245. *For $m \in [0.15, 0.25]$, $\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m - \sqrt{0.65m + 0.02}\right)\right)$ is concave in m*

Lemma B.246. *For $m \in [0.15, 0.25]$, $\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)$ is convex in m.*

Lemma B.247. *For $m \in [0.15, 0.25]$, $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)$ is convex in m.*

Lemmas for Exponential Terms ($\eta = 0.3$) In this section, we include the lemmas regarding the convexity/concavity of the exponential terms encountered during our analysis in lemmas B.125 and B.124. Refer to Section B.5.5 for the proofs of the below lemmas.

Lemma B.248. *For $m \in [0.15, 0.25]$, $\exp\left(-\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ is convex in m.*

Lemma B.249. *For $m \in [0.15, 0.25]$, $\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ is convex in m.*

Lemma B.250. *For $m \in [0.15, 0.25]$,*

$$\exp\left(-2.97186m^3 - 0.3552m^2 - 3.64283m - 0.43539\right) \tag{684}$$

is convex in m.

Lemma B.251. *For $m \in [0.15, 0.25]$,*

$$\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.025}\right)\right) \tag{685}$$

is convex in m.

Lemma B.252. *For $m \in [0.15, 0.25]$, $\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m - \sqrt{0.85m + 0.025}\right)\right)$ is convex in m.*

Lemma B.253. *For $m \in [0.15, 0.25]$, $\exp\left(-\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m - \sqrt{0.85m + 0.025}\right)\right)$ is concave in m.*

Lemma B.254. *For $m \in [0.15, 0.25]$, $\exp\left(\frac{1}{2}\left(3m^2 + 3.67733\right)\left(m - \sqrt{0.85m + 0.02}\right)\right)$ is convex in m.*

Proof for Lemmas in Section B.5.5

Proof of Lemma B.241. To prove the lemma, note that the difference $0.84417m + 0.2161 - \sqrt{0.65m + 0.02}$ is convex in m. So, we can evaluate the difference at $m = 0.15, 0.25$, both of which are negative.

This completes the proof. □

Proof of Lemma **??**. To prove the lemma, note that $\sqrt{0.65m + 0.025} - (0.83012m + 0.22548)$ is concave in m. We evaluate the difference at $m = 0.15, 0.25$, both of which are positive. This completes the proof. □

Proof of Lemma **??**. To see that $0.96827m + 0.2452$ is a local lower bound for $\sqrt{0.85m + 0.025}$, note the convexity of the difference, $0.96827m + 0.2452 - \sqrt{0.85m + 0.025}$. So, we can evaluate the difference at both m ends $0.15, 0.25$, both of which are negative.

This completes the proof. □

Proof of Lemma B.243. To see that $0.98125m + 0.2368$ is a local lower bound for $\sqrt{0.85m + 0.02}$, note the convexity of the difference, $0.98125m + 0.2368 - \sqrt{0.85m + 0.02}$. So, we can evaluate the difference at both m ends $0.15, 0.25$, both of which are negative.

This completes the proof. □

Proof for Lemmas in Section B.5.5

Proof of Lemma B.244. To prove the lemma, consider the second partial m of $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$ below, with $\exp\left(\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$ factored out and simplified by rounding down after five digits of precision.

$$13.5m+\frac{-3.56485m^2-0.1755m+0.23818}{\left(0.65m+0.02\right)^{\frac{3}{2}}}+\left(6.75m^2+2.272+\frac{-3.65625m^2-0.09m-0.7384}{\sqrt{0.65m+0.02}}\right)^2 \tag{686}$$

To show the above is positive, we will show that

$$13.5m+\frac{-3.56485m^2-0.1755m+0.23818}{\left(0.65m+0.02\right)^{\frac{3}{2}}} \tag{687}$$

is positive.

Note that we can lower bound the above by

$$13.5\left(0.15\right)+\frac{-3.56485\left(0.25\right)^2-0.1755\left(0.25\right)+0.23818}{\left(0.65\left(0.15\right)+0.02\right)^{\frac{3}{2}}},$$

which is positive.

This completes the proof.

□

Proof of Lemma B.245. Before we start the proof of the lemma, note that we round terms up after five digits of precision frequently throughout for simplicity, which we will accompany with RU.

To prove the lemma, we will show that the second partial m of $\exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$ is negative. Consider the second partial m below, where we factor out $\exp\left(-\frac{1}{2}\left(4.5m^2+4.544\right)\left(m-\sqrt{0.65m+0.02}\right)\right)$ (RU).

$$\left(-13.5m+\frac{3.56485m^2+0.1755m-0.23818}{\left(0.65m+0.02\right)^{\frac{3}{2}}}\right)+\left(6.75m^2+2.272\right)^2$$
$$+2\left(-6.75m^2-2.272\right)\left(\frac{3.65625m^2+0.09m+0.7384}{\sqrt{0.65m+0.02}}\right)+\left(\frac{3.65625m^2+0.09m+0.7384}{\sqrt{0.65m+0.02}}\right)^2 \tag{688}$$

We then upper bound the above by the following

$$\left(-13.5m+\frac{3.56485\left(0.25\right)^2+0.1755\left(0.25\right)-0.23818}{\left(0.65\left(0.15\right)+0.02\right)^{\frac{3}{2}}}\right)+\left(6.75m^2+2.272\right)^2$$
$$+2\left(-6.75m^2-2.272\right)\left(\frac{3.65625\left(0.15\right)^2+0.09\left(0.15\right)+0.7384}{\sqrt{0.65m+0.02}}\right)+\left(\frac{3.65625m^2+0.09m+0.7384}{\sqrt{0.65m+0.02}}\right)^2. \tag{689}$$

We then scale the above by $0.65m+0.02$ resulting in the following expression (RU).

$$\left(-11.26129m^2-3.79046\right)\sqrt{0.65m+0.02}+29.61563m^5+14.27942m^4+20.59493m^3$$
$$-2.75391m^2+3.67813m+0.66264 \tag{690}$$

We then upper bound $\left(-11.26129m^2-3.79046\right)\sqrt{0.65m+0.02}$ by $\left(-11.26129m^2-3.79046\right)\left(0.84417m+0.2161\right)$ (note lemma B.241), yielding the following (RU).

$$29.61563m^5+14.27942m^4+11.08849m^3-5.18634m^2+0.47834m-0.15609 \tag{691}$$

From the above, we can upper bound $29.61563m^5 + 14.27942m^4 + 11.08849m^3$ by $m = 0.25$, producing a quadratic expression (RU) $-2.41421m^2 + 0.47834m - 0.07138$. We upper bound the quadratic by $(-2.41421\,(0.15) + 0.47834)\,(0.25) - 0.07138$, which is negative.

This completes the proof.

□

Proof of Lemma B.246. To prove the lemma, consider the second partial m of $\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)$ with $\exp\left(-\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)$ factored out below and simplified by rounding down after five digits of precision.

$$\left(-13.5m + \frac{-3.56485m^2 - 0.1755m + 0.23818}{(0.65m + 0.02)^{\frac{3}{2}}}\right) + \left(6.75m^2 + 2.272 + \frac{3.65625m^2 + 0.09m + 0.7384}{\sqrt{0.65m + 0.02}}\right)^2 \tag{692}$$

We can lower bound the above by

$$\left(-13.5\,(0.25) + \frac{-3.56485\,(0.25)^2 - 0.1755\,(0.25) + 0.23818}{(0.65\,(0.15) + 0.02)^{\frac{3}{2}}}\right) + \left(6.75\,(0.15)^2 + 2.272 + \frac{3.65625\,(0.15)^2 + 0.09\,(0.15) + 0.7384}{\sqrt{0.65\,(0.25) + 0.02}}\right)^2,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.247. To prove the lemma, we will show that the second partial m of $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)$ is positive. Consider the second partial m below, simplified by rounding down after five digits of precision and factoring out $\exp\left(\frac{1}{2}\left(4.5m^2 + 4.544\right)\left(m + \sqrt{0.65m + 0.02}\right)\right)$.

$$\left(13.5x + \frac{3.56484x^2 + 0.1755x - 0.23818}{(0.65x + 0.02)^{\frac{3}{2}}}\right) + \left(6.75x^2 + 2.272 + \frac{3.65625x^2 + 0.09x + 0.7384}{\sqrt{0.65x + 0.02}}\right)^2 \tag{693}$$

We can lower bound the above by

$$\left(13.5\,(0.15) + \frac{3.56484\,(0.15)^2 + 0.1755\,(0.15) - 0.23818}{(0.65\,(0.15) + 0.02)^{\frac{3}{2}}}\right) + \left(6.75\,(0.15)^2 + 2.272 + \frac{3.65625\,(0.15)^2 + 0.09\,(0.15) + 0.7384}{\sqrt{0.65\,(0.25) + 0.02}}\right)^2,$$

which is positive.

This completes the proof. □

Proof for Lemmas in Section B.5.5

Proof of Lemma B.248. To prove the lemma, we will show that the second partial m of $\exp\left(-\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ is positive. Consider the second partial m below, simplified by rounding down terms after five digits of precision and factoring out $\exp\left(-\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$.

$$\left(-9.40002m + \frac{-4.24470m^2 - 0.15981m + 0.33741}{(0.85m + 0.02)^{\frac{3}{2}}}\right) + \left(4.70001m^2 + 1.875 + \frac{3.32917m^2 + 0.06266m + 0.79687}{\sqrt{0.85m + 0.02}}\right)^2 \quad (694)$$

Note that we can lower bound the above by

$$\left(-9.40002\,(0.25) + \frac{-4.24470\,(0.25)^2 - 0.15981\,(0.25) + 0.33741}{(0.85\,(0.25) + 0.02)^{\frac{3}{2}}}\right) + \left(4.70001\,(0.15)^2 + 1.875 + \frac{3.32917\,(0.15)^2 + 0.06266\,(0.15) + 0.79687}{\sqrt{0.85\,(0.25) + 0.02}}\right)^2,$$

which is positive.

This completes the proof.

□

Proof of Lemma B.249. To prove the lemma, we will show that the second partial m of $\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$ is positive. Consider the second partial m below, simplified by rounding down terms after five digits of precision and factoring out $\exp\left(\frac{1}{2}\left(3.13334m^2 + 3.75\right)\left(m + \sqrt{0.85m + 0.02}\right)\right)$.

$$\left(9.40002m + \frac{4.24469m^2 + 0.15980m - 0.33742}{(0.85m + 0.02)^{\frac{3}{2}}}\right) + \left(4.70001m^2 + 1.875 + \frac{3.32917m^2 + 0.06266m + 0.79687}{\sqrt{0.85m + 0.02}}\right)^2 \quad (695)$$

Note that the above can be lower bounded by

$$\left(9.40002\,(0.15) + \frac{4.24469\,(0.15)^2 + 0.15980\,(0.15) - 0.33742}{(0.85\,(0.15) + 0.02)^{\frac{3}{2}}}\right) + \left(4.70001\,(0.15)^2 + 1.875 + \frac{3.32917\,(0.15)^2 + 0.06266\,(0.15) + 0.79687}{\sqrt{0.85\,(0.25) + 0.02}}\right)^2,$$

which is positive.

This completes the proof.

□

Lemma B.255 (Proof of Lemma B.250)**.** *To prove the lemma, consider the second partial derivative of Equation* **??** *below, after factoring out* $\exp\left(-2.97186m^3 - 0.3552m^2 - 3.64283m - 0.43539\right)$.

$$(-17.83116m - 0.7104) + \left(8.91558m^2 + 0.7104m + 3.64283\right)^2$$

Note that the above can lower bounded by

$$(-17.83116\,(0.25) - 0.7104) + \left(8.91558\,(0.15)^2 + 0.7104\,(0.15) + 3.64283\right)^2,$$

which is positive.

This completes the proof.

Proof of Lemma B.251. To see that Equation 685 convex in m, note the second partial m (simplified by rounding down after five digits of precision with $\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m+\sqrt{0.85+0.025}\right)\right)$ factored out).

$$\left(9.40002m+\frac{4.24469m^2+0.19975m-0.33672}{(0.85m+0.025)^{\frac{3}{2}}}\right) + \left(4.70001m^2+1.875+\frac{3.32917m^2+0.07833m+0.79687}{\sqrt{0.85m+0.025}}\right)^2 \quad (696)$$

Note that we can lower bound the above by

$$\left(9.40002\left(0.15\right)+\frac{4.24469\left(0.15\right)^2+0.19975\left(0.15\right)-0.33672}{(0.85\left(0.15\right)+0.025)^{\frac{3}{2}}}\right) + \left(4.70001\left(0.15\right)^2+1.875+\frac{3.32917\left(0.15\right)^2+0.07833\left(0.15\right)+0.79687}{\sqrt{0.85\left(0.25\right)+0.025}}\right)^2,$$

which is positive.

This completes the proof. □

Proof of Lemma B.252. To prove the lemma, note the second partial m of $\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m-\sqrt{0.85m+0.025}\right)\right)$ below, with $\exp\left(\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m-\sqrt{0.85m+0.025}\right)\right)$ factored out.

$$4.70001\left(2\right)m+\frac{\begin{array}{c}\left(0.85\left(-3.13334\right)\left(2\right)m-\left(0.2125\right)3.13334\left(2\right)m+0.025\left(-3.13334\right)\right)\left(0.85m+0.025\right)\\-0.425\left(0.85\left(-3.13334\right)m^2-\left(0.2125\right)3.13334m^2+0.025\left(-3.13334\right)m-\left(0.2125\right)3.75\right)\end{array}}{(0.85m+0.025)^{\frac{3}{2}}} + \left(4.70001m^2+1.875+\frac{0.85\left(-3.13334\right)m^2-\left(0.2125\right)3.13334m^2+0.025\left(-3.13334\right)m-\left(0.2125\right)3.75}{\sqrt{0.85m+0.025}}\right)^2 \quad (697)$$

To proceed, we show that

$$4.70001\left(2\right)m+\frac{\begin{array}{c}\left(0.85\left(-3.13334\right)\left(2\right)m-\left(0.2125\right)3.13334\left(2\right)m+0.025\left(-3.13334\right)\right)\left(0.85m+0.025\right)\\-0.425\left(0.85\left(-3.13334\right)m^2-\left(0.2125\right)3.13334m^2+0.025\left(-3.13334\right)m-\left(0.2125\right)3.75\right)\end{array}}{(0.85m+0.025)^{\frac{3}{2}}} \quad (698)$$

is positive.

We first simplify the above (rounding down after five digits of precision) to

$$9.40002m+\frac{-4.24470m^2-0.19976m+0.33671}{(0.85m+0.025)^{\frac{3}{2}}} \quad (699)$$

that we can upper bound by

$$9.40002\left(0.15\right)+\frac{-4.24470\left(0.25\right)^2-0.19976\left(0.25\right)+0.33671}{(0.85\left(0.25\right)+0.025)^{\frac{3}{2}}},$$

which is positive.

This completes the proof.

□

Proof. To prove the lemma, note the second partial m of $\exp\left(-\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m-\sqrt{0.85m+0.025}\right)\right)$ below, with $\exp\left(-\frac{1}{2}\left(3.13334m^2+3.75\right)\left(m-\sqrt{0.85m+0.025}\right)\right)$ factored out and simplified by rounding up after five digits of precision.

$$\left(-9.40002m+\frac{4.24470m^2+0.19976m-0.33671}{\left(0.85m+0.025\right)^{\frac{3}{2}}}\right)+\left(4.70001m^2+1.875\right)^2$$
$$-2\left(4.70001m^2+1.875\right)\left(\frac{3.32917m^2+0.07833m+0.79687}{\sqrt{0.85m+0.025}}\right)+\left(\frac{3.32918m^2+0.07834m+0.79688}{\sqrt{0.85m+0.025}}\right)^2 \tag{700}$$

Note that the above can be upper bounded by

$$\left(-9.40002m+\frac{4.24470m^2+0.19976m-0.33671}{\left(0.85\left(0.25\right)+0.025\right)^{\frac{3}{2}}}\right)+\left(4.70001m^2+1.875\right)^2$$
$$-2\left(4.70001m^2+1.875\right)\left(\frac{3.32917m^2+0.07833m+0.79687}{\sqrt{0.85\left(0.25\right)+0.025}}\right)+\left(\frac{3.32918m^2+0.07834m+0.79688}{\sqrt{0.85\left(0.15\right)+0.025}}\right)^2, \tag{701}$$

which can be simplified to (after rounding up after five digits of precision) to $30.55400m^4+1.90965m^3+48.14402m^2-7.45811m-1.36117$.

Note that $30.55400m^4+1.90965m^3+48.14402m^2-7.45811m-1.36117$ can be upper bounded by

$$30.55400\left(0.25\right)^4+1.90965\left(0.25\right)^3+\left(48.14402\left(0.25\right)-7.45811\right)\left(0.25\right)-1.36117,$$

which is negative.

This completes the proof.

□

Proof of Lemma B.254. To prove the lemma, consider the second partial m of $\exp\left(\frac{1}{2}\left(3m^2+3.67733\right)\left(m-\sqrt{0.85m+0.02}\right)\right)$ below, with $\exp\left(\frac{1}{2}\left(3m^2+3.67733\right)\left(m-\sqrt{0.85m+0.02}\right)\right)$ factored out.

$$9m+\frac{\left(-6.375m-0.06\right)\left(0.85m+0.02\right)-0.425\left(-3.1875m^2-0.06m-3.67733\left(0.2125\right)\right)}{\left(0.85m+0.02\right)^{\frac{3}{2}}}$$
$$+\left(4.5m^2+1.838665+\frac{-3.1875m^2-0.06m-3.67733\left(0.2125\right)}{\sqrt{0.85m+0.02}}\right)^2 \tag{702}$$

From the above, we'll show that

$$9m+\frac{\left(-6.375m-0.06\right)\left(0.85m+0.02\right)-0.425\left(-3.1875m^2-0.06m-3.67733\left(0.2125\right)\right)}{\left(0.85m+0.02\right)^{\frac{3}{2}}} \tag{703}$$

is positive.

Consider Equation 703 simplified, by rounding down after five digits of precision.

$$9m+\frac{-4.06407m^2-0.153m+0.33090}{\left(0.85m+0.02\right)^{\frac{3}{2}}} \tag{704}$$

We can lower bound the above by

$$9\left(0.15\right)+\frac{-4.06407\left(0.25\right)^2-0.153\left(0.25\right)+0.33090}{\left(0.85\left(0.25\right)+0.02\right)^{\frac{3}{2}}},$$

which is positive.

This completes the proof.

□

B.5.6 Additional Simulations of Asymptotic Setting

In this section, we provide additional simulations of our model in the asymptotic case, for $m \in [0.1, 0.4]$ and $\eta = \frac{1}{8}, \frac{1}{4}$. Our simulations show that how the fixed point $\left(\|\tilde{q}\|_2^2, \tilde{c}^2\right)$ appears to transition from a stable to an unstable fixed point, where, in the unstable case, trajectories initialized closed to the fixed seems to orbit around it. When the trajectory orbits around the fixed point, we observe significant oscillations in the loss and sharpness. For each simulation, we show the loss, sharpness, and system trajectories. We also provide zoomed-in versions of the plots to make values that the system jumps between more clear.

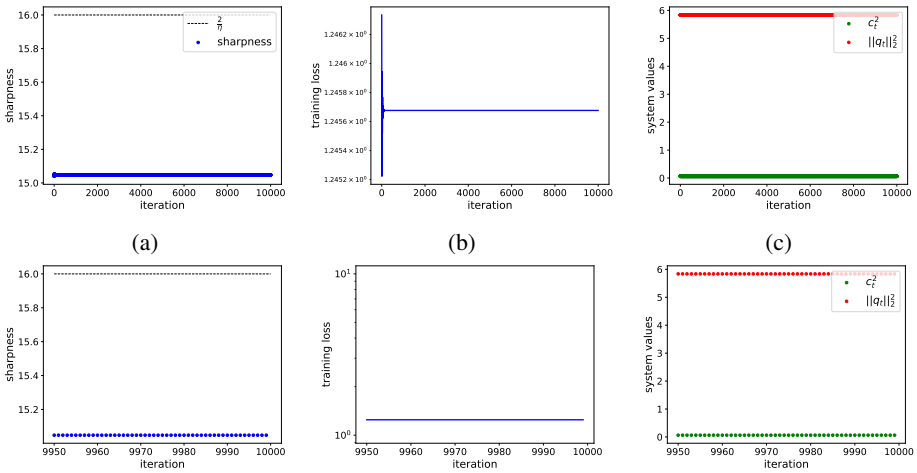


Figure 26: We show the sharpness (a), training loss (b), system iterate evolution (c) for $m = 0.10, \eta = \frac{1}{8}, \|q_0\|_2^2 = 5.836, c_0^2 = 0.064$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system iterate evolution (f).

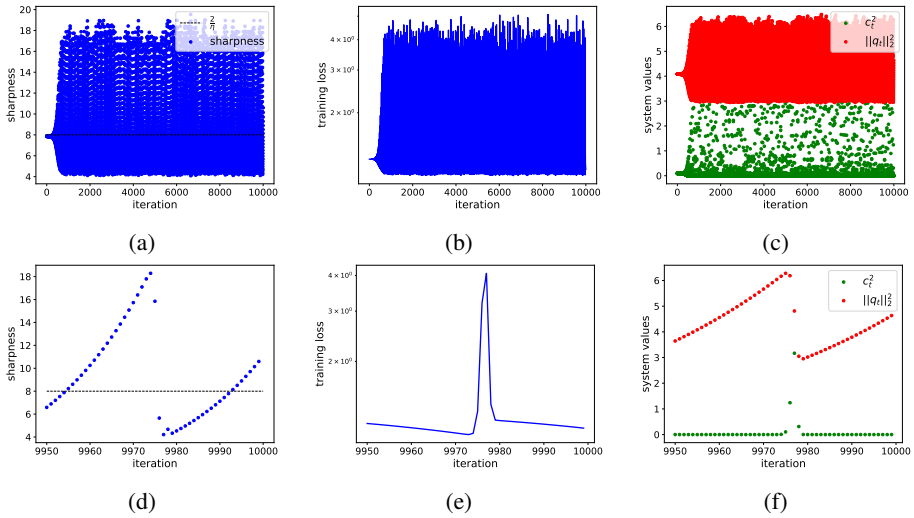


Figure 27: We show the sharpness (a), training loss (b), system iterate evolution (c) for $m = 0.10, \eta = \frac{1}{4}, \|q_0\|_2^2 = 4.083, c_0^2 = 0.093$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system iterate evolution (f).

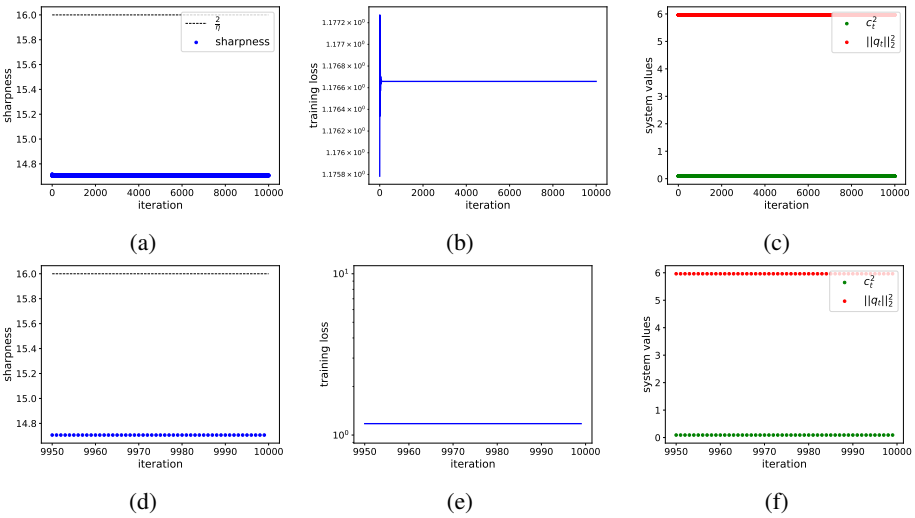


Figure 28: We show the sharpness (a), training loss (b), system iterate evolution (c) for $m = 0.15, \eta = \frac{1}{8}, \|q_0\|_2^2 = 5.964, c_0^2 = 0.091$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system iterate evolution (f).

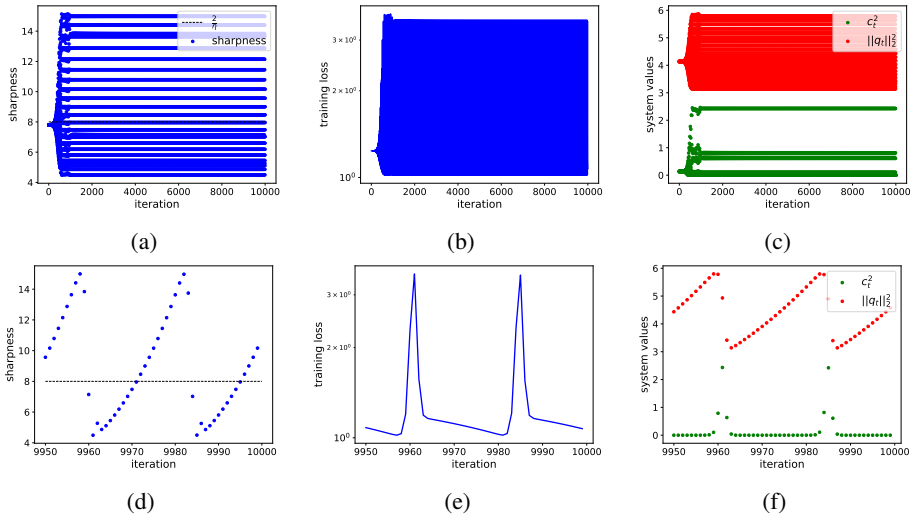


Figure 29: We show the sharpness (a), training loss (b), system trajectory (c) for $m = 0.15, \eta = \frac{1}{4}, \|q_0\|_2^2 = 4.137, c_0^2 = 0.134$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

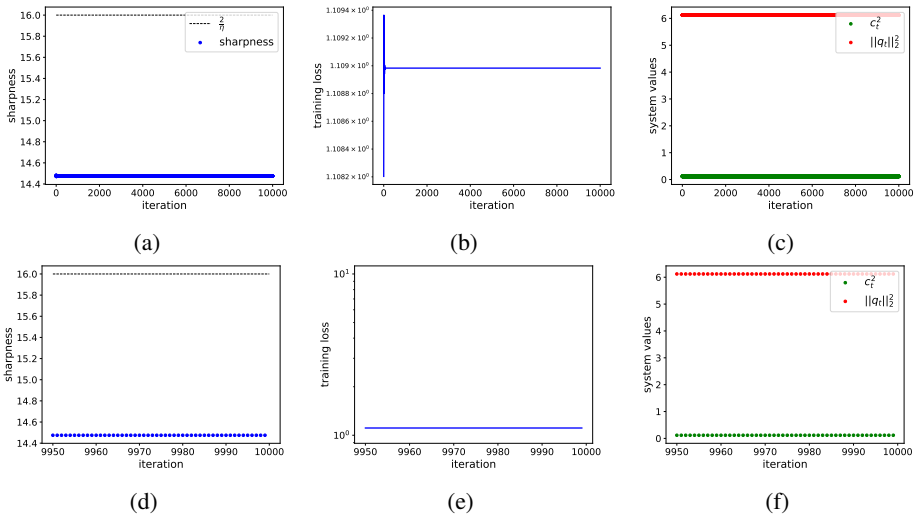


Figure 30: We show the sharpness (a), training loss (b), system trajectory (c) for $m = 0.20, \eta = \frac{1}{8}, \|q_0\|_2^2 = 6.121, c_0^2 = 0.117$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

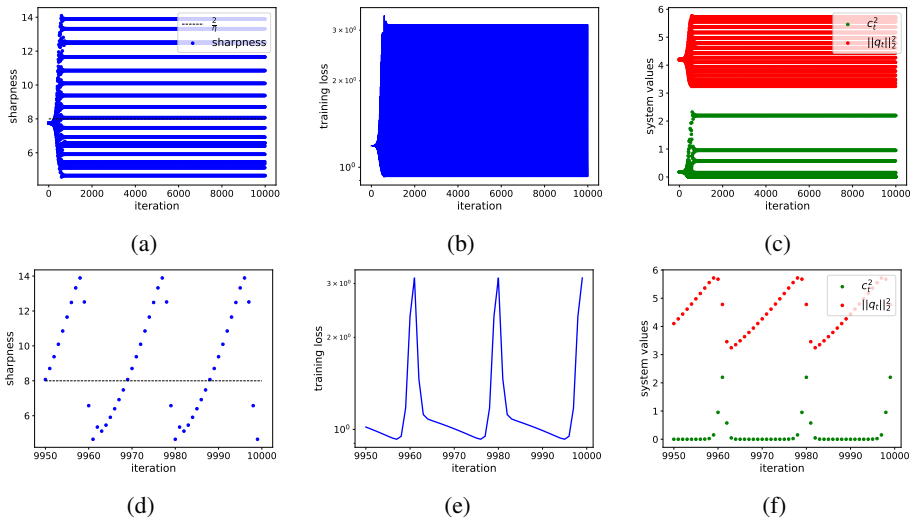


Figure 31: We show the sharpness, training loss, system trajectory for $m = 0.20, \eta = \frac{1}{4}, \|q_0\|_2^2 = 4.200, c_0^2 = 0.174$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

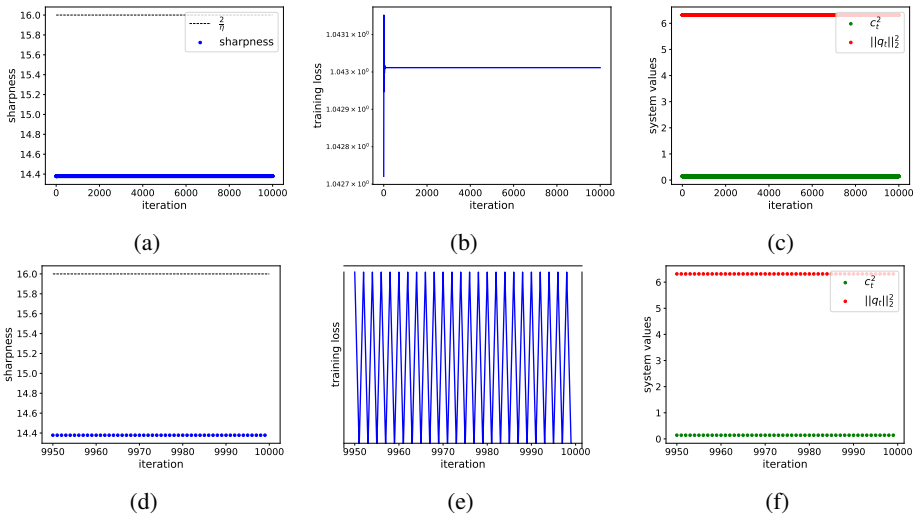


Figure 32: We show the sharpness, training loss, system trajectory for $m = 0.25, \eta = \frac{1}{8}, \|q_0\|_2^2 = 6.315, c_0^2 = 0.142$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

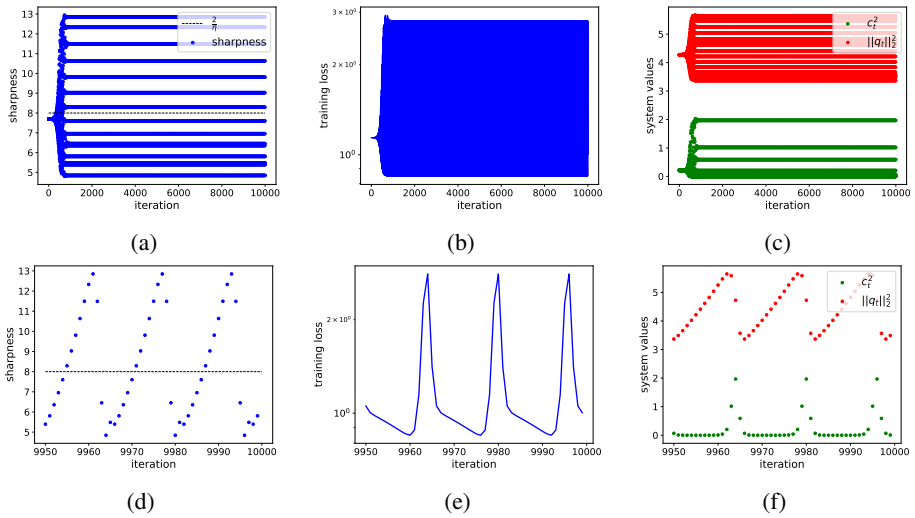


Figure 33: We show the sharpness, training loss, system trajectory for $m = 0.25, \eta = \frac{1}{4}, \|q_0\|_2^2 = 4.274, c_0^2 = 0.211$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

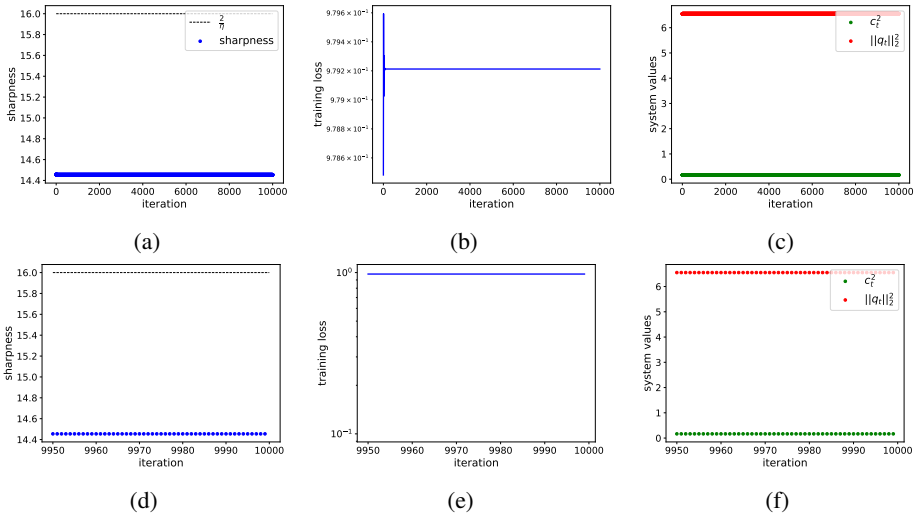


Figure 34: We show the sharpness, training loss, system trajectory for $m = 0.30, \eta = \frac{1}{8}, \|q_0\|_2^2 = 6.553, c_0^2 = 0.166$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

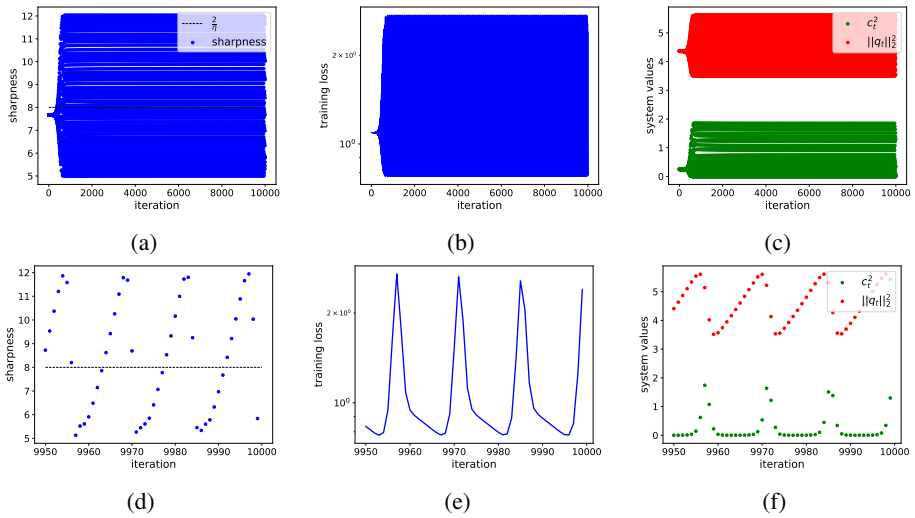


Figure 35: We show the sharpness, training loss, system trajectory for $m = 0.30, \eta = \frac{1}{4}, \|q_0\|_2^2 = 4.359, c_0^2 = 0.247$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

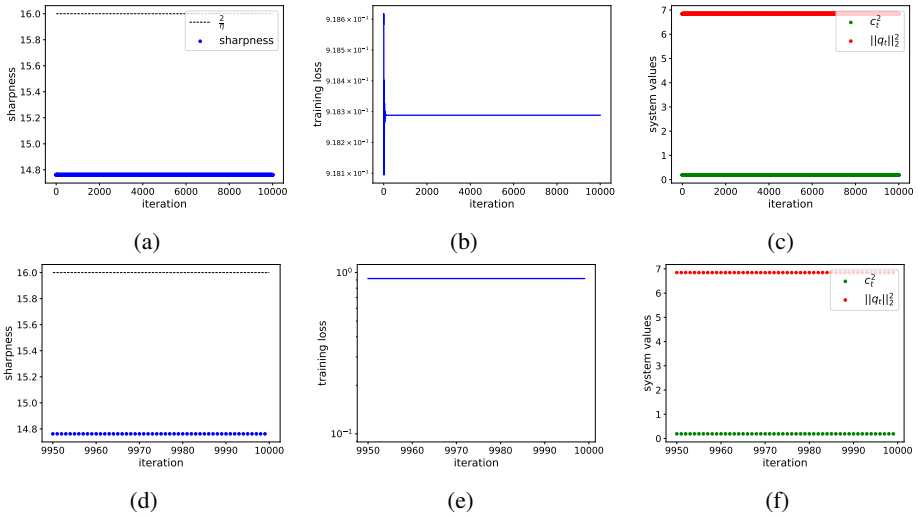


Figure 36: We show the sharpness, training loss, system trajectory for $m = 0.35, \eta = \frac{1}{8}, \|q_0\|_2^2 = 6.847, c_0^2 = 0.191$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

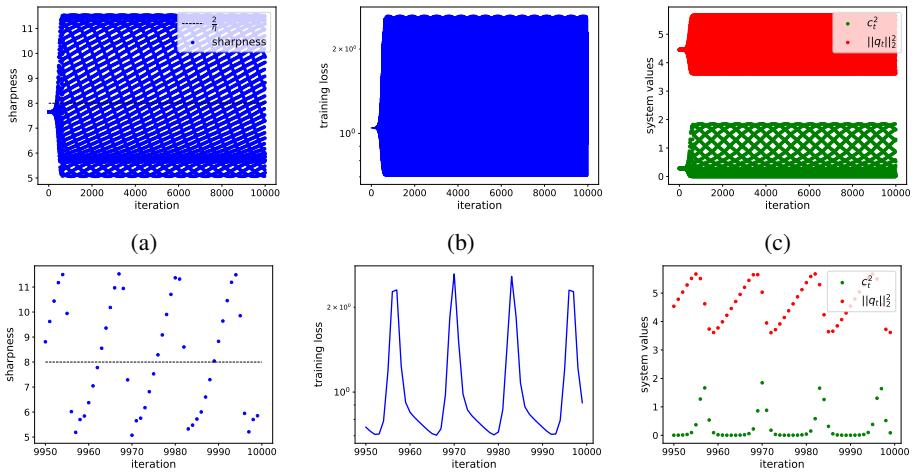


Figure 37: We show the sharpness, training loss, system trajectory for $m = 0.35, \eta = \frac{1}{4}, \|q_0\|_2^2 = 4.458, c_0^2 = 0.282$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

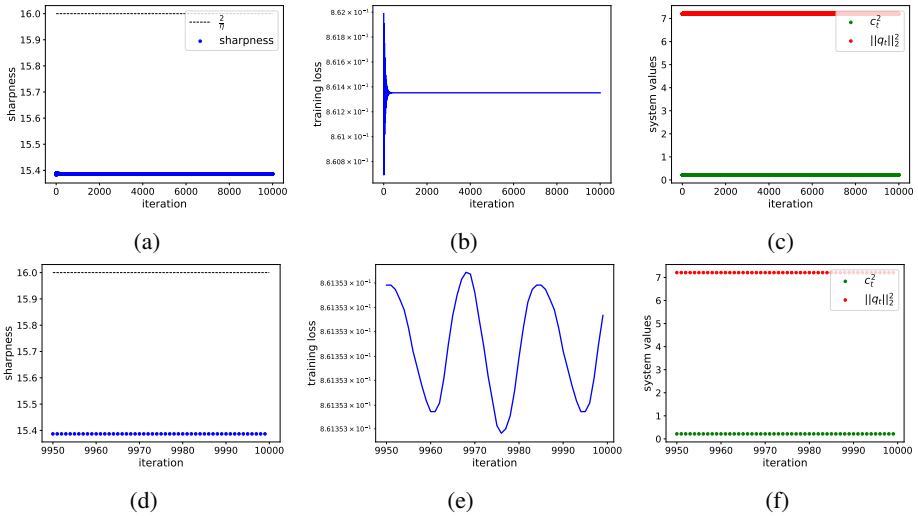


Figure 38: We show the sharpness, training loss, system trajectory for $m = 0.40, \eta = \frac{1}{8}, \|q_0\|_2^2 = 7.210, c_0^2 = 0.217$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).

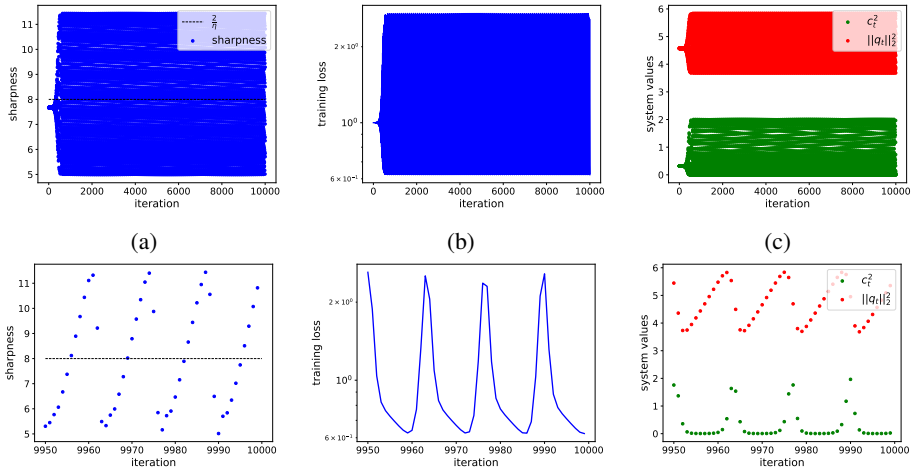


Figure 39: We show the sharpness, training loss, system trajectory for $m = 0.40, \eta = \frac{1}{4}, \|q_0\|_2^2 = 4.572, c_0^2 = 0.316$. We also provide zoomed-in versions of the sharpness (d), training loss (e), and system trajectory (f).