ON THE CONVERGENCE OF LOW-PRECISION LORA TRAINING

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ABSTRACT

Low-rank adaptation (LoRA) has become a standard approach for parameter-efficient fine-tuning, substantially reducing the number of trainable parameters and the optimizer memory required during training. Quantization offers a further route to resource-efficient fine-tuning, but many existing quantized LoRA methods focus on weight-only or deployment-oriented compression, leaving unclear how low precision affects optimization when it is used throughout training. In this work, we initiate a systematic theoretical study of low-precision LoRA training in which the data, pretrained weights, adapters, gradients, and intermediate matrix multiplications are quantized during optimization. For spectrally initialized LoRA under a low-rank linear model with sub-Gaussian data and relative quantization error, we establish finite-horizon convergence upper bounds that consist of a full-precision linear convergence term plus a quantization-dependent additive error floor. Our results show that low-precision LoRA can retain convergence behavior close to its full-precision counterpart, provided that the mantissa length grows only logarithmically with the inverse of the full-precision contraction rate.

1 INTRODUCTION

Low-rank adaptation (LoRA) has emerged as the most prevalent paradigm for parameter-efficient fine-tuning (PEFT) of large-scale pretrained models (Hu et al., 2022). By freezing the pretrained weights and learning low-rank updates through a pair of smaller adapter matrices, LoRA dramatically reduces the number of trainable parameters and the associated optimizer memory overhead. However, as parameter counts continue to scale, reducing the number of trainable parameters alone does not remove the computational and memory cost of the full training execution graph, including forward activations, backward gradients, and matrix multiplications (Chavan et al., 2024). Efficient acceleration and memory reduction of LoRA fine-tuning are therefore essential to enable timely, scalable, and cost-effective deployment of large pretrained models under practical computational constraints.

Quantization presents a compelling solution to these resource constraints. A growing body of empirical research demonstrates that quantizing pretrained weights can significantly reduce the memory footprint of fine-tuning without compromising downstream performance (Dettmers et al., 2023; Xu et al., 2023; Guo et al., 2023; Li et al., 2023; Kim et al., 2024; Qin et al., 2024). However, many existing methods are largely static, weight-only, or deployment-oriented: they compress the frozen backbone while keeping the LoRA adapters, gradients, and intermediate matrix multiplication operations in higher precision. This leaves a critical gap in our understanding of training-time low precision: Can the LoRA training execution graph, including forward and backward arithmetic, be quantized while preserving the behavior of full-precision adaptation? While recent work such as Choi et al. (2025) has sought to accelerate LoRA through low-precision arithmetic, such approaches often rely on specialized reformulations of the forward and backward passes and limit quantization primarily to weights and activations. Consequently, it remains unclear when low-precision arithmetic can be used throughout LoRA training without fundamentally changing its optimization behavior.

Theoretically, a growing body of literature has sought to establish a rigorous foundation for LoRA's empirical success. Beyond analyzing expressive power (Zeng & Lee, 2023) and optimization landscapes (Jang et al., 2024; Kim et al., 2025; Liu et al., 2025), recent works investigate the training dynamics and convergence guarantees of LoRA (Malinovsky et al., 2024; Sokolov et al., 2025;

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Xu et al., 2025; Mu & Klabjan, 2025; Zhang et al., 2025b). From the perspective of continuous gradient flow, Xu et al. (2025) studied LoRA under gradient flow for matrix factorization, revealing an alignment phase where the singular vectors of the LoRA weights gradually orient themselves to correct the misalignment between the pre-trained model and the target task. For discrete gradient descent, Mu & Klabjan (2025) proves that LoRA gradient descent converges to a stationary point at rate $O(1/\log T)$, assuming only that the original loss function is Lipschitz smooth and lower bounded. Zhang et al. (2025b) proves linear convergence for spectrally initialized LoRA under sub-Gaussian downstream features and mean-squared loss. However, these theoretical frameworks are confined to the full-precision setting. They do not explain whether the alignment and contraction mechanisms behind LoRA convergence are stable when the quantities used in the forward and backward passes are quantized. This leaves a fundamental question unanswered: *How does quantization alter the optimization behavior and convergence guarantees of LoRA?*

In this work, we initiate a theoretical study of low-precision LoRA training. We analyze gradient descent (GD) for multi-output linear regression with a frozen pretrained matrix $\mathbf{W}^\natural$ and a rank-r^* target update $\Delta = \mathbf{W}^* - \mathbf{W}^\natural$, learned through LoRA adapters $\mathbf{A} \in \mathbb{R}^{d\times r}$ and $\mathbf{B} \in \mathbb{R}^{r\times k}$. To model training-time low-precision arithmetic, we quantize every tensor entering the matrix multiplications in the forward and backward passes: data features, pretrained weights, adapter factors, output residuals, and intermediate activations. Our goal is to bound the approximation error $R_t = \|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$ and to identify precision regimes under which the low-precision recursion preserves the linear convergence behavior of its full-precision counterpart.

Our theoretical results. Under sub-Gaussian downstream data, a low-rank target condition $r^* \le r$, spectral initialization, and coordinate-wise relative quantization error, we prove finite-horizon convergence rates for low-precision LoRA. Informally, writing ϵ for the worst relative quantization error, our results take the form

$$R_T \lesssim \sqrt{r}\left[(1-\rho)^T + D(\epsilon)\right], \quad D(\epsilon) = \begin{cases} \epsilon\sqrt{d}, & r = r^*, \\ \epsilon\sqrt{d} + \left(T\eta\epsilon\sqrt{d}\right)^2, & r > r^*, \end{cases} \tag{1}$$

where ρ is the full-precision contraction factor determined by the stepsize η and the spectrum of Δ. The first term is the same geometric decay that appears in full-precision LoRA. The second term is a quantization-dependent floor. Our analysis shows that exact rank LoRA ($r = r^*$) and over-parameterized LoRA ($r > r^*$) have qualitatively different low-precision behavior: both retain the same linear contraction term and a static $\epsilon\sqrt{d}$ floor, but over-parameterization introduces an additional horizon-dependent term $(T\eta\epsilon\sqrt{d})^2$ caused by quantization drift in non-signal rank directions. The formal statements are given in Theorems 4.2 and 4.3.

Our contributions. We summarize the main contributions as follows.

- We formulate a training-time low-precision LoRA recursion that quantizes not only frozen weights, but also data features, adapter factors, residuals, and intermediate products in both forward and backward matrix multiplications. For this recursion, we prove convergence bounds for the approximation error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$ between the learned LoRA update and the target update, showing that quantization affects LoRA convergence through an additive error floor.
- We prove that in the exact rank regime $r = r^*$, low-precision LoRA has a horizon-independent additive floor (Theorem 4.2), while in the over rank regime $r > r^*$, the floor contains an additional horizon-dependent accumulated term (Theorem 4.3). This separates the role of signal-rank directions from surplus rank directions and shows that over-parameterized LoRA requires stricter precision control.
- We derive explicit precision conditions under which the low-precision LoRA convergence rate matches the full-precision rate in both exact rank and over rank cases. These conditions translate the convergence bounds into bit-level requirements on the mantissa length of the quantized tensors. Theoretical results show that over rank case require more bits, i.e., higher precision, to recover the full-precision performance compared with exact rank case.

Notations. For two positive-valued functions $f(x)$ and $g(x)$, we write $f(x) \lesssim g(x)$ (or equivalently, $f(x) \le O(g(x))$) or $f(x) \gtrsim g(x)$ (or equivalently, $f(x) \ge \Omega(g(x))$) if $f(x) \le cg(x)$ or $f(x) \ge cg(x)$ holds for some absolute (if not otherwise specified) constant $c > 0$ respectively. We write $f(x) \eqsim g(x)$ (or equivalently, $f(x) = \Theta(g(x))$) if $f(x) \lesssim g(x) \lesssim f(x)$. We use $\|\cdot\|_{op}$ to denote the operator norm and $\|\cdot\|_F$ to denote the Frobenius norm for matrices.

2 Related Work

Empirical quantized variants of LoRA. One prominent line of research focuses on weight-only quantized LoRA, which reduces memory by quantizing the frozen backbone while keeping activations, gradients, adapters, and most training arithmetic in higher precision. QLoRA (Dettmers et al., 2023) pioneered this direction through 4-bit NF4 weight quantization and double quantization, while QA-LoRA (Xu et al., 2023) integrated quantization-aware adaptation with LoRA so that the fine-tuned update can be merged into a quantized model. Follow-up methods improve ultra-low-bit fine-tuning by changing which low-dimensional variables are trained or how adaptation capacity is allocated (Kim et al., 2023; 2024; Rajabzadeh et al., 2024; Yin et al., 2024). Other works reduce quantization mismatch through initialization or error compensation: LoftQ (Li et al., 2023) initializes LoRA adapters jointly with quantized base weights, LQ-LoRA (Guo et al., 2023) decomposes pretrained weights into a quantized component and a high-precision low-rank residual, and subsequent methods preserve information, calibrate initialization, or compensate quantization residuals (Qin et al., 2024; Deng et al., 2025). Closest in spirit to our setting, RoLoRA (Huang et al., 2024) studies rotated outlier-free models for weight-activation quantization, and FALQON (Choi et al., 2025) accelerates LoRA fine-tuning by merging adapters into an FP8-quantized backbone and reformulating forward and backward computation with quantized weights and activations.

Theoretical analysis of LoRA. Theoretical studies of LoRA have analyzed its expressive power (Zeng & Lee, 2023), optimization landscape, and training dynamics. In the Neural Tangent Kernel (NTK) regime, Jang et al. (2024) showed that LoRA has no spurious local minima when the parameterized rank satisfies $r \gtrsim \sqrt{N}$, and Kim et al. (2025) extended landscape analysis to nonlinear settings, showing that a local minimizer is either a small-rank global minimizer or a large-magnitude spurious minimizer. Liu et al. (2025) compared LoRA and full-rank fine-tuning through the Polyak-Łojasiewicz (PL*) region and found that this benign region can be much smaller for LoRA. Beyond landscapes, recent work studies LoRA trajectories and convergence. RAC-LoRA (Malinovsky et al., 2024) and Bernoulli-LoRA (Sokolov et al., 2025) analyze variants that train one adapter while freezing the other as a fixed projection. For standard LoRA, Zhang et al. (2025b) proved that gradient descent aligns adapters with the top singular subspaces of the one-step full fine-tuning gradient, motivating spectral initialization and yielding linear convergence for $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$. From a continuous-time perspective, Xu et al. (2025) studied matrix-factorization gradient flow and identified an alignment phase, while Mu & Klabjan (2025) proved an $O(1/\log T)$ stationary-point rate for LoRA gradient descent under smoothness and lower-boundedness assumptions.

Theoretical analysis of low-precision training. A fundamental line of work establishes convergence guarantees for quantized training algorithms. Early analyses studied SGD with quantized gradients, often under unbiased quantizers and fixed quantization schemes (De Sa et al., 2015; Alistarh et al., 2017; Jiang & Agrawal, 2018); Faghri et al. (2020) extended this direction to adaptive quantization. To handle biased quantizers, subsequent work introduced error-feedback mechanisms that compensate quantization bias while preserving convergence (Karimireddy et al., 2019; Zheng et al., 2019; Koloskova et al., 2019). Complementary results analyze quantized model weights (Markov et al., 2023), vector quantized SGD (Gandikota et al., 2021), and quantized adaptive optimizers such as Adam (Kingma & Ba, 2014) and Muon (Jordan et al., 2024) (Tang et al., 2025). Beyond optimization, Zhang et al. (2025a) established generalization bounds for low-precision SGD in linear regression, and Zhang et al. (2026) derived precision scaling laws under Gaussian sketched linear regression. These works provide important tools for understanding low-precision training, but they do not address the coupled bilinear dynamics and training quantization effects that arise in LoRA.

3 Theoretical Setup

3.1 Low-precision LoRA training

We study LoRA fine-tuning for multi-output linear regression with a frozen pretrained matrix $\mathbf{W}^\natural \in \mathbb{R}^{d\times k}$. Let $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_N]^\top \in \mathbb{R}^{N\times d}$ and $\widetilde{\mathbf{Y}} = [\widetilde{\mathbf{y}}_1, \ldots, \widetilde{\mathbf{y}}_N]^\top \in \mathbb{R}^{N\times k}$ denote the downstream design and label matrices. LoRA learns a rank-r update $\mathbf{AB}$, where $\mathbf{A} \in \mathbb{R}^{d\times r}$ and $\mathbf{B} \in \mathbb{R}^{r\times k}$, by minimizing

$$\widetilde{L}(\mathbf{A}, \mathbf{B}) := \frac{1}{2N} \left\| \widetilde{\mathbf{X}}(\mathbf{W}^\natural + \mathbf{AB}) - \widetilde{\mathbf{Y}} \right\|_F^2 . \tag{2}$$

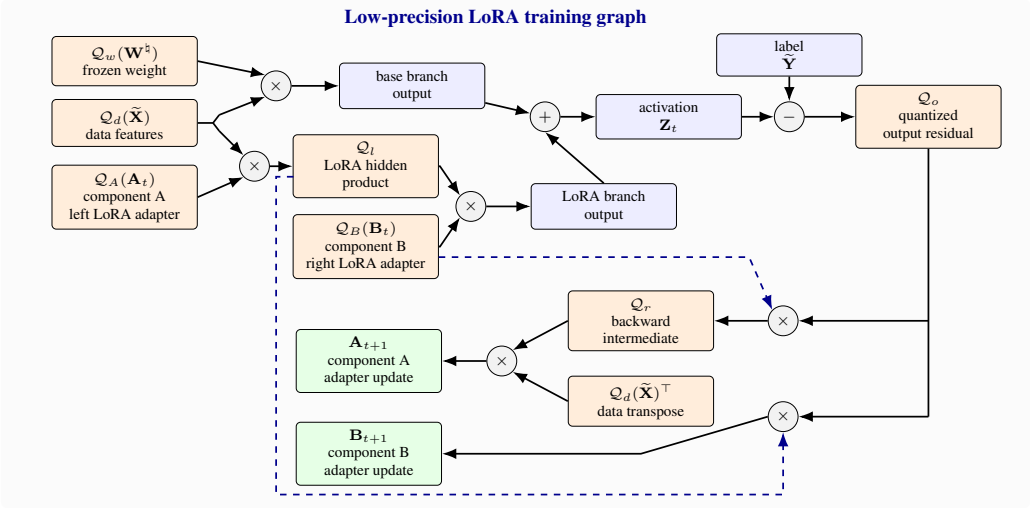

Figure 1: Low-precision LoRA training architecture analyzed in this paper. The forward pass forms the quantized residual; the backward pass proceeds from right to left.

We work in the data-rich regime $N \gg d$ and keep $r = \Theta(1)$, following Zhang et al. (2025b). With stepsizes $\eta_1, \eta_2 > 0$, the corresponding full-precision gradient descent step is

$$\begin{aligned} \mathbf{A}_{t+1} &= \mathbf{A}_t - \frac{\eta_1}{N} \widetilde{\mathbf{X}}^\top \left(\widetilde{\mathbf{X}}(\mathbf{W}^\natural + \mathbf{A}_t\mathbf{B}_t) - \widetilde{\mathbf{Y}} \right) \mathbf{B}_t^\top, \\ \mathbf{B}_{t+1} &= \mathbf{B}_t - \frac{\eta_2}{N} \mathbf{A}_t^\top \widetilde{\mathbf{X}}^\top \left(\widetilde{\mathbf{X}}(\mathbf{W}^\natural + \mathbf{A}_t\mathbf{B}_t) - \widetilde{\mathbf{Y}} \right). \end{aligned} \tag{3}$$

To model training-time low-precision arithmetic, we quantize every tensor that enters a matrix multiplication. Define coordinate-wise quantization operations $\mathcal{Q}_d, \mathcal{Q}_A, \mathcal{Q}_B, \mathcal{Q}_o$ and $\mathcal{Q}_w$ for data features, model parameters $\mathbf{A}$ and $\mathbf{B}$, output gradients and pretrained weight $\mathbf{W}^\natural$ respectively. We would like to highlight that under above quantizers, standard dual-branch LoRA requires multiplication among quantized parameters, quantized data and quantized output gradients. Hence, we further introduce intermediate quantization operations $\mathcal{Q}_l$ and $\mathcal{Q}_r$. Specifically, in the forward pass, the intermediate hidden state $\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t) \in \mathbb{R}^{N\times r}$ must be quantized by $\mathcal{Q}_l$ before it can be multiplied by $\mathcal{Q}_B(\mathbf{B}_t)$. Similarly, during the backward pass, the intermediate hidden state from the multiplication between quantized output gradient and quantized parameter $\mathcal{Q}_B(\mathbf{B}_t)$ must then be quantized by $\mathcal{Q}_r$ before the final multiplication with the input features $\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top$. Figure 1 summarizes the resulting low-precision execution graph.

Formally, denote the forward activation

$$\mathbf{Z}_t = \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_w(\mathbf{W}^\natural) + \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right), \tag{4}$$

then the quantized GD updates are given by:

$$\begin{aligned} \mathbf{A}_{t+1} &= \mathbf{A}_t - \frac{\eta_1}{N} \mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_r\left(\mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right), \\ \mathbf{B}_{t+1} &= \mathbf{B}_t - \frac{\eta_2}{N} \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t)\right)^\top \mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right). \end{aligned} \tag{quantized GD}$$

Spectral initialization. We initialize the adapters using the spectral construction of Zhang et al. (2025b). Specifically, let the compact singular value decomposition of $\mathbf{G}_f^\natural = -\frac{1}{N}\widetilde{\mathbf{X}}^\top\left(\widetilde{\mathbf{X}}\mathbf{W}^\natural - \widetilde{\mathbf{Y}}\right)$ be $\mathbf{G}_f^\natural = \mathbf{U}_\mathbf{G}\mathbf{S}_\mathbf{G}\mathbf{V}_\mathbf{G}^\top$, and let $\mathbf{U}_r \in \mathbb{R}^{d\times r}$ and $\mathbf{V}_r \in \mathbb{R}^{k\times r}$ denote the top-r left and right singular matrices, and let $\mathbf{S}_r \in \mathbb{R}^{r\times r}$ denote the top-r sub-matrix of $\mathbf{S}_\mathbf{G}$. Then

$$\mathbf{A}_0 = \mathbf{U}_r\mathbf{S}_r^{1/2}, \quad \mathbf{B}_0 = \mathbf{S}_r^{1/2}\mathbf{V}_r^\top. \tag{spectral initialization}$$

We would like to remark that, under data model equation 5 with Δ defined in equation 6, $\mathbf{G}_f^\natural = \frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}\Delta$. Under Assumption 3.1, $\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}$ is close to the identity when N is large, so the leading singular spaces of $\mathbf{G}_f^\natural$ approximate those of Δ.

3.2 Data model

Following Zhang et al. (2025b), we assume that each downstream label is generated by a ground-truth linear predictor $\mathbf{W}^* \in \mathbb{R}^{d\times k}$:

$$\widetilde{\mathbf{y}} = \mathbf{W}^{*\top}\widetilde{\mathbf{x}}, \qquad \widetilde{\mathbf{Y}} = \widetilde{\mathbf{X}}\mathbf{W}^*. \tag{5}$$

The update that LoRA should recover is therefore

$$\Delta := \mathbf{W}^* - \mathbf{W}^\natural. \tag{6}$$

We impose the standard low-rank and concentration assumptions used in the full-precision LoRA and matrix factorization analysis (Zhang et al., 2025b).

Assumption 3.1. Assume that $\mathrm{rank}(\Delta) = r^* \le r = \Theta(1)$ and that $\{\widetilde{\mathbf{x}}_i\}_{i=1}^N$ are i.i.d. isotropic, centered sub-Gaussian random vectors.

For a general matrix $\mathbf{M}$, let $\lambda_i(\mathbf{M})$ denote its i-th largest singular value. For Δ, denote $\lambda_i^* = \lambda_i(\Delta)$ and condition number $\kappa = \lambda_1^*/\lambda_{r^*}^*$.

The goal is to derive the convergence rate for the error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2$. We remark that this error also imply generalization performance, because the generalization error for a new data $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ satisfies $\mathbb{E}_{\widetilde{\mathbf{x}}}\left[\left\|\widetilde{\mathbf{y}} - (\mathbf{W}^\natural + \mathbf{A}_t\mathbf{B}_t)\widetilde{\mathbf{x}}\right\|_F\right] = \|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2$ under the noiseless linear data model and Assumption 3.1.

3.3 Quantization

We model the quantization operator $\mathcal{Q}$ and its error properties focusing on floating point quantization, which is widely adopted in practice. Following Tang et al. (2025), we assume the absence of underflow and overflow. This is well justified: low-precision LLM training commonly employs engineering techniques such as per-tensor or per-channel scaling, which ensure that post-quantization values remain within the representable range (Peng et al., 2023; Fishman et al., 2024). Under this condition, the relative quantization error decays exponentially with the number of mantissa bits—a property that is intrinsic to floating-point representations, which can be summarized as the following assumption.

Assumption 3.2. Assume that for each $i = d, A, B, o, w, l, r$, there exists ϵ_i such that the coordinate-wise quantization $\mathcal{Q}_i$ satisfies

$$|\mathcal{Q}_i(u) - u| \le \epsilon_i|u|, \quad \text{for any } u \in \mathbb{R},$$

where $\epsilon_i = \Theta(2^{-M_i})$, and M_i is the mantissa length of the target floating-point format.

4 Main Theory

In this section, we establish theoretical bounds for the approximation error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$. For simplicity, we consider $\eta_1 = \eta_2 =: \eta > 0$ and denote $\epsilon_{\max} = \max_{i=d,A,B,o,w,l,r}\epsilon_i$. We also assume that the leading signal quantities and the LoRA rank under consideration are at constant level.

Assumption 4.1. Assume that $\lambda_1^*, \lambda_{r^*}^*, \|\mathbf{W}^\natural\|_{op} = \Theta(1)$ and $r = O(1)$.

Assumption 4.1 is used only to simplify the main-text rates. Without this constant-level simplification, the same analytic framework yields spectrum-explicit guarantees; see Corollary B.6 for the exact rank case and Corollary B.10 for the general-rank case.

4.1 Exact Rank

In this section, we consider convergence in case of exact rank $r = r^*$.

Theorem 4.2 (Convergence rate in exact rank case)**.** *Under Assumptions 3.1, 3.2, and 4.1, consider the exact rank case $r = r^*$ with the spectral initialization equation spectral initialization and the quantized GD iteration equation quantized GD using $\eta_1 = \eta_2 = \eta > 0$. Suppose $0 < \eta \lesssim \frac{\lambda_1^*}{\kappa^3(1+\lambda_1^*)^3}$, $0 \le \epsilon_{\max} \lesssim \frac{1}{T\sqrt{d}}$, for an integer iteration horizon $T > 0$. Then, with probability at least $1 - e^{-\Omega(N)}$,*

$$\|\mathbf{A}_T\mathbf{B}_T - \Delta\|_F \lesssim \sqrt{r^*}\left[\left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^T + \epsilon_{\max}\sqrt{d}\right].$$

The bound in Theorem 4.2 can be illustrated as follows: the first term represents the geometric convergence characteristic of full-precision LoRA under spectral initialization (Zhang et al., 2025b); the second term establishes an additive quantization floor: once the optimization error diminishes, the convergence is bottlenecked by the perturbation level introduced by low-precision arithmetic.

Crucially, under the conditions of Theorem 4.2, quantization does not degrade the linear convergence rate. Intuitively, this linear contraction is governed by the preservation of nondegenerate signal within the learned LoRA factors along the true left and right singular spaces of Δ, rather than being directly dictated by coordinate-wise rounding errors. Spectral initialization guarantees that these aligned factors are well-conditioned from the outset. Although quantization injects minor perturbations into the gradient updates, the small-error perturbation ensures that the smallest aligned singular values do not collapse. Consequently, while low-precision arithmetic dictates the size of the limiting neighborhood (reflected in the additive term $\epsilon_{\max}\sqrt{d}$), it leaves the geometric decay factor $(1 - \eta\lambda_{r^*}^*/(64\kappa))^T$ intact, provided the perturbations remain within the specified small-error regime.

Furthermore, the $\sqrt{d}$ factor arises as a norm-conversion artifact rather than an inherent optimization instability. The quantizers are defined via coordinate-wise relative errors, whereas our proof bounds operator-norm perturbations to avoid the complexity of directly tracking the fourth-order term $\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top\mathbf{A}_t^\top$ in the Frobenius norm. Translating between these norms necessitates inequalities of the form $\|\mathbf{M}\|_F \le \sqrt{d}\|\mathbf{M}\|_{op}$ for d-dimensional matrices. Thus, a fixed relative quantization error intrinsically induces a larger worst-case Frobenius perturbation in higher dimensions.

Denote $\rho = \eta\lambda_{r^*}^*/(64\kappa)$. A direct implication of Theorem 4.2 is that if $\epsilon_{\max} \lesssim \frac{(1-\rho)^T}{\sqrt{d}}$, the quantization error is absorbed into the geometric term. In this regime, the convergence of the low-precision iterates asymptotically matches the full-precision rate $\mathcal{O}\left(\sqrt{r^*}(1-\rho)^T\right)$.

Bit-level reading. Let $M_{\min} := \min_{i=d,A,B,o,w,l,r} M_i$ denote the smallest mantissa length among the quantized tensors. Since $\epsilon_{\max} = \Theta(2^{-M_{\min}})$ up to format-dependent constants, the condition for the quantization floor to be absorbed by the full-precision term is

$$M_{\min} \gtrsim \frac{1}{2}\log_2 d + T\log_2\frac{1}{1-\rho}.$$

4.2 General Rank

We next consider the over-parameterized case $r \ge r^*$.

Theorem 4.3 (Convergence rate in general-rank case)**.** *Under Assumptions 3.1, 3.2, and 4.1, consider the general-rank case $r \ge r^*$ with the spectral initialization equation spectral initialization and the quantized GD iteration equation quantized GD using $\eta_1 = \eta_2 = \eta > 0$. Suppose $0 < \eta \lesssim \frac{1}{\kappa(1+\lambda_1^*)}$, $0 \le \epsilon_{\max} \lesssim \frac{\lambda_{r^*}^*}{\sqrt{d}\kappa(\lambda_1^*+\kappa^2)(1+T\eta)^{3/2}}$, for an integer iteration horizon $T > 0$. Then, with probability at least $1 - \exp(-\Omega(\frac{N}{(1+T\eta)^2}))$,*

$$\|\mathbf{A}_T\mathbf{B}_T - \Delta\|_F \lesssim \sqrt{r}\left[\left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^T + \epsilon_{\max}\sqrt{d} + \left(T\eta\epsilon_{\max}\sqrt{d}\right)^2\right].$$

Theorem 4.3 demonstrates that over-parameterization preserves the same linear optimization rate as the exact rank setting, but alters the additive quantization floor. Beyond the iteration-independent contribution $\epsilon_{\max}\sqrt{d}$, the general-rank bound introduces an iteration-dependent term $(T\eta\epsilon_{\max}\sqrt{d})^2$. This additional term stems from the non-signal subspace created by the surplus rank directions

($r > r^*$). In the exact rank case, all learned factors align strictly with the true left and right singular spaces of Δ. In contrast, the over-parameterized factors contain orthogonal components that carry no signal from Δ and, consequently, do not benefit from the geometric contraction governing the signal-space dynamics. As a result, quantization perturbations can accumulate within this unregularized non-signal subspace over the effective horizon $T\eta$, yielding the T-dependent floor. When the effective horizon is short (e.g., $T\eta \lesssim 1$), this accumulation is dominated by the static floor, and Theorem 4.3 recovers the additive-error structure of Theorem 4.2, scaling with $\sqrt{r}$ rather than $\sqrt{r^*}$.

Recalling the contraction factor $\rho = \eta\frac{\lambda^*_{r^*}}{64\kappa}$, we can establish a sufficient condition for the low-precision iterates to asymptotically match the full-precision convergence rate:

$$\epsilon_{\max} \lesssim \frac{1}{\sqrt{d}} \min\left\{(1-\rho)^T,\ \frac{(1-\rho)^{T/2}}{T\eta}\right\}.$$

Here, the first argument of the minimum guarantees that the static quantization floor is absorbed into the geometric term, while the second controls the accumulated non-signal-subspace floor. Compared to the exact rank setting, maintaining the optimal rate in the over-parameterized regime imposes a more restrictive upper bound on the quantization error, as the precision must simultaneously suppress both the direct perturbation and its cumulative drift in the orthogonal directions.

Bit-level reading. A sufficient condition to match the full-precision linear convergence rate under the floating point quantization error model is

$$M_{\min} \gtrsim \frac{1}{2}\log_2 d + \max\left\{T\log_2\frac{1}{1-\rho},\ \log_2(T\eta) + \frac{T}{2}\log_2\frac{1}{1-\rho}\right\}.$$

4.3 Numerical Experiment

We complement the preceding bounds with a controlled synthetic experiment using a 4090 GPU.

Experiment setup. We generate noiseless multi-output linear regression data with $N = 4096$ samples and dimensions $d = k = 64$. The covariates are i.i.d. centered isotropic Gaussian, $\widetilde{\mathbf{x}}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$. The target update Δ has rank $r^* = 4$, with random orthonormal singular vectors and singular values geometrically spaced between 1 and 0.6. We compare LoRA ranks $r \in \{4, 8, 16, 32\}$, covering the exact rank case and three over rank cases. The adapters are initialized by the spectral initialization equation spectral initialization. We use constant stepsize $\eta = 0.08$ and average over eight random seeds. We record the relative recovery error $E_t = \|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F / \|\Delta\|_F$.

Observed behaviors. Figure 2 summarizes three diagnostics. Panel (a) measures the early log-linear convergence rate by fitting $\log E_t$ over $0 \le t \le 200$ and normalizing by the corresponding full-precision slope. For $\epsilon = 10^{-8}$, the normalized slopes are 0.99997, 0.99997, 0.99997, and 0.99996 for $r = 4, 8, 16, 32$, respectively. Hence sufficiently fine quantization recovers the full-precision linear phase across all ranks, in line with the common contraction factor in Theorems 4.2 and 4.3. As quantization error level increases, the trajectory departs from the full-precision log-linear phase earlier in the over rank cases. This aligns with our theory that over rank case requires smaller quantization error level to match the full-precision convergence rate. Panel (b) reports the terminal recovery error after $T = 1200$ iterations. The error grows approximately linearly with ϵ over the displayed smaller precision range, consistent with the additive $\epsilon_{\max}\sqrt{d}$ term in the bounds. This also supports the qualitative prediction that surplus directions in over rank case make the limiting neighborhood larger, even though the geometric contraction term itself has the same rate. Panel (c) isolates the horizon-dependent over rank effect by plotting $E_T(r, \epsilon) - E_T(r^*, \epsilon)$ with $r > r^*$ at $\epsilon = 10^{-4}$. As T increases from 100 to 1200, the excess error grows with T, validating the finite-horizon accumulated term in Theorem 4.3. Overall, the experiment supports and probes the mechanisms appearing in Theorems 4.2 and 4.3.

5 Proof Sketch

We sketch the main proof ingredients behind Theorems 4.2 and 4.3. The full derivations are deferred to the appendix. The argument follows the full-precision spectral-initialization analysis of Zhang et al. (2025b), but with a new perturbation analysis for the training-time quantizers in equation quantized GD.

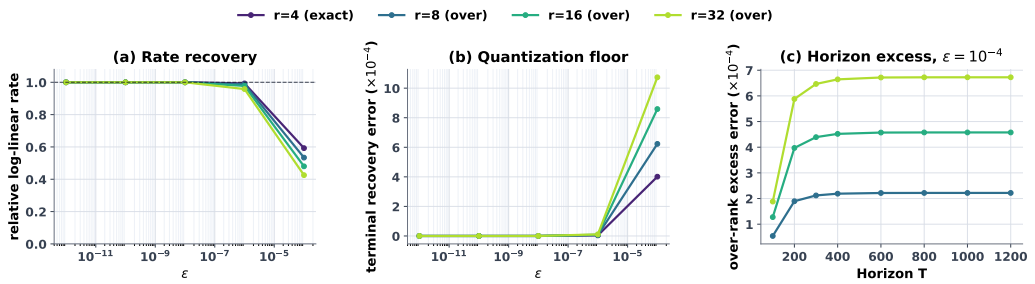

Figure 2: Numerical verification of the theoretical results. Left: sufficiently small ϵ preserves the full-precision log-linear rate, while larger ϵ makes over rank runs leave the full-precision phase earlier. Middle: the terminal recovery error grows with ϵ and with surplus rank. Right: the over rank excess error grows with the horizon.

Step 1: Error decomposition. Let $\Delta = \mathbf{U}\mathbf{S}^*\mathbf{V}^\top$ be the compact SVD of the target update, and let $\mathbf{U}_\perp, \mathbf{V}_\perp$ denote orthogonal complements. We decompose the LoRA factors into signal and non-signal coordinates as reparameterized updates: $\mathbf{A}_t^{\mathbf{U}} = \mathbf{U}^\top \mathbf{A}_t$, $\mathbf{A}_t^{\perp} = \mathbf{U}_\perp^\top \mathbf{A}_t$, $\mathbf{B}_t^{\mathbf{V}} = \mathbf{B}_t\mathbf{V}$, $\mathbf{B}_t^{\perp} = \mathbf{B}_t\mathbf{V}_\perp$. The signal-space error and the three off-signal blocks are $\mathbf{R}_t^* = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\mathbf{V}} - \mathbf{S}^*$, $\mathbf{R}_t^{(1)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}$, $\mathbf{R}_t^{(2)} = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}$, $\mathbf{R}_t^{(3)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}$. Lemma A.2 shows that

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op} \le \|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op} + \|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}.$$

Thus the proof reduces to controlling the aligned block, the two mixed misalignment blocks, and the purely orthogonal block. The final Frobenius guarantees follow by applying rank-dependent conversions to these blocks; see Corollaries B.6 and B.10.

Step 2: Derive reparameterized updates and error norm propagation. The quantized update is first rewritten as the full-precision LoRA recursion plus explicit residual terms.

$$\mathbf{A}_{t+1} = \mathbf{A}_t - \eta(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{B}_t^\top - \eta(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{B}_t^\top + \eta\mathbf{E}_{t,\mathbf{A}},$$

$$\mathbf{B}_{t+1} = \mathbf{B}_t - \eta\mathbf{A}_t^\top(\mathbf{A}_t\mathbf{B}_t - \Delta) - \eta\mathbf{A}_t^\top(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_t\mathbf{B}_t - \Delta) + \eta\mathbf{E}_{t,\mathbf{B}},$$

where $\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} = N^{-1}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})$. The first correction in each line is exactly the full-precision LoRA descent step. The second correction is the covariance distortion caused by quantizing the data. The residuals $\mathbf{E}_{t,\mathbf{A}}$ and $\mathbf{E}_{t,\mathbf{B}}$ collect all remaining low-precision effects: quantized adapters, quantized residuals, quantized frozen weights, and the intermediate product quantizers.

Projecting the separated updates onto $\mathbf{U}, \mathbf{U}_\perp, \mathbf{V}$ and $\mathbf{V}_\perp$ yields the reparameterized dynamics in Lemma A.3. For example, the signal part of the $\mathbf{A}$-update has the form

$$\mathbf{A}_{t+1}^{\mathbf{U}} = \mathbf{A}_t^{\mathbf{U}} - \eta\mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top - \eta\mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top + \text{low-precision and covariance residuals},$$

with analogous recursions for $\mathbf{A}_{t+1}^{\perp}, \mathbf{B}_{t+1}^{\mathbf{V}}$, and $\mathbf{B}_{t+1}^{\perp}$. Multiplying these reparameterized factor updates gives one-step recursions for $\mathbf{R}_t^*, \mathbf{R}_t^{(1)}, \mathbf{R}_t^{(2)}$, and $\mathbf{R}_t^{(3)}$. Then we can summarize the resulting norm propagation with

$$\mathcal{M}_t = \max\{\|\mathbf{R}_t^*\|_{op}, \|\mathbf{R}_t^{(1)}\|_{op}, \|\mathbf{R}_t^{(2)}\|_{op}\}, \qquad s_t = \min\{\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\},$$

Lemma A.4 shows that the dominant part of the recursion is

$$\mathcal{M}_{t+1} \le (1 - \eta s_t)\mathcal{M}_t + \text{perturbation terms}.$$

The perturbation terms consist of higher-order residual products, the covariance error $\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\|_{op}$, the low-precision residuals $\|\mathbf{E}_{t,\mathbf{A}}\|_{op}$ and $\|\mathbf{E}_{t,\mathbf{B}}\|_{op}$, and the non-signal components $\mathbf{A}_t^{\perp}, \mathbf{B}_t^{\perp}, \mathbf{R}_t^{(3)}$.

Step 3: Upper bound the quantization perturbations. The propagation lemma is deterministic once the quantization-induced perturbations are known. Lemma B.1 supplies these bounds by using

the relative quantization assumption $|\mathcal{Q}_i(u) - u| \leq \epsilon_i |u|$ to control $\|\mathbf{E}_{t,\mathbf{A}}\|_{op}$ and $\|\mathbf{E}_{t,\mathbf{B}}\|_{op}$ by a polynomial in the current adapter norms and the quantization levels ϵ_i. Informally,

$$\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \leq \mathrm{Poly}(\epsilon_i, d, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}, \|\mathbf{W}^\natural\|_{op}, \|\Delta\|_{op}).$$

As long as $\|\mathbf{A}_t\|_{op}$ and $\|\mathbf{B}_t\|_{op}$ stay bounded, the update residuals are bounded by a single quantization envelope. Under the constant-level Assumption 4.1, this envelope is of order $\epsilon_{\max}\sqrt{d}$.

Exact rank versus over rank. A key point is that the purely orthogonal block $\mathbf{R}_t^{(3)} = \mathbf{A}_t^\perp \mathbf{B}_t^\perp$ is second order in the perpendicular errors, but the perpendicular errors themselves behave very differently in the exact rank and over rank regimes. In the exact rank case $r = r^*$, the learned adapter has no surplus rank directions beyond the true signal subspace. Once the aligned components $\mathbf{A}_t^{\mathrm{U}}$ and $\mathbf{B}_t^{\mathbf{V}}$ keep singular values bounded below, the perpendicular components $\mathbf{A}_t^\perp$ and $\mathbf{B}_t^\perp$ inherit the same signal-space contraction (see Lemma A.3). Thus, in exact rank, perpendicular errors are repeatedly damped and only remain at the quantization floor. Since $\mathbf{R}_t^{(3)}$ is the product of two perpendicular components, it is even smaller and does not create a new horizon-dependent leading term. Corollary B.4 shows that for $\mathbf{M}_t \in \{\mathbf{A}_t, \mathbf{B}_t\}$,

$$\|\mathbf{M}_{t+1}^\perp\|_{op} \lesssim (1 - \eta s_t)\|\mathbf{M}_t^\perp\|_{op} + \text{perturbation terms}.$$

However, the over rank case is different. When $r > r^*$, the extra adapter directions are not associated with nonzero singular directions of Δ. Along these surplus directions, there is no signal curvature that contracts the perpendicular components. Corollary B.8 therefore gives only a tube-type recursion:

$$\|\mathbf{A}_{t+1}^\perp\|_{op} \lesssim \|\mathbf{A}_t^\perp\|_{op} + \text{perturbation terms}, \qquad \|\mathbf{B}_{t+1}^\perp\|_{op} \lesssim \|\mathbf{B}_t^\perp\|_{op} + \text{perturbation terms}.$$

The absence of contraction $(1 - \eta s_t)$ and $(1 - \eta s_t)$ is the main technical difference. Quantization perturbations in surplus directions can accumulate over the effective horizon $t\eta$. Consequently, $\mathbf{R}_t^{(3)} = \mathbf{A}_t^\perp \mathbf{B}_t^\perp$ contributes an additional accumulated non-signal floor in the over rank theorem.

Step 4: Induction tube. The induction tube tracks four quantities simultaneously: the signal/mixed error $\mathcal{M}_t$, the adapter norms $\|\mathbf{A}_t\|_{op}$ and $\|\mathbf{B}_t\|_{op}$, the aligned singular values of $\mathbf{A}_t^{\mathbf{U}}$ and $\mathbf{B}_t^{\mathbf{V}}$, and the perpendicular quantities $\mathbf{A}_t^\perp, \mathbf{B}_t^\perp, \mathbf{R}_t^{(3)}$. The bounded adapter norms are needed so the quantization perturbation bounds from Lemma B.1 remain valid. The lower bounds on the aligned singular values are needed so the contraction coefficient $1 - \eta s_t$ remains strictly below one. The perpendicular tube is needed to control the terms that couple the signal and non-signal subspaces.

In the exact rank case, Theorem B.5 closes this tube using the contracting perpendicular recursions from Corollary B.4. By simplifying bounds, Corollary B.7 proves Theorem 4.2:

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \lesssim \sqrt{r^*}\left[(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa})^t + \epsilon_{\max}\sqrt{d}\right].$$

In the over rank case, Theorem B.9 closes a wider finite-horizon tube using Corollary B.8. The signal error still contracts at the same rate, but the perpendicular components are controlled by accumulation rather than contraction. Simplified Corollary B.11 proves Theorem 4.3

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \lesssim \sqrt{r}\left[(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa})^t + \epsilon_{\max}\sqrt{d} + (t\eta\epsilon_{\max}\sqrt{d})^2\right],$$

6 Conclusion and Limitations

We formulate a low-precision training LoRA framework and establish convergence bounds for the approximation error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$ and reveal that quantization affects LoRA convergence through an additive error floor. Theoretical results show a critical divergence in exact rank ($r = r^*$) and over rank ($r > r^*$) cases. Compared with the horizon-independent additive error floor in exact rank case, over rank case contains an additional horizon-dependent accumulated error. Building on this theory, we derive explicit precision conditions under which the low-precision LoRA convergence rate matches the full-precision rate, demonstrating that more bit widths are required in case of over rank.

This study also leaves several limitations for future work: extend the theory to larger quantization errors regime, extend the linear model to practical non-linear cases, extend the analysis of gradient descent to practical optimizers such as adaptive methods.

AI USE STATEMENT

Generative AI tools were employed to support manuscript drafting and editing, LaTeX formatting, and to provide feedback on the methodological descriptions and interpretation of experimental results. All AI-assisted suggestions were revised under the authors' direction. The authors take full responsibility for the final content, including all scientific claims, experimental results, and references.

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APPENDIX

Detailed proofs are contained in the Appendix. Theorem 4.2 follows from Corollary B.7, and Theorem 4.3 follows from Corollary B.11.

Proof roadmap. The appendix is organized as follows. Section A rewrites the quantized LoRA recursion as full-precision LoRA dynamics plus perturbation terms, decomposes the error into signal and non-signal blocks, and proves the one-step propagation inequalities used by both main results. Section B bounds the quantization perturbations, states the initialization guarantees, and proves the finite-horizon induction arguments. The exact-rank proof is given in Section B.3.1; the general-rank proof is given in Section B.3.2. Section C contains a standalone operator-norm contraction lemma used in the propagation analysis.

A ALGEBRAIC REDUCTIONS AND ONE-STEP PROPAGATION

This section contains deterministic reductions that are common to both the exact-rank and general-rank proofs. We first separate the quantized update into the full-precision LoRA update plus low-precision residuals. We then decompose the approximation error into blocks aligned with the singular subspaces of the target update Δ, and derive one-step propagation bounds for these blocks.

A.1 SEPARATING THE QUANTIZED UPDATE

We first focus on the parameter update rule. Define the first-step negative gradient of full fine-tuning in quantized data space:

$$\mathbf{G}^{\natural} = -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \left[\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^{\natural} - \widetilde{\mathbf{Y}}\right]. \tag{7}$$

Define the quantized data covariance as $\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} = \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})$.

Intermediate quantization errors. To isolate the effects of quantization, we define the following quantization errors.

$$\mathbf{E}_t^{(r)} := \mathcal{Q}_r\left(\mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right) - \mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top,$$
$$\mathbf{E}_t^{(l)} := \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right),$$
$$\mathbf{E}_t^{(o)} := \mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right) - \left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right),$$
$$\mathbf{E}^{(w)} := \mathcal{Q}_w(\mathbf{W}^{\natural}) - \mathbf{W}^{\natural}, \quad \mathbf{E}_t^{(A)} := \mathcal{Q}_A(\mathbf{A}_t) - \mathbf{A}_t, \quad \mathbf{E}_t^{(B)} := \mathcal{Q}_B(\mathbf{B}_t) - \mathbf{B}_t, \quad \mathbf{E}^{(X)} = \mathcal{Q}_d(\widetilde{\mathbf{X}}) - \widetilde{\mathbf{X}}.$$

Then we are ready to simplify the parameter update rule, which is summarized as follows.

Lemma A.1 (Parameter update rule)**.** *The quantized gradient descent (quantized GD) can be written as*

$$\mathbf{A}_{t+1} = \mathbf{A}_t + \eta_1\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right)\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top - \eta_1\mathbf{H}_{t,\mathbf{A}},$$
$$\mathbf{B}_{t+1} = \mathbf{B}_t + \eta_2\left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right) - \eta_2\mathbf{H}_{t,\mathbf{B}},$$

where $\mathbf{E}_t^{(T)} = -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}^{(X)}\mathbf{W}^* - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right)$, *and*

$$\mathbf{H}_{t,\mathbf{A}} = \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top + \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)},$$
$$\mathbf{H}_{t,\mathbf{B}} = \frac{1}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right) + \frac{1}{N}\mathbf{E}_t^{(l)^\top}\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right).$$

Proof. We first consider the update of parameter $\mathbf{A}_t$. By (quantized GD),

$$\begin{aligned}\mathbf{A}_{t+1} =&\mathbf{A}_t - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_r\left(\mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right)\\ =&\mathbf{A}_t - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathbf{E}_t^{(r)} + \mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right]\\ =&\mathbf{A}_t - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathbf{E}_t^{(r)} + \left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right].\end{aligned}$$

Recall that

$$\begin{aligned}\mathbf{Z}_t =&\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_w(\mathbf{W}^\natural) + \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\\ =&\mathcal{Q}_d(\widetilde{\mathbf{X}})(\mathbf{W}^\natural + \mathbf{E}^{(w)}) + \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right),\end{aligned}$$

and utilizing the first-step negative gradient of full fine-tuning (7), we have

$$\begin{aligned}\mathbf{A}_{t+1} =&\mathbf{A}_t - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathbf{E}_t^{(r)} + \left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right]\\ =&\mathbf{A}_t + \eta_1\left(\mathbf{G}^\natural - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right)\right)\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top\\ &-\frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)}.\end{aligned} \tag{8}$$

Next, we consider the update of parameter $\mathbf{B}_t$. By (quantized GD),

$$\begin{aligned}\mathbf{B}_{t+1} =&\mathbf{B}_t - \frac{\eta_2}{N}\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t)\right)^\top\mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\\ =&\mathbf{B}_t - \frac{\eta_2}{N}\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t) + \mathbf{E}_t^{(l)}\right)^\top\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\\ =&\mathbf{B}_t - \frac{\eta_2}{N}\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})(\mathbf{A}_t + \mathbf{E}_t^{(A)}) + \mathbf{E}_t^{(l)}\right)^\top\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right).\end{aligned}$$

Expanding the terms and once again utilizing the first-step negative gradient of full fine-tuning (7), we obtain:

$$\begin{aligned}\mathbf{B}_{t+1} =&\mathbf{B}_t + \eta_2\left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top\left(\mathbf{G}^\natural - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right)\right)\\ &-\frac{\eta_2}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\\ &-\frac{\eta_2}{N}{\mathbf{E}_t^{(l)}}^\top\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right).\end{aligned} \tag{9}$$

We then analyze the dynamics of $\mathbf{A}_t$ (8) and $\mathbf{B}_t$ (9) by dealing with $\mathbf{G}^\natural$. Noticing that

$$\begin{aligned}\mathbf{G}^\natural =&-\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^\natural - \widetilde{\mathbf{Y}}\right]\\ =&-\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^\natural - \widetilde{\mathbf{X}}\mathbf{W}^*\right]\\ =&-\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^\natural - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^* + (\mathcal{Q}_d(\widetilde{\mathbf{X}}) - \widetilde{\mathbf{X}})\mathbf{W}^*\right]\\ =&-\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[-\mathcal{Q}_d(\widetilde{\mathbf{X}})\Delta + (\mathcal{Q}_d(\widetilde{\mathbf{X}}) - \widetilde{\mathbf{X}})\mathbf{W}^*\right]\\ =&\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}^{(X)}\mathbf{W}^*,\end{aligned}$$

denoting

$$\mathbf{E}_t^{(T)} = -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}^{(X)}\mathbf{W}^* - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right),$$

it follows by (8) and (9) that

$$\begin{aligned}\mathbf{A}_{t+1} =& \mathbf{A}_t + \eta_1\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right)\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top\\ &\underbrace{-\frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)}}_{-\eta_1\mathbf{H}_{t,\mathbf{A}}},\end{aligned} \tag{10}$$

$$\mathbf{B}_{t+1} = \mathbf{B}_t + \eta_2 \left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right) \underbrace{-\frac{\eta_2}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right) - \frac{\eta_2}{N}{\mathbf{E}_t^{(l)}}^\top\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)}_{-\eta_2 \mathbf{H}_{t,\mathbf{B}}}. \tag{11}$$

□

A.2 Signal and Non-Signal Error Decomposition

To avoid the complicated analysis for $\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top\mathbf{A}_t^\top$, we analyze the operator norm $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op}$ and the Frobenius norm is controlled by

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \leq \sqrt{\mathrm{rank}(\mathbf{A}_t\mathbf{B}_t) + r^*}\,\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op}.$$

We then perform error decomposition for $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op}$. To proceed, we first perform SVD for Δ:

$$\Delta = \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}^*\widetilde{\mathbf{V}}^\top = [\mathbf{U} \quad \mathbf{U}_\perp]\begin{bmatrix}\mathbf{S}^* & \mathbf{0}_{r^*\times(k-r^*)} \\ \mathbf{0}_{(d-r^*)\times r^*} & \mathbf{0}_{(d-r^*)\times(k-r^*)}\end{bmatrix}\begin{bmatrix}\mathbf{V}^\top \\ \mathbf{V}_\perp^\top\end{bmatrix}. \tag{12}$$

For simplicity, we also define the following notations:

$$\mathbf{A}_t^{\mathbf{U}} = \mathbf{U}^\top\mathbf{A}_t, \quad \mathbf{B}_t^{\mathbf{V}} = \mathbf{B}_t\mathbf{V}, \quad \mathbf{A}_t^\perp = \mathbf{U}_\perp^\top\mathbf{A}_t, \quad \mathbf{B}_t^\perp = \mathbf{B}_t\mathbf{V}_\perp.$$

We then decompose the error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op}$ via (12).

Lemma A.2 (Error decomposition)**.**

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op} \leq \|\mathbf{R}_t^*\|_{op} + \left\|\mathbf{A}_t^\perp\mathbf{B}_t^{\mathbf{V}}\right\|_{op} + \left\|\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^\perp\right\|_{op} + \left\|\mathbf{A}_t^\perp\mathbf{B}_t^\perp\right\|_{op},$$

where $\mathbf{R}_t^* = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\mathbf{V}} - \mathbf{S}^*$.

Proof. Noticing that $\mathbf{I}_k = \mathbf{V}\mathbf{V}^\top + \mathbf{V}_\perp\mathbf{V}_\perp^\top$, it holds

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op} \leq \|(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}\|_{op} + \|(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}_\perp\|_{op}. \tag{13}$$

For the first term, noticing that $\mathbf{I}_d = \mathbf{U}\mathbf{U}^\top + \mathbf{U}_\perp\mathbf{U}_\perp^\top$, we have

$$(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V} = \mathbf{U}\underbrace{(\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\mathbf{V}} - \mathbf{S}^*)}_{\mathbf{R}_t^*} + \mathbf{U}_\perp\mathbf{A}_t^\perp\mathbf{B}_t^{\mathbf{V}}.$$

It follows that

$$\|(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}\|_{op} \leq \|\mathbf{R}_t^*\|_{op} + \left\|\mathbf{A}_t^\perp\mathbf{B}_t^{\mathbf{V}}\right\|_{op}. \tag{14}$$

For the second term, similarly, we have

$$(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}_\perp = \mathbf{U}\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^\perp + \mathbf{U}_\perp\mathbf{A}_t^\perp\mathbf{B}_t^\perp.$$

It follows that

$$\|(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}_\perp\|_{op} \leq \left\|\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^\perp\right\|_{op} + \left\|\mathbf{A}_t^\perp\mathbf{B}_t^\perp\right\|_{op}. \tag{15}$$

Together with (13), (14) and (15), we have

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op} \leq \|\mathbf{R}_t^*\|_{op} + \left\|\mathbf{A}_t^\perp\mathbf{B}_t^{\mathbf{V}}\right\|_{op} + \left\|\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^\perp\right\|_{op} + \left\|\mathbf{A}_t^\perp\mathbf{B}_t^\perp\right\|_{op}.$$

□

A.3 One-Step Propagation of the Decomposed Errors

We then derive update rule for reparameterized updates $\mathbf{A}_t^{\mathbf{U}}, \mathbf{A}_t^\perp, \mathbf{B}_t^{\mathbf{V}}, \mathbf{B}_t^\perp$. Denote

$$\mathbf{R}_t^{(1)} = \mathbf{A}_t^\perp\mathbf{B}_t^{\mathbf{V}}, \quad \mathbf{R}_t^{(2)} = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^\perp, \quad \mathbf{R}_t^{(3)} = \mathbf{A}_t^\perp\mathbf{B}_t^\perp.$$

Lemma A.3 (Update rule of reparameterized iterates)**.** *The dynamics of the reparameterized components are given by:*

$$\begin{aligned}
\mathbf{A}_{t+1}^{\mathbf{U}} =&\mathbf{A}_t^{\mathbf{U}} - \eta_1 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top - \eta_1 \mathbf{R}_t^{(2)} (\mathbf{B}_t^\perp)^\top + \eta_1 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^\perp)^\top\right]. \\
\mathbf{A}_{t+1}^{\perp} =&\mathbf{A}_t^{\perp} - \eta_1 \mathbf{R}_t^{(1)} (\mathbf{B}_t^{\mathbf{V}})^\top - \eta_1 \mathbf{R}_t^{(3)} (\mathbf{B}_t^\perp)^\top + \eta_1 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^\perp)^\top\right]. \\
\mathbf{B}_{t+1}^{\mathbf{V}} =&\mathbf{B}_t^{\mathbf{V}} - \eta_2 (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* - \eta_2 (\mathbf{A}_t^\perp)^\top \mathbf{R}_t^{(1)} + \eta_2 \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
&-\eta_2 \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^\perp)^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)}\right). \\
\mathbf{B}_{t+1}^{\perp} =&\mathbf{B}_t^{\perp} - \eta_2 (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} - \eta_2 (\mathbf{A}_t^\perp)^\top \mathbf{R}_t^{(3)} + \eta_2 \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp \\
&-\eta_2 \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^\perp)^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)}\right).
\end{aligned}$$

Here, with $\mathbf{E}_t^{(T)}, \mathbf{H}_{t,\mathbf{A}}$ and $\mathbf{H}_{t,\mathbf{B}}$ defined in Lemma A.1, $\mathbf{E}_{t,\mathbf{A}}$ and $\mathbf{E}_{t,\mathbf{B}}$ satisfy

$$\begin{aligned}
\mathbf{E}_{t,\mathbf{A}} &= \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \Delta {\mathbf{E}_t^{(B)}}^\top + \mathbf{E}_t^{(T)} \left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top + \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \mathbf{A}_t \mathbf{B}_t \mathbf{B}_t^\top - \mathbf{H}_{t,\mathbf{A}}, \\
\mathbf{E}_{t,\mathbf{B}} &= (\mathbf{E}_t^{(A)})^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \Delta + (\mathbf{A}_t + \mathbf{E}_t^{(A)})^\top \mathbf{E}_t^{(T)} + \mathbf{A}_t^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \mathbf{A}_t \mathbf{B}_t - \mathbf{H}_{t,\mathbf{B}}.
\end{aligned}$$

Proof. By Lemma A.1, the update rule of parameter $\mathbf{A}_t$ satisfies

$$\mathbf{A}_{t+1} = \mathbf{A}_t + \eta_1 \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \Delta + \mathbf{E}_t^{(T)}\right) \left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top - \eta_1 \mathbf{H}_{t,\mathbf{A}}.$$

Denote the low-precision counterpart of $\mathbf{H}_{t,\mathbf{A}}$ as $\mathbf{E}_{t,\mathbf{A}}^{(H)} = \mathbf{H}_{t,\mathbf{A}} - \frac{1}{N} \mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}}) \mathbf{A}_t \mathbf{B}_t \mathbf{B}_t^\top$. Then

$$\begin{aligned}
\mathbf{A}_{t+1} =&\mathbf{A}_t + \eta_1 \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \Delta + \mathbf{E}_t^{(T)}\right) \left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top - \frac{\eta_1}{N} \mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}}) \mathbf{A}_t \mathbf{B}_t \mathbf{B}_t^\top - \eta_1 \mathbf{E}_{t,\mathbf{A}}^{(H)} \\
=&\mathbf{A}_t + \eta_1 \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \Delta \mathbf{B}_t^\top - \eta_1 \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \mathbf{A}_t \mathbf{B}_t \mathbf{B}_t^\top + \eta_1 \underbrace{\left[\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \Delta {\mathbf{E}_t^{(B)}}^\top + \mathbf{E}_t^{(T)} \left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top - \mathbf{E}_{t,\mathbf{A}}^{(H)}\right]}_{\mathbf{E}_{t,\mathbf{A}},\ \text{low-precision part of } \mathbf{A}_{t+1}} \\
=&\mathbf{A}_t - \eta_1 (\mathbf{A}_t \mathbf{B}_t - \Delta) \mathbf{B}_t^\top - \eta_1 \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) (\mathbf{A}_t \mathbf{B}_t - \Delta) \mathbf{B}_t^\top + \eta_1 \mathbf{E}_{t,\mathbf{A}}.
\end{aligned} \tag{16}$$

Next, we analyze $\mathbf{A}_t^{\mathbf{U}} = \mathbf{U}^\top \mathbf{A}_t$ and $\mathbf{A}_t^\perp = \mathbf{U}_\perp^\top \mathbf{A}_t$ respectively. By equation 16, we have

$$\begin{aligned}
\mathbf{U}^\top \mathbf{A}_{t+1} =&\mathbf{U}^\top \mathbf{A}_t - \eta_1 \mathbf{U}^\top (\mathbf{A}_t \mathbf{B}_t - \Delta)(\mathbf{V}\mathbf{V}^\top + \mathbf{V}_\perp \mathbf{V}_\perp^\top) \mathbf{B}_t^\top \\
&-\eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) (\mathbf{A}_t \mathbf{B}_t - \Delta)(\mathbf{V}\mathbf{V}^\top + \mathbf{V}_\perp \mathbf{V}_\perp^\top) \mathbf{B}_t^\top + \eta_1 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \\
=&\mathbf{U}^\top \mathbf{A}_t - \eta_1 (\mathbf{U}^\top \mathbf{A}_t \mathbf{B}_t \mathbf{V} - \mathbf{U}^\top \Delta \mathbf{V})(\mathbf{B}_t \mathbf{V})^\top - \eta_1 (\mathbf{U}^\top \mathbf{A}_t \mathbf{B}_t \mathbf{V}_\perp - \mathbf{U}^\top \Delta \mathbf{V}_\perp)(\mathbf{B}_t \mathbf{V}_\perp)^\top + \eta_1 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) (\mathbf{A}_t \mathbf{B}_t \mathbf{V} - \Delta \mathbf{V})(\mathbf{B}_t \mathbf{V})^\top - \eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) (\mathbf{A}_t \mathbf{B}_t \mathbf{V}_\perp - \Delta \mathbf{V}_\perp)(\mathbf{B}_t \mathbf{V}_\perp)^\top.
\end{aligned}$$

Substituting the definition of $\mathbf{R}_t^*,\ \mathbf{R}_t^{(i)},\ i = 1, 2, 3$, it holds

$$\begin{aligned}
\mathbf{U}^\top \mathbf{A}_{t+1} =&\mathbf{U}^\top \mathbf{A}_t - \eta_1 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top - \eta_1 \mathbf{R}_t^{(2)} (\mathbf{B}_t^\perp)^\top + \eta_1 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^\perp)^\top\right].
\end{aligned}$$

Regarding $\mathbf{U}_\perp^\top \mathbf{A}_t$, we can similarly deduce that

$$\begin{aligned}
\mathbf{A}_{t+1}^{\perp} =&\mathbf{A}_t^{\perp} - \eta_1 \mathbf{R}_t^{(1)} (\mathbf{B}_t^{\mathbf{V}})^\top - \eta_1 \mathbf{R}_t^{(3)} (\mathbf{B}_t^\perp)^\top + \eta_1 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^\perp)^\top\right].
\end{aligned}$$

Regarding the update rule of parameter $\mathbf{B}_t$, by Lemma A.1,

$$\mathbf{B}_{t+1} = \mathbf{B}_t + \eta_2 \left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right) - \eta_2 \mathbf{H}_{t,\mathbf{B}}.$$

Similarly, we extract the primary term from $\mathbf{H}_{t,\mathbf{B}}$ and define its low-precision residual as $\mathbf{E}_{t,\mathbf{B}}^{(H)} = \mathbf{H}_{t,\mathbf{B}} - \mathbf{A}_t^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \mathbf{A}_t \mathbf{B}_t$, yielding:

$$\begin{aligned}\mathbf{B}_{t+1} =& \mathbf{B}_t + \eta_2 \left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right) - \eta_2 \mathbf{A}_t^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \mathbf{A}_t \mathbf{B}_t - \eta_2 \mathbf{E}_{t,\mathbf{B}}^{(H)} \\ =& \mathbf{B}_t - \eta_2 \mathbf{A}_t^\top (\mathbf{A}_t \mathbf{B}_t - \Delta) - \eta_2 \mathbf{A}_t^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)(\mathbf{A}_t \mathbf{B}_t - \Delta) + \eta_2 \mathbf{E}_{t,\mathbf{B}},\end{aligned} \tag{17}$$

where $\mathbf{E}_{t,\mathbf{B}} = (\mathbf{E}_t^{(A)})^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + (\mathbf{A}_t + \mathbf{E}_t^{(A)})^\top \mathbf{E}_t^{(T)} - \mathbf{E}_{t,\mathbf{B}}^{(H)}$.

Next, the analysis for $\mathbf{B}_t^{\mathbf{V}}$ and $\mathbf{B}_t^{\perp}$ can also be deduced. Specifically, it holds

$$\begin{aligned}\mathbf{B}_{t+1}^{\mathbf{V}} =& \mathbf{B}_t^{\mathbf{V}} - \eta_2 \mathbf{A}_t^\top (\mathbf{U}\mathbf{U}^\top + \mathbf{U}_\perp \mathbf{U}_\perp^\top)(\mathbf{A}_t \mathbf{B}_t - \Delta)\mathbf{V} \\ & - \eta_2 \mathbf{A}_t^\top (\mathbf{U}\mathbf{U}^\top + \mathbf{U}_\perp \mathbf{U}_\perp^\top) \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)(\mathbf{A}_t \mathbf{B}_t - \Delta)\mathbf{V} + \eta_2 \mathbf{E}_{t,\mathbf{B}}\mathbf{V} \\ =& \mathbf{B}_t^{\mathbf{V}} - \eta_2 (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* - \eta_2 (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} + \eta_2 \mathbf{E}_{t,\mathbf{B}}\mathbf{V} \\ & - \eta_2 \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)}\right),\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}_{t+1}^{\perp} =& \mathbf{B}_t^{\perp} - \eta_2 (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} - \eta_2 (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} + \eta_2 \mathbf{E}_{t,\mathbf{B}}\mathbf{V}_\perp \\ & - \eta_2 \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)}\right).\end{aligned}$$

□

Lemma A.4 (Error norm propagation)**.** *For a general matrix $\mathbf{M}$, let $\lambda_i(\mathbf{M})$ denote its i-th largest singular value; for a symmetric positive semi-definite matrix $\mathbf{P}$, let $\lambda_{\min}(\mathbf{P})$ denote its smallest eigenvalue. Denote $\mathcal{M}_t = \max\left\{\|\mathbf{R}_t^*\|_{op}, \left\|\mathbf{R}_t^{(1)}\right\|_{op}, \left\|\mathbf{R}_t^{(2)}\right\|_{op}\right\}$, $\delta = \left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right\|_{op}$ and λ_i as the i-th singular value. Assume additionally that*

$$\eta_1 \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}^2 + \|\mathbf{B}_t^{\perp}\|_{op}^2\right) + \eta_2 \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}^2 + \|\mathbf{A}_t^{\perp}\|_{op}^2\right) \leq 1.$$

It holds

$$\begin{aligned}\|\mathbf{R}_{t+1}^*\|_{op} \leq & \left(1 - \eta_1 \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta_2 \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right)\mathcal{M}_t \\ & + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t \\ & + \eta_1\eta_2 \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\ & + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\ & + \eta_1\eta_2 \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t^2 \\ & + \eta_1\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t + \eta_2\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t + \eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\ & + \eta_1\delta \|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t + \eta_2\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t + \eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\ & + 2\eta_2\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t + \eta_1\|\mathbf{B}_t^{\perp}\|_{op}\mathcal{M}_t + \eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t \\ & + 2\eta_1\eta_2\delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t.\end{aligned}$$

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(1)}\|_{op} \le & \left(1-\eta_1\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})-\eta_2\lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top})\right)\mathcal{M}_t \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t \\
&+\eta_1\eta_2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]\mathcal{M}_t \\
&+\eta_1\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+2\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t \\
&+2\eta_1\eta_2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t.
\end{aligned}
$$

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(2)}\|_{op} \le & \left(1-\eta_1\lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp})-\eta_2\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right)\mathcal{M}_t \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\eta_2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\right)\mathcal{M}_t+\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]\mathcal{M}_t \\
&+\eta_1\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\left(\mathcal{M}_t+\|\mathbf{R}_t^{(3)}\|_{op}\right) \\
&+\eta_1\eta_2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\mathcal{M}_t+\|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}
$$

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(3)}\|_{op} \le & \left(1-\eta_1\lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp})-\eta_2\lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top})\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t \\
&+\eta_1\eta_2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t+\left(\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t^2+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}^2\right) \\
&+\eta_1\eta_2\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\eta_2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)(\mathcal{M}_t+\|\mathbf{R}_t^{(3)}\|_{op}).
\end{aligned}
$$

Proof. We prove by Lemma A.3 and Lemma C.1. In the sequel, all quantization/covariance perturbation terms (including mixed terms such as $-\eta_1\eta_2\mathbf{U}^{\top}\mathbf{E}_{t,\mathbf{A}}(\mathbf{A}_t^{\mathbf{U}})^{\top}\mathbf{R}_t^*$) are upper bounded additively

by sub-multiplicativity of $\|\cdot\|_{op}$. We first write the update rule for $\mathbf{R}_t^*$.

$$
\begin{aligned}
\mathbf{R}_{t+1}^* =& \mathbf{A}_{t+1}^{\mathbf{U}}\mathbf{B}_{t+1}^{\mathbf{V}} - \mathbf{S}^* \\
=& \mathbf{R}_t^* - \eta_1 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{B}_t^{\mathbf{V}} - \eta_1 \mathbf{R}_t^{(2)} (\mathbf{B}_t^{\perp})^\top \mathbf{B}_t^{\mathbf{V}} + \eta_1 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{B}_t^{\mathbf{V}} \\
& - \eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{B}_t^{\mathbf{V}} \\
& - \eta_2 \mathbf{A}_t^{\mathbf{U}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* + \eta_1\eta_2 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* + \eta_1\eta_2 \mathbf{R}_t^{(2)} (\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* - \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* \\
& - \eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (-\eta_2 (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^*) \\
& - \eta_2 \mathbf{A}_t^{\mathbf{U}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} + \eta_1\eta_2 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} + \eta_1\eta_2 \mathbf{R}_t^{(2)} (\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} - \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} \\
& - \eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (-\eta_2 (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)}) \\
& + \eta_2 \mathbf{A}_t^{\mathbf{U}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V} - \eta_1\eta_2 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V} - \eta_1\eta_2 \mathbf{R}_t^{(2)} (\mathbf{B}_t^{\perp})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V} + \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
& - \eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \eta_2 \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
& - \eta_2 \mathbf{A}_{t+1}^{\mathbf{U}} \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)}\right).
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(1)}$,

$$
\begin{aligned}
\mathbf{R}_{t+1}^{(1)} =& \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\mathbf{V}} \\
=& \mathbf{R}_t^{(1)} - \eta_1 \mathbf{R}_t^{(1)} (\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{B}_t^{\mathbf{V}} - \eta_1 \mathbf{R}_t^{(3)} (\mathbf{B}_t^{\perp})^\top \mathbf{B}_t^{\mathbf{V}} + \eta_1 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{B}_t^{\mathbf{V}} \\
& - \eta_1 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{B}_t^{\mathbf{V}} \\
& - \eta_2 \mathbf{A}_t^{\perp} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* + \eta_1\eta_2 \mathbf{R}_t^{(1)} (\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* + \eta_1\eta_2 \mathbf{R}_t^{(3)} (\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* - \eta_1\eta_2 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* \\
& + \eta_1\eta_2 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* \\
& - \eta_2 \mathbf{A}_t^{\perp} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} + \eta_1\eta_2 \mathbf{R}_t^{(1)} (\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} + \eta_1\eta_2 \mathbf{R}_t^{(3)} (\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} - \eta_1\eta_2 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} \\
& + \eta_1\eta_2 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} \\
& + \eta_2 \mathbf{A}_t^{\perp} \mathbf{E}_{t,\mathbf{B}} \mathbf{V} - \eta_1\eta_2 \mathbf{R}_t^{(1)} (\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V} - \eta_1\eta_2 \mathbf{R}_t^{(3)} (\mathbf{B}_t^{\perp})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V} + \eta_1\eta_2 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
& - \eta_1\eta_2 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
& - \eta_2 \mathbf{A}_{t+1}^{\perp} \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)}\right).
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(2)}$,

$$
\begin{aligned}
\mathbf{R}_{t+1}^{(2)} =& \mathbf{A}_{t+1}^{\mathbf{U}}\mathbf{B}_{t+1}^{\perp} \\
=& \mathbf{R}_t^{(2)} - \eta_1 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{B}_t^{\perp} - \eta_1 \mathbf{R}_t^{(2)} (\mathbf{B}_t^{\perp})^\top \mathbf{B}_t^{\perp} + \eta_1 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{B}_t^{\perp} \\
& - \eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{B}_t^{\perp} \\
& - \eta_2 \mathbf{A}_t^{\mathbf{U}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} + \eta_1\eta_2 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} + \eta_1\eta_2 \mathbf{R}_t^{(2)} (\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} - \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& + \eta_1\eta_2 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& - \eta_2 \mathbf{A}_t^{\mathbf{U}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} + \eta_1\eta_2 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} + \eta_1\eta_2 \mathbf{R}_t^{(2)} (\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} - \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_1\eta_2 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_2 \mathbf{A}_t^{\mathbf{U}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp - \eta_1\eta_2 \mathbf{R}_t^* (\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp - \eta_1\eta_2 \mathbf{R}_t^{(2)} (\mathbf{B}_t^{\perp})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp + \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp \\
& - \eta_1\eta_2 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp \\
& - \eta_2 \mathbf{A}_{t+1}^{\mathbf{U}} \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)}\right).
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(3)}$,

$$
\begin{aligned}
\mathbf{R}_{t+1}^{(3)} =& \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp} \\
=& \mathbf{R}_t^{(3)} - \eta_1 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{B}_t^{\perp} - \eta_1 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top \mathbf{B}_t^{\perp} + \eta_1 \mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{B}_t^{\perp} \\
& - \eta_1 \mathbf{U}_{\perp}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{B}_t^{\perp} \\
& - \eta_2 \mathbf{A}_t^{\perp}(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} + \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} + \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} - \eta_1\eta_2 \mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& + \eta_1\eta_2 \mathbf{U}_{\perp}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& - \eta_2 \mathbf{A}_t^{\perp}(\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} + \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} + \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} - \eta_1\eta_2 \mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_1\eta_2 \mathbf{U}_{\perp}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_2 \mathbf{A}_t^{\perp} \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} - \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} - \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} + \eta_1\eta_2 \mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} \\
& - \eta_1\eta_2 \mathbf{U}_{\perp}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} \\
& - \eta_2 \mathbf{A}_{t+1}^{\perp} \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_{\perp}^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)}\right).
\end{aligned}
$$

Denote $\delta = \left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right\|_{op}$, we then upper bound $\left\|\mathbf{R}_{t+1}^{*}\right\|_{op}, \left\|\mathbf{R}_{t+1}^{(1)}\right\|_{op}, \left\|\mathbf{R}_{t+1}^{(2)}\right\|_{op}, \left\|\mathbf{R}_{t+1}^{(3)}\right\|_{op}$ using Lemma C.1. To simplify bounds, we also denote $\mathcal{M}_t = \max\left\{\left\|\mathbf{R}_t^{*}\right\|_{op}, \left\|\mathbf{R}_t^{(1)}\right\|_{op}, \left\|\mathbf{R}_t^{(2)}\right\|_{op}\right\}$.
Regarding $\mathbf{R}_t^{*}$,

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{*}\|_{op} \leq& \left(1 - \eta_1 \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta_2 \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right) \|\mathbf{R}_t^{*}\|_{op} \\
& + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \\
& + \eta_1\eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{*}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_1\eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \\
& + \eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{*}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
& + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
& + \eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{*}\|_{op}^2 + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{R}_t^{*}\|_{op} \\
& + \eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{*}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} + \eta_1\eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \\
& + \eta_1 \delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{*}\|_{op}\right) \\
& + \eta_1\eta_2 \delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} + \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \\
& + \eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{*}\|_{op}\right) \\
& + \eta_1\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right) \left(\|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \\
& + \eta_1\eta_2 \delta^2 \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \\
& \cdot \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right).
\end{aligned}
$$

Using the definition of $\mathcal{M}_t$, we have

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^*\|_{op} \leq & \left(1-\eta_1\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})-\eta_2\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right)\mathcal{M}_t \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t \\
&+\eta_1\eta_2\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t^2 \\
&+\eta_1\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+2\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t \\
&+2\eta_1\eta_2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t.
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(1)}$,

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(1)}\|_{op} \leq & \left(1-\eta_1\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})-\eta_2\lambda_{\min}(\mathbf{A}_t^{\perp}{\mathbf{A}_t^{\perp}}^{\top})\right)\|\mathbf{R}_t^{(1)}\|_{op} \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{*}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{*}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\|\mathbf{R}_t^{*}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{*}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op} \\
&+\eta_1\delta\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right]\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{*}\|_{op}\right) \\
&+\eta_1\eta_2\delta\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right]\left(\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}+\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\right) \\
&+\eta_1\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \\
&+\eta_1\eta_2\delta^2\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right] \\
&\cdot\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op}\right).
\end{aligned}
$$

Using the definition of $\mathcal{M}_t$, we have

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(1)}\|_{op} \leq & \left(1-\eta_1\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})-\eta_2\lambda_{\min}(\mathbf{A}_t^{\perp}{\mathbf{A}_t^{\perp}}^{\top})\right)\mathcal{M}_t \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t \\
&+\eta_1\eta_2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]\mathcal{M}_t \\
&+\eta_1\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+2\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t \\
&+2\eta_1\eta_2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t.
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(2)}$,

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(2)}\|_{op} \leq & \left(1-\eta_1\lambda_{\min}({\mathbf{B}_t^{\perp}}^{\top}\mathbf{B}_t^{\perp})-\eta_2\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right)\|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{*}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{*}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\|\mathbf{R}_t^{*}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{*}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1\delta\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right]\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\right) \\
&+\eta_1\eta_2\delta\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right]\left(\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{*}\|_{op}\right) \\
&+\eta_1\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \\
&+\eta_1\eta_2\delta^2\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right] \\
&\cdot\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}
$$

Using the definition of $\mathcal{M}_t$, we have

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(2)}\|_{op} \leq & \left(1-\eta_1\lambda_{\min}({\mathbf{B}_t^{\perp}}^{\top}\mathbf{B}_t^{\perp})-\eta_2\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right)\mathcal{M}_t \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\eta_2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\right)\mathcal{M}_t+\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]\mathcal{M}_t \\
&+\eta_1\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\left(\mathcal{M}_t+\|\mathbf{R}_t^{(3)}\|_{op}\right) \\
&+\eta_1\eta_2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\mathcal{M}_t+\|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(3)}$,

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(3)}\|_{op} \le & \left(1-\eta_1\lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp})-\eta_2\lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top})\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1\eta_2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}+\eta_1\eta_2\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\delta\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right]\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\right) \\
&+\eta_1\eta_2\delta\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right]\left(\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\right) \\
&+\eta_1\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \\
&+\eta_1\eta_2\delta^2\left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op})+\|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op})\right] \\
&\cdot\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}
$$

Using the definition of $\mathcal{M}_t$, we have

$$
\begin{aligned}
\|\mathbf{R}_{t+1}^{(3)}\|_{op} \le & \left(1-\eta_1\lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp})-\eta_2\lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top})\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t \\
&+\eta_1\eta_2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t+\left(\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t^2+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}^2\right) \\
&+\eta_1\eta_2\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\eta_2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)(\mathcal{M}_t+\|\mathbf{R}_t^{(3)}\|_{op}).
\end{aligned}
$$

□

B Convergence Analysis

This section turns the deterministic propagation bounds from Section A into the finite-horizon guarantees used in the main text. The proof has three layers. First, Lemma B.1 bounds the low-precision residuals $\mathbf{E}_{t,\mathbf{A}}$ and $\mathbf{E}_{t,\mathbf{B}}$ in terms of the quantization levels and iterate norms. Second, Lemmas B.2 and B.3 provide the high-probability initialization and covariance control needed to start the induction. Third, Sections B.3.1 and B.3.2 prove separate finite-horizon inductions for $r = r^*$ and $r \ge r^*$, respectively, and convert the resulting operator-norm block bounds into Frobenius-norm error bounds.

B.1 Quantization Perturbation Bounds

Lemma B.1 (Quantization-error norm bounds under operator norm)**.** *Suppose for any $i \in \{d, A, B, o, w, l, r\}$, there exists a constant ϵ_i such that quantization $\mathcal{Q}_i$ satisfies Assumption 3.2 with ϵ_i, then we have*

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N} \cdot (\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \left(\epsilon_1\|\widetilde{\mathbf{Y}}\|_F + \epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \epsilon_2\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right) \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N} \cdot \left[\epsilon_3(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)^2 \cdot \|\mathbf{A}_t\|_F + \epsilon_B\|\mathbf{B}_t\|_F\left(\|\Delta\|_{op} + \|\mathbf{A}_t\|_{op}\left(2\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F\right)\right)\right],\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\Delta\|_{op}\|\mathbf{A}_t\|_F \cdot \epsilon_A \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left[\epsilon_o(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F) + \epsilon_l(1+\epsilon_A)(1+\epsilon_o)\|\mathbf{A}_t\|_F\right]\|\widetilde{\mathbf{Y}}\|_F \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)[\epsilon_w + (1+\epsilon_w)\epsilon_o]\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\|\mathbf{A}_t\|_F\epsilon_l(1+\epsilon_w)(1+\epsilon_o)(1+\epsilon_A)\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot [\epsilon_l(1+\epsilon_A) + \epsilon_A + \epsilon_o(1+\epsilon_l)(1+\epsilon_A)] \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \epsilon_A \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}^2\|\mathbf{B}_t\|_F \cdot \epsilon_B \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F^2(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \epsilon_l(1+\epsilon_o)(1+\epsilon_A)^2(1+\epsilon_l).\end{aligned}$$

where $\mathbf{E}_{t,\mathbf{A}}, \mathbf{E}_{t,\mathbf{B}}$ are defined in Lemma A.3 and

$$\begin{aligned}\epsilon_1 &= \epsilon_r(1+\epsilon_o) + \epsilon_o, \\ \epsilon_2 &= (1+\epsilon_w)[\epsilon_o + \epsilon_r(1+\epsilon_o)] + \epsilon_w, \\ \epsilon_3 &= [\epsilon_o + \epsilon_r(1+\epsilon_o)](1+\epsilon_l)(1+\epsilon_A) + \epsilon_l(1+\epsilon_A) + \epsilon_A.\end{aligned}$$

Proof. Recall that by equation 16 and equation 17,

$$\begin{aligned}\mathbf{E}_{t,\mathbf{A}} &= \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta(\mathbf{E}_t^{(B)})^\top + \mathbf{E}_t^{(T)}(\mathbf{B}_t + \mathbf{E}_t^{(B)})^\top - \mathbf{E}_{t,\mathbf{A}}^{(H)}, \\ \mathbf{E}_{t,\mathbf{B}} &= (\mathbf{E}_t^{(A)})^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + (\mathbf{A}_t + \mathbf{E}_t^{(A)})^\top\mathbf{E}_t^{(T)} - \mathbf{E}_{t,\mathbf{B}}^{(H)},\end{aligned}$$

where $\mathbf{E}_t^{(T)} = -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}^{(X)}\mathbf{W}^* - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right)$, and

$$\begin{aligned}\mathbf{E}_{t,\mathbf{A}}^{(H)} =& \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top + \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)} \\ & - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top, \\ \mathbf{E}_{t,\mathbf{B}}^{(H)} =& \frac{1}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right) + \frac{1}{N}{\mathbf{E}_t^{(l)}}^\top\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right) \\ & - \mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t.\end{aligned}$$

We then prove by the fact under Assumption 3.2: for any matrix $\mathbf{M}$ and quantizer $\mathcal{Q}_i$,

$$\|\mathcal{Q}_i(\mathbf{M}) - \mathbf{M}\|_{op} \leq \epsilon_i\|\mathbf{M}\|_F, \quad \|\mathcal{Q}_i(\mathbf{M})\|_{op} \leq \|\mathbf{M}\|_{op} + \epsilon_i\|\mathbf{M}\|_F. \tag{18}$$

Specifically, it holds

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \leq & \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\Delta\|_{op}\|\mathbf{E}_t^{(B)}\|_{op} + \|\mathbf{E}_t^{(T)}\|_{op}\|\mathbf{B}_t + \mathbf{E}_t^{(B)}\|_{op} + \|\mathbf{E}_{t,\mathbf{A}}^{(H)}\|_{op} \\ \leq & \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\Delta\|_{op}\epsilon_B\|\mathbf{B}_t\|_F + \|\mathbf{E}_t^{(T)}\|_{op}(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) + \|\mathbf{E}_{t,\mathbf{A}}^{(H)}\|_{op},\end{aligned} \tag{19}$$

and

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \leq & \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\Delta\|_{op}\|\mathbf{E}_t^{(A)}\|_{op} + \|\mathbf{E}_t^{(T)}\|_{op}\|\mathbf{A}_t + \mathbf{E}_t^{(A)}\|_{op} + \|\mathbf{E}_{t,\mathbf{B}}^{(H)}\|_{op} \\ \leq & \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\Delta\|_{op}\epsilon_A\|\mathbf{A}_t\|_F + \|\mathbf{E}_t^{(T)}\|_{op}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F) + \|\mathbf{E}_{t,\mathbf{B}}^{(H)}\|_{op}.\end{aligned} \tag{20}$$

It remains to bound the operator norm of $\mathbf{E}_t^{(T)}$, $\mathbf{E}_{t,\mathbf{A}}^{(H)}$ and $\mathbf{E}_{t,\mathbf{B}}^{(H)}$. By definition,

$$\begin{aligned}\|\mathbf{E}_t^{(T)}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left(\|\mathbf{E}^{(X)}\|_{op}\|\mathbf{W}^*\|_{op} + \|\mathbf{E}_t^{(o)}\|_{op} + \|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{E}^{(w)}\|_{op}\right) \\ \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left(\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \|\mathbf{E}_t^{(o)}\|_{op} + \epsilon_w\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right) \\ \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left(\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \epsilon_o\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right) + \epsilon_w\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right).\end{aligned} \tag{21}$$

Moreover, it holds

$$\begin{aligned}\|\mathbf{Z}_t\|_F = & \left\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_w(\mathbf{W}^\natural) + \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\right\|_F \\ \leq & \left[\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathcal{Q}_w(\mathbf{W}^\natural)\|_F + \left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\right\|_F\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\|_{op}\right] \\ \leq & \|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\left[(1+\epsilon_w)\|\mathbf{W}^\natural\|_F + (1+\epsilon_l)(1+\epsilon_A)\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)\right].\end{aligned} \tag{22}$$

Regarding

$$\begin{aligned}\mathbf{E}_{t,\mathbf{A}}^{(H)} = & \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top + \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)} \\ & -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top \\ = & \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top \\ & +\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top + \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)},\end{aligned}$$

we have

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{A}}^{(H)}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op}\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\|_{op}^2 \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left(\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_{op}\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \mathbf{B}_t\mathbf{B}_t^\top\|_{op} + \|\mathbf{E}_t^{(r)}\|_{op}\right).\end{aligned} \tag{23}$$

Moreover, it holds

$$\begin{aligned}& \left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \\ \leq & \left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right) + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \\ \leq & \left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right\|_{op} + \left\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \\ \leq & \epsilon_l\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathcal{Q}_A\left(\mathbf{A}_t\right)\|_F + \epsilon_A\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F \\ \leq & \left(\epsilon_l(1+\epsilon_A) + \epsilon_A\right)\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F,\end{aligned} \tag{24}$$

$$\begin{aligned}\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \mathbf{B}_t\mathbf{B}_t^\top\|_{op} \leq & \|\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \mathcal{Q}_B\left(\mathbf{B}_t\right)\mathbf{B}_t^\top\|_{op} \\ & + \|\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathbf{B}_t^\top - \mathbf{B}_t\mathbf{B}_t^\top\|_{op} \\ \leq & \epsilon_B\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\|_{op}\|\mathbf{B}_t\|_F + \epsilon_B\|\mathbf{B}_t\|_{op}\|\mathbf{B}_t\|_F \\ \leq & \epsilon_B\|\mathbf{B}_t\|_F\left(2\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F\right),\end{aligned} \tag{25}$$

and

$$\|\mathbf{E}_t^{(r)}\|_{op} \leq \epsilon_r \|\mathcal{Q}_o(\mathbf{Z}_t - \widetilde{\mathbf{Y}})\mathcal{Q}_B(\mathbf{B}_t)^\top\|_F \leq \epsilon_r(1+\epsilon_o)\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right)(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F). \tag{26}$$

Therefore, together with equation 19, equation 21, equation 22, equation 23, equation 24, equation 25 and equation 26, we have

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N} \cdot (\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \left(\epsilon_1\|\widetilde{\mathbf{Y}}\|_F + \epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \epsilon_2\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right) \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N} \cdot \left[\epsilon_3(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)^2 \cdot \|\mathbf{A}_t\|_F + \epsilon_B\|\mathbf{B}_t\|_F\left(\|\Delta\|_{op} + \|\mathbf{A}_t\|_{op}\left(2\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F\right)\right)\right],\end{aligned}$$

where

$$\begin{aligned}\epsilon_1 &= \epsilon_r(1+\epsilon_o) + \epsilon_o, \\ \epsilon_2 &= (1+\epsilon_w)[\epsilon_o + \epsilon_r(1+\epsilon_o)] + \epsilon_w, \\ \epsilon_3 &= [\epsilon_o + \epsilon_r(1+\epsilon_o)](1+\epsilon_l)(1+\epsilon_A) + \epsilon_l(1+\epsilon_A) + \epsilon_A.\end{aligned}$$

Regarding $\mathbf{E}_{t,\mathbf{B}}^{(H)}$, similarly, write

$$\begin{aligned}\mathbf{E}_{t,\mathbf{B}}^{(H)} = & \frac{1}{N}\mathcal{Q}_A(\mathbf{A}_t)^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t)\right)\mathcal{Q}_B(\mathbf{B}_t) - \mathcal{Q}_A(\mathbf{A}_t)^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathcal{Q}_B(\mathbf{B}_t) \\ & + \mathcal{Q}_A(\mathbf{A}_t)^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathcal{Q}_B(\mathbf{B}_t) - \mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathcal{Q}_B(\mathbf{B}_t) \\ & + \mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathcal{Q}_B(\mathbf{B}_t) - \mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t \\ & + \frac{1}{N}\mathbf{E}_t^{(l)\top}\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right),\end{aligned}$$

then we have

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{B}}^{(H)}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\|\mathcal{Q}_A(\mathbf{A}_t)\|_{op}\|\mathcal{Q}_B(\mathbf{B}_t)\|_{op}\left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \\ & + \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\mathbf{A}_t\|_{op}\|\mathcal{Q}_B(\mathbf{B}_t)\|_{op}\|\mathcal{Q}_A(\mathbf{A}_t) - \mathbf{A}_t\|_{op} \\ & + \|\mathbf{A}_t\|_{op}^2\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\mathcal{Q}_B(\mathbf{B}_t) - \mathbf{B}_t\|_{op} \\ & + \frac{\|\mathbf{E}_t^{(l)}\|_{op}}{N}(1+\epsilon_o)\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right).\end{aligned} \tag{27}$$

Moreover, it holds

$$\begin{aligned}\left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \leq & \epsilon_l\left\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t)\right\|_F + \epsilon_A\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F \\ \leq & [\epsilon_l(1+\epsilon_A) + \epsilon_A]\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F,\end{aligned} \tag{28}$$

and

$$\|\mathbf{E}_t^{(l)}\|_{op} \leq \epsilon_l(1+\epsilon_A)\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F. \tag{29}$$

Together with equation 20, equation 21, equation 22, equation 27, equation 28 and equation 29, we have

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\Delta\|_{op}\|\mathbf{A}_t\|_F \cdot \epsilon_A \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)\left(\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \epsilon_o\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right) + \epsilon_w\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right) \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)[\epsilon_l(1+\epsilon_A) + \epsilon_A] \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \epsilon_A \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}^2\|\mathbf{B}_t\|_F \cdot \epsilon_B \\ & + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\|\mathbf{A}_t\|_F(1+\epsilon_o)\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right) \cdot \epsilon_l(1+\epsilon_A).\end{aligned}$$

Merging terms and using

$$\|\mathbf{Z}_t\|_F \le \|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\left[(1+\epsilon_w)\|\mathbf{W}^\natural\|_F + (1+\epsilon_l)(1+\epsilon_A)\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)\right],$$

we have

$$\begin{aligned}
\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\Delta\|_{op}\|\mathbf{A}_t\|_F\cdot\epsilon_A \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left[\epsilon_o(\|\mathbf{A}_t\|_{op}+\epsilon_A\|\mathbf{A}_t\|_F)+\epsilon_l(1+\epsilon_A)(1+\epsilon_o)\|\mathbf{A}_t\|_F\right]\|\widetilde{\mathbf{Y}}\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op}+\epsilon_A\|\mathbf{A}_t\|_F)\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op}+\epsilon_A\|\mathbf{A}_t\|_F)[\epsilon_w+(1+\epsilon_w)\epsilon_o]\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\|\mathbf{A}_t\|_F\epsilon_l(1+\epsilon_w)(1+\epsilon_o)(1+\epsilon_A)\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F(\|\mathbf{A}_t\|_{op}+\epsilon_A\|\mathbf{A}_t\|_F)(\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)\cdot[\epsilon_l(1+\epsilon_A)+\epsilon_A+\epsilon_o(1+\epsilon_l)(1+\epsilon_A)] \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)\cdot\epsilon_A \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}^2\|\mathbf{B}_t\|_F\cdot\epsilon_B \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F^2(\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)\cdot\epsilon_l(1+\epsilon_o)(1+\epsilon_A)^2(1+\epsilon_l).
\end{aligned}$$

□

B.2 Initialization and Covariance Control

Lemma B.2 (Lemma C.11 in Zhang et al. (2025b))**.** *Consider equation spectral initialization, under Assumption 3.1, let* $\kappa = \frac{\lambda_1(\Delta)}{\lambda_{r^*}(\Delta)} := \frac{\lambda_1^*}{\lambda_{r^*}^*}$*, then with probability at least* $1-2C\exp(-\epsilon^2 N)$ *for a universal constant* $C>0$*, we have* $\left\|\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}-\mathbf{I}\right\|_{op}\le\epsilon$ *and then*

$$\|\mathbf{A}_0\mathbf{B}_0-\Delta\|_{op}\le\left\|\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}-\mathbf{I}\right\|_{op}\|\Delta\|_{op}\le\epsilon\|\Delta\|_{op}.$$

Moreover, if Assumption 3.2 holds for $\mathcal{Q}_d$ *with parameter* ϵ_d*, then on the same event,*

$$\delta=\left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}-\mathbf{I}_d\right\|_{op}\le\epsilon+\frac{2\epsilon_d\|\widetilde{\mathbf{X}}\|_{op}\|\widetilde{\mathbf{X}}\|_F+\epsilon_d^2\|\widetilde{\mathbf{X}}\|_F^2}{N}\le\epsilon+(1+\epsilon)\left(2\epsilon_d\sqrt{d}+\epsilon_d^2 d\right):=\delta_q.$$

Proof. The first claim is Lemma C.11 in Zhang et al. (2025b). For the quantized covariance bound, write

$$\mathcal{Q}_d(\widetilde{\mathbf{X}})=\widetilde{\mathbf{X}}+\mathbf{E}^{(X)}.$$

By Assumption 3.2,

$$\|\mathbf{E}^{(X)}\|_{op}\le\|\mathbf{E}^{(X)}\|_F\le\epsilon_d\|\widetilde{\mathbf{X}}\|_F.$$

Expanding $\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}$ gives

$$\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}-\mathbf{I}_d=\left(\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}-\mathbf{I}_d\right)+\frac{1}{N}\left(\widetilde{\mathbf{X}}^\top\mathbf{E}^{(X)}+\mathbf{E}^{(X)\top}\widetilde{\mathbf{X}}+\mathbf{E}^{(X)\top}\mathbf{E}^{(X)}\right).$$

Therefore, on the event $\left\|\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}-\mathbf{I}_d\right\|_{op}\le\epsilon$,

$$\delta\le\epsilon+\frac{2\|\widetilde{\mathbf{X}}\|_{op}\|\mathbf{E}^{(X)}\|_{op}+\|\mathbf{E}^{(X)}\|_{op}^2}{N}\le\epsilon+\frac{2\epsilon_d\|\widetilde{\mathbf{X}}\|_{op}\|\widetilde{\mathbf{X}}\|_F+\epsilon_d^2\|\widetilde{\mathbf{X}}\|_F^2}{N}.$$

Since $\|\widetilde{\mathbf{X}}\|_{op}^2/N\le 1+\epsilon$ and $\|\widetilde{\mathbf{X}}\|_F\le\sqrt{d}\|\widetilde{\mathbf{X}}\|_{op}$ on the same event, the simplified bound follows. □

B.3 Finite-Horizon Induction Arguments

For simplicity, we consider $\eta_1 = \eta_2 = \eta$. The finite-horizon analysis below follows the same three-step pattern in both rank regimes. First, an induction-ready propagation corollary packages Lemma A.4 into a compact recursion with explicit envelopes. Second, a finite-horizon induction theorem proves that the iterates stay in a bounded tube, the signal singular values remain bounded away from zero, and the decomposed errors contract up to quantization-dependent envelopes. Third, a Frobenius-error corollary converts the operator-norm block bounds into the approximation error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$, followed by a simplified corollary that yields the corresponding main-text theorem under the constant-level assumptions.

Lemma B.3 (Induction base bounds for full-precision spectral initialization)**.** *Consider spectral initialization equation spectral initialization. Let* $\widehat{\mathbf{\Sigma}} = \frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}$ *and write* $\lambda_i^* = \lambda_i(\Delta)$*. Under Assumption 3.1, for* $\epsilon \in (0,1)$*, with probability at least* $1 - 2C\exp(-\epsilon^2 N)$ *for a universal constant* $C > 0$*, the following bounds hold:*

$$
\begin{aligned}
\|\mathbf{A}_0\mathbf{B}_0 - \Delta\|_{op} &\le \epsilon\lambda_1^*, \\
\|\mathbf{A}_0\|_{op} = \|\mathbf{B}_0\|_{op} &\le \sqrt{(1+\epsilon)\lambda_1^*}, \\
\|\mathbf{A}_0^\perp\|_{op} &\le \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \qquad \|\mathbf{B}_0^\perp\|_{op} = 0, \\
\|\mathbf{A}_0^{\mathbf{U}}\|_{op},\ \|\mathbf{B}_0^{\mathbf{V}}\|_{op} &\le \sqrt{(1+\epsilon)\lambda_1^*}, \\
\lambda_{r^*}(\mathbf{A}_0^{\mathbf{U}}) &\ge \sqrt{(1-\epsilon)\lambda_{r^*}^*} - \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \\
\lambda_{r^*}(\mathbf{B}_0^{\mathbf{V}}) &\ge \sqrt{(1-\epsilon)\lambda_{r^*}^*}, \\
\|\mathbf{R}_0^*\|_{op} &\le \epsilon\lambda_1^*, \qquad \|\mathbf{R}_0^{(1)}\|_{op} \le \epsilon\lambda_1^*, \\
\|\mathbf{R}_0^{(2)}\|_{op} &= 0, \qquad \|\mathbf{R}_0^{(3)}\|_{op} = 0.
\end{aligned}
$$

Proof. Since $\widetilde{\mathbf{Y}} = \widetilde{\mathbf{X}}\mathbf{W}^*$, the full-precision first gradient satisfies $\mathbf{G}_f^\natural = \widehat{\mathbf{\Sigma}}\Delta$. On the event $\|\widehat{\mathbf{\Sigma}} - \mathbf{I}_d\|_{op} \le \epsilon$, all eigenvalues of $\widehat{\mathbf{\Sigma}}$ lie in $[1-\epsilon, 1+\epsilon]$. Hence, by singular-value inequalities applied to $\widehat{\mathbf{\Sigma}}\Delta$,

$$(1-\epsilon)\lambda_i^* \le \lambda_i(\mathbf{G}_f^\natural) \le (1+\epsilon)\lambda_i^*, \qquad i \le r^*.$$

Since $\mathrm{rank}(\mathbf{G}_f^\natural) \le r^* \le r$, the top-$r$ spectral factorization reconstructs $\mathbf{G}_f^\natural$ exactly, so $\mathbf{A}_0\mathbf{B}_0 = \mathbf{G}_f^\natural$. This gives

$$\|\mathbf{A}_0\mathbf{B}_0 - \Delta\|_{op} = \|(\widehat{\mathbf{\Sigma}} - \mathbf{I}_d)\Delta\|_{op} \le \epsilon\lambda_1^*,$$

and $\|\mathbf{A}_0\|_{op} = \|\mathbf{B}_0\|_{op} = \sqrt{\lambda_1(\mathbf{G}_f^\natural)} \le \sqrt{(1+\epsilon)\lambda_1^*}$.

Next, $\mathbf{G}_f^\natural\mathbf{V}_\perp = \widehat{\mathbf{\Sigma}}\Delta\mathbf{V}_\perp = \mathbf{0}$. Therefore the nonzero right singular vectors of $\mathbf{G}_f^\natural$ lie in the span of $\mathbf{V}$, and the zero singular directions do not contribute to $\mathbf{B}_0$; hence $\mathbf{B}_0^\perp = \mathbf{B}_0\mathbf{V}_\perp = \mathbf{0}$. If $\mathbf{V}_{r^*}$ and $\mathbf{S}_{r^*}$ denote the nonzero right singular vectors and singular values of $\mathbf{G}_f^\natural$, then

$$\mathbf{A}_0^\perp = \mathbf{U}_\perp^\top\mathbf{G}_f^\natural\mathbf{V}_{r^*}\mathbf{S}_{r^*}^{-1/2} = \mathbf{U}_\perp^\top(\widehat{\mathbf{\Sigma}} - \mathbf{I}_d)\Delta\mathbf{V}_{r^*}\mathbf{S}_{r^*}^{-1/2},$$

where we used $\mathbf{U}_\perp^\top\Delta = \mathbf{0}$. Thus

$$\|\mathbf{A}_0^\perp\|_{op} \le \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}.$$

The upper bounds for $\mathbf{A}_0^{\mathbf{U}}$ and $\mathbf{B}_0^{\mathbf{V}}$ follow from the bounds on $\|\mathbf{A}_0\|_{op}$ and $\|\mathbf{B}_0\|_{op}$. Since $\mathbf{B}_0^\perp = \mathbf{0}$, the nonzero singular values of $\mathbf{B}_0^{\mathbf{V}}$ equal those of $\mathbf{B}_0$, and therefore $\lambda_{r^*}(\mathbf{B}_0^{\mathbf{V}}) \ge \sqrt{(1-\epsilon)\lambda_{r^*}^*}$. For $\mathbf{A}_0^{\mathbf{U}}$, Weyl's singular-value perturbation inequality gives

$$\lambda_{r^*}(\mathbf{A}_0^{\mathbf{U}}) \ge \lambda_{r^*}(\mathbf{A}_0) - \|\mathbf{A}_0^\perp\|_{op} \ge \sqrt{(1-\epsilon)\lambda_{r^*}^*} - \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}.$$

Finally, because $\mathbf{A}_0\mathbf{B}_0 = \mathbf{G}_f^\natural$,

$$\mathbf{R}_0^* = \mathbf{U}^\top(\mathbf{G}_f^\natural - \Delta)\mathbf{V}, \qquad \mathbf{R}_0^{(1)} = \mathbf{U}_\perp^\top(\mathbf{G}_f^\natural - \Delta)\mathbf{V},$$
$$\mathbf{R}_0^{(2)} = \mathbf{U}^\top\mathbf{G}_f^\natural\mathbf{V}_\perp, \qquad \mathbf{R}_0^{(3)} = \mathbf{U}_\perp^\top\mathbf{G}_f^\natural\mathbf{V}_\perp.$$

The first two are bounded by $\epsilon\lambda_1^*$, and the last two are zero by $\mathbf{G}_f^\natural\mathbf{V}_\perp = \mathbf{0}$. The final high-probability statement is exactly the covariance concentration event from Lemma B.2. □

B.3.1 Exact Rank

The exact-rank proof uses the fact that $\mathbf{B}_t^{\mathbf{V}}$ is square and remains well-conditioned. This allows $\mathbf{A}_t^\perp$ to be controlled through $\mathbf{R}_t^{(1)} = \mathbf{A}_t^\perp\mathbf{B}_t^{\mathbf{V}}$, so the non-signal drift remains at the static quantization scale. The proof proceeds through an induction-ready propagation corollary, a finite-horizon induction theorem, and two corollaries converting the induction to the simplified main-text bound.

Corollary B.4 (Exact-rank induction-ready propagation with separated envelopes)**.** *Consider $r = r^*$. Assume the hypotheses of Lemma A.4 with $\eta_1 = \eta_2 = \eta$, and define*

$$P_t = \max\{1, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}\}, \quad q_t = \max\{\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\},$$
$$e_t = \mathcal{M}_t, \quad s_t = \min\left\{\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\right\}, \quad \delta = \left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right\|_{op}.$$

If $r = r^*$, and if nonnegative numbers $\alpha_t, \beta_t, \zeta_t$ satisfy

$$\|\mathbf{A}_t^\perp\|_{op} \le \alpha_t, \qquad \|\mathbf{B}_t^\perp\|_{op} \le \beta_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \zeta_t,$$

then, with

$$K_t^{\mathrm{sep}} = e_t + \alpha_t + \beta_t + \zeta_t,$$

define

$$\widetilde{\Phi}_t = C\eta P_t(\alpha_t + \beta_t)(e_t + \zeta_t) + C\eta^2 P_t q_t K_t^{\mathrm{sep}} + C\eta P_t q_t + C\eta^2 q_t^2$$
$$+ C\eta^2 P_t^2 (K_t^{\mathrm{sep}})^2 + C\eta\delta P_t\left(P_t + \eta P_t K_t^{\mathrm{sep}} + \eta q_t\right) K_t^{\mathrm{sep}} + C\eta^2\delta^2 P_t^2 (K_t^{\mathrm{sep}})^2,$$

for a universal constant $C > 0$, it holds

$$\mathcal{M}_{t+1} \le (1 - \eta s_t)e_t + \widetilde{\Phi}_t,$$
$$\|\mathbf{A}_{t+1}^\perp\|_{op} \le (1 - \eta s_t)\alpha_t + \eta q_t + C\eta\delta\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right],$$
$$\|\mathbf{B}_{t+1}^\perp\|_{op} \le (1 - \eta s_t)\beta_t + \eta q_t + C\eta\delta P_t(P_t\beta_t + \zeta_t).$$

Moreover,

$$\|\mathbf{R}_{t+1}^{(3)}\|_{op} = \|\mathbf{A}_{t+1}^\perp\mathbf{B}_{t+1}^\perp\|_{op} \le \|\mathbf{A}_{t+1}^\perp\|_{op}\,\|\mathbf{B}_{t+1}^\perp\|_{op}.$$

Proof. Apply Lemma A.4 with $\eta_1 = \eta_2 = \eta$. Throughout the proof we use the envelope facts

$$\|\mathbf{A}_t^{\mathbf{U}}\|_{op}, \|\mathbf{A}_t^\perp\|_{op}, \|\mathbf{B}_t^{\mathbf{V}}\|_{op}, \|\mathbf{B}_t^\perp\|_{op} \le P_t,$$
$$\|\mathbf{R}_t^*\|_{op}, \|\mathbf{R}_t^{(1)}\|_{op}, \|\mathbf{R}_t^{(2)}\|_{op} \le e_t, \quad \|\mathbf{A}_t^\perp\|_{op} \le \alpha_t, \quad \|\mathbf{B}_t^\perp\|_{op} \le \beta_t, \quad \|\mathbf{R}_t^{(3)}\|_{op} \le \zeta_t,$$

and $\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le q_t$. The leading coefficients in the first three residual recursions are

$$1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}),$$
$$1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta\lambda_{\min}(\mathbf{A}_t^\perp{\mathbf{A}_t^\perp}^\top),$$
$$1 - \eta\lambda_{\min}({\mathbf{B}_t^\perp}^\top\mathbf{B}_t^\perp) - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}).$$

By the step-size hypothesis these coefficients are nonnegative, and each is at most $1 - \eta s_t$ after dropping the nonnegative term not included in s_t.

The deterministic first-order non-signal groups from Lemma A.4 are

$$\eta\|\mathbf{B}_t^\perp\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{A}_t^\perp\|_{op}e_t,$$
$$\eta\|\mathbf{B}_t^\perp\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{A}_t^\perp\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t,$$
$$\eta\|\mathbf{B}_t^\perp\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{A}_t^\perp\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}.$$

They are bounded by

$$C\eta P_t(\alpha_t+\beta_t)(e_t+\zeta_t).$$

The one-quantization and pure-quantization groups are

$$\begin{aligned}
&\eta^2\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t\\
&\quad+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op},\\
&\eta^2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)e_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\\
&\quad+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op},\\
&\eta^2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\\
&\quad+\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}.
\end{aligned}$$

Using $\|\mathbf{A}_t^{\perp}\|_{op},\|\mathbf{B}_t^{\perp}\|_{op}\le P_t$ in the direct ηq_t terms, they are bounded by

$$C\eta^2P_tq_tK_t^{\mathrm{sep}}+C\eta P_tq_t+C\eta^2q_t^2,$$

since $e_t+\zeta_t\le K_t^{\mathrm{sep}}$. The quadratic residual groups are

$$\begin{aligned}
&\eta^2\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op}\right)e_t^2,\\
&\eta^2\left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)e_t+\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]e_t,\\
&\eta^2\left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\right)e_t+\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]e_t.
\end{aligned}$$

Each displayed product has two residual or separated non-signal factors and at most two parameter factors, so they are bounded by

$$C\eta^2P_t^2(K_t^{\mathrm{sep}})^2,$$

where we used $e_t+\zeta_t\le K_t^{\mathrm{sep}}$. The covariance-linear groups in the first three recursions are

$$\begin{aligned}
&\eta\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t\\
&\quad+\eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&\quad+2\eta\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t+\eta\|\mathbf{B}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)e_t,\\
&\eta\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t\\
&\quad+\eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&\quad+2\eta\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t+\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)e_t,\\
&\eta\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t\\
&\quad+\eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&\quad+\eta\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t+\eta\|\mathbf{B}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\left(e_t+\|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}$$

They are bounded by

$$C\eta\delta P_t\left(P_t+\eta P_tK_t^{\mathrm{sep}}+\eta q_t\right)K_t^{\mathrm{sep}}.$$

The covariance-quadratic groups have the common form

$$\eta^2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)e_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(e_t+\|\mathbf{R}_t^{(3)}\|_{op}\right),$$

with the last factor equal to e_t in the first two recursions. Hence, they are bounded by

$$C\eta^2\delta^2P_t^2K_t^{\mathrm{sep}}(e_t+\zeta_t)\le C\eta^2\delta^2P_t^2(K_t^{\mathrm{sep}})^2.$$

Combining the leading contraction and the five perturbation groups gives

$$\begin{aligned}
\mathcal{M}_{t+1}&\le(1-\eta s_t)e_t+C\eta P_t(\alpha_t+\beta_t)(e_t+\zeta_t)+C\eta^2P_tq_tK_t^{\mathrm{sep}}+C\eta P_tq_t+C\eta^2q_t^2\\
&\quad+C\eta^2P_t^2(K_t^{\mathrm{sep}})^2+C\eta\delta P_t\left(P_t+\eta P_tK_t^{\mathrm{sep}}+\eta q_t\right)K_t^{\mathrm{sep}}+C\eta^2\delta^2P_t^2(K_t^{\mathrm{sep}})^2\\
&=(1-\eta s_t)e_t+\widetilde{\Phi}_t.
\end{aligned}$$

It remains to prove the sharpened perpendicular recursions. For $\mathbf{A}_{t+1}^{\perp}$, the deterministic full-precision part of the $\mathbf{A}_{t+1}^{\perp}$ update equals

$$\mathbf{A}_t^{\perp}\left[\mathbf{I}_r - \eta \mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top - \eta \mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top\right].$$

Since $r = r^*$, $\mathbf{B}_t^{\mathbf{V}}$ is square. The step-size condition in Lemma A.4 implies

$$0 \preceq \mathbf{I}_r - \eta \mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top - \eta \mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top \preceq \mathbf{I}_r.$$

Therefore,

$$\begin{aligned}&\left\|\mathbf{I}_r - \eta \mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top - \eta \mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top\right\|_{op}\\ =&1 - \eta\lambda_{\min}\left(\mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top + \mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top\right) \le 1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}).\end{aligned}$$

The remaining terms in the $\mathbf{A}_{t+1}^{\perp}$ update are the quantization term and the covariance term:

$$\|\mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}}\|_{op} \le q_t$$

and

$$\begin{aligned}&\left\|(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right\|_{op}\\ \le&\left(\|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right)\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right)\|\mathbf{B}_t^{\perp}\|_{op}\\ \le&C\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right],\end{aligned}$$

where the last line uses $\mathbf{R}_t^{(1)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}$ and $\mathbf{R}_t^{(2)} = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}$. Combining these bounds gives

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op} \le \left(1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\right)\|\mathbf{A}_t^{\perp}\|_{op} + \eta q_t + C\eta\delta\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right].$$

Since $s_t \le \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})$, this implies the displayed bound with $1 - \eta s_t$.

For $\mathbf{B}_{t+1}^{\perp}$, the deterministic full-precision part is

$$\left[\mathbf{I}_r - \eta(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{A}_t^{\mathbf{U}} - \eta(\mathbf{A}_t^{\perp})^\top\mathbf{A}_t^{\perp}\right]\mathbf{B}_t^{\perp}.$$

Since $r = r^*$, $\mathbf{A}_t^{\mathbf{U}}$ is square, and the same step-size condition gives

$$\left\|\mathbf{I}_r - \eta(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{A}_t^{\mathbf{U}} - \eta(\mathbf{A}_t^{\perp})^\top\mathbf{A}_t^{\perp}\right\|_{op} \le 1 - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}).$$

The quantization term is bounded by q_t. Its covariance term only involves the right-perpendicular residual blocks:

$$\begin{aligned}&\left\|\left[(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top\mathbf{U}_{\perp}^\top\right](\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})\right\|_{op}\\ \le&\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right)\\ \le&C\delta P_t(P_t\beta_t + \zeta_t),\end{aligned}$$

where $\|\mathbf{R}_t^{(2)}\|_{op} \le \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op} \le P_t\beta_t$. Hence

$$\|\mathbf{B}_{t+1}^{\perp}\|_{op} \le \left(1 - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right)\|\mathbf{B}_t^{\perp}\|_{op} + \eta q_t + C\eta\delta P_t(P_t\beta_t + \zeta_t).$$

Since $s_t \le \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})$, this implies the displayed bound with $1 - \eta s_t$.

Finally, $\mathbf{R}_{t+1}^{(3)} = \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}$ by definition, so the product bound follows from submultiplicativity. □

Next, we are ready to state the main induction lemma. For simplicity, denote $\epsilon_{\max} = \max\{\epsilon_d, \epsilon_A, \epsilon_B, \epsilon_o, \epsilon_w, \epsilon_l, \epsilon_r\}$.

Theorem B.5 (Exact-rank finite-horizon induction)**.** *Under Assumptions 3.1 and 3.2, consider the spectral initialization equation spectral initialization and the quantized GD iteration equation quantized GD with $\eta_1 = \eta_2 = \eta > 0$. Assume $r = r^*$ and write $\lambda_i^* = \lambda_i(\Delta)$ and $\kappa = \lambda_1^*/\lambda_{r^*}^*$. There are*

universal constants $c_0 \in (0,1)$ and $C > 1$ such that the following holds. For any $0 < \epsilon < 1$ and $\tau \in \mathbb{N}_0$, define

$$C_\epsilon = 1 + (1 + 2\sqrt{\lambda_1^*})^2 + \lambda_1^* + \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \qquad \delta_q = \epsilon + (1+\epsilon)\left(2\epsilon_{\max}\sqrt{d} + \epsilon_{\max}^2 d\right).$$

Let

$$\begin{aligned}\mathfrak{e}_q =& C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r^*})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2(1+2\sqrt{\lambda_1^*}) \\ &\cdot\Big[(1+\sqrt{r^*})\|\mathbf{W}^*\|_F + \sqrt{d}\|\mathbf{W}^*\|_{op} + (1+\sqrt{r^*})\|\mathbf{W}^\natural\|_F + \sqrt{r^*}\lambda_1^* + r^*(1+2\sqrt{\lambda_1^*})^2\Big],\end{aligned} \tag{30}$$

let

$$\Phi_q = C\left[\frac{(1+2\sqrt{\lambda_1^*})(1+\eta)}{\lambda_{r^*}^*/(64\kappa)} + \frac{(1+2\sqrt{\lambda_1^*})(1+\eta(1+2\sqrt{\lambda_1^*})+\delta_q)}{\left(\lambda_{r^*}^*/(64\kappa)\right)^2}\right]C_\epsilon^2\mathfrak{e}_q + C\mathfrak{e}_q + \frac{C\eta}{\lambda_{r^*}^*/(64\kappa)}\mathfrak{e}_q^2, \tag{31}$$

and let

$$\Phi_A(\tau) = C\eta\delta_q(1+2\sqrt{\lambda_1^*})\sum_{j=0}^{\tau-1}\left(1-\eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^{\tau-1-j}\left[\left(1-\eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^j\epsilon\lambda_1^* + \Phi_q\right] + \Phi_q.$$

Set

$$P_\star = 1 + 2\sqrt{\lambda_1^*}, \qquad \gamma = \frac{\lambda_{r^*}^*}{64\kappa}, \qquad L_\perp = \frac{4\sqrt{\lambda_1^*}}{\lambda_{r^*}^*},$$

and

$$E_\star = \epsilon\lambda_1^* + \Phi_q, \qquad \Xi_\star = 2 + L_\perp(1+\Phi_q).$$

Suppose

$$\delta_q \le c_0\min\left\{\frac{1}{\kappa}, \frac{\lambda_{r^*}^*}{\kappa(\lambda_1^*+\kappa^2)}, \frac{\gamma}{(1+2\sqrt{\lambda_1^*})C_\epsilon}\right\}, \quad 0 \le \epsilon_{\max} \le 1, \tag{32}$$

$$\eta \le \min\left\{\frac{c_0}{\kappa\lambda_1^*}, \frac{c_0}{\lambda_1^*}, \frac{c_0\gamma}{(1+2\sqrt{\lambda_1^*})^2C_\epsilon^2}, \frac{1}{4(1+2\sqrt{\lambda_1^*})^2}\right\},$$

$$\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}, \quad 0 < \epsilon \le \min\left\{\frac{1}{2}, c_0\frac{\gamma\lambda_{r^*}^*}{\lambda_1^*\sqrt{\lambda_1^*}P_\star}\right\},$$

$$\max\left\{\frac{\Phi_q}{\lambda_{r^*}^*}, \frac{\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*}, \frac{\tau\eta(1+\delta_q)(1+2\sqrt{\lambda_1^*})\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*}\right\} \le c_0, \quad C_\epsilon\Phi_q \le c_0P_\star,$$

$$\max\left\{P_\star\Xi_\star^2E_\star,\ \eta P_\star\Xi_\star E_\star,\ \eta P_\star^2\Xi_\star^2E_\star,\ \delta_qP_\star(P_\star+\eta P_\star\Xi_\star E_\star+\eta\mathfrak{e}_q)\Xi_\star,\ \eta\delta_q^2P_\star^2\Xi_\star^2E_\star\right\} \le c_0\gamma.$$

Then with probability at least $1 - 2C'\exp(-\epsilon^2 N)$ for a universal constant $C' > 0$, on the same high-probability event, for every integer $0 \le t \le \tau$, the following bounds hold.

- ***(H1) Error:***

$$\begin{aligned}\mathcal{M}_t &\le \left(1-\eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^t\epsilon\lambda_1^* + \Phi_q, \\ \|\mathbf{R}_t^{(3)}\|_{op} &\le \Phi_q\left[\left(1-\eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^t\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t)\right].\end{aligned} \tag{33}$$

- ***(H2) Parameter upper bounds:***

$$\|\mathbf{A}_t\|_{op} \le 1 + 2\sqrt{\lambda_1^*}, \qquad \|\mathbf{B}_t\|_{op} \le 1 + 2\sqrt{\lambda_1^*}. \tag{34}$$

- ***(H3) Signal lower bounds:***

$$\lambda_{r^*}(\mathbf{A}_t^{\mathbf{U}}) \geq \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}}, \qquad \lambda_{r^*}(\mathbf{B}_t^{\mathbf{V}}) \geq \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}}. \tag{35}$$

- ***(H4) Approximate alignment:***

$$\begin{aligned} \|\mathbf{A}_t^{\perp}\|_{op} \leq & \left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^t \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t), \\ \|\mathbf{B}_t^{\perp}\|_{op} \leq & \Phi_q. \end{aligned} \tag{36}$$

Proof. By Lemma B.2, with probability at least $1 - 2C' \exp(-\epsilon^2 N)$,

$$\delta = \left\| \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d \right\|_{op} \leq \delta_q.$$

The same concentration bound gives

$$\|\widetilde{\mathbf{X}}\|_{op} \leq \sqrt{N(1+\epsilon)}, \qquad \|\widetilde{\mathbf{X}}\|_F \leq \sqrt{Nd(1+\epsilon)}.$$

By Assumption 3.2,

$$\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op} \leq \|\widetilde{\mathbf{X}}\|_{op} + \epsilon_{\max}\|\widetilde{\mathbf{X}}\|_F \leq \sqrt{N(1+\epsilon)}(1 + \epsilon_{\max}\sqrt{d}).$$

For simplicity, we use the theorem notation

$$\rho = \eta\gamma.$$

We first prove the base case at $t = 0$. By Lemma B.3, we have

$$\mathcal{M}_0 \leq \epsilon\lambda_1^*, \qquad \|\mathbf{A}_0^{\perp}\|_{op} \leq \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \qquad \mathbf{B}_0^{\perp} = \mathbf{0}, \qquad \mathbf{R}_0^{(3)} = \mathbf{0},$$

The bound $0 < \epsilon \leq 1/2$ gives

$$\|\mathbf{A}_0\|_{op},\ \|\mathbf{B}_0\|_{op} \leq \sqrt{(1+\epsilon)\lambda_1^*} \leq 1 + 2\sqrt{\lambda_1^*}.$$

The initial-perpendicular smallness condition in equation 32 gives

$$\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} \leq c_0 \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}.$$

Since $\epsilon \leq 1/2$ and $\lambda_{r^*}^* \leq \lambda_1^*$,

$$\sqrt{(1-\epsilon)\lambda_{r^*}^*} \geq \frac{1}{\sqrt{2}}\sqrt{\lambda_{r^*}^*} \geq \frac{1}{\sqrt{2}}\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}.$$

Choose the universal constant $c_0 \leq 1/\sqrt{2} - 1/2$. Lemma B.3 then gives the stronger base signal bound

$$\lambda_{r^*}(\mathbf{A}_0^{\mathbf{U}}),\ \lambda_{r^*}(\mathbf{B}_0^{\mathbf{V}}) \geq \frac{\lambda_{r^*}^*}{2\sqrt{\lambda_1^*}},$$

and hence (H2) and (H3) hold at $t = 0$.

Assume the induction claims hold for every time j with $0 \leq j \leq t < \tau$. Use the notation of Corollary B.4, we have

$$\begin{aligned} P_t &= \max\{1, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}\} \leq 1 + 2\sqrt{\lambda_1^*}, \\ e_t &= \mathcal{M}_t \leq \left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^t \epsilon\lambda_1^* + \Phi_q, \\ s_t &= \min\left\{\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\right\} \geq \frac{(\lambda_{r^*}^*)^2}{16\lambda_1^*} = 4\frac{\lambda_{r^*}^*}{64\kappa}. \end{aligned} \tag{37}$$

Define

$$E_t = (1-\rho)^t \epsilon \lambda_1^* + \Phi_q, \qquad a_t = (1-\rho)^t \frac{\epsilon \lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t), \qquad b_t = \Phi_q, \qquad z_t = a_t b_t.$$

By (H4) and the exact-rank identity $\mathbf{R}_t^{(3)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}$,

$$\|\mathbf{A}_t^{\perp}\|_{op} \le a_t, \qquad \|\mathbf{B}_t^{\perp}\|_{op} \le b_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le z_t.$$

Since $r = r^*$, (H2) gives

$$\|\mathbf{A}_t\|_F,\ \|\mathbf{B}_t\|_F \le \sqrt{r^*}P_\star, \qquad \|\mathbf{A}_t\|_{op},\ \|\mathbf{B}_t\|_{op} \le P_\star.$$

Moreover

$$\|\Delta\|_{op} = \lambda_1^*, \qquad \|\widetilde{\mathbf{Y}}\|_F = \|\widetilde{\mathbf{X}}\mathbf{W}^*\|_F \le \sqrt{N(1+\epsilon)}\|\mathbf{W}^*\|_F,$$

and $\epsilon_1, \epsilon_2, \epsilon_3 \le C\epsilon_{\max}(1+\epsilon_{\max})^3$. Substituting these estimates, $\epsilon_i \le \epsilon_{\max}$, and the displayed data bounds into Lemma B.1 yields

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \le & C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r^*})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2 P_\star \\ & \cdot\Big[(1+\sqrt{r^*})\|\mathbf{W}^*\|_F + \sqrt{d}\|\mathbf{W}^*\|_{op} + (1+\sqrt{r^*})\|\mathbf{W}^\natural\|_F + \sqrt{r^*}\lambda_1^* + r^* P_\star^2\Big].\end{aligned}$$

The displayed bound for $\mathbf{E}_{t,\mathbf{B}}$ in Lemma B.1 gives, with the same substitutions,

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le & C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r^*})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2 P_\star \\ & \cdot\Big[(1+\sqrt{r^*})\|\mathbf{W}^*\|_F + \sqrt{d}\|\mathbf{W}^*\|_{op} + (1+\sqrt{r^*})\|\mathbf{W}^\natural\|_F + \sqrt{r^*}\lambda_1^* + r^* P_\star^2\Big].\end{aligned}$$

Thus

$$q_t = \max\{\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\} \le \mathfrak{e}_q.$$

We record the consequences of the envelopes and smallness hypotheses that will be used below:

$$P_\star \le C_\epsilon, \qquad P_\star^2 \le C_\epsilon, \qquad \delta_q P_\star C_\epsilon \le c_0\gamma, \qquad C_\epsilon \Phi_q \le c_0 P_\star, \qquad q_t \le \gamma\Phi_q, \qquad \eta q_t \le \rho\Phi_q. \tag{38}$$

The first two inequalities follow from the definition of C_ϵ, and the next two are explicit assumptions in equation 32. Also, by choosing the universal constant in equation 31 larger than the constants in the quantization estimate, the bound $q_t \le \mathfrak{e}_q$ gives

$$\Phi_q \ge C\left(\frac{P_\star}{\gamma}q_t + q_t + \frac{\eta}{\gamma}q_t^2\right), \tag{39}$$

which implies the last two inequalities in equation 38.

We now turn to the case at $t+1$. The step-size hypothesis needed for Corollary B.4 is satisfied because

$$\eta\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}^2 + \|\mathbf{B}_t^{\perp}\|_{op}^2\right) + \eta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}^2 + \|\mathbf{A}_t^{\perp}\|_{op}^2\right) \le 4\eta(1+2\sqrt{\lambda_1^*})^2 \le 1.$$

Since $r = r^*$, $\mathbf{B}_t^{\mathbf{V}}$ is square. By (H3),

$$\|(\mathbf{B}_t^{\mathbf{V}})^{-1}\|_{op} \le L_\perp.$$

Therefore the identity $\mathbf{R}_t^{(1)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}$ gives the sharper exact-rank envelope

$$\|\mathbf{A}_t^{\perp}\|_{op} \le L_\perp E_t, \qquad \|\mathbf{B}_t^{\perp}\|_{op} \le \Phi_q, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le L_\perp E_t \Phi_q.$$

(H1) for $\mathcal{M}_{t+1}$. Apply Corollary B.4 with the sharper envelopes

$$\alpha_t = L_\perp E_t, \qquad \beta_t = \Phi_q, \qquad \zeta_t = L_\perp E_t \Phi_q.$$

Then

$$\mathcal{M}_{t+1} \le (1-4\rho)e_t + \widetilde{\Phi}_t,$$

where [1]

$$\begin{aligned}\widetilde{\Phi}_t \leq & C\eta P_t(L_\perp E_t + \Phi_q)(E_t + L_\perp E_t \Phi_q) + C\eta P_t q_t + C\eta^2 P_t q_t \Xi_\star E_t + C\eta^2 q_t^2 \\ & + C\eta^2 P_t^2 \Xi_\star^2 E_t^2 + C\eta\delta_q P_t \left(P_t + \eta P_t \Xi_\star E_t + \eta q_t\right) \Xi_\star E_t + C\eta^2 \delta_q^2 P_t^2 \Xi_\star^2 E_t^2,\end{aligned}$$

where we used $E_t \leq E_\star$, $\Phi_q \leq E_t$, and hence

$$e_t + L_\perp E_t + \Phi_q + L_\perp E_t \Phi_q \leq \Xi_\star E_t.$$

The quantization-only terms are absorbed as follows. Choose the universal constant in equation 31 so that the first two terms below are each at most $\rho\Phi_q/4$. Then equation 39, $P_t \leq P_\star$, and $E_t \leq E_\star$ give

$$\begin{aligned}C\eta P_t q_t &\leq \frac{\rho\Phi_q}{4}, \\ C\eta^2 q_t^2 &\leq \frac{\rho\Phi_q}{4}, \\ C\eta^2 P_t q_t \Xi_\star E_t &\leq C\frac{\eta P_\star \Xi_\star E_\star}{\gamma}\rho\Phi_q \leq Cc_0\rho\Phi_q,\end{aligned}$$

where the last inequality uses $\eta P_\star \Xi_\star E_\star \leq c_0\gamma$ from equation 32. Choose c_0 so that $Cc_0 \leq 1/2$. Then

$$C\eta P_t q_t + C\eta^2 q_t^2 + C\eta^2 P_t q_t \Xi_\star E_t \leq \rho\Phi_q.$$

The last smallness condition in equation 32 gives the term-by-term bounds

$$\begin{aligned}C\eta P_t(L_\perp E_t + \Phi_q)(E_t + L_\perp E_t\Phi_q) &\leq Cc_0\rho E_t, \\ C\eta^2 P_t^2 \Xi_\star^2 E_t^2 &\leq Cc_0\rho E_t, \\ C\eta\delta_q P_t(P_t + \eta P_t \Xi_\star E_t + \eta q_t)\Xi_\star E_t &\leq Cc_0\rho E_t, \\ C\eta^2\delta_q^2 P_t^2 \Xi_\star^2 E_t^2 &\leq Cc_0\rho E_t.\end{aligned}$$

Choose c_0 so that the four $Cc_0\rho E_t$ terms sum to at most $3\rho E_t$. Then

$$\widetilde{\Phi}_t \leq 3\rho E_t + \rho\Phi_q \leq 3\rho(1-\rho)^t\epsilon\lambda_1^* + 4\rho\Phi_q.$$

Combining this with $e_t \leq (1-\rho)^t\epsilon\lambda_1^* + \Phi_q$ in equation 37 gives

$$\mathcal{M}_{t+1} \leq (1-4\rho)\left[(1-\rho)^t\epsilon\lambda_1^* + \Phi_q\right] + 3\rho(1-\rho)^t\epsilon\lambda_1^* + 4\rho\Phi_q = (1-\rho)^{t+1}\epsilon\lambda_1^* + \Phi_q.$$

(H4) for $\mathbf{A}_{t+1}^\perp$ and $\mathbf{B}_{t+1}^\perp$. For the separated perpendicular induction, we return to the envelopes

$$\alpha_t = a_t, \qquad \beta_t = b_t = \Phi_q, \qquad \zeta_t = z_t = a_t\Phi_q.$$

The definition of $\Phi_A(t)$ and the geometric sum bound $\sum_{j=0}^{t-1}(1-\rho)^{t-1-j}\eta \leq 1/\gamma$ give

$$\begin{aligned}a_t &\leq \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \frac{C\delta_q P_\star}{\gamma}(\epsilon\lambda_1^* + \Phi_q) + \Phi_q \\ &\leq CC_\epsilon.\end{aligned}$$

The last inequality follows from equation 32, $0 < \epsilon \leq 1/2$, and the definition of C_ϵ. The perpendicular bounds in Corollary B.4 imply

$$\|\mathbf{A}_{t+1}^\perp\|_{op} \leq (1-4\rho)a_t + \eta q_t + C\eta\delta_q\left[P_t(e_t + P_t a_t) + (P_t b_t + z_t)b_t\right].$$

Since $P_t \leq P_\star$, $a_t \leq CC_\epsilon$, and equation 38 holds, the first covariance term is absorbed into the contraction:

$$C\eta\delta_q P_t^2 a_t \leq C\eta\delta_q P_\star C_\epsilon a_t \leq Cc_0\eta\gamma a_t \leq \rho a_t,$$

where c_0 is fixed so that the displayed universal factor satisfies $Cc_0 \leq 1$. For the remaining covariance contribution, $E_t \geq \Phi_q$ and

$$(P_\star\Phi_q + a_t\Phi_q)\Phi_q \leq CC_\epsilon\Phi_q^2 \leq Cc_0 P_\star\Phi_q \leq P_\star E_t.$$

[1] Here we slightly abuse the notation of C by selecting the maximum of the constant C in Theorem B.5 and constant C in Corollary B.4.

Hence

$$C\eta\delta_q\left[P_te_t + (P_tb_t + z_t)b_t\right] \le C\eta\delta_q P_\star E_t.$$

Hence

$$\|\mathbf{A}_{t+1}^\perp\|_{op} \le (1-\rho)a_t + C\eta\delta_q(1+2\sqrt{\lambda_1^*})\left[(1-\rho)^t\epsilon\lambda_1^* + \Phi_q\right] + \eta q_t.$$

By equation 38, $\eta q_t \le \rho\Phi_q$. The definition of $\Phi_A(t+1)$ yields

$$\|\mathbf{A}_{t+1}^\perp\|_{op} \le (1-\rho)^{t+1}\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t+1).$$

The same corollary gives

$$\|\mathbf{B}_{t+1}^\perp\|_{op} \le (1-\eta s_t)\|\mathbf{B}_t^\perp\|_{op} + \eta q_t + C\eta\delta_q P_t(P_tb_t + z_t).$$

Since $b_t = \Phi_q$, $z_t = a_t\Phi_q$, $a_t \le CC_\epsilon$, and $P_t \le P_\star$, equation 38 gives

$$C\eta\delta_q P_t(P_tb_t + z_t) \le C\eta\delta_q P_\star C_\epsilon\Phi_q \le Cc_0\eta\gamma\Phi_q \le \rho\Phi_q$$

with the same choice of c_0. By $s_t \ge 4\gamma$ and $\eta q_t \le \rho\Phi_q$, the last two terms are at most $2\rho\Phi_q$. Thus

$$\|\mathbf{B}_{t+1}^\perp\|_{op} \le (1-4\rho)\Phi_q + 2\rho\Phi_q \le \Phi_q.$$

(H1) for $\mathbf{R}_{t+1}^{(3)}$. We use the identity

$$\mathbf{R}_{t+1}^{(3)} = \mathbf{A}_{t+1}^\perp\mathbf{B}_{t+1}^\perp.$$

The two (H4) bounds just proved imply

$$\|\mathbf{R}_{t+1}^{(3)}\|_{op} \le \Phi_q\left[(1-\rho)^{t+1}\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t+1)\right],$$

which is the second (H1) claim at time $t+1$.

(H2) and (H3) at $t+1$. It remains to keep the aligned factors in the same bounded and nonsingular tube. For every $0 \le j \le t$, (H1) and (H3) at time j imply that $\mathbf{B}_j^{\mathbf{V}}$ is invertible and

$$\|\mathbf{A}_j^\perp\|_{op} = \|\mathbf{R}_j^{(1)}(\mathbf{B}_j^{\mathbf{V}})^{-1}\|_{op} \le L_\perp\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right].$$

Together with the (H4) bound $\|\mathbf{B}_j^\perp\|_{op} \le \Phi_q$, this gives

$$\|\mathbf{R}_j^{(3)}\|_{op} = \|\mathbf{A}_j^\perp\mathbf{B}_j^\perp\|_{op} \le L_\perp\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right]\Phi_q \le C\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right],$$

and hence

$$\|\mathbf{A}_j\mathbf{B}_j - \Delta\|_{op} \le C\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right].$$

Projecting the component updates in Lemma A.3 back to the ambient coordinates gives

$$\mathbf{A}_{j+1} - \mathbf{A}_j = -\eta(\mathbf{A}_j\mathbf{B}_j - \Delta)\mathbf{B}_j^\top - \eta(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_j\mathbf{B}_j - \Delta)\mathbf{B}_j^\top + \eta\mathbf{E}_{j,\mathbf{A}},$$
$$\mathbf{B}_{j+1} - \mathbf{B}_j = -\eta\mathbf{A}_j^\top(\mathbf{A}_j\mathbf{B}_j - \Delta) - \eta\mathbf{A}_j^\top(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_j\mathbf{B}_j - \Delta) + \eta\mathbf{E}_{j,\mathbf{B}}.$$

Taking operator norms, using $\delta \le \delta_q$, the parameter bounds, and the preceding error estimate gives

$$\|\mathbf{A}_{j+1} - \mathbf{A}_j\|_{op} + \|\mathbf{B}_{j+1} - \mathbf{B}_j\|_{op} \le C\eta(1+\delta_q)(1+2\sqrt{\lambda_1^*})\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right] + C\eta q_j.$$

The preceding quantization estimate applies at each $j \le t$, so $q_j \le \mathfrak{e}_q$. Summing this display from $j = 0$ to t gives

$$\|\mathbf{A}_{t+1} - \mathbf{A}_0\|_{op} + \|\mathbf{B}_{t+1} - \mathbf{B}_0\|_{op}$$
$$\le C(1+\delta_q)P_\star\left[\frac{\epsilon\lambda_1^*}{\gamma} + \tau\eta\Phi_q\right] + C\tau\eta\mathfrak{e}_q.$$

Here $\sum_{j=0}^t \eta(1-\rho)^j \le 1/\gamma$ was used. The $C\mathfrak{e}_q$ term in equation 31 gives $\mathfrak{e}_q \le C\Phi_q$. Also $\delta_q \le 1$ and $P_\star \ge 1$. Hence the two smallness conditions

$$\frac{P_\star\epsilon\lambda_1^*}{\gamma} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}, \qquad \frac{\tau\eta(1+\delta_q)P_\star\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*} \le c_0$$

imply, provided c_0 is chosen so that the preceding universal constant times c_0 is at most 1, that

$$\|\mathbf{A}_{t+1} - \mathbf{A}_0\|_{op} + \|\mathbf{B}_{t+1} - \mathbf{B}_0\|_{op} \le c_0 \min\left\{\sqrt{\lambda_1^*}, \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}\right\}.$$

Therefore, by Lemma B.3, the stronger base signal bound proved above, and Weyl's singular-value inequality,

$$\|\mathbf{A}_{t+1}\|_{op},\ \|\mathbf{B}_{t+1}\|_{op} \le \sqrt{(1+\epsilon)\lambda_1^*} + c_0\sqrt{\lambda_1^*} \le 1 + 2\sqrt{\lambda_1^*},$$

and

$$\lambda_{r^*}(\mathbf{A}_{t+1}^{\mathbf{U}}),\ \lambda_{r^*}(\mathbf{B}_{t+1}^{\mathbf{V}}) \ge \frac{\lambda_{r^*}^*}{2\sqrt{\lambda_1^*}} - c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \ge \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}},$$

because $c_0 \le 1/4$. This closes (H2) and (H3) at time $t+1$, and hence the induction. □

Corollary B.6 (Whole Frobenius error from the exact-rank induction)**.** *For every fixed $t \ge 0$, under the conditions of Theorem B.5 with $\tau = t$, on the same high-probability event,*

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le \sqrt{r^*}\left(3\mathcal{E}_t^2 + \Phi_q^2\mathcal{A}_t^2\right)^{1/2} \le \sqrt{r^*}\left(\sqrt{3}\,\mathcal{E}_t + \Phi_q\mathcal{A}_t\right),$$

where, with $\gamma = \lambda_{r^*}^*/(64\kappa)$,

$$\mathcal{E}_t = (1-\eta\gamma)^t\epsilon\lambda_1^* + \Phi_q \le (1-\eta\gamma)^t\lambda_{r^*}^* + \Phi_q, \qquad \mathcal{A}_t = (1-\eta\gamma)^t\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t).$$

Proof. The orthogonal decompositions induced by $\mathbf{U}, \mathbf{U}_\perp, \mathbf{V}, \mathbf{V}_\perp$ give

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2 = \|\mathbf{R}_t^*\|_F^2 + \|\mathbf{R}_t^{(1)}\|_F^2 + \|\mathbf{R}_t^{(2)}\|_F^2 + \|\mathbf{R}_t^{(3)}\|_F^2.$$

Because $r = r^*$ in Theorem B.5, each block on the right-hand side has rank at most r^*. Hence $\|\mathbf{X}\|_F \le \sqrt{r^*}\|\mathbf{X}\|_{op}$ for each of these four blocks. The induction proved in Theorem B.5 gives, on the same high-probability event,

$$\|\mathbf{R}_t^*\|_{op},\ \|\mathbf{R}_t^{(1)}\|_{op},\ \|\mathbf{R}_t^{(2)}\|_{op} \le \mathcal{E}_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \Phi_q\mathcal{A}_t.$$

Substituting these estimates into the orthogonal block identity yields the first displayed inequality. The second follows from $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$. $(1-\eta\gamma)^t\epsilon\lambda_1^* + \Phi_q \le (1-\eta\gamma)^t\lambda_{r^*}^* + \Phi_q$ holds by $\epsilon \le \frac{1}{\kappa}$. □

Corollary B.7 (Simplified whole Frobenius error in exact-rank case)**.** *Under Assumptions 3.1 and 3.2, consider the spectral initialization equation spectral initialization and the quantized GD iteration equation quantized GD with $\eta_1 = \eta_2 = \eta > 0$. Assume $r = r^*$, write $\lambda_i^* = \lambda_i(\Delta)$, $\kappa = \lambda_1^*/\lambda_{r^*}^*$, and*

$$\gamma = \frac{\lambda_{r^*}^*}{64\kappa}.$$

Suppose the signal quantities are at constant level: λ_1^*, $1/\lambda_{r^*}^*$, κ, r^*, and $\|\mathbf{W}^\natural\|_{op}$ are bounded by absolute constants. Define

$$P_\star = 1 + 2\sqrt{\lambda_1^*}, \qquad C_0 = 1 + P_\star^2 + \lambda_1^* + \frac{\lambda_1^*}{\sqrt{\lambda_{r^*}^*/2}}.$$

There exist constants $c \in (0,1)$ and $C > 1$, depending only on the preceding constant-level bounds and the universal constants in Theorem B.5, such that the following holds. Choose ϵ, η satisfying

$$0 < \epsilon \le c\min\left\{1,\ \frac{1}{\kappa},\ \frac{\lambda_{r^*}^*}{\kappa(\lambda_1^* + \kappa^2)},\ \frac{\gamma}{P_\star C_0},\ \frac{\gamma\lambda_{r^*}^*}{P_\star\lambda_1^*\sqrt{\lambda_1^*}},\ \frac{(\lambda_{r^*}^*)^{3/2}}{(\lambda_1^*)^{3/2}}\right\},$$

and

$$0 < \eta \le c\min\left\{1,\ \frac{1}{\kappa\lambda_1^*},\ \frac{1}{\lambda_1^*},\ \frac{1}{P_\star^2},\ \frac{\gamma}{P_\star^2 C_0^2}\right\}.$$

For any $t > 0$, if the low-precision errors obey

$$t\epsilon_{\max}\sqrt{d} \le c,$$

then with probability at least $1 - C\exp(-c\epsilon^2 N)$,

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le C\sqrt{r^*}\left[(1-\eta\gamma)^t + \epsilon_{\max}\sqrt{d}\right].$$

In particular, if additionally

$$\epsilon_{\max}\sqrt{d} \le c(1-\eta\gamma)^t,$$

then the low-precision iterates match the full-precision linear rate:

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le C\sqrt{r^*}(1-\eta\gamma)^t.$$

Proof. Let c_0 denote the small universal constant in Theorem B.5, and write

$$z = \epsilon_{\max}\sqrt{d}.$$

Throughout the proof, constants denoted by C depend only on the constant-level bounds in the statement and on the universal constants in Theorem B.5. By taking the statement constant c small enough, the displayed assumptions imply $\epsilon \le 1/2$, $\eta \le 1$, $z \le 1$, and $\delta_q \le 1$.

We first record the simplified envelopes entering Theorem B.5. Since $\epsilon \le 1/2$, the definition of C_ϵ gives

$$C_\epsilon \le C_0.$$

Also,

$$\delta_q = \epsilon + (1+\epsilon)(2z+z^2) \le \epsilon + 5z.$$

The constant-level assumptions imply

$$\|\Delta\|_F \le \sqrt{r^*}\lambda_1^* = \mathcal{O}(1), \qquad \|\mathbf{W}^*\|_{op} \le \|\mathbf{W}^\natural\|_{op} + \lambda_1^* = \mathcal{O}(1),$$

and, because $\mathbf{W}^\natural \in \mathbb{R}^{d\times k}$,

$$\|\mathbf{W}^\natural\|_F \le \sqrt{d}\,\|\mathbf{W}^\natural\|_{op} = \mathcal{O}(\sqrt{d}), \qquad \|\mathbf{W}^*\|_F \le \|\mathbf{W}^\natural\|_F + \|\Delta\|_F = \mathcal{O}(\sqrt{d}).$$

Substituting these bounds and $z \le 1$ into equation 30 gives

$$\mathfrak{e}_q \le Cz.$$

Then equation 31, $\eta \le 1$, and $\delta_q \le 1$ give

$$\Phi_q \le Cz.$$

Consequently

$$E_\star = \epsilon\lambda_1^* + \Phi_q \le C(\epsilon + z), \qquad \Xi_\star = 2 + L_\perp(1+\Phi_q) \le C.$$

We now verify equation 32. The first covariance-smallness condition follows from the displayed bound on ϵ, the low-precision bound $z \le c$, and the smallness of c:

$$\delta_q \le \epsilon + 5z \le c_0 \min\left\{\frac{1}{\kappa}, \frac{\lambda_{r^*}^*}{\kappa(\lambda_1^* + \kappa^2)}\right\}.$$

The assumptions also give $0 < \epsilon \le 1/2$ and $0 \le \epsilon_{\max} \le z \le 1$. The displayed step-size condition gives

$$\eta \le \min\left\{\frac{c_0}{\kappa\lambda_1^*}, \frac{c_0}{\lambda_1^*}, \frac{1}{4P_\star^2}\right\}$$

after decreasing c if necessary. Moreover,

$$\delta_q P_\star C_\epsilon \le C(\epsilon+z)P_\star C_0 \le c_0\gamma, \qquad \eta P_\star^2 C_\epsilon^2 \le cP_\star^2 C_0^2 \frac{\gamma}{P_\star^2 C_0^2} \le c_0\gamma.$$

The two initialization smallness requirements follow directly from the ϵ condition:

$$\frac{P_\star\epsilon\lambda_1^*}{\gamma} \le c\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}},$$

and

$$\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} \le \sqrt{2}\,\epsilon\frac{\lambda_1^*}{\sqrt{\lambda_{r^*}^*}} \le c\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}.$$

The Φ_q conditions are consequences of $\Phi_q \le Cz$, $z \le c$, and $tz \le c$:

$$C_\epsilon\Phi_q \le CC_0 z \le c_0 P_\star,$$

$$\frac{\Phi_q}{\lambda_{r^*}^*} \le Cz \le c_0, \qquad \frac{\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*} \le Cz \le c_0,$$

and

$$\frac{t\eta(1+\delta_q)P_\star\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*} \le Ctz \le c_0.$$

Finally, using $E_\star \le C(\epsilon+z)$, $\Xi_\star \le C$, $\delta_q \le C(\epsilon+z)$, and $\mathfrak{e}_q \le Cz$, the last maximum in equation 32 satisfies

$$P_\star\Xi_\star^2 E_\star, \quad \eta P_\star\Xi_\star E_\star, \quad \eta P_\star^2\Xi_\star^2 E_\star \le C(\epsilon+z) \le c_0\gamma,$$
$$\delta_q P_\star(P_\star + \eta P_\star\Xi_\star E_\star + \eta\mathfrak{e}_q)\Xi_\star \le C(\epsilon+z) \le c_0\gamma,$$

and

$$\eta\delta_q^2 P_\star^2\Xi_\star^2 E_\star \le C(\epsilon+z) \le c_0\gamma.$$

Thus every condition in Theorem B.5 holds, so Corollary B.6 applies.

For $t=0$, the summation defining $\Phi_A(0)$ is empty, so $\Phi_A(0) = \Phi_q \le Cz$. The exact definition of $\Phi_A(t)$ and $\gamma^{-1} = \mathcal{O}(1)$ give

$$\Phi_A(t) \le C\left[\eta t(1-\eta\gamma)^{t-1} + \epsilon_{\max}\sqrt{d}\right].$$

The first envelope satisfies, for every $0 \le t$,

$$\mathcal{E}_t \le C\left[(1-\eta\gamma)^t + \epsilon_{\max}\sqrt{d}\right],$$

For the second envelope, the case $t=0$ gives

$$\Phi_q\mathcal{A}_0 \le C\epsilon_{\max}\sqrt{d}.$$

For $1 \le t$,

$$\Phi_q\mathcal{A}_t \le C\epsilon_{\max}\sqrt{d}\left[(1-\eta\gamma)^t + \eta t(1-\eta\gamma)^{t-1} + \epsilon_{\max}\sqrt{d}\right] \le C\epsilon_{\max}\sqrt{d},$$

because $\eta t(1-\eta\gamma)^{t-1} \le \gamma^{-1} = \mathcal{O}(1)$ and $\epsilon_{\max}\sqrt{d} \le c$. Substituting these estimates into Corollary B.6 proves the claim.

If $\epsilon_{\max}\sqrt{d} \le c(1-\eta\gamma)^t$, then for every $0 \le t$,

$$\epsilon_{\max}\sqrt{d} \le c(1-\eta\gamma)^t.$$

Absorbing this term into the geometric term in the preceding display gives the final full-precision-rate statement. □

B.3.2 General Rank

The general-rank proof follows the same template as the exact-rank proof but cannot invert the signal block to eliminate all surplus directions. Instead, it tracks explicit envelopes for $\mathbf{A}_t^\perp$, $\mathbf{B}_t^\perp$, and $\mathbf{R}_t^{(3)} = \mathbf{A}_t^\perp\mathbf{B}_t^\perp$. This produces the additional horizon-dependent term in the final Frobenius bound.

Corollary B.8 (Induction-ready propagation with separated envelopes)**.** *Assume the hypotheses of Lemma A.4 with $\eta_1 = \eta_2 = \eta$, and define*

$$P_t = \max\{1, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}\}, \quad q_t = \max\{\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\},$$
$$e_t = \mathcal{M}_t, \quad s_t = \min\left\{\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\right\}, \quad \delta = \left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right\|_{op}.$$

If nonnegative numbers $\alpha_t, \beta_t, \zeta_t$ satisfy

$$\|\mathbf{A}_t^{\perp}\|_{op} \le \alpha_t, \qquad \|\mathbf{B}_t^{\perp}\|_{op} \le \beta_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \zeta_t,$$

then, with

$$K_t^{\mathrm{sep}} = e_t + \alpha_t + \beta_t + \zeta_t, \qquad H_t = e_t + \zeta_t,$$

define

$$\begin{aligned}\widetilde{\Phi}_t &= C\eta P_t(\alpha_t+\beta_t)H_t + C\eta^2 P_t q_t K_t^{\mathrm{sep}} + C\eta P_t q_t + C\eta^2 q_t^2 \\ &\quad + C\eta^2 P_t^2 H_t\left(K_t^{\mathrm{sep}} + (K_t^{\mathrm{sep}})^2\right) + C\eta\delta P_t\left(P_t + K_t^{\mathrm{sep}} + \eta P_t K_t^{\mathrm{sep}} + \eta q_t\right) H_t \\ &\quad + C\eta^2\delta^2 P_t^2 H_t\left(K_t^{\mathrm{sep}} + (K_t^{\mathrm{sep}})^2\right),\end{aligned}$$

for a universal constant $C > 0$, it holds

$$\begin{aligned}\mathcal{M}_{t+1} &\le (1-\eta s_t)e_t + \widetilde{\Phi}_t, \\ \|\mathbf{A}_{t+1}^{\perp}\|_{op} &\le \alpha_t + \eta q_t + C\eta\delta\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right], \\ \|\mathbf{B}_{t+1}^{\perp}\|_{op} &\le \beta_t + \eta q_t + C\eta\delta P_t(P_t\beta_t + \zeta_t).\end{aligned}$$

Moreover,

$$\|\mathbf{R}_{t+1}^{(3)}\|_{op} = \|\mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}\|_{op} \le \|\mathbf{A}_{t+1}^{\perp}\|_{op}\,\|\mathbf{B}_{t+1}^{\perp}\|_{op}.$$

In particular, if $q_t = 0$ and $\beta_t = \zeta_t = 0$, then the displayed bounds give $\|\mathbf{B}_{t+1}^{\perp}\|_{op} = 0$ and $\|\mathbf{R}_{t+1}^{(3)}\|_{op} = 0$.

Proof. Apply Lemma A.4 with $\eta_1 = \eta_2 = \eta$. Throughout the proof we use

$$\|\mathbf{A}_t^{\mathbf{U}}\|_{op}, \|\mathbf{A}_t^{\perp}\|_{op}, \|\mathbf{B}_t^{\mathbf{V}}\|_{op}, \|\mathbf{B}_t^{\perp}\|_{op} \le P_t,$$

$$\|\mathbf{R}_t^{*}\|_{op}, \|\mathbf{R}_t^{(1)}\|_{op}, \|\mathbf{R}_t^{(2)}\|_{op} \le e_t, \quad \|\mathbf{A}_t^{\perp}\|_{op} \le \alpha_t, \quad \|\mathbf{B}_t^{\perp}\|_{op} \le \beta_t, \quad \|\mathbf{R}_t^{(3)}\|_{op} \le \zeta_t,$$

and $\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le q_t$. The leading coefficients in the first three residual recursions are

$$\begin{aligned}&1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \\ &1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta\lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top}), \\ &1 - \eta\lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp}) - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}).\end{aligned}$$

By the step-size hypothesis these coefficients are nonnegative, and each is at most $1 - \eta s_t$ after dropping the nonnegative term not included in s_t.

The deterministic first-order non-signal groups from Lemma A.4 are

$$\begin{aligned}&\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}e_t, \\ &\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t, \\ &\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}.\end{aligned}$$

Therefore, they are bounded by

$$C\eta P_t(\alpha_t+\beta_t)H_t,$$

where $H_t = e_t + \zeta_t$. The one-quantization and pure-quantization groups are

$$\begin{aligned}&\eta^2\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t \\ &\quad + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}, \\ &\eta^2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)e_t + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \\ &\quad + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}, \\ &\eta^2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \\ &\quad + \eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}.\end{aligned}$$

They are bounded by

$$C\eta^2 P_t q_t K_t^{\mathrm{sep}} + C\eta P_t q_t + C\eta^2 q_t^2.$$

The quadratic residual groups are

$$\eta^2 \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op}\right) e_t^2,$$

$$\eta^2 \left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right) e_t + \left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right] e_t,$$

$$\eta^2 \left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\right) e_t + \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right] e_t.$$

They are bounded by

$$C\eta^2 P_t^2 H_t \left(K_t^{\mathrm{sep}} + (K_t^{\mathrm{sep}})^2\right).$$

The covariance-linear groups are

$$\begin{aligned}
&\eta\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}e_t + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) e_t \\
&\quad + \eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}e_t + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&\quad + 2\eta\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{B}_t^{\perp}\|_{op}e_t + \eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) e_t, \\
&\eta\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}e_t + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) e_t \\
&\quad + \eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}e_t + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&\quad + 2\eta\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op} + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) e_t, \\
&\eta\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) e_t \\
&\quad + \eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&\quad + \eta\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{B}_t^{\perp}\|_{op}e_t + \eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\left(e_t + \|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}$$

In the third residual recursion the leading $\|\mathbf{B}_t^{\perp}\|_{op}$ term is kept as a separated non-signal factor, and all other displayed factors are bounded by the common envelopes. Thus they are bounded by

$$C\eta\delta P_t \left(P_t + K_t^{\mathrm{sep}} + \eta P_t K_t^{\mathrm{sep}} + \eta q_t\right) H_t$$

The covariance-quadratic groups have the common form

$$\eta^2\delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) e_t + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(e_t + \|\mathbf{R}_t^{(3)}\|_{op}\right),$$

with the last factor equal to e_t in the first two recursions. Since $\beta_t\zeta_t \le K_t^{\mathrm{sep}} H_t$, the envelope reduction is bounded by

$$C\eta^2\delta^2 P_t^2 H_t \left(K_t^{\mathrm{sep}} + (K_t^{\mathrm{sep}})^2\right).$$

Combining these estimates proves

$$\mathcal{M}_{t+1} \le (1 - \eta s_t) e_t + \widetilde{\Phi}_t.$$

It remains to prove the separated perpendicular recursions. For $\mathbf{A}_{t+1}^{\perp}$, the deterministic full-precision part of the $\mathbf{A}_{t+1}^{\perp}$ update equals

$$\mathbf{A}_t^{\perp}\left[\mathbf{I}_r - \eta\mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^{\top} - \eta\mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^{\top}\right].$$

The step-size hypothesis implies that the bracketed matrix lies between $\mathbf{0}$ and $\mathbf{I}_r$ in Loewner order, so its operator norm is at most one. Thus this part contributes at most α_t. The quantization term is bounded by q_t. The covariance factor satisfies

$$\begin{aligned}
&\left\|(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^{\top} + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^{\top}\right\|_{op} \\
\le &\left(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right)\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right)\|\mathbf{B}_t^{\perp}\|_{op} \\
\le &C\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right],
\end{aligned}$$

where we used $\mathbf{R}_t^{(1)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}$ and $\mathbf{R}_t^{(2)} = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}$. Combining these bounds gives

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op} \le \alpha_t + \eta q_t + C\eta\delta\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right].$$

For $\mathbf{B}_{t+1}^{\perp}$, the deterministic full-precision part is

$$\left[\mathbf{I}_r - \eta(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{A}_t^{\mathbf{U}} - \eta(\mathbf{A}_t^{\perp})^\top\mathbf{A}_t^{\perp}\right]\mathbf{B}_t^{\perp},$$

whose contribution is at most β_t by the same step-size argument. The quantization term is bounded by q_t, and the covariance factor satisfies

$$\begin{aligned}
&\left\|\left[(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top\mathbf{U}_{\perp}^\top\right](\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})\right\|_{op}\\
\le&\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right)\\
\le&C\delta P_t(P_t\beta_t + \zeta_t),
\end{aligned}$$

where $\|\mathbf{R}_t^{(2)}\|_{op} \le \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op} \le P_t\beta_t$. Hence

$$\|\mathbf{B}_{t+1}^{\perp}\|_{op} \le \beta_t + \eta q_t + C\eta\delta P_t(P_t\beta_t + \zeta_t).$$

Finally, $\mathbf{R}_{t+1}^{(3)} = \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}$ by definition, so the product bound follows from submultiplicativity. □

Theorem B.9 (General-rank finite-horizon induction)**.** *Under Assumptions 3.1 and 3.2, consider the spectral initialization equation spectral initialization and the quantized GD iteration equation quantized GD with $\eta_1 = \eta_2 = \eta > 0$. Assume $r \ge r^*$, write $\lambda_i^* = \lambda_i(\Delta)$ and $\kappa = \lambda_1^*/\lambda_{r^*}^*$, and set $\epsilon_{\max} = \max\{\epsilon_d, \epsilon_A, \epsilon_B, \epsilon_o, \epsilon_w, \epsilon_l, \epsilon_r\}$. There are universal constants $c_0 \in (0,1)$ and $C > 1$ such that the following holds. For any $0 < \epsilon < 1$ and $\tau \in \mathbb{N}_0$, define*

$$\delta_q = \epsilon + (1+\epsilon)\left(2\epsilon_{\max}\sqrt{d} + \epsilon_{\max}^2 d\right), \qquad P_\star = 1 + 2\sqrt{\lambda_1^*}, \qquad \gamma = \frac{\lambda_{r^*}^*}{64\kappa}, \qquad \rho = \eta\gamma.$$

Let

$$\begin{aligned}
\mathfrak{e}_{q,r} =&C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star\\
&\cdot\left[(1+\sqrt{r})\|\mathbf{W}^*\|_F + \sqrt{d}\|\mathbf{W}^*\|_{op} + (1+\sqrt{r})\|\mathbf{W}^\natural\|_F + \sqrt{r\lambda_1^*} + rP_\star^2\right].
\end{aligned} \tag{40}$$

Define deterministic envelopes $\{\mathcal{E}_t, \mathcal{A}_t, \mathcal{B}_t, \mathcal{Z}_t, \Omega_t\}$ recursively by

$$\mathcal{E}_0 = \epsilon\lambda_1^*, \qquad \mathcal{A}_0 = \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \qquad \mathcal{B}_0 = 0, \qquad \mathcal{Z}_0 = 0.$$

For $0 \le t \le \tau$, set

$$\mathcal{K}_t = \mathcal{E}_t + \mathcal{A}_t + \mathcal{B}_t + \mathcal{Z}_t, \qquad \mathcal{H}_t = \mathcal{E}_t + \mathcal{Z}_t.$$

For $0 \le t < \tau$, define

$$\begin{aligned}
\Omega_t =&C\eta P_\star(\mathcal{A}_t + \mathcal{B}_t)\mathcal{H}_t + C\eta^2P_\star\mathfrak{e}_{q,r}\mathcal{K}_t + C\eta P_\star\mathfrak{e}_{q,r} + C\eta^2\mathfrak{e}_{q,r}^2\\
&+ C\eta^2P_\star^2\mathcal{H}_t(\mathcal{K}_t + \mathcal{K}_t^2) + C\eta\delta_qP_\star\left(P_\star + \mathcal{K}_t + \eta P_\star\mathcal{K}_t + \eta\mathfrak{e}_{q,r}\right)\mathcal{H}_t + C\eta^2\delta_q^2P_\star^2\mathcal{H}_t(\mathcal{K}_t + \mathcal{K}_t^2),\\
\mathcal{E}_{t+1} =&(1-4\rho)\mathcal{E}_t + \Omega_t,\\
\mathcal{A}_{t+1} =&\mathcal{A}_t + \eta\mathfrak{e}_{q,r} + C\eta\delta_q\left[P_\star(\mathcal{E}_t + P_\star\mathcal{A}_t) + (P_\star\mathcal{B}_t + \mathcal{Z}_t)\mathcal{B}_t\right],\\
\mathcal{B}_{t+1} =&\mathcal{B}_t + \eta\mathfrak{e}_{q,r} + C\eta\delta_qP_\star(P_\star\mathcal{B}_t + \mathcal{Z}_t), \qquad \mathcal{Z}_{t+1} = \mathcal{A}_{t+1}\mathcal{B}_{t+1}.
\end{aligned} \tag{41}$$

Suppose

$$\begin{aligned}
&0 < \epsilon \le \frac{1}{2}, \qquad 0 \le \epsilon_{\max} \le 1,\\
&\eta \le \min\left\{\frac{c_0}{\kappa\lambda_1^*}, \frac{1}{4P_\star^2}\right\}, \qquad \mathcal{A}_0 \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}},\\
&C(1+\delta_q)P_\star\sum_{j=0}^{\tau-1}\eta(3\mathcal{E}_j + \mathcal{Z}_j) + C\tau\eta\mathfrak{e}_{q,r} \le c_0\min\left\{\sqrt{\lambda_1^*}, \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}\right\},
\end{aligned} \tag{42}$$

Then with probability at least $1 - 2C' \exp(-\epsilon^2 N)$ *for a universal constant* $C' > 0$*, on the same high-probability event, for every integer* $0 \le t \le \tau$*,*

$$\begin{aligned} \mathcal{M}_t \le \mathcal{E}_t, \qquad & \|\mathbf{R}_t^{(3)}\|_{op} \le \mathcal{Z}_t, \\ \|\mathbf{A}_t^{\perp}\|_{op} \le \mathcal{A}_t, \qquad & \|\mathbf{B}_t^{\perp}\|_{op} \le \mathcal{B}_t, \\ \|\mathbf{A}_t\|_{op} \le P_\star, \qquad & \|\mathbf{B}_t\|_{op} \le P_\star, \\ \lambda_{r^*}(\mathbf{A}_t^{\mathbf{U}}) \ge \frac{\lambda^*_{r^*}}{4\sqrt{\lambda_1^*}}, \qquad & \lambda_{r^*}(\mathbf{B}_t^{\mathbf{V}}) \ge \frac{\lambda^*_{r^*}}{4\sqrt{\lambda_1^*}}. \end{aligned} \tag{43}$$

Proof. By Lemma B.2, with probability at least $1 - 2C' \exp(-\epsilon^2 N)$,

$$\delta = \left\| \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d \right\|_{op} \le \delta_q, \qquad \|\widetilde{\mathbf{X}}\|_{op} \le \sqrt{N(1+\epsilon)}, \qquad \|\widetilde{\mathbf{X}}\|_F \le \sqrt{Nd(1+\epsilon)}.$$

On this event, Assumption 3.2 gives

$$\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op} \le \sqrt{N(1+\epsilon)}(1 + \epsilon_{\max}\sqrt{d}).$$

We prove equation 43 by induction.

For $t = 0$, Lemma B.3 gives

$$\mathcal{M}_0 \le \mathcal{E}_0, \qquad \|\mathbf{A}_0^{\perp}\|_{op} \le \mathcal{A}_0, \qquad \mathbf{B}_0^{\perp} = \mathbf{0}, \qquad \mathbf{R}_0^{(3)} = \mathbf{0}.$$

Also,

$$\|\mathbf{A}_0\|_{op},\ \|\mathbf{B}_0\|_{op} \le \sqrt{(1+\epsilon)\lambda_1^*} \le P_\star.$$

Since $\epsilon \le 1/2$ and $\lambda^*_{r^*} \le \lambda_1^*$,

$$\sqrt{(1-\epsilon)\lambda^*_{r^*}} \ge \frac{1}{\sqrt{2}} \frac{\lambda^*_{r^*}}{\sqrt{\lambda_1^*}}.$$

Choosing $c_0 \le 1/\sqrt{2} - 1/2$ in equation 42, Lemma B.3 and the condition on $\mathcal{A}_0$ imply the stronger base signal bound

$$\lambda_{r^*}(\mathbf{A}_0^{\mathbf{U}}),\ \lambda_{r^*}(\mathbf{B}_0^{\mathbf{V}}) \ge \frac{\lambda^*_{r^*}}{2\sqrt{\lambda_1^*}}.$$

This proves the base case, since the theorem only requires the weaker lower bound with constant $1/4$.

Envelope consequences at time t**.** Assume the claims hold for every time $0 \le j \le t < \tau$. Then, in the notation of Corollary B.8,

$$P_t = \max\{1, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}\} \le P_\star,\ e_t = \mathcal{M}_t \le \mathcal{E}_t,\ \|\mathbf{A}_t^{\perp}\|_{op} \le \mathcal{A}_t,\ \|\mathbf{B}_t^{\perp}\|_{op} \le \mathcal{B}_t,\ \|\mathbf{R}_t^{(3)}\|_{op} \le \mathcal{Z}_t,$$

and

$$s_t = \min\left\{\lambda^2_{r^*}(\mathbf{A}_t^{\mathbf{U}}), \lambda^2_{r^*}(\mathbf{B}_t^{\mathbf{V}})\right\} \ge \frac{(\lambda^*_{r^*})^2}{16\lambda_1^*} = 4\gamma.$$

The rank of $\mathbf{A}_t$ and $\mathbf{B}_t$ is at most r, so the induction hypothesis also gives

$$\|\mathbf{A}_t\|_F,\ \|\mathbf{B}_t\|_F \le \sqrt{r} P_\star.$$

Consequently,

$$\|\mathbf{A}_t\|_{op} + \epsilon_A \|\mathbf{A}_t\|_F,\ \|\mathbf{B}_t\|_{op} + \epsilon_B \|\mathbf{B}_t\|_F \le P_\star(1 + \epsilon_{\max}\sqrt{r}),$$

and the same estimate with $\mathbf{A}_t$ or $\mathbf{B}_t$ replaced by its Frobenius norm costs at most another factor $\sqrt{r}$. Also,

$$\|\Delta\|_{op} = \lambda_1^*, \qquad \|\widetilde{\mathbf{Y}}\|_F = \|\widetilde{\mathbf{X}}\mathbf{W}^*\|_F \le \sqrt{N(1+\epsilon)}\|\mathbf{W}^*\|_F,$$

and

$$\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N} \le \frac{\sqrt{1+\epsilon}(1 + \epsilon_{\max}\sqrt{d})}{\sqrt{N}}, \qquad \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N} \le (1+\epsilon)(1 + \epsilon_{\max}\sqrt{d})^2.$$

Quantization-error bound. Since $\epsilon_i \le \epsilon_{\max}$ and $\epsilon_1, \epsilon_2, \epsilon_3 \le C\epsilon_{\max}(1+\epsilon_{\max})^3$, the first group in the bound for $\mathbf{E}_{t,\mathbf{A}}$ in Lemma B.1 satisfies

$$\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)\left(\epsilon_1\|\widetilde{\mathbf{Y}}\|_F+\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op}+\epsilon_2\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right)$$
$$\le C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star\left(\|\mathbf{W}^*\|_F+\sqrt{d}\|\mathbf{W}^*\|_{op}+\|\mathbf{W}^\natural\|_F\right).$$

The second group in the same bound satisfies

$$\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\left[\epsilon_3(\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)^2\|\mathbf{A}_t\|_F+\epsilon_B\|\mathbf{B}_t\|_F\left(\|\Delta\|_{op}+\|\mathbf{A}_t\|_{op}(2\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)\right)\right]$$
$$\le C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star\left(\sqrt{r}\,\lambda_1^*+rP_\star^2\right).$$

Hence

$$\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \le C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star$$
$$\cdot\left[(1+\sqrt{r})\|\mathbf{W}^*\|_F+\sqrt{d}\|\mathbf{W}^*\|_{op}+(1+\sqrt{r})\|\mathbf{W}^\natural\|_F+\sqrt{r}\lambda_1^*+rP_\star^2\right].$$

For $\mathbf{E}_{t,\mathbf{B}}$, the terms containing $\|\Delta\|_{op}$ are bounded by the $\sqrt{r}\lambda_1^*$ term, the two response terms by $(1+\sqrt{r})\|\mathbf{W}^*\|_F$, the data-error term by $\sqrt{d}\|\mathbf{W}^*\|_{op}$, the two $\mathbf{W}^\natural$ terms by $(1+\sqrt{r})\|\mathbf{W}^\natural\|_F$, and the remaining parameter-only terms by $rP_\star^2$. Thus Lemma B.1 gives

$$\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star$$
$$\cdot\left[(1+\sqrt{r})\|\mathbf{W}^*\|_F+\sqrt{d}\|\mathbf{W}^*\|_{op}+(1+\sqrt{r})\|\mathbf{W}^\natural\|_F+\sqrt{r}\lambda_1^*+rP_\star^2\right].$$

Increasing the universal constant C in equation 40 if necessary, this yields

$$q_t = \max\{\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\} \le \mathfrak{e}_{q,r}. \tag{44}$$

The same argument applies at every earlier time $0 \le j \le t$ under the induction hypotheses, so $q_j \le \mathfrak{e}_{q,r}$ for all such j.

Residual bound for $\mathcal{M}_{t+1}$. The step-size hypothesis in Corollary B.8 holds because

$$\eta\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}^2+\|\mathbf{B}_t^{\perp}\|_{op}^2\right)+\eta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}^2+\|\mathbf{A}_t^{\perp}\|_{op}^2\right) \le 4\eta P_\star^2 \le 1.$$

Apply Corollary B.8 with

$$\alpha_t = \mathcal{A}_t, \qquad \beta_t = \mathcal{B}_t, \qquad \zeta_t = \mathcal{Z}_t.$$

Since every term in $\widetilde{\Phi}_t$ has nonnegative coefficients and is monotone in $P_t, q_t, e_t, \delta, \alpha_t, \beta_t, \zeta_t$, the induction bounds, $\delta \le \delta_q$, and equation 44 imply

$$\widetilde{\Phi}_t \le \Omega_t.$$

Moreover $1-4\rho \ge 0$ after taking c_0 small enough, because $\eta \le c_0/(\kappa\lambda_1^*)$ and $\gamma = \lambda_{r^*}^*/(64\kappa) \le \lambda_1^*$. Therefore Corollary B.8 gives

$$\mathcal{M}_{t+1} \le (1-\eta s_t)e_t + \widetilde{\Phi}_t \le (1-4\rho)\mathcal{E}_t + \Omega_t = \mathcal{E}_{t+1}.$$

Perpendicular and $\mathbf{R}_{t+1}^{(3)}$ bounds. The perpendicular bounds in the same corollary give

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op} \le \mathcal{A}_t + \eta\mathfrak{e}_{q,r} + C\eta\delta_q\left[P_\star(\mathcal{E}_t+P_\star\mathcal{A}_t)+(P_\star\mathcal{B}_t+\mathcal{Z}_t)\mathcal{B}_t\right] = \mathcal{A}_{t+1},$$

and

$$\|\mathbf{B}_{t+1}^{\perp}\|_{op} \le \mathcal{B}_t + \eta\mathfrak{e}_{q,r} + C\eta\delta_q P_\star(P_\star\mathcal{B}_t+\mathcal{Z}_t) = \mathcal{B}_{t+1}.$$

Therefore

$$\|\mathbf{R}_{t+1}^{(3)}\|_{op} = \|\mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}\|_{op} \le \mathcal{A}_{t+1}\mathcal{B}_{t+1} = \mathcal{Z}_{t+1}.$$

Bounded and nonsingular tube at $t+1$. It remains to keep the iterates inside the bounded, nonsingular tube. For every $0 \le j \le t$, the already proved block bounds give

$$\|\mathbf{A}_j\mathbf{B}_j - \Delta\|_{op} \le \|\mathbf{R}_j^*\|_{op} + \|\mathbf{R}_j^{(1)}\|_{op} + \|\mathbf{R}_j^{(2)}\|_{op} + \|\mathbf{R}_j^{(3)}\|_{op} \le 3\mathcal{E}_j + \mathcal{Z}_j.$$

Projecting the component updates in Lemma A.3 back to the ambient coordinates gives

$$\mathbf{A}_{j+1} - \mathbf{A}_j = -\eta(\mathbf{A}_j\mathbf{B}_j - \Delta)\mathbf{B}_j^\top - \eta(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_j\mathbf{B}_j - \Delta)\mathbf{B}_j^\top + \eta\mathbf{E}_{j,\mathbf{A}},$$
$$\mathbf{B}_{j+1} - \mathbf{B}_j = -\eta\mathbf{A}_j^\top(\mathbf{A}_j\mathbf{B}_j - \Delta) - \eta\mathbf{A}_j^\top(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_j\mathbf{B}_j - \Delta) + \eta\mathbf{E}_{j,\mathbf{B}}.$$

Hence, using $P_j \le P_\star$ and $q_j \le \mathfrak{e}_{q,r}$,

$$\|\mathbf{A}_{j+1} - \mathbf{A}_j\|_{op} + \|\mathbf{B}_{j+1} - \mathbf{B}_j\|_{op} \le C\eta(1+\delta_q)P_\star(3\mathcal{E}_j + \mathcal{Z}_j) + C\eta\mathfrak{e}_{q,r}.$$

Summing from $j = 0$ to t and using equation 42 gives

$$\|\mathbf{A}_{t+1} - \mathbf{A}_0\|_{op} + \|\mathbf{B}_{t+1} - \mathbf{B}_0\|_{op} \le c_0 \min\left\{\sqrt{\lambda_1^*}, \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}\right\}.$$

Therefore, by Lemma B.3 and Weyl's singular-value inequality,

$$\|\mathbf{A}_{t+1}\|_{op},\ \|\mathbf{B}_{t+1}\|_{op} \le \sqrt{(1+\epsilon)\lambda_1^*} + c_0\sqrt{\lambda_1^*} \le P_\star,$$

and, after taking c_0 small enough,

$$\lambda_{r^*}(\mathbf{A}_{t+1}^{\mathbf{U}}),\ \lambda_{r^*}(\mathbf{B}_{t+1}^{\mathbf{V}}) \ge \frac{\lambda_{r^*}^*}{2\sqrt{\lambda_1^*}} - c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \ge \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}}.$$

This closes the induction. □

Corollary B.10 (Whole Frobenius error from the general-rank induction)**.** *Under the conditions of Theorem B.9, on the same high-probability event, for every integer $0 \le t \le \tau$,*

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le \left(3r^*\mathcal{E}_t^2 + r\mathcal{Z}_t^2\right)^{1/2} \le \sqrt{3r^*}\,\mathcal{E}_t + \sqrt{r}\,\mathcal{Z}_t.$$

Proof. The orthogonal decompositions induced by $\mathbf{U}, \mathbf{U}_\perp, \mathbf{V}, \mathbf{V}_\perp$ give

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2 = \|\mathbf{R}_t^*\|_F^2 + \|\mathbf{R}_t^{(1)}\|_F^2 + \|\mathbf{R}_t^{(2)}\|_F^2 + \|\mathbf{R}_t^{(3)}\|_F^2.$$

The first three blocks have rank at most r^*, while $\mathbf{R}_t^{(3)} = \mathbf{A}_t^\perp\mathbf{B}_t^\perp$ has rank at most r. Hence $\|\mathbf{X}\|_F \le \sqrt{\mathrm{rank}(\mathbf{X})}\|\mathbf{X}\|_{op}$ gives

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2 \le r^*\left(\|\mathbf{R}_t^*\|_{op}^2 + \|\mathbf{R}_t^{(1)}\|_{op}^2 + \|\mathbf{R}_t^{(2)}\|_{op}^2\right) + r\|\mathbf{R}_t^{(3)}\|_{op}^2.$$

Theorem B.9 gives

$$\|\mathbf{R}_t^*\|_{op},\ \|\mathbf{R}_t^{(1)}\|_{op},\ \|\mathbf{R}_t^{(2)}\|_{op} \le \mathcal{E}_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \mathcal{Z}_t.$$

Substituting these estimates proves the first inequality, and the second follows from $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$. □

Corollary B.11 (Simplified whole Frobenius error, general rank)**.** *Under Assumptions 3.1 and 3.2, consider the spectral initialization equation spectral initialization and the quantized GD iteration equation quantized GD with $\eta_1 = \eta_2 = \eta > 0$. Assume $r \ge r^*$, write $\lambda_i^* = \lambda_i(\Delta)$, $\kappa = \lambda_1^*/\lambda_{r^*}^*$, and*

$$\gamma = \frac{\lambda_{r^*}^*}{64\kappa}.$$

Suppose the signal quantities are at constant level: λ_1^*, $1/\lambda_{r^*}^*$, κ, r, and $\|\mathbf{W}^\natural\|_{op}$ are bounded by absolute constants. Define

$$P_\star = 1 + 2\sqrt{\lambda_1^*}.$$

There exist constants $c \in (0,1)$ and $C > 1$, depending only on the preceding constant-level bounds and the universal constants in Theorem B.9, such that the following holds. For an integer $t \geq 0$, define

$$z = \epsilon_{\max}\sqrt{d}, \qquad T_t = t\eta, \qquad D_t = z + \epsilon T_t z + (T_t z)^2.$$

Choose ϵ, η satisfying

$$0 < \epsilon \leq c \min\left\{1, \frac{1}{\kappa}, \frac{\lambda^*_{r^*}}{\kappa(\lambda^*_1 + \kappa^2)}, \frac{\gamma}{P_\star^2}, \frac{(\lambda^*_{r^*})^{3/2}}{(\lambda^*_1)^{3/2}}\right\},$$

and

$$0 < \eta \leq c \min\left\{1, \frac{1}{\kappa\lambda^*_1}, \frac{1}{\lambda^*_1}, \frac{1}{P_\star^2}\right\}.$$

If the low-precision errors and horizon obey

$$z \leq c \min\left\{1, \frac{1}{\kappa}, \frac{\lambda^*_{r^*}}{\kappa(\lambda^*_1 + \kappa^2)}, \frac{\gamma}{P_\star^2}\right\},$$

$$T_t(\epsilon + z) \leq c, \qquad \epsilon + T_t z + D_t \leq c\frac{\gamma}{P_\star},$$

and

$$\epsilon + T_t z + T_t D_t \leq \frac{c}{P_\star} \min\left\{\sqrt{\lambda^*_1}, \frac{\lambda^*_{r^*}}{\sqrt{\lambda^*_1}}\right\},$$

then with probability at least $1 - C\exp(-c\epsilon^2 N)$, on the same high-probability event,

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \leq C\sqrt{r}\left[(1-\eta\gamma)^t + D_t\right].$$

In particular, if additionally

$$D_t \leq c(1-\eta\gamma)^t,$$

then the low-precision iterates match the full-precision linear rate at time t:

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \leq C\sqrt{r}(1-\eta\gamma)^t.$$

Proof. Let c_0 denote the small universal constant in Theorem B.9. Throughout the proof, constants denoted by C depend only on the constant-level bounds in the statement and on the universal constants in Theorem B.9. We apply Theorem B.9 with horizon $\tau = t$.

By taking the statement constant c small enough, the displayed assumptions imply $\epsilon \leq 1/2$, $\eta \leq 1$, $0 \leq \epsilon_{\max} \leq z \leq 1$, and the covariance perturbation satisfies

$$\delta_q = \epsilon + (1+\epsilon)(2z + z^2) \leq \epsilon + 5z.$$

The constant-level assumptions imply

$$\|\Delta\|_F \leq \sqrt{r}\,\lambda^*_1 = \mathcal{O}(1), \qquad \|\mathbf{W}^*\|_{op} \leq \|\mathbf{W}^\natural\|_{op} + \lambda^*_1 = \mathcal{O}(1),$$

and, because $\mathbf{W}^\natural \in \mathbb{R}^{d\times k}$,

$$\|\mathbf{W}^\natural\|_F \leq \sqrt{d}\,\|\mathbf{W}^\natural\|_{op} = \mathcal{O}(\sqrt{d}), \qquad \|\mathbf{W}^*\|_F \leq \|\mathbf{W}^\natural\|_F + \|\Delta\|_F = \mathcal{O}(\sqrt{d}).$$

Substituting these bounds and $z \leq c$ into equation 40 gives

$$\mathfrak{e}_{q,r} \leq Cz.$$

We first record explicit bounds for the recursive envelopes in Theorem B.9. For $0 \leq s \leq t$, set $T_s = s\eta$ and

$$D_s = z + \epsilon T_s z + (T_s z)^2.$$

We claim that, after decreasing c if necessary,

$$\mathcal{E}_s \leq C\left[(1-\eta\gamma)^s\epsilon + D_s\right], \quad \mathcal{A}_s \leq C(\epsilon + T_s z), \quad \mathcal{B}_s \leq CT_s z, \quad \mathcal{Z}_s \leq C\left[\epsilon T_s z + (T_s z)^2\right]. \tag{45}$$

The bounds hold at $s = 0$ by the initialization in Theorem B.9. Suppose they hold up to time $s < t$. The assumptions imply

$$\mathcal{K}_s \le C\left[(1-\eta\gamma)^s\epsilon + \epsilon + T_t z + D_t\right], \qquad \mathcal{A}_s + \mathcal{B}_s \le C(\epsilon + T_t z).$$

Hence the displayed smallness assumptions, together with $\delta_q \le C(\epsilon + z)$ and $\mathfrak{e}_{q,r} \le Cz$, give

$$P_\star(\mathcal{A}_s + \mathcal{B}_s) \le c_0\gamma, \qquad \eta P_\star^2(\mathcal{K}_s + \mathcal{K}_s^2) \le c_0\gamma,$$

and

$$\delta_q P_\star(P_\star + \mathcal{K}_s + \eta P_\star \mathcal{K}_s + \eta\mathfrak{e}_{q,r}) + \eta\delta_q^2 P_\star^2(\mathcal{K}_s + \mathcal{K}_s^2) \le c_0\gamma,$$

after decreasing c by a universal factor. Here the first estimate uses $\epsilon + T_t z + D_t \le c\gamma/P_\star$, the second also uses $\eta \le c/P_\star^2$, and the covariance estimate also uses $\epsilon + z \le c\gamma/P_\star^2$. Substituting equation 45 into the recursions equation 41, using $\mathfrak{e}_{q,r} \le Cz$ and $\delta_q \le C(\epsilon + z)$, and applying the discrete Gronwall inequality with $T_t(\epsilon + z) \le c$, gives

$$\begin{aligned}
\mathcal{A}_{s+1} &\le \mathcal{A}_s + C\eta z + C\eta(\epsilon + z)(\mathcal{E}_s + \mathcal{A}_s + \mathcal{B}_s^2 + \mathcal{Z}_s\mathcal{B}_s) \le C(\epsilon + T_{s+1}z),\\
\mathcal{B}_{s+1} &\le \mathcal{B}_s + C\eta z + C\eta(\epsilon + z)(\mathcal{B}_s + \mathcal{Z}_s) \le CT_{s+1}z.
\end{aligned}$$

Therefore

$$\mathcal{Z}_{s+1} = \mathcal{A}_{s+1}\mathcal{B}_{s+1} \le C\left[\epsilon T_{s+1}z + (T_{s+1}z)^2\right].$$

For the signal-block envelope, the same substitutions yield

$$\Omega_s \le C\eta z + c_0\eta\gamma\left(\mathcal{E}_s + \mathcal{Z}_s\right).$$

Taking c_0 small enough so that the coefficient of $\mathcal{E}_s$ is absorbed into the contraction, and using the discrete convolution bound $\sum_{\ell=0}^{s}\eta(1-\eta\gamma)^\ell \le C/\gamma = C$, while bounding the convolved $\mathcal{Z}_s$ term by its monotone envelope, gives

$$\mathcal{E}_{s+1} \le C\left[(1-\eta\gamma)^{s+1}\epsilon + z + \epsilon T_{s+1}z + (T_{s+1}z)^2\right].$$

This proves equation 45.

We now verify the conditions of Theorem B.9. We have already observed that $0 < \epsilon \le 1/2$ and $0 \le \epsilon_{\max} \le 1$. The displayed step-size condition gives

$$\eta \le \min\left\{\frac{c_0}{\kappa\lambda_1^*}, \frac{1}{4P_\star^2}\right\}$$

after decreasing c if necessary. The initialization condition follows from

$$\mathcal{A}_0 = \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}},$$

using the explicit upper bound on ϵ. Finally, by equation 45,

$$\sum_{s=0}^{t-1}\eta(3\mathcal{E}_s + \mathcal{Z}_s) + T_t\mathfrak{e}_{q,r} \le C\left[\epsilon + T_t z + T_t D_t\right].$$

Here we used $\sum_{s=0}^{t-1}\eta(1-\eta\gamma)^s \le C/\gamma = C$. Since $\delta_q \le C(\epsilon + z) \le 1$ after decreasing c, the displayed tube smallness assumption, with c decreased by another universal factor if necessary, gives the finite-horizon tube condition in equation 42.

Thus Corollary B.10 applies. Combining it with equation 45 at time t and using $r^* \le r$ and $\mathcal{Z}_t \le CD_t$ gives

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le C\sqrt{r}\left[(1-\eta\gamma)^t\epsilon + D_t\right].$$

Since $\epsilon \le 1$, the first displayed claim follows. If $D_t \le c(1-\eta\gamma)^t$, then the low-precision terms are absorbed into the geometric term, proving the final statement. □

C TECHNICAL LEMMAS

Lemma C.1 (Operator norm contraction)**.** *For any compatible matrices* $\mathbf{X}, \mathbf{P}, \mathbf{Q}$ *and scalars* $\eta_1, \eta_2 \geq 0$ *such that* $\eta_1\|\mathbf{P}\|_{op}^2 + \eta_2\|\mathbf{Q}\|_{op}^2 \leq 1$*, we have*

$$\left\|\mathbf{X} - \eta_1\mathbf{X}\mathbf{P}^\top\mathbf{P} - \eta_2\mathbf{Q}\mathbf{Q}^\top\mathbf{X}\right\|_{op} \leq \left(1 - \eta_1\lambda_{\min}(\mathbf{P}^\top\mathbf{P}) - \eta_2\lambda_{\min}(\mathbf{Q}\mathbf{Q}^\top)\right)\|\mathbf{X}\|_{op}.$$

Proof. Let $\mathcal{L}(\mathbf{X}) = \mathbf{X} - \eta_1\mathbf{X}\mathbf{P}^\top\mathbf{P} - \eta_2\mathbf{Q}\mathbf{Q}^\top\mathbf{X}$ denote the linear mapping. Since $\mathbf{P}^\top\mathbf{P}$ and $\mathbf{Q}\mathbf{Q}^\top$ are real symmetric and positive semi-definite (PSD) matrices, they admit orthogonal diagonalizations: $\mathbf{P}^\top\mathbf{P} = \mathbf{V}_P\mathbf{\Lambda}_P\mathbf{V}_P^\top$ and $\mathbf{Q}\mathbf{Q}^\top = \mathbf{V}_Q\mathbf{\Lambda}_Q\mathbf{V}_Q^\top$, where $\mathbf{V}_P, \mathbf{V}_Q$ are orthogonal matrices and $\mathbf{\Lambda}_P, \mathbf{\Lambda}_Q$ are diagonal matrices containing the non-negative eigenvalues.

Applying this decomposition to the linear mapping and multiplying by $\mathbf{V}_Q^\top$ on the left and $\mathbf{V}_P$ on the right, we obtain

$$\mathbf{V}_Q^\top\mathcal{L}(\mathbf{X})\mathbf{V}_P = \widetilde{\mathbf{X}} - \eta_1\widetilde{\mathbf{X}}\mathbf{\Lambda}_P - \eta_2\mathbf{\Lambda}_Q\widetilde{\mathbf{X}},$$

where we define $\widetilde{\mathbf{X}} = \mathbf{V}_Q^\top\mathbf{X}\mathbf{V}_P$. Since orthogonal transformations preserve the operator norm, we have $\|\widetilde{\mathbf{X}}\|_{op} = \|\mathbf{X}\|_{op}$ and $\|\mathbf{V}_Q^\top\mathcal{L}(\mathbf{X})\mathbf{V}_P\|_{op} = \|\mathcal{L}(\mathbf{X})\|_{op}$.

The mapping can now be expressed as a Hadamard product (element-wise multiplication):

$$\mathbf{V}_Q^\top\mathcal{L}(\mathbf{X})\mathbf{V}_P = \mathbf{C} \circ \widetilde{\mathbf{X}},$$

where the coefficient matrix $\mathbf{C}$ has entries $C_{i,j} = 1 - \eta_2(\mathbf{\Lambda}_Q)_{ii} - \eta_1(\mathbf{\Lambda}_P)_{jj}$.

Let $a_i = \eta_2(\mathbf{\Lambda}_Q)_{ii}$ and $b_j = \eta_1(\mathbf{\Lambda}_P)_{jj}$. By the assumption $\eta_1\|\mathbf{P}\|_{op}^2 + \eta_2\|\mathbf{Q}\|_{op}^2 \leq 1$, we know $a_i \geq 0, b_j \geq 0$ and $\max_i(a_i) + \max_j(b_j) \leq 1$. We define the maximum contraction factor as $\mu_{\max} = 1 - \min_i(a_i) - \min_j(b_j) = 1 - \eta_2\lambda_{\min}(\mathbf{Q}\mathbf{Q}^\top) - \eta_1\lambda_{\min}(\mathbf{P}^\top\mathbf{P})$. If $\mu_{\max} = 0$, then $\max_i(a_i) + \max_j(b_j) = \min_i(a_i) + \min_j(b_j) = 1$, so $C_{i,j} = 0$ for every i, j and the contraction map is zero; the claim is immediate.

To bound the operator norm of the Hadamard product, we normalize the coefficients. Let $s_i = \frac{a_i - \min(a)}{\mu_{\max}}$ and $t_j = \frac{b_j - \min(b)}{\mu_{\max}}$. It is straightforward to verify that $s_i \geq 0$, $t_j \geq 0$, and $s_i + t_j \leq 1$. We can then rewrite $C_{i,j}$ as:

$$C_{i,j} = \mu_{\max} \cdot (1 - s_i - t_j).$$

Crucially, the term $(1 - s_i - t_j)$ can be represented as an integral of the product of two indicator functions over $x \in [0, 1]$:

$$1 - s_i - t_j = \int_0^1 \mathbb{1}_{\{x \geq s_i\}} \cdot \mathbb{1}_{\{x \leq 1 - t_j\}} dx.$$

This integral representation reveals that $\mathbf{C}$ is a Schur multiplier. We can substitute this back into the Hadamard product:

$$\mathbf{C} \circ \widetilde{\mathbf{X}} = \mu_{\max} \int_0^1 \mathbf{D}_1(x)\widetilde{\mathbf{X}}\mathbf{D}_2(x)dx,$$

where $\mathbf{D}_1(x) = \text{diag}(\mathbb{1}_{\{x \geq s_i\}})$ and $\mathbf{D}_2(x) = \text{diag}(\mathbb{1}_{\{x \leq 1 - t_j\}})$ are diagonal matrices with entries being either 0 or 1. Consequently, their operator norms are strictly bounded by 1, i.e., $\|\mathbf{D}_1(x)\|_{op} \leq 1$ and $\|\mathbf{D}_2(x)\|_{op} \leq 1$ for any $x \in [0, 1]$.

Finally, taking the operator norm on both sides and applying the triangle inequality for integrals (Minkowski's inequality) alongside the sub-multiplicativity of the operator norm, we obtain:

$$\begin{aligned}
\|\mathcal{L}(\mathbf{X})\|_{op} &= \|\mathbf{C} \circ \widetilde{\mathbf{X}}\|_{op} \\
&\leq \mu_{\max} \int_0^1 \|\mathbf{D}_1(x)\|_{op}\|\widetilde{\mathbf{X}}\|_{op}\|\mathbf{D}_2(x)\|_{op}dx \\
&\leq \mu_{\max} \int_0^1 1 \cdot \|\mathbf{X}\|_{op} \cdot 1 dx \\
&= \mu_{\max}\|\mathbf{X}\|_{op}.
\end{aligned}$$

Substituting the definition of $\mu_{\max}$ completes the proof. □