# CONSTRAINT-BASED FRAMEWORK FOR BOUNDING PROBABILITIES OF CAUSATION UNDER PARTIAL STRUCTURAL ASSUMPTIONS AND AUXILIARY STATISTICAL INFORMATION

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## ABSTRACT

Probabilities of causation (PoCs) are fundamental to individual level explanation and decision making, yet they are inherently counterfactual and not point identifiable from data in general. Existing bounds either disregard available auxiliary information, require complete structural assumptions and auxiliary statistical information, or rely on restrictive binary settings, limiting their practical use. In real world applications, structural assumptions and auxiliary statistical information are often incomplete but still informative. This paper proposes a unified constraint based framework for bounding PoCs using partial structural assumptions and partial auxiliary statistical information. We show how partial structural assumptions and partial auxiliary statistical information can be systematically incorporated as constraints in an optimization formulation, even when such information is available only for subsets of variables or covariate values. The resulting framework yields tighter and formally valid bounds without full identifiability and naturally accommodates multiple sources of incomplete structural and statistical information within a single optimization program. This approach extends the applicability of PoCs to realistic settings where both structural assumptions and auxiliary statistical information are incomplete.

## 1 INTRODUCTION

Advanced decisions in medicine, policy, and law require reasoning about causes of observed effects at the individual level (Lu & Boutilier, 2011; Li & Pearl, 2019; Mueller & Pearl, 2023; Pearl, 2009). Typical questions include whether a treatment caused a patient's recovery, whether an intervention was necessary for an outcome, or whether a policy decision was sufficient to prevent harm. Such questions concern counterfactual relationships and cannot be answered using population level causal effects alone, even when such effects are identified from randomized experiments.

Probabilities of causation (PoCs) (Pearl, 1999; Galles & Pearl, 1998; Halpern, 2000; Tian & Pearl, 2000) within Structural Causal Models (Pearl, 2009) provide a formal framework for addressing these individualized causal questions. Well studied examples include the probability of necessity (PN), the probability of sufficiency (PS), and the probability of necessity and sufficiency (PNS) (Pearl, 1999). These quantities are central to applications such as treatment prioritization, legal attribution of responsibility, and personalized decision making. However, PoCs are generally not point identifiable from data because they depend on joint counterfactual events that cannot be simultaneously observed. As a result, research in this area has focused on deriving informative bounds rather than point estimates (Tian & Pearl, 2000; Mueller et al., 2022; Li & Pearl, 2024; Shu et al., 2026).

Classical results show that sharp bounds on PoCs can be obtained by combining experimental and observational data while making no assumptions about the underlying data generating process (Tian & Pearl, 2000). Although these bounds are valid under minimal assumptions, they are often wide and may be insufficient for practical decision making. Subsequent work demonstrated that additional structural assumptions encoded in fully specified causal diagrams, together with the auxiliary statistical information implied by those assumptions, can substantially tighten these bounds (Mueller et al., 2022; Zhang et al., 2022; Dawid et al., 2017).



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Despite their theoretical importance, existing structural based approaches have limited practical applicability. They typically assume a fully specified causal graph (Zhang et al., 2022) and are largely restricted to binary treatments and outcomes (Mueller et al., 2022). More importantly, existing methods derive separate theorems for each causal structure and the corresponding auxiliary statistical information (Mueller et al., 2022). Consequently, incorporating new structural or statistical information generally requires deriving new theoretical results rather than extending a unified framework. In practice, however, both structural assumptions and auxiliary statistical information are often incomplete but still informative. Domain knowledge may provide only selected causal relationships, qualitative ordering constraints, or limited information about confounding mechanisms, while auxiliary statistical information may be available only for subsets of variables or covariate values rather than the full joint distribution. Existing approaches provide no systematic mechanism for incorporating such heterogeneous structural and statistical information within a single optimization framework.

In this paper, we develop a unified constraint based framework for bounding PoCs using partial structural assumptions and partial auxiliary statistical information. Balke (1995); Tian & Pearl (2000) formulated the problem of bounding PoCs as a linear program, but their formulation incorporates only experimental and observational distributions. Building on this foundation, we show how heterogeneous structural assumptions and auxiliary statistical information can be systematically translated into optimization constraints. These constraints may arise from incomplete causal diagrams, background scientific knowledge, or statistical information available only for subsets of variables or covariate values. The resulting framework accommodates multiple sources of incomplete information within a single optimization program, eliminating the need to derive new theorems for each information setting while yielding tighter and formally valid bounds. It subsumes existing graph-based bounds as special cases, extends naturally beyond binary treatments and outcomes, and provides a unified mechanism for combining experimental data, observational data, partial structural assumptions, and partial auxiliary statistical information.

**Contributions.** We propose a constraint based framework for bounding PoCs under incomplete structural and statistical information. The key novelty is a general methodology for translating partial structural assumptions and partial auxiliary statistical information into optimization constraints on counterfactual distributions, enabling multiple sources of incomplete structural and statistical information to be incorporated within a single optimization program. This eliminates the need to derive new theorems for each causal structure. The resulting framework generalizes prior bounds beyond binary treatments and outcomes and flexibly accommodates partial auxiliary statistical information, allowing partial structural assumptions and partial auxiliary statistical information to be incorporated even when they are available only for subsets of variables or covariate values.

## 2 Preliminaries and Related Work

In this section, we review standard definitions of probabilities of causation following (Pearl, 1999). Our analysis is formulated using counterfactual events interpreted under the Structural Causal Model (SCM) semantics (Galles & Pearl, 1998; Halpern, 2000). We use causal diagrams as a representational tool when structural assumptions are available, but the definitions themselves do not depend on a fully specified causal graph.

Let $Y_x = y$ denote the counterfactual statement "$Y$ would take value $y$ if $X$ were set to $x$." For notational convenience, throughout the paper we write $y_x$ for the event $Y_x = y$, $y_{x'}$ for $Y_{x'} = y$, $y'_x$ for $Y_x = y'$, and $y'_{x'}$ for $Y_{x'} = y'$. However, the proposed framework is not restricted to the binary case.

We consider the following three probabilities of causation.

**Definition 1** (Probabilities of causation)**.** Let $X$ and $Y$ be two binary variables in a causal model $M$, let $x$ and $y$ stand for the propositions $X =$ true and $Y =$ true, respectively, and $x'$ and $y'$ for their complements. The three probabilities of causation are defined as (Pearl, 1999):

$$\text{PN} \triangleq P(y'_{x'}|x,y)\text{PS} \triangleq P(y_x|y',x')\text{PNS} \triangleq P(y_x, y'_{x'})$$

where PN, PS, and PNS denote the probability of necessity, sufficiency, and necessity and sufficiency, respectively.

Tian & Pearl (2000) derived tight bounds for PNS, PN, and PS using an optimization-based formulation that places no assumptions on the underlying causal structure beyond consistency with experimental and observational data. These bounds are obtained by solving a linear program over admissible counterfactual distributions, often referred to as Balke's linear programming (Balke, 1995). Li & Pearl (2019) later provided a theoretical proof of these bounds.

The tight bounds for PNS and PN are given by

$$\max\left\{\begin{array}{c}0\\ P(y_x)-P(y_{x'})\\ P(y)-P(y_{x'})\\ P(y_x)-P(y)\end{array}\right\} \le \text{PNS} \le \min\left\{\begin{array}{c}P(y_x)\\ P(y'_{x'})\\ P(x,y)+P(x',y')\\ P(y_x)-P(y_{x'})+\\ +P(x,y')+P(x',y)\end{array}\right\} \tag{1}$$

$$\max\left\{\begin{array}{c}0\\ \frac{P(y)-P(y_{x'})}{P(x,y)}\end{array}\right\} \le \text{PN} \le \min\left\{\begin{array}{c}1\\ \frac{P(y'_{x'})-P(x',y')}{P(x,y)}\end{array}\right\} \tag{2}$$

Bounds for PS follow directly by symmetry, exchanging $x$ with $x'$ and $y$ with $y'$.

For subpopulations defined by a set of observed characteristics $C$, all expressions above can be conditioned on $C = c$. In this work, we focus on a different source of information. Rather than conditioning on observed covariates alone or assuming a fully specified causal diagram, we study how partial causal information can be incorporated as constraints into the underlying optimization problem. This perspective allows probabilities of causation to be bounded more tightly by restricting the feasible set of counterfactual distributions in a principled and modular way. Extensions to PN and PS follow analogously from the same construction.

## 3 Bounds with Partial Causal Information

### 3.1 Non-descendant Covariates

Theorems 4 and 5 in (Mueller et al., 2022) derived bounds on PNS under the assumption that a set of observed variables, $Z$, is available and that $Z$ contains no descendants of the treatment variable $X$. This assumption guarantees that counterfactual quantities such as $P(Y_X|Z)$ are well defined from the available data and that conditioning on $Z$ can impose additional constraints on the joint counterfactual distribution. Under these conditions, the resulting bounds were shown to be contained within the Tian-Pearl bounds. However, the treatment and outcome variables $X$ and $Y$ are restricted to the binary case, and the analysis assumes access to covariate-specific information for all values of $Z$ ($Z$ is a set of covariates), namely $P(Z)$ and $P(Y_x|z)$ for every $z \in Z$.

In the present work, we remove this requirement. We allow treatments and outcomes to take arbitrary finite values and permit partial covariate information to be incorporated in a modular manner. In particular, covariate-specific distributions need not be jointly available. For example, when multiple confounders $Z_1$ and $Z_2$ are present, the available data may consist of $P(Y_x|Z_1)$ and $P(Z_1)$ separately from $P(Y_x|Z_2)$ and $P(Z_2)$. Throughout, we adopt the definition of multivalued PNS introduced in (Shu et al., 2026), which provides a complete and unified characterization for arbitrary discrete treatments and outcomes.

In subsequent theorems, we further show how such covariate information can be combined with information on mediating variables. Confounders and mediators are treated within the same optimization-based formulation and can be incorporated jointly to impose additional constraints on the admissible counterfactual distributions. Rather than relying on variable-specific measurability arguments, we express the available data and structural assumptions as constraints in an optimization problem over counterfactual distributions. This formulation provides a unified treatment of different types of auxiliary information.

The main theorem in this paper gives sufficient conditions under which such constraints are valid and yields corresponding bounds on probabilities of causation. Mueller et al. (2022)'s results on non-descendant covariates and binary variables follow as a special case of this general formulation.

**Theorem 2.** *Given a partial causal diagram G and distribution compatible with G, let $Z_1, ..., Z_m$ be a set of $m$ variables that does not contain any descendant of $X$ in G. Let the variable $X$ has $n$ values $x_1, ..., x_n$ and $Y$ has $n$ values $y_1, ..., y_n$, $k \leq n$, then the probability of necessity and sufficiency(k) $P(y_{1_{x_1}}, ..., y_{k_{x_k}})$ (PNS(k)) is bounded as following:*

$$LB \leq PNS(k) \leq UB$$

*where LB and UB is the min and max solution to the following linear optimization problem, where the variables $p_{j_1 \ldots j_{n+m+1}}$ represent the probability $P(Y_{x_1}, ..., Y_{x_n}, Z_1, ..., Z_m, X)$, where $j_1, ..., j_n, j_{n+m+1} \in [1, n]$ and $j_{n+i} \in [1, |Z_i|]$ for $i \in [1, m]$.*

$$max/min \sum_{j_{k+1}, \ldots, j_{n+m+1}} p_{1 \ldots k j_{k+1} \ldots j_{n+m+1}}$$

$$\textit{Linear constraints:} \sum_{j_1, \ldots, j_{n+m+1}} p_{j_1 \ldots j_{n+m+1}} = 1, \textit{ and } p_{j_1 \ldots j_{n+m+1}} \geq 0,$$

$\forall s, t \in [1, n], r_i \in [1, |Z_i|], Z \subseteq \{Z_1, ..., Z_m\}$:

$$\sum_{\substack{j_1, \ldots, j_{t-1}, j_{t+1}, \ldots, j_n, \\ j_{n+m+1}, j_{n+i} \textit{ for } Z_i \notin Z}} p_{j_1 \ldots j_{t-1} s j_{t+1} \ldots j_n [j_{n+i} \textit{ if } Z_i \notin Z, r_i \textit{ o.w.}] j_{n+m+1}} = P(y_{s_{x_t}}, z_{i r_i} \textit{ for } Z_i \in Z), \quad (3)$$

$$\sum_{\substack{j_1, \ldots j_{t-1}, j_{t+1}, \ldots, j_n, \\ j_{n+i} \textit{ for } Z_i \notin Z}} p_{j_1 \ldots j_{t-1} s j_{t+1} \ldots j_n [j_{n+i} \textit{ if } Z_i \notin Z, r_i \textit{ o.w.}] t} = P(x_t, y_s, z_{i r_i} \textit{ for } Z_i \in Z). \quad (4)$$

The special case of the above Theorem 2 with $n = k = 2$ coincides with Theorem 4 in (Mueller et al., 2022). The constraints in Equations 3 and 4 enumerate all possible combinations when covariates $Z_1, \ldots, Z_m$ are jointly available, thereby yielding the tightest bounds. However, in general, the applicable constraints depend on the availability of data. For example, in many real-world settings, one may only observe covariate-specific data, such as $P(X, Y, Z_i)$ and $P(Y_X, Z_i)$ for each $i \in [1, n]$, without access to the full joint distributions $P(X, Y, Z_1, \ldots, Z_n)$ or $P(Y_X, Z_1, \ldots, Z_n)$. Even in this case, the proposed Theorem 2 still integrates information across covariates, rather than treating them separately, and thus produces tighter bounds, as formalized in the following corollary.

**Corollary 3.** *Given a partial causal diagram G and distribution compatible with G, let $Z_1, ..., Z_m$ be a set of $m$ variables that does not contain any descendant of $X$ in G. Let the variable $X$ has $n$ values $x_1, ..., x_n$ and $Y$ has $n$ values $y_1, ..., y_n$, $k \leq n$, then the probability of necessity and sufficiency(k) $P(y_{1_{x_1}}, ..., y_{k_{x_k}})$ (PNS(k)) is bounded as following:*

$$LB \leq PNS(k) \leq UB$$

*where LB and UB is the min and max solution to the following linear optimization problem, where the variables $p_{j_1 \ldots j_{n+m+1}}$ represent the probability $P(Y_{x_1}, ..., Y_{x_n}, Z_1, ..., Z_m, X)$, where $j_1, ..., j_n, j_{n+m+1} \in [1, n]$ and $j_{n+i} \in [1, |Z_i|]$ for $i \in [1, m]$.*

$$max/min \sum_{j_{k+1}, \ldots, j_{n+m+1}} p_{1 \ldots k j_{k+1} \ldots j_{n+m+1}}$$

$$\textit{Linear constraints:} \sum_{j_1, \ldots, j_{n+m+1}} p_{j_1 \ldots j_{n+m+1}} = 1, \textit{ and } p_{j_1 \ldots j_{n+m+1}} \geq 0,$$

$\forall s, t \in [1, n], i \in [1, .., m], r_i \in [1, |Z_i|]$:

$$\sum_{j_1, \ldots j_{t-1}, j_{t+1}, \ldots, j_{n+i-1}, j_{n+i+1} \ldots, j_{n+m+1}} p_{j_1 \ldots j_{t-1} s j_{t+1} \ldots j_{n+i-1} r_i j_{n+i+1} \ldots j_{n+m+1}} = P(y_{s_{x_t}}, z_{i r_i}), \quad (5)$$

$$\sum_{j_1, \ldots j_{t-1}, j_{t+1}, \ldots, j_{n+i-1}, j_{n+i+1} \ldots, j_{n+m}} p_{j_1 \ldots j_{t-1} s j_{t+1} \ldots j_{n+i-1} r_i j_{n+i+1} \ldots j_{n+m} t} = P(x_t, y_s, z_{i r_i}). \quad (6)$$

The above Corollary 3 represents a special case in which only covariate-specific data are available. Indeed, if joint distributions over any subset of covariates are also available (e.g., $P(X, Y, Z_1, Z_2)$), such information can be incorporated into Theorem 2 to obtain tighter bounds.

### 3.2 BACKDOOR SET WITH MEDIATORS

The back-door criterion (Pearl, 1993; 2022) is a special case of non-descendant covariates, to which Theorem 2 can be applied to obtain bounds on $PNS(k)$. However, incorporating additional information from mediators can further tighten these bounds. Theorem 6 in (Mueller et al., 2022) provides a binary illustration of this idea in a toy example with an empty back-door set. The following theorem systematically characterizes bounds on $PNS(k)$ in the presence of such additional information.

Given a causal diagram $G$ and distribution compatible with $G$, let $Z$ be a set of variables satisfying the back-door criterion (Pearl, 1993) in $G$, then the PNS is bounded as follows:

**Theorem 4.** *Given a causal diagram G and distribution compatible with G, let $Z_1, ..., Z_m$ be a set of $m$ variables satisfying the back-door criterion (Pearl, 1993) in G, and $W$ is a mediator of $X$ and $Y$. Let the variable $X$ has $n$ values $x_1, ..., x_n$ and $Y$ has $n$ values $y_1, ..., y_n$, $k \leq n$, then the probability of necessity and sufficiency(k) $P(y_{1_{x_1}}, ..., y_{k_{x_k}})$ (PNS(k)) is bounded as following:*

$$LB \leq PNS(k) \leq UB$$

*where LB and UB is the min and max solution to the following optimization problem, where the variables $p_{j_1 ... j_{2n+m+2}}$ represent the probability $P(Y_{x_1}, ..., Y_{x_n}, W_{x_1}, ..., W_{x_n}, Z_1, ..., Z_m, W, X)$, where $j_1, ..., j_n, j_{2n+m+2} \in [1, n]$ and $j_{2n+i} \in [1, |Z_i|]$ for $i \in [1, m]$, and $j_{2n+m+1}, j_{n+j} \in [1, |W|]$ for $j \in [1, n]$.*

$$max/min \sum_{j_{k+1}, ..., j_{2n+m+2}} p_{1...k j_{k+1} ... j_{2n+m+2}}$$

$$\textit{Constraints:} \sum_{j_1, ..., j_{2n+m+2}} p_{j_1 ... j_{2n+m+2}} = 1, \textit{ and } p_{j_1 ... j_{2n+m+2}} \geq 0,$$

$\forall s, t \in [1, n], r_i \in [1, |Z_i|], Z \subseteq \{Z_1, ..., Z_m\}, u \in [1, |W|]$:

$$\sum_{\substack{j_1, ... j_{t-1}, j_{t+1}, ... j_{2n}, \\ j_{2n+m+1}, j_{2n+m+2}, j_{2n+i} \textit{ for } Z_i \notin Z}} p_{\substack{j_1 ... j_{t-1} s j_{t+1} ... j_{2n} \\ [j_{2n+i} \textit{ if } Z_i \notin Z, r_i \textit{ o.w.}] j_{2n+m+1} j_{2n+m+2}}} = P(y_{s_{x_t}}, z_{ir_i} \textit{ for } Z_i \in Z), \quad (7)$$

$$\sum_{\substack{j_1, ... j_{t-1}, j_{t+1}, ..., j_{2n}, \\ j_{2n+i} \textit{ for } Z_i \notin Z}} p_{j_1 ... j_{t-1} s j_{t+1} ... j_{2n} [j_{2n+i} \textit{ if } Z_i \notin Z, r_i \textit{ o.w.}] u t} = P(x_t, y_s, w_u, z_{ir_i} \textit{ for } Z_i \in Z), \quad (8)$$

$\forall s, t, v \in [1, n], u \in [1, |W|], z_i \in [1, |Z_i|]$ *for* $i \in [1, m]$: *let*

$$\begin{aligned}
a &= \sum_{\substack{j_1, ... j_{t-1}, j_{t+1}, ..., j_{n+t-1}, \\ j_{n+t+1}, ..., j_{2n}, j_{2n+m+1}, j_{2n+m+2}}} p_{j_1 ... j_{t-1} s j_{t+1} ... j_{n+t-1} u j_{n+t+1} ... j_{2n} z_1 ... z_m j_{2n+m+1} j_{2n+m+2}}, \\
b &= \sum_{\substack{j_1, ... j_{t-1}, j_{t+1}, ..., j_{n+t-1}, \\ j_{n+t+1}, ..., j_{2n}, j_{2n+m+1}}} p_{j_1 ... j_{t-1} s j_{t+1} ... j_{n+t-1} u j_{n+t+1} ... j_{2n} z_1 ... z_m j_{2n+m+1} v}, \\
c &= \sum_{j_1, ..., j_{n+t-1}, j_{n+t+1}, ..., j_{2n}, j_{2n+m+1}, j_{2n+m+2}} p_{j_1 ... j_{n+t-1} u j_{n+t+1} ... j_{2n} z_1 ... z_m j_{2n+m+1} j_{2n+m+2}}, \\
d &= \sum_{j_1, ..., j_{n+t-1}, j_{n+t+1}, ..., j_{2n}, j_{2n+m+1}} p_{j_1 ... j_{n+t-1} u j_{n+t+1} ... j_{2n} z_1 ... z_m j_{2n+m+1} v},
\end{aligned}$$

$$a \times c = b \times d, \quad (9)$$

$\forall s, t, t' \in [1, n], t < t', u, v \in [1, |W|], z_i \in [1, |Z_i|]$ *for* $i \in [1, m]$*: let*

$$e = \sum_{\substack{j_1,\dots j_{t-1},j_{t+1},\dots,j_{n+t-1},\\ j_{n+t+1},\dots,j_{2n},j_{2n+m+1},j_{2n+m+2}}} p_{j_1\dots j_{t-1}sj_{t+1}\dots j_{n+t-1}uj_{n+t+1}\dots j_{2n}z_1\dots z_m j_{2n+m+1}j_{2n+m+2}},$$

$$f = \sum_{\substack{j_1,\dots j_{t-1},j_{t+1},\dots,j_{n+t-1},j_{n+t+1},\dots,j_{n+t'-1},\\ j_{n+t'+1},\dots,j_{2n},j_{2n+m+1},j_{2n+m+2}}} p_{\substack{j_1\dots j_{t-1}sj_{t+1}\dots j_{n+t-1}uj_{n+t+1}\dots j_{n+t'-1}v,\\ j_{n+t'+1}\dots j_{2n}z_1\dots z_m j_{2n+m+1}j_{2n+m+2}}}$$

$$g = \sum_{j_1,\dots,j_{n+t-1},j_{n+t+1},\dots,j_{2n},j_{2n+m+1},j_{2n+m+2}} p_{j_1\dots j_{n+t-1}uj_{n+t+1}\dots j_{2n}z_1\dots z_m j_{2n+m+1}j_{2n+m+2}},$$

$$h = \sum_{\substack{j_1,\dots j_{n+t-1},j_{n+t+1},\dots,j_{n+t'-1},\\ j_{n+t'+1},\dots,j_{2n},j_{2n+m+1},j_{2n+m+2}}} p_{\substack{j_1\dots j_{n+t-1}uj_{n+t+1}\dots j_{n+t'-1}v\\ j_{n+t'+1}\dots j_{2n}z_1\dots z_m j_{2n+m+1}j_{2n+m+2}}},$$

$$e \times g = f \times h. \tag{10}$$

**Remark.** Note that, similar to Theorem 2, Equations 7 and 8 are optional and depend on data availability. Equations 9 enforce the conditional independence $Y_x \perp\!\!\!\perp X \mid W_x, Z_1, ..., Z_m$ for all $x \in X$, while Equations 10 enforce $Y_x \perp\!\!\!\perp W_{x'} \mid W_x, Z_1, ..., Z_m$ for all $x, x' \in X$ with $x \neq x'$. The detailed proofs of all the theorems are provided in the appendix. Moreover, the causal graph $G$ may be a partial causal graph, provided that $Z_1, \dots, Z_m$ form a valid back-door set. Finally, $W$ may represent a collection of mediators, either by introducing a vector of mediators $W_1, W_2, \dots$, (and extend Theorem 4) or by applying the method of (Li & Pearl, 2022) to construct an equivalent mediator $W'$ with an expanded state space that jointly represents the mediator set.

**Remark (Scope of PoCs).** The above theorems are stated in terms of PNS mainly for convenience of presentation. Since PNS is defined through a joint counterfactual event, it represents the most demanding case and makes clear how partial causal information constrains the space of admissible counterfactual distributions. The same set of constraints, however, applies to other probabilities of causation (PoCs). In particular, other PoCs can be obtained from the same optimization program by changing only the objective function. For example, in Theorem 2, bounds on PN can be derived by replacing the objective with $\max / \min \sum_{j_{k+1},\dots,j_{n+m+1}} p_{1\dots k\, j_{k+1}\dots j_{n+m+1}\, 1}$, while leaving all constraints unchanged. No additional structural assumptions are required for these extensions.

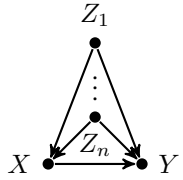
## 4 Simulation Results

Since Mueller et al. (2022) already provides toy examples in simple binary settings, this paper focuses on simulated comparing the proposed Theorems 2, 3, and 4 with the Tian–Pearl bounds in Equation 1, as well as the Mueller–Li–Pearl bounds in (Mueller et al., 2022). Although the proposed bounds strictly generalize the Tian–Pearl and Mueller–Li–Pearl bounds by accommodating multi-valued treatments and outcomes, our comparisons are restricted to binary settings, since both the Tian–Pearl and Mueller–Li–Pearl bounds are only defined for binary variables.

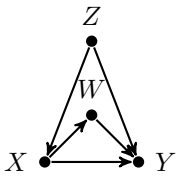
### 4.1 Multiple non-descendant covariates with covariate-specific data

We first consider settings with varying numbers of non-descendant covariates, as illustrated in Figure 1a. As discussed earlier, the available information consists only of covariate-specific data, namely $P(X, Y, Z_i)$ and $P(Y_X, Z_i)$ for each covariate $Z_i$.

Following (Mueller et al., 2022), we randomly generate $1000$ sample distributions compatible with the causal diagrams in Figure 1a, for $n = 1, \dots, 6$, corresponding to Corollary 3 (the observational distributions are generated following the CPT generation in Appendix E of (Mueller et al., 2022), and the experimental distributions are computed using the adjustment formula in (Pearl, 1993) based on the generated observational distributions). The Mueller–Li–Pearl bounds are computed using the best single covariate, since no joint covariate distributions are available. Specifically, for each sample distribution, the bounds are computed separately for each covariate, and the tightest lower and upper bounds among them are selected.




(a) $Z_1, \ldots, Z_n$ are non-descendants of $X$.




(b) $Z$ is the backdoor set and $W$ is the mediator.

Figure 1: Causal structures used in the experiments.

Table 1: Performance comparison of bounds with multiple non-descendant covariates using covariate-specific data.

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ |
|---|---|---|---|---|---|---|
| ↑ TP lb | 0.0263 | 0.0407 | 0.0453 | 0.0478 | 0.0436 | 0.0402 |
| ↓ TP ub | 0.0251 | 0.0421 | 0.0447 | 0.0478 | 0.0422 | 0.0398 |
| ↑ MLP lb | 0 | 0.0007 | 0.0020 | 0.0033 | 0.0037 | 0.0043 |
| ↓ MLP ub | 0 | 0.0009 | 0.0020 | 0.0033 | 0.0035 | 0.0039 |
| gap (TP) | 0.2136 | 0.2690 | 0.3101 | 0.3469 | 0.3693 | 0.3965 |
| gap (MLP) | 0.1622 | 0.1878 | 0.2242 | 0.2578 | 0.2907 | 0.3247 |
| gap (Col. 3) | 0.1622 | 0.1862 | 0.2201 | 0.2512 | 0.2835 | 0.3165 |
| # imp. TP | 844 | 974 | 994 | 1000 | 1000 | 1000 |
| # imp. MLP | 0 | 638 | 774 | 880 | 913 | 937 |

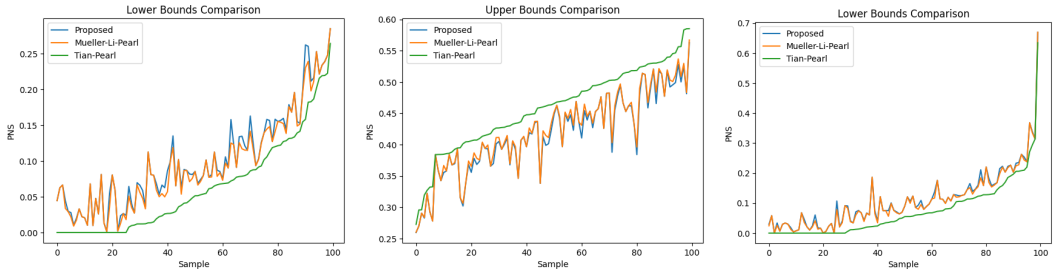
For each sampled distribution $i$, let $[a_i, b_i]$ denote the bounds obtained using the proposed theorem, $[c_i, d_i]$ the Tian–Pearl bounds, and $[e_i, f_i]$ the Mueller–Li–Pearl bounds. For each causal diagram, we report the following summary statistics, as shown in Table 1:

- Average increase in the Tian–Pearl (TP) lower bound (lb): $\frac{1}{1000}\sum(a_i - c_i)$
- Average decrease in the Tian–Pearl upper bound (ub): $\frac{1}{1000}\sum(d_i - b_i)$
- Average increase in the Mueller–Li–Pearl (MLP) lower bound (lb): $\frac{1}{1000}\sum(a_i - e_i)$
- Average decrease in the Mueller–Li–Pearl upper bound (ub): $\frac{1}{1000}\sum(f_i - b_i)$
- Average gap of the Tian–Pearl bounds: $\frac{1}{1000}\sum(d_i - c_i)$
- Average gap of the Mueller–Li–Pearl bounds: $\frac{1}{1000}\sum(f_i - e_i)$
- Average gap using Corollary 3: $\frac{1}{1000}\sum(b_i - a_i)$
- Number of sample distributions for which Corollary 3 improves upon the Tian–Pearl bounds: $\sum g_i$, where $g_i = 1$ if $(a_i > c_i)$ or $(b_i < d_i)$, and $g_i = 0$ otherwise
- Number of sample distributions for which Corollary 3 improves upon the Mueller–Li–Pearl bounds: $\sum h_i$, where $h_i = 1$ if $(a_i > e_i)$ or $(b_i < f_i)$, and $h_i = 0$ otherwise

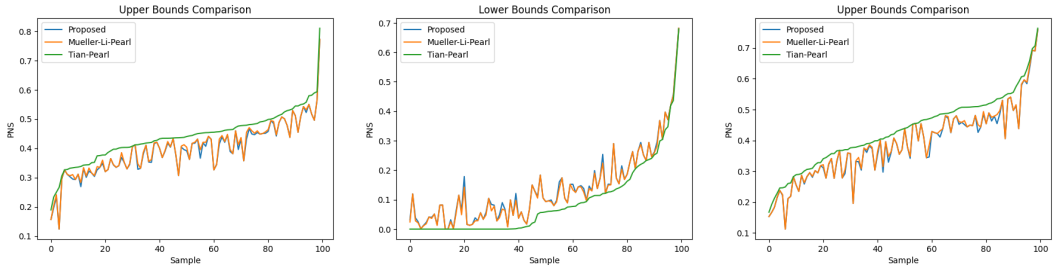
For each $n = 4, \ldots, 6$, 100 out of 1,000 sampled distributions are randomly selected, sorted according to their lower and upper Tian–Pearl bounds, and then used to plot the Tian–Pearl, Müller–Li–Pearl, and proposed bounds (Figures 2a–2f).

Table 1 demonstrates that incorporating multiple non-descendant covariates through Corollary 3 consistently tightens the bounds relative to both the Tian–Pearl (TP) and Mueller–Li–Pearl (MLP) baselines. Compared with the TP bounds, the proposed method achieves substantial average improvements in both the lower and upper bounds across all values of $n$, with mean lower-bound increases and upper-bound decreases of approximately $0.02$–$0.04$. These improvements translate into a marked reduction of the bound width: while the average TP gap increases from $0.2136$ to $0.3965$ as $n$ grows, the gap under Corollary 3 remains consistently smaller, increasing only from $0.1622$ to $0.3165$.

Relative to the MLP bounds, Corollary 3 yields modest but steadily growing improvements as the number of covariates increases. Although no improvement is possible when $n = 1$, the average gains in both the lower and upper bounds increase monotonically with $n$, and the resulting gaps are uniformly smaller than those of MLP. Correspondingly, the proportion of sample distributions benefiting from Corollary 3 increases rapidly with $n$, exceeding $99\%$ relative to TP for $n \geq 3$, and reach-




(a) PNS lower bounds for $n = 6$. (b) PNS upper bounds for $n = 6$. (c) PNS lower bounds for $n = 5$.




(d) PNS upper bounds for $n = 5$. (e) PNS lower bounds for $n = 4$. (f) PNS upper bounds for $n = 4$.

Figure 2: PNS bounds under Figure 1a.

Table 2: Performance comparison for the mediator case with a back-door set under the causal structure in Figure 1b.

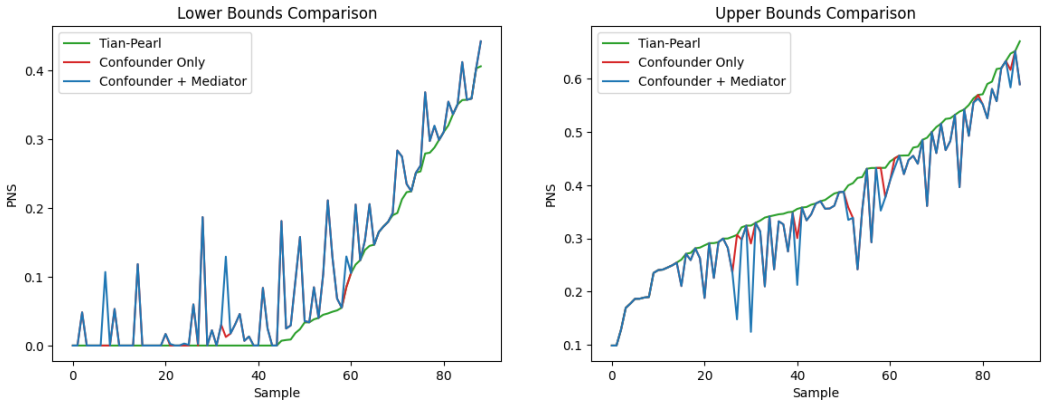
| ↑ TP lb | ↓ TP ub | ↑ MLP lb | ↓ MLP ub | gap (TP) |
|---|---|---|---|---|
| 0.0268 | 0.0258 | 0.0057 | 0.0051 | 0.2833 |
| gap (MLP) | gap (Thm 4) | # impr. TP | # impr. MLP | |
| 0.2417 | 0.2308 | 782 | 245 | |

ing over $90\%$ relative to MLP by $n = 5$. These results confirm that aggregating covariate-specific information across multiple non-descendant covariates can substantially tighten bounds even in the absence of joint covariate distributions.

### 4.2 Backdoor Set with Mediators

As for the mediator case with backdoor set, we compare the proposed Theorem 4 with the Tian–Pearl bounds and the Mueller–Li–Pearl bounds under the causal structure shown in Figure 1b, which includes one confounder and one mediator. In this setting, the Tian–Pearl bounds do not exploit either additional covariates, the Mueller–Li–Pearl bounds utilize only the confounder, and the proposed theorem incorporates information from both the confounder and the mediator. Following the same simulation protocol as before, we generate $1000$ compatible sample distributions. We report the same summary statistics as before, as shown in Table 2.

The results in Table 2 indicate that incorporating mediator information consistently tightens the bounds. Compared to the Tian–Pearl bounds, Theorem 4 achieves a substantial reduction in the average bound gap, highlighting the benefit of exploiting auxiliary structure beyond using only treatment and outcome variables, without considering any covariates. Although the Mueller–Li–Pearl bounds already improve upon Tian–Pearl by conditioning on the confounder, their framework does not allow confounder and mediator information to be incorporated simultaneously. In contrast, the proposed formulation flexibly integrates both sources of information through additional constraints, yielding the smallest average gap among all methods. Moreover, Theorem 4 improves upon the Tian–Pearl bounds for the majority of sampled distributions and also provides nontrivial improvements over the Mueller–Li–Pearl bounds. Due to space limitations, we restrict attention to a single mediator; investigating the effect of multiple mediators is left for future work.




(a) PNS lower bounds comparison.

(b) PNS upper bounds comparison.

Figure 3: PNS bounds comparison under Figure 1b.

We further randomly select 100 out of the 1000 sampled distributions, sort them by their Tian–Pearl lower and upper bounds, and plot the corresponding Tian–Pearl, Mueller–Li–Pearl, and proposed bounds in Figures 3a and 3b.

## 5 Conclusion

This paper studies the problem of bounding probabilities of causation when structural assumptions and auxiliary statistical information are incomplete but informative. Rather than assuming either no structural information or a fully specified causal diagram, we propose a constraint-based framework that systematically incorporates partial structural assumptions and partial auxiliary statistical information as constraints on counterfactual distributions. This formulation provides a unified way to combine experimental data, observational data, and heterogeneous structural and statistical information within a single optimization framework.

Our results extend existing bounds in several directions. The proposed framework subsumes the classical Tian–Pearl bounds and existing graph-based bounds as special cases, naturally extends to multivalued treatments and outcomes, and flexibly incorporates partial covariate and mediator information within a unified optimization program. In each case, the resulting bounds remain formally valid under incomplete structural assumptions and partial auxiliary statistical information.

Simulation results show that systematically incorporating partial structural assumptions and partial auxiliary statistical information consistently yields tighter bounds than existing baselines. Overall, this work demonstrates that meaningful progress on individualized causal reasoning is possible without full causal identification and extends the applicability of probabilities of causation to realistic settings with incomplete structural assumptions and partial auxiliary statistical information.

## 6 Limitations and Future Work.

A limitation of our framework is computational. The programs in Theorems 2 and 3 are linear and can be solved efficiently with standard linear programming solvers, even for moderately sized discrete domains. However, incorporating mediator information in Theorem 4 introduces nonlinear (bilinear) constraints induced by conditional independence assumptions, resulting in a nonconvex optimization problem that may increase runtime as the state spaces grow. An important direction for future work is to develop more scalable formulations that preserve most of the tightening effect while improving computational efficiency, for example through lower-dimensional mediator representations, equivalent linear formulations, or tractable relaxations with certified bounds.

### AI USE STATEMENT

In this work, we used generative AI tools for polish writing and finding related work. We have not used generative AI tools for research ideation or execution, draft sections of the paper, generating synthetic datasets, and proving mathematical claims. We have reviewed all AI-assisted work. We checked manually for all the suggested related work and all the polished writing. We take responsibility for the final content of this work, including text, claims or artifacts produced with the aid of generative AI.

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# A Appendix

## A.1 Proof of Theorem 2

*Proof.* The theorem follows directly from the fact that the variables $p_{j_1 \ldots j_{n+m+1}}$ represent the joint probability $P(Y_{x_1}, \ldots, Y_{x_n}, Z_1, \ldots, Z_m, X)$. Equation 3 encodes the experimental distribution constraints, while Equation 4 encodes the observational distribution constraints.

□

## A.2 Proof of Corollary 3

*Proof.* Again, the theorem follows directly from the fact that the variables $p_{j_1 \ldots j_{n+m+1}}$ represent the joint probability $P(Y_{x_1}, \ldots, Y_{x_n}, Z_1, \ldots, Z_m, X)$. Equation 5 encodes the experimental distribution constraints, while Equation 6 encodes the observational distribution constraints. □

## A.3 Proof of Theorem 4

*Proof.* The theorem again follows from the fact that the variables $p_{j_1 \ldots j_{2n+m+1}}$ represent the joint probability $P(Y_{x_1}, ..., Y_{x_n}, W_{x_1}, ..., W_{x_n}, Z_1, ..., Z_m, W, X)$, Equation 7 encodes the experimental distribution constraints, while Equation 8 encodes the observational distribution constraints.

Moreover, since $(Z_1, \ldots, Z_m)$ forms a backdoor set, it follows that $Y_x \perp\!\!\!\perp X \mid W_x, Z_1, \ldots, Z_m$ for all $x \in X$, and $Y_x \perp\!\!\!\perp W_{x'} \mid W_x, Z_1, \ldots, Z_m$ for all $x, x' \in X$ with $x \neq x'$. These conditional independences can be read off from the twin network representation (Pearl, 2009). Equation 9 encodes the former independence, while Equation 10 encodes the latter. □

## A.4 Data Generating Process: Multiple Non-descendant Covariates with Covariate-specific Data

### A.4.1 Setup and Notation:

Consider a structural causal model with a binary treatment $X$, a binary outcome $Y$, and $n$ binary covariates as in Figure 1a. Let

$$Z = \{z_1, \ldots, z_n\}$$

denote the full covariate vector, and for a fixed index $k$, let

$$Z_k = Z \setminus \{z_k\}$$

denote the remaining $n - 1$ covariates with $z_k$ removed. So $z_k$ is a single fixed value, while $Z_k$ (uppercase) is the set of everything else. Every $Z_i$ is a parent of both $X$ and $Y$:

$$Z_i \to X, \qquad Z_i \to Y, \qquad X \to Y, \qquad i = 1, \ldots, n.$$

Note that $Z$ satisfies the back-door criterion relative to $(X, Y)$.

### A.4.2 Deriving the adjustment for a single coordinate $z_k$

By definition,

$$P(y_x, z_k) = P(y_x \mid z_k)\, P(z_k).$$

Marginalizing $P(y_x \mid z_k)$ over $Z_k$ and applying the z specific adjustment formula, $P(y \mid do(x), Z) = P(y \mid x, Z)$ gives

$$P(y_x, z_k) = \sum_{Z_k} P(y \mid x, Z)\, P(Z_k \mid z_k)\, P(z_k).$$

Because the $z_i$ are mutually independent, $P(Z_k \mid z_k) = P(Z_k) = \prod_{z_i \in Z_k} P(z_i)$, so the expression collapses to

$$P(y_x, z_k) = \sum_{Z_k} P(y \mid x, Z)\, P(Z), \qquad P(Z) = \prod_{i=1}^{n} P(z_i),$$

which is exactly the quantity the Algorithm 1 below computes: for every value of $Z_k$, rebuild the full vector $Z$ by combining it with the fixed $z_k$, then evaluate $P(y \mid x, Z)$ and $P(Z)$ on that full vector. The independence simplification is what lets the joint $P(Z)$ be computed as a flat product of marginals rather than a conditional chain $P(z_1)\prod_{i\geq 2} P(z_i \mid z_1, \ldots, z_{i-1})$. If the covariates were structurally dependent, the formula above with $P(Z_k|z_k)$ would still hold, but the product of marginals shortcut would not. The data generating process is shown in Algorithm 1.

### A.5 Data Generating Process: Backdoor Set with Mediators

#### A.5.1 Setup

Consider a structural causal model with a binary treatment $X$, a binary outcome $Y$, one covariates $Z$, and one mediator $W$ with the graph in Figure 1b.

The joint factorizes according to the graph as

$$P(x, y, z, w) = P(z) \cdot P(x \mid z) \cdot P(w \mid x) \cdot P(y \mid x, z, w).$$

#### A.5.2 Deriving $P(y_x, z)$

Under $do(X{=}x)$, the truncated factorization deletes only the equation for the intervened variable, $P(x \mid z)$, and leaves every other mechanism unchanged:

$$P(y, z, w \mid do(X{=}x)) = P(z) \cdot P(w \mid x) \cdot P(y \mid x, z, w).$$

Marginalizing out $W$, since it is not part of the target, gives

$$\begin{aligned} P(y_x, z) &= \sum_w P(z) \cdot P(w \mid x) \cdot P(y \mid x, z, w) \\ &= P(z) \sum_w P(w \mid x)\, P(y \mid x, z, w). \end{aligned}$$

.

**Equivalent observational identity.** Note that,

$$P(y \mid x, z) = \frac{P(x, y, z)}{P(x, z)} = \sum_w P(w \mid x)\, P(y \mid x, z, w),$$

which implies that $P(y_x, z) = P(y|x, z)\, P(z)$, consistent with the back-door adjustment since $Z$ is a back-door adjustment set.

Note that we set $\mathrm{low} = 0.05$ and $\mathrm{high} = 0.95$ in Algorithm 2. This choice is not part of the causal modeling assumptions, but rather serves as a numerical safeguard for the nonlinear optimization. Restricting every factor to the interval $[0.05, 0.95]$ keeps each product away from flat or numerically unstable regions, resulting in a better-conditioned optimization landscape for the solver without altering the underlying causal structure being represented. The detailed data generating process is shown in Algorithm 2.

**Algorithm 1:** Random SCM generation

**Input:** $n$ (number of covariates), $\alpha$ (Beta concentration,default to 1 for uniform distribution and seed=43), $n_{\text{sets}}$ (number of samples)

**Output:** Interventional tables $P^{do}_{X=1}, P^{do}_{X=0}$ and observational table $P^{obs}$, each indexed by sample $s$ and covariate $k$

**for** $s \leftarrow 1$ **to** $n_{\text{sets}}$ **do**
  // draw random SCM observational distributions
  **for** $i \leftarrow 1$ **to** $n$ **do**
    // marginal over $\{0,1\}$, independent across $i$
    draw $P(z_i = 1) \sim \text{Beta}(\alpha, \alpha)$;
  **for** *each* $Z \in \{0,1\}^n$ **do**
    // treatment node
    draw $P(X{=}1 \mid Z) \sim \text{Beta}(\alpha, \alpha)$;
  **for** *each* $x \in \{0,1\}$, *each* $Z \in \{0,1\}^n$ **do**
    // outcome node
    draw $P(Y{=}1 \mid X{=}x, Z) \sim \text{Beta}(\alpha, \alpha)$;

  // For each $k$: $Z_k = Z \setminus \{z_k\}$
  **for** $k \leftarrow 1$ **to** $n$ **do**
    **for** *each* $(x, y, z_k) \in \{0,1\}^3$ **do**
      // observational data:
      // $P(x, y, z_k) = \sum_{Z_k} P(Z) \cdot P(x \mid Z) \cdot P(y \mid x, Z)$
      $P^{obs}_{s,k}(x, y, z_k) \leftarrow \sum_{Z_k} P(Z) \cdot P(x \mid Z) \cdot P(y \mid x, Z)$;
    **for** *each* $(x, y, z_k) \in \{0,1\}^3$ **do**
      // interventional data:
      // $P(y_x, z_k) = P(z_k) \sum_{Z_k} P(Z_k \mid z_k)\, P(y \mid x, Z)$
      // $\quad = \sum_{Z_k} P(Z) \cdot P(y \mid x, Z)$ under independence of the $z_i$'s
      tmp $\leftarrow \sum_{Z_k} P(Z) \cdot P(y \mid x, Z)$;
      **if** $x = 1$ **then** append tmp to $P^{do}_{X=1,\, s,k}$;
      **else** append tmp to $P^{do}_{X=0,\, s,k}$;
**return** $P^{do}_{X=1}, P^{do}_{X=0}, P^{obs}$;

**Algorithm 2:** Random SCM generation

**Input:** low, high (uniform range for SCM parameters and seed=42), $n_{\text{sets}}$ (number of samples)
**Output:** Observational table $P^{obs}$ over $(X, Y, Z, W)$, interventional tables $P^{do}_{X=1}, P^{do}_{X=0}$ over $(Z)$

**for** $s \leftarrow 1$ **to** $n_{\text{sets}}$ **do**
  // Draw random SCM parameters
  draw $P(Z{=}1) \sim \text{Uniform}(\text{low}, \text{high})$;
  **for** *each* $z \in \{0, 1\}$ **do**
    // treatment mechanism, depends on $Z$ only
    draw $P(X{=}1 \mid z) \sim \text{Uniform}(\text{low}, \text{high})$;
  **for** *each* $x \in \{0, 1\}$ **do**
    // mediator mechanism, depends on $X$ only
    draw $P(W{=}1 \mid x) \sim \text{Uniform}(\text{low}, \text{high})$;
  **for** *each* $x \in \{0, 1\}$, *each* $z \in \{0, 1\}$, *each* $w \in \{0, 1\}$ **do**
    // outcome mechanism, depends on $X, Z, W$
    draw $P(Y{=}1 \mid x, z, w) \sim \text{Uniform}(\text{low}, \text{high})$;

  // observational joint: direct product over the graph's factorization
  **for** *each* $(x, y, z, w) \in \{0, 1\}^4$ **do**
    $P^{obs}_s(x, y, z, w) \leftarrow P(z) \cdot P(x \mid z) \cdot P(w \mid x) \cdot P(y \mid x, z, w)$;

  // interventional target via the observational identity
  **for** *each* $x \in \{0, 1\}$ **do**
    **for** *each* $z \in \{0, 1\}$ **do**
      // denominator:
      // $P(x, z) = \sum_{y,w} P^{obs}_s(x, y, z, w)$
      $P_{xz} \leftarrow \sum_{y,w} P^{obs}_s(x, y, z, w)$;
      **for** *each* $y \in \{0, 1\}$ **do**
        // numerator:
        // $P(x, y, z) = \sum_{w} P^{obs}_s(x, y, z, w)$
        $P_{xyz} \leftarrow \sum_{w} P^{obs}_s(x, y, z, w)$;
        // $P(y \mid x, z) = P_{xyz}/P_{xz}$, then multiply by $P(z)$
        val $\leftarrow \left(P_{xyz}/P_{xz}\right) \cdot P(z)$;
        **if** $x = 1$ **then** $P^{do}_{X=1,s}(z) \leftarrow$ val;
        **else** $P^{do}_{X=0,s}(z) \leftarrow$ val;
**return** $P^{obs}, P^{do}_{X=1}, P^{do}_{X=0}$;