ANYTIME-VALID DISTRIBUTION SHIFT DETECTION VIA PREDICTIVE RANK MARTINGALES

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ABSTRACT

Many sequential distribution shift detectors update a growing reference set with incoming observations. After a change, this update contaminates the reference set with post-change observations and can weaken subsequent evidence. Keeping the calibration sample fixed mitigates this contamination, but repeated reuse induces dependence among fixed-reference ranks, so arguments based on independent conformal p-values do not apply. We derive the exact conditional null distribution of the next rank given the preceding ranks and use it to construct a predictive rank martingale (PRM). Thresholding a PRM gives distribution-free, finite-sample anytime marginal type I error control. To target specific departures, we apply a pre-specified feature to each rank, center the resulting payoff under the predictive null law, and use Online Newton Step (ONS) to adapt the bet. Order and dispersion features target directional and center-versus-tail changes, respectively. For any Lipschitz feature with nonzero induced contrast under the alternative, we establish a finite-window detection guarantee and show that the test is consistent as the initial calibration size increases. At a fixed calibration size, however, we derive a power ceiling for every distribution-free detection procedure. Across synthetic and real data, our PRM methods achieve better detection performance than conditional conformal test martingale (CCTM).

1 INTRODUCTION

Distribution shift monitoring asks whether an online data stream has departed from the distribution represented by historical data. A monitoring procedure should detect such changes quickly while controlling false alarms. We seek a sequential procedure for which the probability of ever raising a false alarm is at most α.

Conformal test martingales (CTMs) provide anytime-valid tests of exchangeability by betting on sequential conformal p-values (Vovk et al., 2003). In the classical online construction, each incoming observation is added to the reference sample used for subsequent observations. For distribution shift monitoring, this expanding reference creates a vulnerability: after a change, post-change observations enter the comparison sample and can make later observations appear less unusual. The effect is especially pronounced for early changes, when post-change observations quickly form a substantial fraction of the reference sample.

Keeping the historical calibration sample fixed mitigates this contamination. However, repeated calibration reuse induces dependence across the resulting ranks. The classical martingale argument based on independent conformal p-values therefore no longer applies. CCTM addresses this problem by betting on the empirical cumulative distribution function (ECDF) of the fixed calibration sample and using a uniform Dvoretzky–Kiefer–Wolfowitz (DKW) confidence band to account for ECDF estimation error (Shaer et al., 2026). Its probably approximately correct (PAC) calibration-conditional guarantee, however, does not by itself imply marginal type I error control at the same nominal level, and its confidence band can discount the betting signal.

Our key observation is that this dependence has an exact sequential structure: under the null, the next fixed-reference rank has a closed-form predictive distribution conditional on the rank history. We use this law to construct a PRM, and thresholding this PRM provides distribution-free, finite-sample anytime marginal type I error control. The construction follows the test-by-betting principle, which interprets accumulated wealth as evidence against the null (Shafer, 2021).

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The predictive law ensures validity, but detecting a particular shift requires a suitable betting strategy. We therefore use a feature h to score locations on the reference quantile scale and target shifts of interest without specifying a parametric alternative. The choice of h affects power but not validity: every pre-specified h retains the same anytime false-alarm guarantee. We construct Order PRM for directional changes, Dispersion PRM for center-versus-tail changes, and a portfolio PRM that combines their evidence. ONS (Hazan et al., 2007) adapts the direction and magnitude of each bet.

1.1 Our Contributions

Our contributions are:

- We derive the exact sequential predictive null law of fixed-reference ranks and use it to construct a PRM with distribution-free, finite-sample anytime type I error control.
- We construct Order PRM and Dispersion PRM by combining rank features with ONS betting. They target directional and center-versus-tail changes, respectively, and a PRM portfolio covers several pre-specified features while keeping the probability of any false alarm at most α.
- We provide a power analysis of PRMs. We establish a finite-window detection guarantee and show that the test is consistent as the initial calibration size increases. At a fixed calibration size, we also derive a power ceiling that applies to every distribution-free detection procedure.
- Empirically, Order PRM reaches 80% detection earlier than CCTM in all synthetic settings from Shaer et al. (2026), and the PRM portfolio remains close to Order PRM in these settings. Dispersion PRM and the portfolio detect symmetric shifts for which CCTM has near-zero power within the studied horizon.

1.2 Related Work

Our setting is related to classical placement-based two-sample tests, including partially sequential procedures that compare a fixed first sample with sequentially arriving observations (Wolfe, 1977; Orban & Wolfe, 1980; 1982). These studies motivate our use of features on the reference quantile scale, but their procedures use pre-specified sampling rules or finite horizons. We instead embed placement scores in an e-process that remains valid under arbitrary stopping.

Turning to anytime-valid methods, CTMs are constructed by betting on sequential conformal p-values computed using a reference set that grows with the online stream (Vovk et al., 2003; 2005; Fedorova et al., 2012; Volkhonskiy et al., 2017; Vovk et al., 2021). Our construction follows the same test-by-betting principle but addresses the dependence induced by repeatedly comparing online observations with a fixed calibration sample. Other anytime-valid approaches include safe e-processes for binary observations (Ramdas et al., 2022), pairwise betting (Saha & Ramdas, 2024), and prediction-based betting for sequential two-sample and independence testing (Podkopaev & Ramdas, 2023), but use different observation schemes or alternative structures.

Calibration reuse has also been studied in offline conformal multiple testing. Bates et al. (2023) used positive regression dependence among conformal p-values to establish false discovery rate control (Benjamini & Hochberg, 1995; Benjamini & Yekutieli, 2001). Their goal is simultaneous testing, whereas we characterize the sequential conditional law for anytime-valid monitoring.

CCTM is the closest existing method for fixed-reference monitoring (Shaer et al., 2026). It handles calibration uncertainty through a uniform DKW band, whereas we model the dependence induced by calibration reuse through its exact sequential predictive law. This yields a nonnegative martingale and finite-sample anytime marginal type I error control; Appendix B discusses how this guarantee differs from CCTM's PAC calibration-conditional statement. Sequential conditioning is essential, since multiplying marginally valid e-values need not produce an e-process (Vovk et al., 2025).

2 Predictive Rank Martingale Construction

We first derive the predictive null law of fixed-reference ranks and use it to construct a general family of PRMs. We then introduce features that target different departures, convert a chosen feature into a betting process, and use ONS to adapt the betting coefficient. Finally, we combine the processes corresponding to several features in a portfolio.

2.1 PROBLEM SETUP

Let $D_0 = (Y_1, \dots, Y_n)$ be the fixed calibration observations and let $X_1, X_2, \dots$ be the online observations. For clarity, we present the method for scalar observations. High-dimensional observations can be reduced to scalar scores according to a rule fixed before observing the calibration and monitoring data. Under the null, the calibration and online observations are independent and identically distributed (i.i.d.):

$$H_0(P): \qquad Y_1, \dots, Y_n, X_1, X_2, \dots \overset{\text{i.i.d.}}{\sim} P,$$

where P is unknown. We seek a stopping time τ satisfying

$$\sup_P \mathbb{P}_{H_0(P)}(\tau < \infty) \le \alpha,$$

where the probability is taken jointly over D_0 and the online stream.

For continuous observations, define the rank of X_t among the fixed calibration observations as $R_t = 1 + \sum_{i=1}^{n} \mathbf{1}\{Y_i \le X_t\} \in \{1, \dots, n+1\}$. For distributions with atoms, we break ties using independent $W_i, W_t' \sim \mathrm{Unif}(0,1)$. We compare (Y_i, W_i) and (X_t, W_t') lexicographically and define R_t from the resulting order. Let $\mathcal{G}_t = \sigma(R_1, \dots, R_t)$. Reusing the same calibration sample induces dependence across the ranks. This dependence nevertheless admits an exact sequential characterization, which we state next.

2.2 PREDICTIVE RANK LAW AND MARTINGALE CONSTRUCTION

Define $N_{t-1,j} = \sum_{s=1}^{t-1} \mathbf{1}\{R_s = j\}$. Before observing R_t, a bettor may use past ranks to choose an $\mathcal{G}_{t-1}$-measurable distribution q_t over the next rank. The next theorem gives the exact null predictive distribution and shows that comparing q_t with this distribution yields a PRM.

Theorem 1. *For every distribution P, under $H_0(P)$, for every $t \ge 1$,*

$$\mathbb{P}(R_t = j \mid \mathcal{G}_{t-1}) = \pi_{t,j} := \frac{1 + N_{t-1,j}}{n+t}, \qquad j = 1, \dots, n+1. \tag{1}$$

Let $q_t = (q_{t,1}, \dots, q_{t,n+1})$ be any $\mathcal{G}_{t-1}$-measurable probability vector, and define

$$E_t = \frac{q_{t,R_t}}{\pi_{t,R_t}}, \qquad M_t = \prod_{s=1}^{t} E_s, \qquad M_0 = 1.$$

Under $H_0(P)$, $(M_t)_{t \ge 0}$ is a nonnegative martingale, so by Ville's inequality (Ville, 1939),

$$\sup_P \mathbb{P}_{H_0(P)}\left(\sup_{t \ge 0} M_t \ge \frac{1}{\alpha}\right) \le \alpha.$$

The predictive law in (1) coincides with that of a standard Pólya urn (Blackwell & MacQueen, 1973). It characterizes the dependence induced by repeated reuse of the fixed calibration sample: each R_t is computed against the same D_0. Since q_t is a probability vector, $\mathbb{E}_{H_0(P)}(E_t \mid \mathcal{G}_{t-1}) = 1$, so the martingale property follows immediately. Therefore, the stopping rule $\tau = \inf\{t \ge 1 : M_t \ge 1/\alpha\}$ satisfies $\sup_P \mathbb{P}_{H_0(P)}(\tau < \infty) \le \alpha$, giving finite-sample anytime type I error control. The allocation of q_t across ranks affects power but not this guarantee.

To choose q_t, write the factor as $E_t = 1 + \lambda_t Z_t$, where Z_t is bounded and conditionally mean zero under the null. This centering preserves validity, while a nonzero mean under a shift provides signal. The predictable coefficient λ_t controls the direction and magnitude of the bet and is constrained so that $E_t \ge 0$. We next construct Z_t from a feature contrast.

2.3 FEATURES AND TARGETED ALTERNATIVES

To construct Z_t, we use a pre-specified function $h : [0,1] \to \mathbb{R}$ to score whether an observation lies in the lower tail, center, or upper tail of the calibration distribution. Different choices of h weight these regions differently and therefore target different departures. We choose h before observing the calibration and monitoring data. We call h a feature if

$$\int_0^1 h(u)\, du = 0, \qquad \sup_{u \in [0,1]} h(u) - \inf_{u \in [0,1]} h(u) \le 1.$$

As shown below, adding a constant to h does not change the resulting procedure, so the first condition is imposed only for notational convenience. The second condition ensures that the martingale constructed below is nonnegative. Any function bounded on $[0, 1]$ can be shifted and rescaled to satisfy both conditions.

Suppose that the calibration observations follow a continuous distribution P, while an online observation X follows Q. The population signal targeted by h is the contrast:

$$\Delta_h(P, Q) = \mathbb{E}_{X\sim Q} h\{F_P(X)\}.$$

Under the null $Q = P$, $F_P(X) \sim \mathrm{Unif}(0, 1)$, and hence $\Delta_h(P, P) = \int_0^1 h(u)\, du = 0$. Positive and negative values of $\Delta_h(P, Q)$ indicate excess alternative mass in regions where h is respectively large or small. Its magnitude is the population signal available to that feature. $\Delta_h(P, Q) = 0$ need not imply $P = Q$, it only means that the chosen feature has no population signal for that departure.

We first consider the order feature

$$h_{\mathrm{ord}}(u) = u - \frac{1}{2}, \qquad u \in [0, 1].$$

It is the classical Mann–Whitney placement score (Orban & Wolfe, 1980; 1982) and also coincides with the linear reference-quantile score underlying CCTM (Shaer et al., 2026). It assigns negative values to lower reference ranks and positive values to upper reference ranks, with larger magnitude farther from the midpoint.

For the order feature, the contrast has the pairwise interpretation $\Delta_{\mathrm{ord}}(P, Q) = \mathbb{E}_Q F_P(X) - 1/2 = \mathbb{P}(Y \le X) - 1/2$, where $Y \sim P$ is independent of X. It measures whether observations tend to rank above or below reference observations. A rightward shift gives $\Delta_{\mathrm{ord}} > 0$, whereas a leftward shift gives $\Delta_{\mathrm{ord}} < 0$. The order feature therefore targets location and stochastic order changes.

However, the order feature has no population signal for some common shifts. Its lower and upper rank scores can cancel when the change is symmetric. For example, under the Gaussian scale shift $P = \mathcal{N}(0, 1)$ and $Q = \mathcal{N}(0, \sigma^2)$, we have $\Delta_{\mathrm{ord}}(\sigma) = 0$ for every $\sigma > 0$. To detect such changes, we use a score related to the classical Ansari–Bradley dispersion score (Ansari & Bradley, 1960):

$$h_{\mathrm{disp}}(u) = |2u - 1| - \frac{1}{2}, \qquad u \in [0, 1].$$

We call h_{disp} the dispersion feature in our PRM construction. The term $|2u - 1|$ measures distance from the reference median on the rank scale. Subtracting $1/2$ gives zero mean.

For this feature, the dispersion contrast is $\Delta_{\mathrm{disp}}(P, Q) = \mathbb{E}_Q|2F_P(X) - 1| - 1/2$. This contrast compares mass in the reference tails with mass near the reference center. Moving probability mass toward the tails gives $\Delta_{\mathrm{disp}} > 0$, whereas concentrating mass near the reference median gives $\Delta_{\mathrm{disp}} < 0$. The two tails reinforce rather than cancel, so the feature targets scale and other center-versus-tail changes.

For the Gaussian scale shift above, $\Delta_{\mathrm{disp}}(\sigma) = 2\arctan(\sigma)/\pi - 1/2$. This contrast is nonzero for every $\sigma \neq 1$: it is positive for scale expansion and negative for scale contraction. Having defined the feature h, we next construct its betting payoff Z_t.

2.4 Predictively centered feature betting

The contrast $\Delta_h(P, Q)$ is defined in terms of $h\{F_P(X)\}$, but F_P is unknown. For each X_t, the fixed-reference rank gives the empirical placement $(R_t - 1)/n = \widehat{F}_n(X_t)$, so we use $h\{(R_t - 1)/n\}$ instead. Under the null, Equation (1) gives its conditional mean. Subtracting this mean defines the betting payoff:

$$Z_t = h\left(\frac{R_t - 1}{n}\right) - \sum_{j=1}^{n+1} \pi_{t,j} h\left(\frac{j - 1}{n}\right).$$

By (1), $\mathbb{E}_{H_0(P)}(Z_t \mid \mathcal{G}_{t-1}) = 0$. Replacing h by $h + c$ leaves Z_t, and hence the resulting PRM, unchanged because the constant cancels between the observed score and its conditional expectation.

Since the values of h lie in an interval of width at most one, $|Z_t| \le 1$. Under an alternative, the drift of Z_t is linked to $\Delta_h(P, Q)$. Specifically, when P is continuous and h is Lipschitz,

$$\mathbb{E}_{P^n \otimes Q^\infty} Z_t = \frac{n+1}{n+t}\{\Delta_h(P, Q) + o_n(1)\},$$

where $o_n(1) \to 0$ and does not depend on t. Thus, for any fixed t, the expectation approaches $\Delta_h(P, Q)$ as $n \to \infty$. Lemma 10 establishes this relation. Since Z_t is bounded and conditionally centered under the null, this yields the following feature-based specialization of Theorem 1.

Proposition 2. *For any $\mathcal{G}_{t-1}$-measurable $\lambda_t \in [-1, 1]$, define*

$$E_t = 1 + \lambda_t Z_t, \qquad M_t = \prod_{s=1}^{t} E_s, \qquad M_0 = 1.$$

Under $H_0(P)$, $(M_t)_{t\ge 0}$ is a nonnegative martingale and hence an e-process.

As a special case of Theorem 1, the stopping rule $\tau_h = \inf\{t \ge 1 : M_t \ge 1/\alpha\}$ satisfies $\sup_P \mathbb{P}_{H_0(P)}(\tau_h < \infty) \le \alpha$, so every pre-specified feature retains anytime type I error control.

The corresponding probability vector in Theorem 1 is

$$q_{t,j} = \pi_{t,j}\left[1 + \lambda_t\left\{h\left(\frac{j-1}{n}\right) - \sum_{\ell=1}^{n+1} \pi_{t,\ell} h\left(\frac{\ell-1}{n}\right)\right\}\right].$$

The range condition ensures $q_{t,j} \ge 0$, while predictive centering gives $\sum_{j=1}^{n+1} q_{t,j} = 1$. Under an alternative, Z_t estimates the feature contrast $\Delta_h(P, Q)$, while λ_t controls the direction and size of the bet. When λ_t and Z_t have the same sign, $E_t > 1$ and the wealth increases. We therefore adapt λ_t from past observations to favor positive values of $\lambda_t Z_t$ and thereby accumulate wealth under an alternative. The next subsection uses ONS to perform this adaptation.

2.5 Adaptive betting with ONS

The sign and size of the feature signal are unknown, so we choose λ_t from past data. Proposition 2 permits any $\mathcal{G}_{t-1}$-measurable $\lambda_t \in [-1, 1]$. We use ONS (Hazan et al., 2007) to make the cumulative log wealth $\log M_T = \sum_{t=1}^{T} \log(1 + \lambda_t Z_t)$ large.

For stable ONS updates, we restrict the betting coefficient to a predictable interval that keeps every betting factor $E_t = 1 + \lambda_t Z_t$ uniformly away from zero. Write $h_j = h\{(j-1)/n\}$ and define

$$B_t = \max_{1\le j\le n+1}\left|h_j - \sum_{\ell=1}^{n+1} \pi_{t,\ell} h_\ell\right|, \qquad b_t = \begin{cases} \min\{1, 3/(4B_t)\}, & B_t > 0, \\ 1, & B_t = 0, \end{cases}.$$

Here B_t is the largest possible value of $|Z_t|$. The feature range condition gives $B_t \le 1$, so $3/4 \le b_t \le 1$. Initialize $\lambda_1 = 0$. After observing Z_t, let $g_t = Z_t/(1 + \lambda_t Z_t)$, compute b_{t+1} from the updated counts, and update

$$\lambda_{t+1} = \Pi_{[-b_{t+1}, b_{t+1}]}\left(\lambda_t + \frac{(9/2)g_t}{1 + \sum_{s=1}^{t} g_s^2}\right). \tag{2}$$

Here Π denotes projection onto the subscripted interval. Since $|Z_t| \le B_t$, this choice guarantees $1 + \lambda_t Z_t \ge \frac{1}{4}$. Thus every betting factor is uniformly bounded away from zero. The quantity g_t is the derivative of $\log(1 + \lambda Z_t)$ at λ_t. Its sign determines whether the next coefficient λ_{t+1} increases or decreases, while the cumulative squared gradients scale the step size. The update uses only ranks through time t, so λ_t is $\mathcal{G}_{t-1}$-measurable. Moreover, $\lambda_t \in [-1, 1]$. The resulting process therefore remains valid, with $\mathbb{P}_{H_0(P)}(\sup_{t\ge 1} M_t \ge 1/\alpha) \le \alpha$. The next proposition quantifies the log wealth lost by adapting λ_t online relative to a fixed coefficient selected after observing the full path.

Proposition 3. *Let M_T be generated by* (2)*, and define $\underline{b}_T = \min_{1\le t\le T} b_t$. Then, for every path,*

$$\log M_T \ge \sup_{|u|\le \underline{b}_T} \sum_{t=1}^{T} \log(1 + uZ_t) - O(\log T).$$

Thus ONS incurs only logarithmic regret relative to the best fixed coefficient that remains feasible throughout the observed path.

Algorithm 1 summarizes the PRM with ONS betting.

Algorithm 1 PRM with ONS Betting

1: Initialize $M_0 = A_0 = 1$, $\lambda_1 = 0$, and $N_{0,j} = 0$ for $j = 1, \dots, n+1$.
2: **for** $t = 1, 2, \dots$ **do**
3: Compute $\pi_{t,j} = (1 + N_{t-1,j})/(n+t)$ and the projection radius b_t.
4: Rank X_t against D_0 to obtain R_t, and set $Z_t = h((R_t - 1)/n) - \sum_{j=1}^{n+1} \pi_{t,j} h((j-1)/n)$.
5: Update $M_t = M_{t-1}(1 + \lambda_t Z_t)$.
6: **If** $M_t \geq 1/\alpha$, set $\tau_h = t$, reject, and stop.
7: Set $g_t = Z_t/(1 + \lambda_t Z_t)$ and $A_t = A_{t-1} + g_t^2$.
8: Update $N_{t,j} = N_{t-1,j} + \mathbf{1}\{R_t = j\}$ for $j = 1, \dots, n+1$.
9: From N_t, compute π_{t+1} and b_{t+1}, then update λ_{t+1} by (2).
10: **end for**

2.6 PRM PORTFOLIOS

Write $M_{h,t}$ for the wealth process constructed using feature h. In particular, the order and dispersion features yield $M_{\mathrm{ord},t}$ and $M_{\mathrm{disp},t}$, which we call *Order PRM* and *Dispersion PRM*, respectively.

Accordingly, we may choose $h_1, \dots, h_m$, run their PRMs in parallel, and form the weighted average

$$M_t^{\mathrm{mix}} = \sum_{\ell=1}^{m} \omega_\ell M_{h_\ell,t}, \qquad \omega_\ell \geq 0, \quad \sum_{\ell=1}^{m} \omega_\ell = 1.$$

As a convex combination of nonnegative martingales, M_t^{mix} is also a nonnegative martingale and retains the same anytime type I error guarantee. It also satisfies $M_t^{\mathrm{mix}} \geq \omega_\ell M_{h_\ell,t}$ for every ℓ. If $\omega_\ell > 0$ and $\Delta_{h_\ell}(P, Q) \neq 0$ for at least one ℓ, the portfolio retains detection ability. Relative to component ℓ, the mixture loses at most $\log(1/\omega_\ell)$ in log wealth.

For the order and dispersion features, equal weights give the *PRM portfolio* $M_t^{\mathrm{OD}} = \frac{1}{2} M_{\mathrm{ord},t} + \frac{1}{2} M_{\mathrm{disp},t}$. It satisfies $\log M_t^{\mathrm{OD}} \geq \max_{k \in \{\mathrm{ord},\mathrm{disp}\}} \log M_{k,t} - \log 2$. Additional features can be included when other types of shift are plausible.

3 DETECTION GUARANTEES AND FIXED-CALIBRATION LIMITS

The false alarm guarantee in Theorem 1 holds for every calibration size and every monitoring horizon. We next study the detection ability of PRMs with ONS betting. We first establish a finite-window guarantee for any Lipschitz feature with nonzero induced contrast under the alternative. The result applies to both the order and dispersion features and also yields consistency as the initial calibration size increases. We close with limits of distribution-free procedures under marginal and calibration-conditional type I error control.

3.1 DETECTION GUARANTEES

Throughout this subsection, P is continuous, whereas Q may be arbitrary.

Theorem 4. *Let $h : [0,1] \to \mathbb{R}$ be a Lipschitz feature with Lipschitz constant L_h, and write $\Delta_h = \Delta_h(P, Q)$. Let τ_h be the stopping time produced by Algorithm 1 with feature h, and suppose that $\Delta_h \neq 0$. For $\beta \in (0, 1]$, define $r_{h,\beta} = |\Delta_h|^{-2} \log\{e/(\alpha\beta|\Delta_h|)\}$. There is a universal constant $C > 0$ such that, if $n \geq \lceil C \max\{1, L_h^2\} r_{h,\beta} \rceil$, then*

$$\mathbb{P}_{P^n \otimes Q^\infty}(\tau_h \leq \lceil C r_{h,\beta} \rceil) \geq 1 - \beta.$$

The theorem bounds the number of online observations needed for detection. Under its calibration-size condition, with probability at least $1 - \beta$,

$$\tau_h = O\left(|\Delta_h|^{-2} \log \frac{e}{\alpha\beta|\Delta_h|} \right).$$

For delayed changes, Appendix A.11 bounds the delay $\tau_h - \nu$ after ν null observations. Let $r_{\nu,h,\beta} = |\Delta_h|^{-2} \log\{e(\nu+1)/(\alpha\beta|\Delta_h|)\}$. Under the corresponding calibration-size condition, with probability at least $1 - \alpha - \beta$, the procedure does not stop before the change and

$$\tau_h - \nu = O\big(r_{\nu,h,\beta} + \sqrt{\nu r_{\nu,h,\beta}}\big)\,.$$

In particular, setting $\nu = 0$ recovers the immediate-change bound.

Moreover, Theorem 4 implies consistency as $n \to \infty$: the detection probability tends to one. For each n, the calibration sample remains fixed throughout monitoring.

Proposition 5. *Let h be a Lipschitz feature. For each n, let $M_{h,t}^{(n)}$ and $\tau_h^{(n)}$ denote the e-process and stopping time based on a calibration sample of size n. If $\Delta_h \neq 0$, then, for every deterministic sequence $T_n \to \infty$,*

$$\mathbb{P}_{P^n \otimes Q^\infty}(\tau_h^{(n)} \leq T_n) \longrightarrow 1.$$

In particular, $\mathbb{P}_{P^n \otimes Q^\infty}(\tau_h^{(n)} < \infty) \longrightarrow 1$.

Proposition 5 treats one feature. The next corollary extends the result to a finite portfolio when at least one positively weighted feature has a nonzero contrast.

Corollary 6. *Fix a finite collection of Lipschitz features $h_1, \ldots, h_m$ and deterministic weights $\omega_1, \ldots, \omega_m$ such that $\omega_\ell \geq 0$ and $\sum_{\ell=1}^m \omega_\ell = 1$. Define $M_{\mathrm{mix},t}^{(n)} = \sum_{\ell=1}^m \omega_\ell M_{h_\ell,t}^{(n)}$ and $\tau_{\mathrm{mix}}^{(n)} = \inf\{t \geq 1 : M_{\mathrm{mix},t}^{(n)} \geq 1/\alpha\}$. If, for some ℓ, $\omega_\ell > 0$ and $\Delta_{h_\ell}(P, Q) \neq 0$, then, for every deterministic sequence $T_n \to \infty$,*

$$\mathbb{P}_{P^n \otimes Q^\infty}(\tau_{\mathrm{mix}}^{(n)} \leq T_n) \longrightarrow 1.$$

The PRM portfolio is the special case with the order and dispersion features assigned equal weights. It has asymptotic power one whenever either contrast is nonzero. Appendix A.11 extends Proposition 5 and Corollary 6 to delayed changes.

3.2 Limits of fixed calibration

The positive results in Section 3.1 use marginal validity, whereas Shaer et al. (2026) give a PAC calibration-conditional guarantee. We first show that requiring calibration-conditional validity for every null distribution rules out nontrivial power. We then show that, even under the marginal validity requirement, a fixed calibration size limits attainable power.

Theorem 7. *Suppose a sequential procedure satisfies calibration-conditional validity at level α for every distribution P on a standard Borel space:*

$$\mathbb{P}_{H_0(P)}(\tau < \infty \mid D_0) \leq \alpha \qquad P^n\text{-almost surely}.$$

Then, for every fixed calibration sample D_0 and every stream distribution Q,

$$\mathbb{P}_{X_1, X_2, \ldots \sim Q^\infty}\big(\tau < \infty \mid D_0\big) \leq \alpha.$$

Thus calibration-conditional validity leaves no distribution-free power beyond level α. Our method requires only marginal validity and therefore avoids this impossibility. However, distribution-free marginal validity still entails an information limit.

Theorem 8. *Let a sequential test satisfy $\sup_R \mathbb{P}_{H_0(R)}(\tau < \infty) \leq \alpha$. Let TV denote total variation distance. Then, for any P, Q,*

$$\mathbb{P}_{P^n \otimes Q^\infty}(\tau < \infty) \leq \beta_\alpha(P^n, Q^n) \leq \alpha + \mathrm{TV}(P^n, Q^n),$$

where $\beta_\alpha(P^n, Q^n) := \sup_{0 \leq \varphi \leq 1 : \mathbb{E}_{Q^n}\varphi \leq \alpha} \mathbb{E}_{P^n}\varphi$.

To interpret the bound, compare the alternative $D_0 \sim P^n, X_{1:\infty} \sim Q^\infty$ with the null under which both the calibration sample and stream follow Q. The stream has the same law in both experiments; only the calibration sample differs. An infinite stream can reveal Q, but it supplies no additional observations from P. Power is therefore bounded by the best level-α test of Q^n against P^n based on the calibration sample alone.

Remark 9. The TV bound can be vacuous when $\alpha + \mathrm{TV}(P^n, Q^n) \geq 1$. Yet even for a basic Gaussian location shift, no procedure with distribution-free type I error control can attain power one. Specifically, for $P = \mathcal{N}(0,1)$ and $Q = \mathcal{N}(\mu, 1)$, every distribution-free procedure satisfies

$$\mathbb{P}_{P^n \otimes Q^\infty}(\tau < \infty) \leq \Phi\big(\sqrt{n}|\mu| - \Phi^{-1}(1-\alpha)\big) < 1.$$

At a fixed calibration size, unlimited monitoring does not in general yield power one.

4 Synthetic Experiments

We compare five procedures. Order PRM uses Algorithm 1 with $h(u) = u - 1/2$, and Dispersion PRM uses the same algorithm with $h(u) = |2u - 1| - 1/2$. The PRM portfolio is the equal-weight average of these two martingales. CCTM and Standard CTM are from Shaer et al. (2026). Appendix C.1 gives implementation details for these two methods.

We conduct two synthetic studies. The first reruns the nine settings considered by Shaer et al. (2026), covering immediate, delayed, and gradual location shifts. The second considers symmetric shifts with zero order contrast, as discussed in Section 2.3. All experiments use $\alpha = 0.05$; Figure 4 in Appendix C reports the type I error results. Code to reproduce all experiments is included in the supplementary material.

4.1 Gaussian location shifts

We rerun the nine Gaussian settings considered by Shaer et al. (2026). In every setting, the calibration set consists of $n = 1000$ i.i.d. observations from $\mathcal{N}(0,1)$. The immediate settings use $X_t \sim \mathcal{N}(d,1)$ from $t = 1$, with $d \in \{1, 1.5, 2\}$. The delayed settings use $X_t \sim \mathcal{N}(0,1)$ for $t < t_0$ and $X_t \sim \mathcal{N}(2,1)$ for $t \geq t_0$, where $t_0 \in \{200, 600, 4000\}$. The gradual settings use $X_t \sim \mathcal{N}(\lambda t, 1)$, $t = 1, \ldots, 100$, with $\lambda \in \{0.015, 0.03, 0.05\}$. We compare Order PRM, the PRM portfolio, CCTM, and Standard CTM over 1000 repetitions. Figure 1 shows one setting from each regime; Appendix C.3 reports all nine.

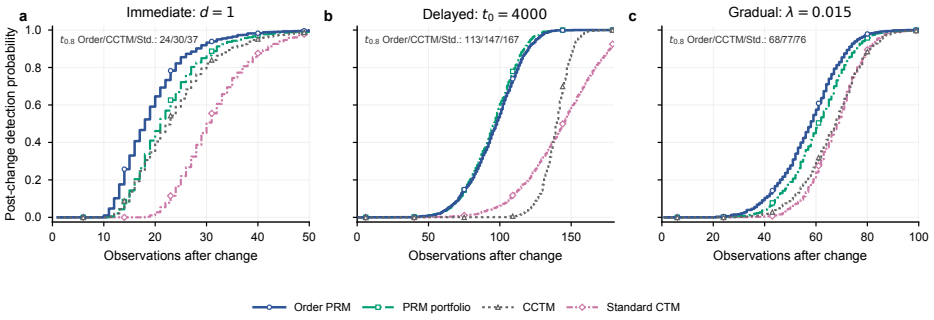


Figure 1: Detection under immediate, delayed, and gradual location shifts from Shaer et al. (2026).

Across all nine settings, Order PRM reaches 80% detection probability earlier than both CCTM and Standard CTM. Relative to CCTM, it requires 10.2%–23.1% fewer observations; relative to Standard CTM, it requires 10.5%–42.1% fewer. The PRM portfolio remains close to Order PRM despite splitting its initial wealth.

4.2 Symmetric shifts with zero order contrast

We next consider three alternatives with zero order contrast. In each repetition, the calibration set contains $n = 1000$ i.i.d. observations from $\mathcal{N}(0,1)$, and the online observations are i.i.d. from the alternative starting at $t = 1$. The alternatives are $\mathcal{N}(0, 1.5^2)$, $\mathsf{Laplace}(0, 1/\sqrt{2})$, and $t_3/\sqrt{3}$.

At $T = 1000$, Order PRM, CCTM, and Standard CTM have near-zero detection probability across all alternatives, whereas Dispersion PRM and PRM portfolio achieve near-one detection probability. Thus a feature with nonzero contrast recovers detection ability when the order contrast is zero.

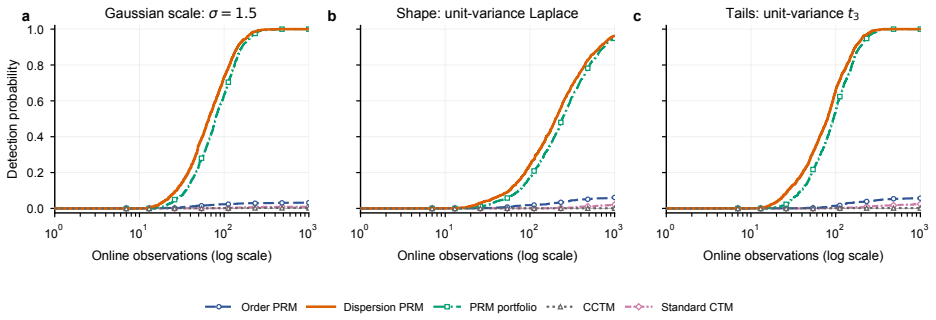


Figure 2: Detection under three symmetric alternatives with zero order contrast.

5 Real Data: CIFAR-10-C

Finally, we evaluate Order PRM, the PRM portfolio, CCTM, and Standard CTM using CIFAR-10 and CIFAR-10-C (Krizhevsky, 2009; Hendrycks & Dietterich, 2019). CIFAR-10-C is constructed by applying 15 corruption types, each at five severity levels, to the CIFAR-10 images. For each image, we use the Shannon entropy of the softmax probabilities produced by a publicly available ResNet-20 pretrained on CIFAR-10 (He et al., 2016) as the scalar monitoring score. For each $n \in \{20, 30, 50\}$, we use n clean images for calibration and a disjoint $5{,}000$-image corrupted stream. We consider all 15 severity-5 corruptions and 10 random index splits, yielding 150 trials per calibration size. Results for severity levels 1–4 are reported in Appendix C.5.

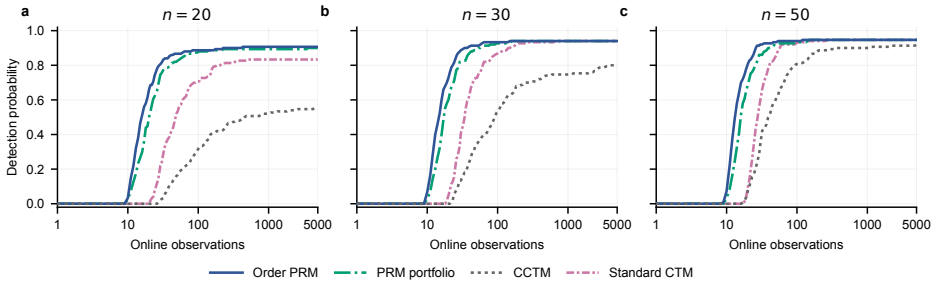


Figure 3: Detection of CIFAR-10-C corruptions at severity 5 for $n = 20, 30, 50$.

Order PRM reaches 80% detection after 28, 23, and 20 observations for $n = 20, 30, 50$, respectively. The PRM portfolio requires 40, 28, and 26 observations, and Standard CTM requires 176, 61, and 40. CCTM does not reach 80% detection within $T = 5{,}000$ for $n = 20$, and requires $3{,}757$ and 94 observations for $n = 30$ and 50. Order PRM and the portfolio reach 80% detection earlier than both baselines for all three calibration sizes.

6 Conclusion

We derive the exact predictive law of fixed-reference ranks and use it to construct PRMs with anytime type I error control. Pre-specified features target different shifts, while ONS adapts the bet. For Lipschitz features with nonzero contrast, we prove finite-window detection guarantees and consistency as the initial calibration size increases. Empirically, Order PRM detects Gaussian shifts and CIFAR-10-C earlier than CCTM, while Dispersion PRM detects symmetric shifts on which Order PRM and CCTM have near-zero power. These gains coexist with fundamental limits: distribution-free calibration-conditional validity permits only trivial power, and a fixed calibration size limits marginally valid procedures.

AI USE STATEMENT

In this work, generative AI tools assisted with drafting, revising, and checking mathematical proofs; writing and modifying experimental code; improving the language and readability of the manuscript; and conducting literature searches. The authors independently verified each theorem and proof step by step. The experimental code was tested, and all numerical results were independently rerun by the authors. The authors reviewed all AI-assisted content and take responsibility for the final content of this work.

REFERENCES

Abdur Rahman Ansari and Ralph A Bradley. Rank-sum tests for dispersions. *The annals of mathematical statistics*, pp. 1174–1189, 1960.

Stephen Bates, Emmanuel Candès, Lihua Lei, Yaniv Romano, and Matteo Sesia. Testing for outliers with conformal p-values. *The Annals of Statistics*, 51(1):149 – 178, 2023. doi: 10.1214/22-AOS2244. URL https://doi.org/10.1214/22-AOS2244.

Yoav Benjamini and Yosef Hochberg. Controlling the false discovery rate: a practical and powerful approach to multiple testing. *Journal of the Royal statistical society: series B (Methodological)*, 57(1):289–300, 1995.

Yoav Benjamini and Daniel Yekutieli. The control of the false discovery rate in multiple testing under dependency. *Annals of statistics*, pp. 1165–1188, 2001.

David Blackwell and James B MacQueen. Ferguson distributions via pólya urn schemes. *The annals of statistics*, 1(2):353–355, 1973.

Valentina Fedorova, Alex Gammerman, Ilia Nouretdinov, and Vladimir Vovk. Plug-in martingales for testing exchangeability on-line. *arXiv preprint arXiv:1204.3251*, 2012.

Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2):169–192, 2007.

Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 770–778, 2016.

Dan Hendrycks and Thomas Dietterich. Benchmarking neural network robustness to common corruptions and perturbations. *arXiv preprint arXiv:1903.12261*, 2019.

Alex Krizhevsky. Learning multiple layers of features from tiny images. 2009. URL https://api.semanticscholar.org/CorpusID:18268744.

John Orban and Douglas A Wolfe. Distribution-free partially sequential piacment procedures. *Communications in Statistics-Theory and Methods*, 9(9):883–904, 1980.

John Orban and Douglas A Wolfe. A class of distribution-free two-sample tests based on placements. *Journal of the American Statistical Association*, 77(379):666–672, 1982.

Aleksandr Podkopaev and Aaditya Ramdas. Sequential predictive two-sample and independence testing. *Advances in neural information processing systems*, 36:53275–53307, 2023.

Aaditya Ramdas, Johannes Ruf, Martin Larsson, and Wouter M Koolen. Testing exchangeability: Fork-convexity, supermartingales and e-processes. *International Journal of Approximate Reasoning*, 141:83–109, 2022.

Aytijhya Saha and Aaditya Ramdas. Testing exchangeability by pairwise betting. In *International Conference on Artificial Intelligence and Statistics*, pp. 4915–4923. PMLR, 2024.

Shalev Shaer, Yarin Bar, Drew Prinster, and Yaniv Romano. Testing for distribution shifts with conditional conformal test martingales. In *Forty-third International Conference on Machine Learning*, 2026. URL https://openreview.net/forum?id=B8sTBdGvqG.

Glenn Shafer. Testing by betting: A strategy for statistical and scientific communication. *Journal of the Royal Statistical Society Series A: Statistics in Society*, 184(2):407–431, 04 2021. ISSN 0964-1998. doi: 10.1111/rssa.12647. URL https://doi.org/10.1111/rssa.12647.

Jean Ville. *Etude critique de la notion de collectif*, volume 3. Gauthier-Villars Paris, 1939.

Denis Volkhonskiy, Evgeny Burnaev, Ilia Nouretdinov, Alexander Gammerman, and Vladimir Vovk. Inductive conformal martingales for change-point detection. In *Conformal and Probabilistic Prediction and Applications*, pp. 132–153. PMLR, 2017.

Vladimir Vovk, Ilia Nouretdinov, and Alexander Gammerman. Testing exchangeability on-line. In *Proceedings of the 20th international conference on machine learning (ICML-03)*, pp. 768–775, 2003.

Vladimir Vovk, Alexander Gammerman, and Glenn Shafer. *Algorithmic learning in a random world*. Springer, 2005.

Vladimir Vovk, Ivan Petej, Ilia Nouretdinov, Ernst Ahlberg, Lars Carlsson, and Alex Gammerman. Retrain or not retrain: Conformal test martingales for change-point detection. In *Conformal and Probabilistic Prediction and Applications*, pp. 191–210. PMLR, 2021.

Vladimir Vovk, Ilia Nouretdinov, and Alex Gammerman. Conformal e-testing. *Pattern Recognition*, 168:111841, 2025.

Douglas A Wolfe. On a class of partially sequential two-sample test procedures. *Journal of the American Statistical Association*, 72(357):202–205, 1977.

The appendix is organized as follows. Appendix A proves the theorems, propositions, and corollaries stated in the main text and extends the detection guarantees to delayed changes. Appendix B examines the CCTM type I error guarantee and gives continuous and discrete counterexamples. Appendix C provides the baseline implementations, simulation settings, and CIFAR-10-C protocol.

A Proofs of Main-Text Results

A.1 Proof of Theorem 1

Proof. The substantive part of the theorem is the predictive rank law (1). Once this law is established, the martingale property and the anytime bound follow by standard arguments.

Suppose first that P is continuous, and define $U_i = F_P(Y_i)$ and $V_t = F_P(X_t)$. Under $H_0(P)$,

$$U_1, \dots, U_n, V_1, V_2, \dots \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(0,1).$$

Since a monotone transformation preserves ranks, the rank of X_t among $Y_1, \dots, Y_n$ equals the rank of V_t among $U_1, \dots, U_n$. Let $U_{(1)} < \cdots < U_{(n)}$ be the order statistics of the calibration uniforms, and set $U_{(0)} = 0$ and $U_{(n+1)} = 1$. These points partition $[0,1]$ into $n+1$ intervals with lengths $W_j = U_{(j)} - U_{(j-1)}$, $j = 1, \dots, n+1$. Up to events of probability zero,

$$R_t = j \quad \Longleftrightarrow \quad V_t \in \big(U_{(j-1)}, U_{(j)}\big).$$

Thus, conditional on $W = (W_1, \dots, W_{n+1})$, we have $\mathbb{P}(R_t = j \mid W) = W_j$. The ranks are conditionally independent because $V_1, V_2, \dots$ are independent, and hence $R_t \mid W \overset{\text{i.i.d.}}{\sim} \mathrm{Categorical}(W_1, \dots, W_{n+1})$. Let $N_{t-1,j} = \sum_{s=1}^{t-1} \mathbf{1}\{R_s = j\}$. The conditional likelihood of the observed rank history is

$$\mathbb{P}(R_1, \dots, R_{t-1} \mid W) = \prod_{j=1}^{n+1} W_j^{N_{t-1,j}}.$$

The spacings of uniform order statistics satisfy $(W_1, \dots, W_{n+1}) \sim \mathrm{Dirichlet}(1, \dots, 1)$. Dirichlet conjugacy therefore gives

$$W \mid \mathcal{G}_{t-1} \sim \mathrm{Dirichlet}\left(1 + N_{t-1,1}, \dots, 1 + N_{t-1,n+1}\right),$$

and hence $\mathbb{E}(W_j \mid \mathcal{G}_{t-1}) = (1 + N_{t-1,j}) / \sum_{\ell=1}^{n+1} (1 + N_{t-1,\ell}) = (1 + N_{t-1,j})/(n+t)$. Finally, the tower property yields

$$\begin{aligned} \mathbb{P}(R_t = j \mid \mathcal{G}_{t-1}) &= \mathbb{E}[\mathbb{P}(R_t = j \mid W, \mathcal{G}_{t-1}) \mid \mathcal{G}_{t-1}] \\ &= \mathbb{E}(W_j \mid \mathcal{G}_{t-1}) \\ &= \frac{1 + N_{t-1,j}}{n+t}, \end{aligned}$$

which proves (1). For atomic P, the same argument applies to the lexicographically ordered pairs used for tie breaking.

Since q_t is $\mathcal{G}_{t-1}$-measurable,

$$\mathbb{E}[E_t \mid \mathcal{G}_{t-1}] = \sum_{j=1}^{n+1} \frac{q_{t,j}}{\pi_{t,j}} \pi_{t,j} = 1.$$

Thus $M_t = \prod_{s=1}^{t} E_s$ is a nonnegative martingale with $M_0 = 1$. Ville's inequality gives

$$\mathbb{P}\left(\sup_{t \geq 0} M_t \geq 1/\alpha \right) \leq \alpha.$$

□

A.2 Proof of Proposition 2

Proof. The range condition gives $|Z_t| \le 1$, so $E_t = 1 + \lambda_t Z_t \ge 0$. Predictive centering gives

$$\mathbb{E}(E_t \mid \mathcal{G}_{t-1}) = 1 + \lambda_t \mathbb{E}(Z_t \mid \mathcal{G}_{t-1}) = 1.$$

Thus $(M_t)_{t\ge 0}$ is a nonnegative martingale and hence an e-process. □

A.3 Proof of Proposition 3

Proof. Write $g_t = Z_t/(1+\lambda_t Z_t)$ and $A_t = 1 + \sum_{s=1}^{t} g_s^2$. The predictable domain gives $|\lambda_t Z_t| \le 3/4$. Hence $1 + \lambda_t Z_t \ge 1/4$,

$$|g_t| \le 4, \qquad A_t \le 1 + 16t.$$

Fix u such that $|u| \le b_t$. Since $|Z_t| \le B_t$, we also have $|uZ_t| \le 3/4$. Therefore

$$\frac{1+uZ_t}{1+\lambda_t Z_t} = 1 + x_t, \qquad x_t = g_t(u - \lambda_t).$$

The preceding bounds imply $x_t \in [-6/7, 6]$. On this interval, $\log(1+x) \le x - x^2/9$. Consequently,

$$\log(1+uZ_t) - \log(1+\lambda_t Z_t) \le g_t(u-\lambda_t) - \frac{1}{9} g_t^2 (u-\lambda_t)^2. \tag{3}$$

If $|u| \le b_{t+1}$ as well, projection in (2) does not increase the distance to u, and

$$\begin{aligned}(\lambda_{t+1} - u)^2 &\le \left(\lambda_t + \frac{9g_t}{2A_t} - u\right)^2 \\ &= (\lambda_t - u)^2 + \frac{9g_t}{A_t}(\lambda_t - u) + \frac{81 g_t^2}{4A_t^2}.\end{aligned}$$

Rearranging this inequality gives

$$g_t(u-\lambda_t) \le \frac{1}{9}\left\{A_t(\lambda_t - u)^2 - A_t(\lambda_{t+1} - u)^2\right\} + \frac{9g_t^2}{4A_t}.$$

Substituting this bound into (3) and using $A_t = A_{t-1} + g_t^2$ gives

$$\begin{aligned}&\log(1+uZ_t) - \log(1+\lambda_t Z_t) \\ &\le \frac{1}{9}\left\{(A_t - g_t^2)(\lambda_t - u)^2 - A_t(\lambda_{t+1} - u)^2\right\} + \frac{9g_t^2}{4A_t} \\ &= \frac{1}{9}\left\{A_{t-1}(\lambda_t - u)^2 - A_t(\lambda_{t+1} - u)^2\right\} + \frac{9g_t^2}{4A_t}.\end{aligned}$$

Fix $T \ge 1$, let $\underline{b}_T = \min_{1\le t\le T} b_t$, and take $|u| \le \underline{b}_T$. For $t = 1, \dots, T-1$, we have $|u| \le b_t$ and $|u| \le b_{t+1}$, so the preceding inequality telescopes to

$$\begin{aligned}&\sum_{t=1}^{T-1}\{\log(1+uZ_t) - \log(1+\lambda_t Z_t)\} \\ &\le \frac{1}{9}\left\{A_0(\lambda_1 - u)^2 - A_{T-1}(\lambda_T - u)^2\right\} + \frac{9}{4}\sum_{t=1}^{T-1}\frac{g_t^2}{A_t} \\ &\le \frac{1}{9} + \frac{9}{4}\log(1+16T).\end{aligned}$$

Indeed, $A_0 = 1$, $\lambda_1 = 0$, and $|u| \le 1$, so the initial quadratic term is at most $1/9$, while

$$\sum_{t=1}^{T-1}\frac{g_t^2}{A_t} \le \log A_{T-1} \le \log(1+16T).$$

For $T = 1$, both sums above are empty and the same bound holds.

It remains to control the final factor. Both u and λ_T belong to $[-b_T, b_T]$, so $|uZ_T| \le 3/4$ and $|\lambda_T Z_T| \le 3/4$. Consequently,

$$\log(1 + uZ_T) - \log(1 + \lambda_T Z_T) \le \log 7.$$

Combining these bounds and maximizing over $|u| \le \underline{b}_T$ gives

$$\sup_{|u| \le \underline{b}_T} \sum_{t=1}^{T} \{\log(1 + uZ_t) - \log(1 + \lambda_t Z_t)\} \le \log 7 + \frac{1}{9} + \frac{9}{4} \log(1 + 16T).$$

This proves Proposition 3.

For the deterministic consequence used below, fix $u \in [-3/4, 3/4]$. Since $b_t \ge 3/4$ for every t, this comparator belongs to both $[-b_t, b_t]$ and $[-b_{t+1}, b_{t+1}]$ at every step. The preceding one-step inequality therefore telescopes through $t = T$. Using $u^2 \le 9/16$ and

$$\sum_{t=1}^{T} \frac{g_t^2}{A_t} = \sum_{t=1}^{T} \frac{A_t - A_{t-1}}{A_t} \le \sum_{t=1}^{T} \log \frac{A_t}{A_{t-1}} = \log A_T$$

gives

$$\begin{aligned} \sup_{u \in [-3/4, 3/4]} \sum_{t=1}^{T} \{\log(1 + uZ_t) - \log(1 + \lambda_t Z_t)\} &\le \frac{1}{16} + \frac{9}{4} \log A_T \\ &\le \frac{1}{16} + \frac{9}{4} \log(1 + 16T). \end{aligned} \tag{4}$$

Since $\sum_{t=1}^{T} \log(1 + \lambda_t Z_t) = \log M_T$, this is the deterministic explicit bound used in the power proofs below. □

A.4 Proof of Theorem 4

We first establish two auxiliary lemmas. The first relates the realized feature contrast to $\Delta_h(P, Q)$, and the second gives the finite-window bound used to prove the theorem.

Write $Z_{h,t}$ for the payoff Z_t constructed from feature h. For finite n, define the realized feature contrast

$$d_{h,n}(D_0) := \mathbb{E}_Q \left[h\left(\frac{R_t - 1}{n} \right) \middle| D_0 \right] - \frac{1}{n+1} \sum_{j=1}^{n+1} h\left(\frac{j-1}{n} \right).$$

Under the i.i.d. alternative, this contrast does not depend on t.

Lemma 10. *Let P be continuous, let Q be arbitrary, let $D_0 \sim P^n$, and let h be L-Lipschitz with zero uniform mean. Then*

$$d_{h,n}(D_0) \xrightarrow{P^n} \Delta_h(P, Q),$$

and, for every $x > 0$,

$$\mathbb{P}_{P^n} \left(|d_{h,n}(D_0) - \Delta_h(P, Q)| > Lx + \frac{L}{n+1} \right) \le 2e^{-2nx^2}. \tag{5}$$

Moreover, for every $t \ge 1$,

$$\mathbb{E}_Q(Z_{h,t} \mid D_0) = \frac{n+1}{n+t} d_{h,n}(D_0).$$

Proof. Abbreviate $g_n(j) = h\{(j-1)/n\}$, and put

$$\bar{h}_n = \frac{1}{n+1} \sum_{j=1}^{n+1} g_n(j), \qquad m_{h,n}(D_0) = \mathbb{E}_Q\{g_n(R_t) \mid D_0\}, \qquad d_{h,n}(D_0) = m_{h,n}(D_0) - \bar{h}_n.$$

The π_t-weighted feature mean can be written as

$$\sum_{j=1}^{n+1} \pi_{t,j} g_n(j) = \frac{(n+1)\bar{h}_n + \sum_{s<t} g_n(R_s)}{n+t}.$$

Conditional on D_0, the ranks are i.i.d. under Q. Taking expectation in the preceding display and subtracting from $m_{h,n}(D_0)$ gives

$$\mathbb{E}_Q(Z_{h,t} \mid D_0) = \frac{n+1}{n+t} d_{h,n}(D_0).$$

Since P is continuous, $(R_t - 1)/n = \widehat{F}_n(X_t)$ almost surely under $P^n \otimes Q$. Hence

$$|m_{h,n}(D_0) - \Delta_h(P, Q)| \le L \|\widehat{F}_n - F_P\|_\infty.$$

To control the grid mean, let $U \sim \mathrm{Unif}(0, 1)$ and set $U_n = k/n$ when $U \in [k/(n+1), (k+1)/(n+1))$. Then U_n is uniform on $\{0, 1/n, \dots, 1\}$ and $|U_n - U| \le 1/(n+1)$. Since $\mathbb{E}h(U) = 0$,

$$|\bar{h}_n| = |\mathbb{E}h(U_n) - \mathbb{E}h(U)| \le \frac{L}{n+1}.$$

The DKW inequality proves (5), which also gives $d_{h,n}(D_0) \xrightarrow{P^n} \Delta_h(P, Q)$.

Finally, boundedness of $d_{h,n}(D_0)$ upgrades convergence in probability to convergence of expectations. Averaging the conditional identity over D_0 gives the relation stated in the main text:

$$\mathbb{E}_{P^n \otimes Q^\infty} Z_{h,t} = \frac{n+1}{n+t} \mathbb{E}_{D_0} d_{h,n}(D_0) = \frac{n+1}{n+t} \{\Delta_h(P, Q) + o_n(1)\}.$$

The remainder does not depend on t because $d_{h,n}(D_0)$ does not. □

Lemma 11. *Let P be continuous, let Q be arbitrary, and let h be an L-Lipschitz feature. Write $\Delta = \Delta_h(P, Q)$, $\delta = |\Delta|$, and*

$$r_T = \frac{1}{16} + \frac{9}{4} \log(1 + 16T).$$

Suppose that $\delta > 0$. If

$$n + 1 \ge \frac{8L}{\delta}, \qquad 1 \le T \le n, \qquad T\delta^2 \ge 128 \left\{ \log \frac{1}{\alpha} + r_T \right\},$$

then

$$\mathbb{P}_{P^n \otimes Q^\infty}(\tau_h > T) \le 2 \exp\left(-\frac{n\delta^2}{32L^2}\right) + \exp\left(-\frac{T\delta^2}{2048}\right). \tag{6}$$

Proof. The concentration bound in Lemma 10, with $x = \delta/(8L)$, gives

$$\mathbb{P}_{P^n}\left\{ |d_{h,n}(D_0) - \Delta| > \frac{\delta}{4} \right\} \le 2 \exp\left(-\frac{n\delta^2}{32L^2}\right),$$

where $n + 1 \ge 8L/\delta$ controls the deterministic grid error. Work on the complementary event, and write $s = \mathrm{sign}(\Delta)$. Then $s\, d_{h,n}(D_0) \ge 3\delta/4$.

Take the fixed comparator $u = s\theta$, where $\theta = \delta/8$. Since h has zero uniform mean and range width at most one, $\delta \le 1$. Hence $|\theta Z_{h,t}| \le 1/8$, and $\log(1 + x) \ge x - x^2$ applies. Conditional on D_0, Lemma 10 gives

$$\mathbb{E}_Q(Z_{h,t} \mid D_0) = \frac{n+1}{n+t} d_{h,n}(D_0).$$

Since $T \le n$, we have $\sum_{t=1}^{T} (n+1)/(n+t) \ge T/2$. The range condition therefore yields

$$\begin{aligned} \mathbb{E}_Q\left[\sum_{t=1}^{T} \log(1 + s\theta Z_{h,t}) \,\middle|\, D_0 \right] &\ge \frac{3\theta\delta T}{8} - \theta^2 T \\ &= \frac{T\delta^2}{32}. \end{aligned} \tag{7}$$

The sign s has already been combined with the drift. Thus the linear term is $3\theta\delta T/8$, including when $\Delta < 0$.

Conditional on D_0, the ranks are independent. Replacing R_j while holding the other ranks fixed changes $Z_{h,j}$ by at most one and each $Z_{h,t}$, $t > j$, by at most $1/(n+t)$. Each comparator log factor is 2θ-Lipschitz as a function of $Z_{h,t}$, so the total change in the comparator log wealth is at most

$$2\theta \left\{ 1 + \sum_{t=j+1}^{T} \frac{1}{n+t} \right\} \le 4\theta.$$

McDiarmid's inequality and (7) imply

$$\mathbb{P}_Q \left(\sum_{t=1}^{T} \log(1 + s\theta Z_{h,t}) < \frac{3T\delta^2}{128} \,\middle|\, D_0 \right) \le \exp\left(-\frac{T\delta^2}{2048} \right).$$

The explicit regret bound in (4) gives, outside this lower-tail event,

$$\log M_{h,T} \ge \frac{3T\delta^2}{128} - r_T \ge \log \frac{1}{\alpha}.$$

Hence $\tau_h \le T$, and averaging over D_0 proves (6). □

We now prove Theorem 4.

Proof. Write

$$\delta = |\Delta_h|, \qquad L = L_h, \qquad K = \max\{1, L^2\}, \qquad \ell_\beta = \log \frac{e}{\alpha\beta\delta}.$$

The feature normalization gives $\delta \le 1$. Hence $\ell_\beta \ge 1 + \log(1/\beta)$, and $r_{h,\beta} = \delta^{-2}\ell_\beta$.

Let

$$T = \left\lceil C\delta^{-2}\ell_\beta \right\rceil, \qquad r_T = \frac{1}{16} + \frac{9}{4}\log(1 + 16T).$$

For $C \ge 1$, we have $T \le 2C\delta^{-2}\ell_\beta$. Since $\log(1/\delta) \le \ell_\beta$ and $\log \ell_\beta \le \ell_\beta$, there is a universal constant c_0 such that

$$r_T \le c_0(1 + \log C)\ell_\beta.$$

Choose a universal C large enough that

$$C \ge 128\{1 + c_0(1 + \log C)\} \qquad \text{and} \qquad C \ge 2048.$$

Then

$$T\delta^2 \ge C\ell_\beta \ge 128 \left\{ \log \frac{1}{\alpha} + r_T \right\}.$$

Now suppose that $n \ge \lceil CK\delta^{-2}\ell_\beta \rceil$. Since $K \ge 1$, we have $T \le n$. Moreover, $K \ge L$, while $\delta \le 1$ and $\ell_\beta \ge 1$, so

$$n + 1 \ge CK\delta^{-2}\ell_\beta \ge \frac{8L}{\delta}.$$

Lemma 11 therefore gives

$$\begin{aligned} \mathbb{P}_{P^n \otimes Q^\infty}(\tau_h > T) &\le 2\exp\left(-\frac{n\delta^2}{32L^2} \right) + \exp\left(-\frac{T\delta^2}{2048} \right) \\ &\le 2\exp\left(-\frac{C\ell_\beta}{32} \right) + \exp\left(-\frac{C\ell_\beta}{2048} \right) \le \beta. \end{aligned}$$

Thus $\mathbb{P}_{P^n \otimes Q^\infty}(\tau_h \le T) \ge 1 - \beta$. □

A.5 Proof of Proposition 5

Proof. Fix any $\eta \in (0,1)$. Applying Theorem 4 with $\beta = \eta$, define

$$T_\eta = \left\lceil C|\Delta_h|^{-2} \log\left\{ \frac{e}{\alpha\eta|\Delta_h|} \right\} \right\rceil.$$

For fixed η, this horizon does not depend on n. For all sufficiently large n, the theorem's calibration-size condition holds, and therefore

$$\mathbb{P}_{P^n \otimes Q^\infty}(\tau_h^{(n)} \le T_\eta) \ge 1 - \eta.$$

Moreover, since $T_n \to \infty$, we have $T_n \ge T_\eta$ for all sufficiently large n. Consequently,

$$\mathbb{P}_{P^n \otimes Q^\infty}(\tau_h^{(n)} \le T_n) \ge \mathbb{P}_{P^n \otimes Q^\infty}(\tau_h^{(n)} \le T_\eta) \ge 1 - \eta.$$

Thus,

$$\liminf_{n\to\infty} \mathbb{P}_{P^n \otimes Q^\infty}(\tau_h^{(n)} \le T_n) \ge 1 - \eta.$$

Letting $\eta \downarrow 0$ shows that the limit inferior is at least one. Since the probability is at most one, it converges to one. □

A.6 Proof of Corollary 6

Proof. Choose ℓ as in the corollary. Since $M_{\mathrm{mix},t}^{(n)} \ge \omega_\ell M_{h_\ell,t}^{(n)}$, a crossing by component ℓ at $1/(\alpha\omega_\ell)$ forces the mixture to cross $1/\alpha$. Proposition 5, applied to h_ℓ at level $\alpha\omega_\ell$, therefore gives

$$\mathbb{P}_{P^n \otimes Q^\infty}\left(\tau_{\mathrm{mix}}^{(n)} \le T_n\right) \longrightarrow 1.$$

□

A.7 Proof of Theorem 7

Proof. Fix $D_0 = d = (d_1, \ldots, d_n)$ and a stream distribution Q. Let $U \sim \nu$ collect the auxiliary randomization independently of the data, and let G_d be the empirical distribution of $d_1, \ldots, d_n$. For $\varepsilon \in (0,1)$, define

$$P_\varepsilon = (1-\varepsilon)Q + \varepsilon G_d.$$

Each coordinate of d is an atom of G_d, so $D_0 = d$ has positive probability under P_ε^n. Calibration-conditional validity for P_ε therefore gives

$$\mathbb{P}_{X_{1:\infty} \sim P_\varepsilon^\infty,\, U \sim \nu}\big(\tau(d, X_{1:\infty}, U) < \infty\big) \le \alpha.$$

For $T \ge 1$, let

$$A_T(d) = \{(x_{1:T}, u) : \text{the procedure stops by time } T \text{ after history } (d, x_{1:T})\}.$$

The preceding bound implies

$$(P_\varepsilon^T \otimes \nu)(A_T(d)) \le \alpha.$$

As $\varepsilon \downarrow 0$, $P_\varepsilon^T \otimes \nu \to Q^T \otimes \nu$ in total variation, so $(Q^T \otimes \nu)(A_T(d)) \le \alpha$. Since $A_T(d)$ increases to the event of eventual rejection, continuity from below gives

$$\mathbb{P}_{Q^\infty,\, U \sim \nu}\big(\tau(d, X_{1:\infty}, U) < \infty\big) = \lim_{T\to\infty} (Q^T \otimes \nu)(A_T(d)) \le \alpha.$$

Because d and Q were arbitrary, this is the claimed bound for every fixed calibration sample. □

A.8 Proof of Theorem 8

Proof. Let U contain any auxiliary randomness and define the test function

$$\varphi(d) = \mathbb{P}_{X_{1:\infty} \sim Q^\infty, U}\{\tau(d, X_{1:\infty}, U) < \infty\}.$$

When $D_0 \sim Q^n$, iterated expectation and validity at the null distribution Q give

$$\begin{aligned} \mathbb{E}_{Q^n}\varphi &= \mathbb{E}_{D_0 \sim Q^n}\left[\mathbb{P}_{X_{1:\infty} \sim Q^\infty, U}\{\tau(D_0, X_{1:\infty}, U) < \infty\}\right] \\ &= \mathbb{P}_{D_0 \sim Q^n,\, X_{1:\infty} \sim Q^\infty, U}(\tau < \infty) \le \alpha. \end{aligned}$$

Thus φ is feasible in the definition of $\beta_\alpha(P^n, Q^n)$. For the target experiment $D_0 \sim P^n$ and $X_{1:\infty} \sim Q^\infty$,

$$\begin{aligned}\mathbb{E}_{P^n}\varphi &= \mathbb{E}_{D_0\sim P^n}\left[\mathbb{P}_{X_{1:\infty}\sim Q^\infty, U}\{\tau(D_0, X_{1:\infty}, U) < \infty\}\right] \\ &= \mathbb{P}_{D_0\sim P^n,\, X_{1:\infty}\sim Q^\infty, U}(\tau < \infty).\end{aligned}$$

The definition of $\beta_\alpha(P^n, Q^n)$ therefore implies

$$\mathbb{P}_{P^n\otimes Q^\infty}(\tau < \infty) \le \beta_\alpha(P^n, Q^n).$$

For every feasible test ψ, the definition of total variation gives

$$\mathbb{E}_{P^n}\psi - \mathbb{E}_{Q^n}\psi \le \mathrm{TV}(P^n, Q^n).$$

Since $\mathbb{E}_{Q^n}\psi \le \alpha$, it follows that

$$\mathbb{E}_{P^n}\psi \le \alpha + \mathrm{TV}(P^n, Q^n).$$

Taking the supremum over all feasible ψ yields

$$\beta_\alpha(P^n, Q^n) \le \alpha + \mathrm{TV}(P^n, Q^n).$$

Combining the two inequalities gives

$$\mathbb{P}_{D_0\sim P^n,\, X_{1:\infty}\sim Q^\infty}(\tau < \infty) \le \beta_\alpha(P^n, Q^n) \le \alpha + \mathrm{TV}(P^n, Q^n).$$

□

A.9 Proof of Remark 9

Proof. By Theorem 8, it remains to evaluate $\beta_\alpha(P^n, Q^n)$. For $P = \mathcal{N}(0, 1)$, $Q = \mathcal{N}(\mu, 1)$, and $\mu > 0$,

$$\log\frac{dP^n}{dQ^n}(y_{1:n}) = -\mu\sum_{i=1}^n y_i + \frac{n\mu^2}{2}.$$

The level-α Neyman–Pearson test therefore rejects when

$$\overline{Y} \le \mu - \frac{\Phi^{-1}(1-\alpha)}{\sqrt{n}}.$$

Thus $\beta_\alpha(P^n, Q^n) = \Phi\{\sqrt{n}\mu - \Phi^{-1}(1-\alpha)\}$. For $\mu < 0$, the same expression holds with $|\mu|$. Hence

$$\mathbb{P}_{P^n\otimes Q^\infty}(\tau < \infty) \le \Phi\big(\sqrt{n}|\mu| - \Phi^{-1}(1-\alpha)\big) < 1.$$

□

A.10 Proof of the Gaussian dispersion formula

For $P = \mathcal{N}(0, 1)$ and $Q = \mathcal{N}(0, \sigma^2)$, with $\sigma > 0$, write $X = \sigma Z$, where $Z \sim \mathcal{N}(0, 1)$. Symmetry and $\Phi(-x) = 1 - \Phi(x)$ give $\mathbb{E}\Phi(\sigma Z) = 1/2$, hence $\Delta_{\mathrm{ord}} = 0$. Also,

$$\begin{aligned}\mathbb{E}|2\Phi(\sigma Z) - 1| &= 2\int_0^\infty \{2\Phi(\sigma z) - 1\}\phi(z)\, dz \\ &= \frac{2}{\pi}\arctan(\sigma).\end{aligned}$$

The last identity follows from rotational symmetry of two independent standard Gaussians. Thus $\Delta_{\mathrm{disp}}(\sigma) = 2\arctan(\sigma)/\pi - 1/2$. Since $\Delta_{\mathrm{disp}}(1) = 0$ and $\Delta'_{\mathrm{disp}}(\sigma) = 2/\{\pi(1+\sigma^2)\} > 0$, the contrast is zero only at $\sigma = 1$.

The same sign interpretation holds for a common symmetric location scale family. Let P and Q be the laws of $m + \sigma_P Z$ and $m + \sigma_Q Z$, where $\sigma_P, \sigma_Q > 0$ and Z has a continuous, strictly increasing distribution function G symmetric about zero. With $r = \sigma_Q/\sigma_P$, $F_P(X) = G(rZ)$ for $X \sim Q$, and

$$|2G(rZ) - 1| = 2G(r|Z|) - 1$$

is strictly increasing in r almost surely. Because its expectation is $1/2$ at $r = 1$, Δ_{disp} is positive when $\sigma_Q > \sigma_P$ and negative when $\sigma_Q < \sigma_P$. Without this common form, its sign need not represent a scale ordering.

A.11 DELAYED CHANGE RESULTS

We extend the immediate-change result to a shift after ν null observations and bound the subsequent detection delay.

Theorem 12. *Let P be continuous, Q be arbitrary, and $h : [0,1] \to \mathbb{R}$ be a Lipschitz feature with constant L_h and $\Delta_h = \Delta_h(P,Q) \neq 0$. For an integer $\nu \geq 0$, suppose that $D_0 \sim P^n$, that $X_1, \ldots, X_\nu$ are i.i.d. from P, and that $X_{\nu+1}, X_{\nu+2}, \ldots$ are i.i.d. from Q, with all observations independent.*

There exist universal constants $C_0, C_1 > 0$ with the following property. For every $\beta \in (0,1)$, define

$$T_{h,\beta}(\nu) := \left\lceil C_1 \left[\frac{1}{|\Delta_h|^2} \log\left\{ \frac{e(\nu+1)}{\alpha\beta|\Delta_h|} \right\} + \sqrt{\frac{\nu}{|\Delta_h|^2} \log\left\{ \frac{e(\nu+1)}{\alpha\beta|\Delta_h|} \right\}} \right] \right\rceil.$$

Let τ_h be the stopping time produced by Algorithm 1 with feature h. Suppose that $n \geq \lceil C_0 \max\{1, L_h^2\} |\Delta_h|^{-2} \log\{e(\nu+1)/(\alpha\beta|\Delta_h|)\} \rceil$. Then

$$\mathbb{P}_{P^n \otimes Q^\infty}\{\tau_h \leq \nu + T_{h,\beta}(\nu)\} \geq 1 - \beta,$$
$$\mathbb{P}_{P^n \otimes Q^\infty}\{\nu < \tau_h \leq \nu + T_{h,\beta}(\nu)\} \geq 1 - \alpha - \beta.$$

Corollary 13. *Under the setting of Theorem 12, take $h = h_{\mathrm{ord}}$. For every $\beta \in (0,1)$, if*

$$n \geq \left\lceil C_0 |\Delta_{\mathrm{ord}}|^{-2} \log\left\{ \frac{e(\nu+1)}{\alpha\beta|\Delta_{\mathrm{ord}}|} \right\} \right\rceil,$$

then

$$\mathbb{P}_{P^n \otimes Q^\infty}\{\tau_{\mathrm{ord}} \leq \nu + T_{h_{\mathrm{ord}},\beta}(\nu)\} \geq 1 - \beta,$$
$$\mathbb{P}_{P^n \otimes Q^\infty}\{\nu < \tau_{\mathrm{ord}} \leq \nu + T_{h_{\mathrm{ord}},\beta}(\nu)\} \geq 1 - \alpha - \beta.$$

Proposition 14. *Let P be continuous, let Q be arbitrary, fix an integer $\nu \geq 0$, and let h be a Lipschitz feature with $\Delta_h(P,Q) \neq 0$. For each n, let $M_{h,t}^{(n)}$ and $\tau_h^{(n)}$ denote the e-process and stopping time based on a calibration sample of size n. Then, for every deterministic sequence $T_n \to \infty$,*

$$\mathbb{P}_{P^n \otimes Q^\infty}\left\{ \tau_h^{(n)} \leq \nu + T_n \right\} \longrightarrow 1$$

and

$$\mathbb{P}_{P^n \otimes Q^\infty}\left\{ \tau_h^{(n)} \leq \nu + T_n \,\middle|\, \tau_h^{(n)} > \nu \right\} \longrightarrow 1.$$

Moreover,

$$\liminf_{n\to\infty} \mathbb{P}_{P^n \otimes Q^\infty}\left\{ \nu < \tau_h^{(n)} \leq \nu + T_n \right\} \geq 1 - \alpha.$$

In particular, $\mathbb{P}_{P^n \otimes Q^\infty}(\tau_h^{(n)} < \infty) \to 1$.

Corollary 15. *Let P be continuous, let Q be arbitrary, and fix an integer $\nu \geq 0$. Fix the features and weights in Corollary 6, and let $M_{\mathrm{mix},t}^{(n)}$ and $\tau_{\mathrm{mix}}^{(n)}$ be defined there. If, for some ℓ, $\omega_\ell > 0$ and $\Delta_{h_\ell}(P,Q) \neq 0$, then, for every deterministic sequence $T_n \to \infty$,*

$$\mathbb{P}_{P^n \otimes Q^\infty}\left\{ \tau_{\mathrm{mix}}^{(n)} \leq \nu + T_n \right\} \longrightarrow 1$$

and

$$\mathbb{P}_{P^n \otimes Q^\infty}\left\{ \tau_{\mathrm{mix}}^{(n)} \leq \nu + T_n \,\middle|\, \tau_{\mathrm{mix}}^{(n)} > \nu \right\} \longrightarrow 1.$$

Moreover,

$$\liminf_{n\to\infty} \mathbb{P}_{P^n \otimes Q^\infty}\left\{ \nu < \tau_{\mathrm{mix}}^{(n)} \leq \nu + T_n \right\} \geq 1 - \alpha.$$

A.12 Proof of Theorem 12

Proof. Let $\mathbb{P}_\nu = \mathbb{P}_{P^{n+\nu}\otimes Q^\infty}$. Throughout the proof, abbreviate

$$\Delta = \Delta_h(P,Q), \qquad \delta = |\Delta|, \qquad L = L_h, \qquad K = \max\{1, L^2\}.$$

The feature normalization gives $\delta \le \min\{1, L\}$ and $L\delta \le K$; in particular, $L > 0$. For the proof, continue the ONS recursion after its first crossing; this leaves τ_h unchanged. Fix a deterministic integer $T \ge 1$, and put

$$\bar{v}_n = \frac{1}{n+1}\sum_{j=1}^{n+1} h\left(\frac{j-1}{n}\right), \qquad V_t = h\left(\frac{R_t - 1}{n}\right).$$

Conditional on D_0, define

$$m_P = \mathbb{E}_P(V_t \mid D_0), \qquad m_Q = \mathbb{E}_Q(V_t \mid D_0),$$

and

$$d_P = m_P - \bar{v}_n, \qquad d_Q = m_Q - \bar{v}_n.$$

The quantity d_Q is precisely the realized contrast $d_{h,n}(D_0)$ in Lemma 10, while d_P is the same quantity with $Q = P$.

Let

$$\mathcal{H}_\nu := \sigma(D_0, R_1, \dots, R_\nu), \qquad D := n + \nu + 1,$$

and define the realized signal at the change point by

$$d_\nu := m_Q - \frac{(n+1)\bar{v}_n + \sum_{s=1}^{\nu} V_s}{D}.$$

Writing

$$S_\nu := \sum_{s=1}^{\nu}(V_s - m_P),$$

where $S_0 = 0$, gives the exact decomposition

$$d_\nu = d_Q - \frac{\nu d_P + S_\nu}{D}. \tag{8}$$

Assume for the moment that

$$n + 1 \ge \frac{32L}{\delta}. \tag{9}$$

Lemma 10, applied with $x = \delta/(32L)$ first to Q and then to P, gives

$$\mathbb{P}_{P^n}\left(|d_Q - \Delta| > \frac{\delta}{16}\right) \le 2\exp\left(-\frac{n\delta^2}{512L^2}\right),$$
$$\mathbb{P}_{P^n}\left(|d_P| > \frac{\delta}{16}\right) \le 2\exp\left(-\frac{n\delta^2}{512L^2}\right).$$

Conditional on D_0, the variables $V_1, \dots, V_\nu$ are independent, have common mean m_P, and lie in an interval of width at most one. Hence, for $\nu \ge 1$, Hoeffding's inequality gives

$$\mathbb{P}_\nu\left(|S_\nu| > \frac{D\delta}{8} \,\middle|\, D_0\right) \le 2\exp\left(-\frac{D^2\delta^2}{32\nu}\right).$$

For $\nu = 0$, $S_0 = 0$, so no Hoeffding bound is needed.

Define the $\mathcal{H}_\nu$-measurable event

$$\mathcal{E}_\nu := \left\{|d_Q - \Delta| \le \frac{\delta}{16}\right\} \cap \left\{|d_P| \le \frac{\delta}{16}\right\} \cap \left\{|S_\nu| \le \frac{D\delta}{8}\right\}.$$

On $\mathcal{E}_\nu$, (8) yields

$$|d_\nu - \Delta| \le \frac{\delta}{16} + \frac{\nu}{D}\frac{\delta}{16} + \frac{\delta}{8} \le \frac{\delta}{4}.$$

Therefore, with $s = \operatorname{sign}(\Delta)$,

$$sd_\nu \geq \frac{3\delta}{4} \qquad \text{on } \mathcal{E}_\nu. \tag{10}$$

A union bound and $D^2/\nu \geq 4(n+1)$ for $\nu \geq 1$, together with $K = \max\{1, L^2\}$, give

$$\mathbb{P}_\nu(\mathcal{E}_\nu^c) \leq 6 \exp\left(-\frac{n\delta^2}{512K}\right).$$

Conditional on $\mathcal{H}_\nu$, the post-change ranks are i.i.d. and $\mathbb{E}_\nu(V_{\nu+m} \mid \mathcal{H}_\nu) = m_Q$. Thus, for every $m \geq 1$,

$$\begin{aligned}\mathbb{E}_\nu\left(Z_{h,\nu+m} \,|\, \mathcal{H}_\nu\right) &= m_Q - \frac{(n+1)\bar{v}_n + \sum_{s=1}^{\nu} V_s + (m-1)m_Q}{D+m-1} \\ &= \frac{D}{D+m-1} d_\nu. \end{aligned} \tag{11}$$

To account for the bets placed before the change, we need a suffix regret bound for the current ONS state. Write

$$g_t = \frac{Z_{h,t}}{1+\lambda_t Z_{h,t}}, \qquad A_t = 1 + \sum_{j=1}^{t} g_j^2.$$

The predictable domain gives $|\lambda_t Z_{h,t}| \leq 3/4$, so

$$|g_t| \leq 4, \qquad A_t \leq 1 + 16t.$$

Retaining the endpoint terms in the proof of Proposition 3 gives, for every $u \in [-3/4, 3/4]$,

$$\log(1+uZ_{h,t}) - \log(1+\lambda_t Z_{h,t}) \leq \frac{1}{9}\left\{A_{t-1}(\lambda_t - u)^2 - A_t(\lambda_{t+1} - u)^2\right\} + \frac{9}{4}\frac{g_t^2}{A_t}. \tag{12}$$

For $u \in [-3/4, 3/4]$, write

$$L_{\nu,T}(u) := \sum_{m=1}^{T} \log\{1 + uZ_{h,\nu+m}\}.$$

Summing (12) from $t = \nu+1$ to $t = \nu+T$, and using

$$\sum_{t=\nu+1}^{\nu+T} \frac{g_t^2}{A_t} \leq \log\left(\frac{A_{\nu+T}}{A_\nu}\right),$$

gives

$$\begin{aligned}\log M_{h,\nu+T} - \log M_{h,\nu} \geq L_{\nu,T}(u) - \frac{A_\nu}{9}(\lambda_{\nu+1} - u)^2 \\ - \frac{9}{4}\log\left(\frac{A_{\nu+T}}{A_\nu}\right).\end{aligned} \tag{13}$$

Applying (12) from $t = 1$ to $t = \nu$ with comparator $u = 0$, and using $\lambda_1 = 0$, yields

$$-\log M_{h,\nu} \leq -\frac{A_\nu}{9}\lambda_{\nu+1}^2 + \frac{9}{4}\log A_\nu.$$

Consequently,

$$\frac{A_\nu}{9}\lambda_{\nu+1}^2 \leq \log M_{h,\nu} + \frac{9}{4}\log A_\nu. \tag{14}$$

Combining (13) and (14), then using $2\lambda_{\nu+1}u - u^2 \geq -\lambda_{\nu+1}^2 - 2u^2$, $A_t \leq 1 + 16t$, and (14) again, gives the following pathwise bound on $\{\tau_h > \nu\}$, where $M_{h,\nu} < 1/\alpha$:

$$\begin{aligned}\log M_{h,\nu+T} \geq L_{\nu,T}(u) - \log\frac{1}{\alpha} - \frac{9}{2}\log\{1 + 16(\nu+T)\} \\ - \frac{2(1+16\nu)}{9}u^2,\end{aligned} \tag{15}$$

Suppose now that

$$T \le D = n + \nu + 1. \tag{16}$$

Put

$$\theta := \frac{\delta T}{16(\nu + T + 1)}.$$

Because $\delta \le 1$, we have $\theta \le 1/16$. Hence $\log(1+x) \ge x - x^2$ applies to $x = s\theta Z_{h,\nu+m}$. On $\mathcal{E}_\nu$, (10) and (11) imply

$$\begin{aligned}\mathbb{E}_\nu \left\{ L_{\nu,T}(s\theta) \,|\, \mathcal{H}_\nu \right\} &\ge \theta s d_\nu \sum_{m=1}^{T} \frac{D}{D+m-1} - \theta^2 T \\ &\ge \frac{3\theta\delta T}{8} - \theta^2 T.\end{aligned}$$

The last inequality uses $\sum_{m=1}^{T} D/(D+m-1) \ge T/2$, which follows from $T \le D$. Since

$$T + \frac{2(1+16\nu)}{9} \le 4(\nu + T + 1),$$

the choice of θ gives

$$\begin{aligned}\mathbb{E}_\nu \left\{ L_{\nu,T}(s\theta) \,|\, \mathcal{H}_\nu \right\} - \frac{2(1+16\nu)}{9}\theta^2 \\ \ge \frac{\delta^2 T^2}{128(\nu + T + 1)} \qquad \text{on } \mathcal{E}_\nu.\end{aligned} \tag{17}$$

Conditional on $\mathcal{H}_\nu$, regard $L_{\nu,T}(s\theta)$ as a function of the independent post-change ranks. Replacing $R_{\nu+j}$ changes its own innovation by at most one and each later innovation $Z_{h,\nu+m}$ by at most $1/(D+m-1)$. Since

$$\sup_{|z|\le 1} \left| \frac{d}{dz} \log(1 + s\theta z) \right| \le \frac{\theta}{1-\theta} < 2\theta,$$

changing $R_{\nu+j}$ changes $L_{\nu,T}(s\theta)$ by at most

$$2\theta \left\{ 1 + \sum_{m=j+1}^{T} \frac{1}{D+m-1} \right\} \le 4\theta,$$

where the last inequality again uses $T \le D$. The sum of squared bounded differences is at most $16T\theta^2$. McDiarmid's inequality and (17) therefore give, on $\mathcal{E}_\nu$,

$$\mathbb{P}_\nu \left(L_{\nu,T}(s\theta) - \frac{2(1+16\nu)}{9}\theta^2 < \frac{\delta^2 T^2}{256(\nu + T + 1)} \,\middle|\, \mathcal{H}_\nu \right) \le \exp\left(-\frac{T\delta^2}{2048} \right). \tag{18}$$

Suppose also that

$$\frac{\delta^2 T^2}{256(\nu + T + 1)} \ge 2\log\frac{1}{\alpha} + \frac{9}{2}\log\{1 + 16(\nu + T)\}. \tag{19}$$

If $\tau_h > \nu + T$, then (15) applies. On $\mathcal{E}_\nu$ and the complement of the lower-tail event in (18), (15) and (19) imply $\log M_{h,\nu+T} \ge \log(1/\alpha)$, a contradiction. Hence, whenever (9), (16), and (19) hold,

$$\mathbb{P}_\nu(\tau_h > \nu + T) \le 6\exp\left(-\frac{n\delta^2}{512K} \right) + \exp\left(-\frac{T\delta^2}{2048} \right). \tag{20}$$

We now verify the grid, window, and crossing conditions for $T = T_{h,\beta}(\nu)$. Put

$$\ell = \log\left\{ \frac{e(\nu+1)}{\alpha\beta\delta} \right\}, \qquad r = \frac{\ell}{\delta^2}.$$

Then $\ell \ge 1$ and $r \ge 1$. Because $L\delta \le K$, the calibration-size condition implies (9) whenever $C_0 \ge 32$.

Let

$$T = \left\lceil C_1\{r + \sqrt{\nu r}\} \right\rceil .$$

For now, let $C_1 \geq 1$ and suppose that

$$C_0 \geq C_1 + 1 + \frac{C_1^2}{4}.$$

Since $n \geq C_0 K r \geq C_0 r$, writing $T \leq C_1\{r + \sqrt{(\nu+1)r}\} + 1$ gives

$$n + \nu + 1 - T \geq r \left\{ \frac{\nu+1}{r} - C_1\sqrt{\frac{\nu+1}{r}} + C_0 - C_1 - 1 \right\} \geq 0.$$

Thus $T \leq n + \nu + 1 = D$, proving (16).

Next, let $T_0 = C_1\{r + \sqrt{\nu r}\}$. Since $t \mapsto t^2/(\nu + 1 + t)$ is increasing,

$$\frac{T^2}{\nu + 1 + T} \geq \frac{T_0^2}{\nu + 1 + T_0}.$$

Because $r \geq 1$,

$$\nu + 1 + T_0 \leq (C_1 + 1)(\sqrt{\nu} + \sqrt{r})^2.$$

Hence

$$\frac{T_0^2}{\nu + 1 + T_0} \geq \frac{C_1^2}{1 + C_1} r \geq \frac{C_1}{2} r.$$

Therefore,

$$\frac{\delta^2 T^2}{256(\nu + 1 + T)} \geq \frac{C_1}{512}\ell. \tag{21}$$

On the other hand, using $\sqrt{(\nu+1)r} \leq (\nu + 1 + r)/2$ and $r \geq 1$,

$$1 + 16(\nu + T) \leq 24 C_1(\nu + r + 2)$$

for $C_1 \geq 1$. Furthermore,

$$\log(\nu + 1) \leq \ell, \qquad \log r = \log \ell + 2 \log \frac{1}{\delta} \leq 3\ell.$$

It follows that

$$\log\{1 + 16(\nu + T)\} \leq \{6 + \log(24 C_1)\}\ell. \tag{22}$$

Choose $C_1 \geq 2048$ large enough that

$$\frac{C_1}{512} \geq 2 + \frac{9}{2}\{6 + \log(24 C_1)\}.$$

Then choose

$$C_0 \geq \max\left\{1536,\ C_1 + 1 + \frac{C_1^2}{4}\right\}.$$

Combining (21) and (22), and using $\log(1/\alpha) \leq \ell$, proves (19). The calibration condition and the definition of T now give

$$6 \exp\left(-\frac{n\delta^2}{512K}\right) \leq 6e^{-3\ell}, \qquad \exp\left(-\frac{T\delta^2}{2048}\right) \leq e^{-\ell}.$$

Because $\ell \geq \log(e/\beta)$,

$$6e^{-3\ell} + e^{-\ell} \leq \left(6e^{-3} + e^{-1}\right)\beta < \beta.$$

It follows from (20) that

$$\mathbb{P}_\nu\left\{\tau_h > \nu + T_{h,\beta}(\nu)\right\} \leq \beta,$$

which proves the first conclusion of the theorem.

Under $\mathbb{P}_\nu$, the joint law of $(D_0, X_1, \ldots, X_\nu)$ is the same as under $H_0(P)$. The anytime-validity result in Proposition 2 therefore gives

$$\mathbb{P}_\nu(\tau_h \leq \nu) = \mathbb{P}_{H_0(P)}(\tau_h \leq \nu) \leq \alpha.$$

Combining this bound with the first conclusion gives

$$\mathbb{P}_\nu\left\{\nu < \tau_h \leq \nu + T_{h,\beta}(\nu)\right\} \geq 1 - \alpha - \beta,$$

which proves the second conclusion. □

A.13 Proof of Corollary 13

Proof. The order feature has Lipschitz constant one, so the result follows directly from Theorem 12. □

A.14 Proof of Proposition 14

Proof. Fix $\eta \in (0,1)$. The horizon $T_{h,\eta}(\nu)$ in Theorem 12 does not depend on n. For all sufficiently large n, the calibration-size condition holds, and $T_n \geq T_{h,\eta}(\nu)$. The first conclusion of the theorem therefore gives

$$\liminf_{n\to\infty} \mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_h^{(n)} \leq \nu + T_n\right\} \geq 1-\eta.$$

Since η is arbitrary, the claimed unconditional convergence follows.

For the conditional statement, apply the theorem with $\beta = \eta(1-\alpha)$. The observations up to time ν have the null law, so anytime validity gives

$$\mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_h^{(n)} > \nu\right\} \geq 1-\alpha.$$

For all sufficiently large n,

$$\mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_h^{(n)} > \nu + T_n \,\middle|\, \tau_h^{(n)} > \nu\right\} \leq \frac{\beta}{1-\alpha} = \eta.$$

This proves the conditional convergence. Finally,

$$\mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\nu < \tau_h^{(n)} \leq \nu + T_n\right\} = \mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_h^{(n)} > \nu\right\} - \mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_h^{(n)} > \nu + T_n\right\}.$$

The second term tends to zero, so the displayed identity and anytime validity give the claimed lower bound. The eventual rejection claim follows from the first convergence proved above. □

A.15 Proof of Corollary 15

Proof. Choose ℓ satisfying the conditions in the corollary, and define

$$\widetilde{\tau}_\ell^{(n)} = \inf\left\{t \geq 1 : M_{h_\ell,t}^{(n)} \geq \frac{1}{\alpha\omega_\ell}\right\}.$$

Since $M_{\mathrm{mix},t}^{(n)} \geq \omega_\ell M_{h_\ell,t}^{(n)}$, we have $\tau_{\mathrm{mix}}^{(n)} \leq \widetilde{\tau}_\ell^{(n)}$. Applying Proposition 14 to h_ℓ at level $\alpha\omega_\ell$ gives

$$\mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_{\mathrm{mix}}^{(n)} > \nu + T_n\right\} \longrightarrow 0.$$

Because the mixture is an e-process and the observations through time ν have the null law,

$$\mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_{\mathrm{mix}}^{(n)} > \nu\right\} \geq 1-\alpha.$$

Hence

$$\mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_{\mathrm{mix}}^{(n)} > \nu + T_n \,\middle|\, \tau_{\mathrm{mix}}^{(n)} > \nu\right\} \leq \frac{\mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\{\tau_{\mathrm{mix}}^{(n)} > \nu + T_n\}}{1-\alpha} \longrightarrow 0.$$

The final claim follows from

$$\mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\nu < \tau_{\mathrm{mix}}^{(n)} \leq \nu + T_n\right\} = \mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_{\mathrm{mix}}^{(n)} > \nu\right\} - \mathbb{P}_{P^{n+\nu}\otimes Q^\infty}\left\{\tau_{\mathrm{mix}}^{(n)} > \nu + T_n\right\}.$$

□

B From CCTM's PAC Guarantee to Marginal Type I Error

This section explains why the CCTM calibration-conditional guarantee does not imply marginal type I error control at the same nominal level and gives continuous and discrete counterexamples.

B.1 From the conditional statement to a marginal bound

Theorem 3.1 of Shaer et al. (2026) displays the statement

$$\mathbb{P}_{D_0}\left\{\mathbb{P}_{H_0}\left(\exists t \geq 1 : S_t \geq \frac{1}{\alpha} \,\middle|\, \mathcal{F}_{t-1}, D_0\right) \leq \alpha\right\} \geq 1-\delta,$$

where $\mathcal{F}_{t-1} = \sigma(\widehat{F}_0(X_1), \ldots, \widehat{F}_0(X_{t-1}))$. This notation is not formally well defined because the crossing event ranges over all t, whereas the conditioning field $\mathcal{F}_{t-1}$ changes with t. The intended statement should instead be written as

$$\mathbb{P}_{D_0}\left\{\mathbb{P}_{H_0(P)}\left(\sup_{t\geq 1} S_t \geq \frac{1}{a} \,\middle|\, D_0\right) \leq a\right\} \geq 1-\delta.$$

The conditional crossing probability is at most a for a set of calibration samples with probability at least $1-\delta$, and at most one otherwise. Hence

$$\mathbb{P}_{H_0(P)}(\tau < \infty) \leq a(1-\delta) + \delta.$$

Using this bound, marginal level α requires $\delta < \alpha$ and $a \leq (\alpha-\delta)/(1-\delta)$. The synthetic experiments of Shaer et al. (2026) use $\alpha = 0.05$ and $\delta = 0.1$. With $a = 0.05$, the bound is 0.145, and no nonnegative choice of a makes it at most 0.05.

B.2 A continuous-null example

The preceding conversion shows that the internal conditional level need not equal the marginal type I error. This difference can lead to an actual violation of the nominal level.

Proposition 16. *Consider CCTM under the continuous null $P = \mathrm{Unif}(0,1)$. At*

$$n = 5, \qquad \delta = 0.9, \qquad k = 10^{-6}, \qquad \alpha = 0.01,$$

its infinite-horizon null rejection probability is at least

$$0.0122385070 > \alpha.$$

This example shows that CCTM does not control marginal type I error for all allowed parameter choices, even under a continuous null.

Proof. Let $W = (W_0, \ldots, W_n)$ be the spacings made by the ordered calibration sample on $[0,1]$. Under the continuous uniform null, $W \sim \mathrm{Dirichlet}(1, \ldots, 1)$. Conditional on W, the fixed ECDF value $\widehat{F}_0(X_t) = j/n$ has probability W_j, independently over t.

Put

$$\epsilon = \sqrt{\frac{\log(2/\delta)}{2n}}, \qquad \mathcal{C} = \{1/2 + \sqrt{1+k^2}\,\epsilon\}^{-1}.$$

For the two fixed ONS comparators $\eta = \pm 1/2$, define

$$b_j^{\pm} = 1 + \mathcal{C}\left[\pm\frac{1}{2}\left(\frac{j}{n} - \frac{1}{2}\right) - \sqrt{\frac{1}{4}+k^2}\,\epsilon\right], \qquad \ell_j^{\pm} = \log b_j^{\pm}.$$

The conditional long-run log growth of comparator $\pm 1/2$ is $L_{\pm}(W) = \sum_{j=0}^{n} W_j \ell_j^{\pm}$. If $L_{\pm}(W) > 0$, the conditional strong law makes that comparator's log wealth grow linearly. The pathwise $O(\log T)$ ONS regret bound then makes the analyzed CCTM wealth cross every fixed threshold almost surely.

For distinct coefficients $c_0, \ldots, c_n$ and a vector uniform on the simplex, direct integration of the simplex slice gives

$$\mathbb{P}\left(\sum_{j=0}^{n} c_j W_j > x\right) = \sum_{j:c_j>x} \frac{(c_j - x)^n}{\prod_{r\neq j}(c_j - c_r)}.$$

One derivation writes $W_j = E_j / \sum_r E_r$ for independent unit-rate exponentials and uses partial fractions for the resulting signed weighted sum. At $n = 5$, $\delta = 0.9$, and $k = 10^{-6}$, substitution of $c_j = \ell_j^+$ and $x = 0$ gives

$$\mathbb{P}\{L_+(W) > 0\} = 0.0061192535.$$

Symmetry gives the same probability for $L_-(W) > 0$. These two events are disjoint: for every j, $b_j^+ b_j^- < 1$, so $L_+(W) + L_-(W) < 0$. Consequently the unconditional probability of a positive-growth comparator, and hence the infinite-horizon null rejection probability, is at least

$$2(0.0061192535) = 0.0122385070 > 0.01.$$

□

B.3 A DISCRETE-NULL EXAMPLE

CCTM computes

$$\widehat{F}_0(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{Y_i \le x\}.$$

Under a continuous null, $\widehat{F}_0(X_t)$ is marginally uniform on $\{0, 1/n, \ldots, 1\}$. This property can fail in the presence of ties. We tested the null

$$Y_i \sim \text{Bernoulli}(1/2), \qquad X_t \sim \text{Bernoulli}(1/2),$$

with $n = 1000$, $T = 200$, $\alpha = 0.05$, $\delta = 0.1$, and 5000 repetitions. There is no distribution shift. Table 1 reports the results. CCTM rejects in every run, while Order PRM remains below the nominal level.

Table 1: Tie counterexample under a Bernoulli null.

Method	False-alarm probability	Standard error
Order PRM	0.0344	0.0026
CCTM	1.0000	0.0000

The released CCTM implementation exhibits severe empirical type I error inflation under this null. Together with Proposition 16, this gives counterexamples under both continuous and discrete nulls.

C EXPERIMENTAL DETAILS

This section specifies the two baselines, the type I error simulation, and the synthetic and CIFAR-10-C protocols.

C.1 CCTM AND STANDARD CTM

Algorithm 2 combines Algorithms 1 and 2 of Shaer et al. (2026).

Algorithm 2 CCTM

Require: Fixed reference set $D_0 = \{X_i^0\}_{i=1}^n$, stream $X_1, \ldots, X_T$, test level α, confidence bound level δ, and smoothing parameter $k > 0$
Ensure: Rejection time $\tau_{\mathrm{CCTM}} \in \{1, \ldots, T\} \cup \{\infty\}$
1: Estimate $\widehat{F}_0(x) = n^{-1} \sum_{i=1}^n \mathbf{1}\{X_i^0 \le x\}$.
2: Set $\epsilon(x) = \epsilon_n = \sqrt{\log(2/\delta)/(2n)}$, and set $\mathcal{C} = \{0.5 + \sqrt{1+k^2} \max_{\hat{p} \in [0,1]} \epsilon(\hat{p})\}^{-1}$.
3: Initialize $S_0 = 1$, $\eta_1 = 0$, $a_0 = 1$, and $\tau_{\mathrm{CCTM}} = \infty$.
4: **for** $t = 1, \ldots, T$ **do**
5: Set $\widehat{p}_t = \widehat{F}_0(X_t)$ and evaluate $\epsilon(\widehat{p}_t)$.
6: Set $g_t = \mathcal{C}\{\eta_t(\widehat{p}_t - 0.5) - \sqrt{\eta_t^2 + k^2}\epsilon(\widehat{p}_t)\}$.
7: Update $S_t = S_{t-1}(1 + g_t)$.
8: **if** $S_t \ge 1/\alpha$ **then**
9: Set $\tau_{\mathrm{CCTM}} = t$ and return τ_{CCTM}.
10: **end if**
11: Set $g_t' = \mathcal{C}\{\widehat{p}_t - 0.5 - \eta_t \epsilon(\widehat{p}_t)/\sqrt{\eta_t^2 + k^2}\}$.
12: Set $z_t = g_t'/(1 + g_t)$ and $a_t = a_{t-1} + z_t^2$.
13: Set $\eta_{t+1} = \max\{-1/2, \min(\eta_t + 4z_t/a_t, 1/2)\}$.
14: **end for**
15: Return $\tau_{\mathrm{CCTM}} = \infty$.

Our simulations use the implementation released with that paper. Relative to Algorithm 2, the released code uses $(0.5 + (1 + k)\epsilon_n)^{-1}$ in place of $\mathcal{C}$, replaces the ONS coefficient 4 by $2/\{2 - \log(3)\}$, and applies the clipping rule in Section 3.4 of Shaer et al. (2026). We use $D = 0.5$ and $k = 10^{-6}$, with $\delta = C = 0.1$ in the synthetic experiments and $\delta = C = 0.05$ in the CIFAR-10-C experiment.

Algorithm 3 gives the Standard CTM. Its randomized conformal value is from Section 2.2 of Shaer et al. (2026), and its ONS update is from the released implementation.

Algorithm 3 Standard CTM

Require: Reference set $D_0 = \{X_i^0\}_{i=1}^n$, stream $X_1, \ldots, X_T$, test level α, betting bound D, and clipping threshold C
Ensure: Rejection time $\tau_{\mathrm{CTM}} \in \{1, \ldots, T\} \cup \{\infty\}$
1: Initialize $\mathcal{D}_0 = D_0$, $S_0 = 1$, $\eta_1 = 0$, $a_0 = 1$, and $\tau_{\mathrm{CTM}} = \infty$.
2: **for** $t = 1, \ldots, T$ **do**
3: Draw $U_t \sim \mathrm{Unif}(0,1)$ independently and set $p_t = \{\sum_{Z \in \mathcal{D}_{t-1}} \mathbf{1}\{Z < X_t\} + U_t[1 + \sum_{Z \in \mathcal{D}_{t-1}} \mathbf{1}\{Z = X_t\}]\}/(n + t)$.
4: Set $\mathcal{D}_t = \mathcal{D}_{t-1} \cup \{X_t\}$ and $\widetilde{\eta}_t = \eta_t \mathbf{1}\{|\eta_t| \ge C\}$.
5: Update $S_t = S_{t-1}\{1 + \widetilde{\eta}_t(p_t - 0.5)\}$.
6: **if** $S_t \ge 1/\alpha$ **then**
7: Set $\tau_{\mathrm{CTM}} = t$ and return τ_{CTM}.
8: **end if**
9: Set $v_t = 2(p_t - 0.5)$, $z_t = v_t/(1 + \eta_t v_t)$, and $a_t = a_{t-1} + z_t^2$.
10: Set $\eta_{t+1} = \Pi_{[-D,D]}\{\eta_t + [2/\{2 - \log(3)\}]z_t/a_t\}$.
11: **end for**
12: Return $\tau_{\mathrm{CTM}} = \infty$.

We use $D = 0.5$ and $C = 0.1$ for Standard CTM.

C.2 Type I error simulation

Under the null, the calibration observations and the entire online stream are i.i.d. from $\mathcal{N}(0,1)$, so no change occurs. We use $\alpha = 0.05$, monitor 20000 online observations, and estimate the rejection probability from 10000 repetitions for $n \in \{10, 500, 1000, \ldots, 5000\}$.

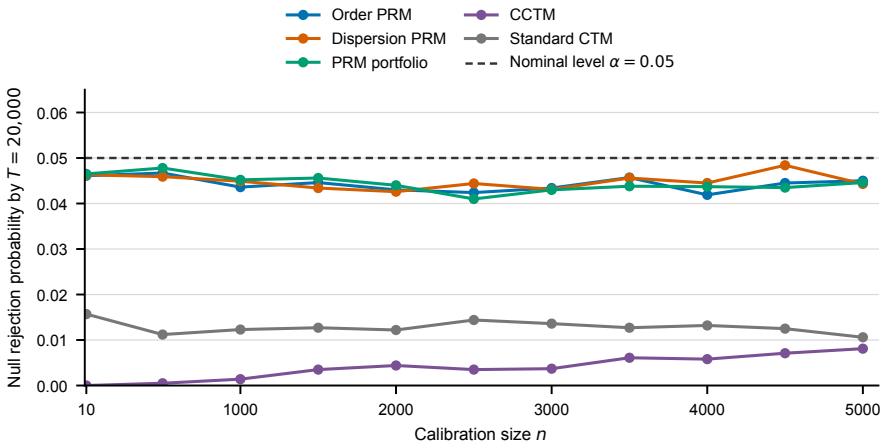


Figure 4: Null rejection probabilities across calibration sizes.

In this Gaussian null experiment, none of the five estimated rejection probabilities exceeds the nominal level (0.05). The PRMs are closer to the nominal level than CCTM and Standard CTM.

C.3 Complete results for the Gaussian location experiments

Table 2 reports the complete results for the nine Gaussian settings described in the main text. Within each repetition, all four methods receive the same calibration sample and online stream. For delayed changes, $t_{0.8}$ is computed from the post-change detection curve conditional on no alarm by the change point.

Table 2: Time to 80% detection probability in the nine Gaussian settings of Shaer et al. (2026). Smaller is better. Reduction is measured relative to CCTM.

Setting	Order PRM	Portfolio	CCTM	Standard CTM	Reduction
Immediate, $d = 1$	**24**	27	30	37	**20.0%**
Immediate, $d = 1.5$	**15**	17	17	24	**11.8%**
Immediate, $d = 2$	**11**	12	13	19	**15.4%**
Delayed, $t_0 = 200$	**27**	**27**	34	38	**20.6%**
Delayed, $t_0 = 600$	43	**42**	55	61	**21.8%**
Delayed, $t_0 = 4000$	113	**111**	147	167	**23.1%**
Gradual, $\lambda = 0.015$	**68**	71	77	76	**11.7%**
Gradual, $\lambda = 0.03$	**44**	46	49	52	**10.2%**
Gradual, $\lambda = 0.05$	**33**	35	37	40	**10.8%**

C.4 CIFAR-10-C protocol

The image experiment uses a publicly available ResNet-20 model pretrained on CIFAR-10. For each image, we use the Shannon entropy of its softmax probabilities as the scalar monitoring score. We use all 15 main corruptions at severity 5. We randomly permute the 10,000 test-image indices. The first 50 indices provide the largest clean calibration sample, and the next 5000 indices provide the corrupted stream. The same indices are used across methods. For the larger calibration sizes $n \in \{100, 500, 1000\}$, the first 1000 indices of each permutation form the largest calibration sample, and the next 5000 form the corrupted stream. In each experiment, the calibration set at size n consists of the first n indices of its largest calibration sample. The monitoring indices are shared across calibration sizes within each experiment.

All four methods receive the same entropy-score stream in every trial.

C.5 ADDITIONAL CIFAR-10-C SEVERITY LEVELS

Figures 5–8 report the detection curves at severity levels 1–4. All other settings are unchanged.

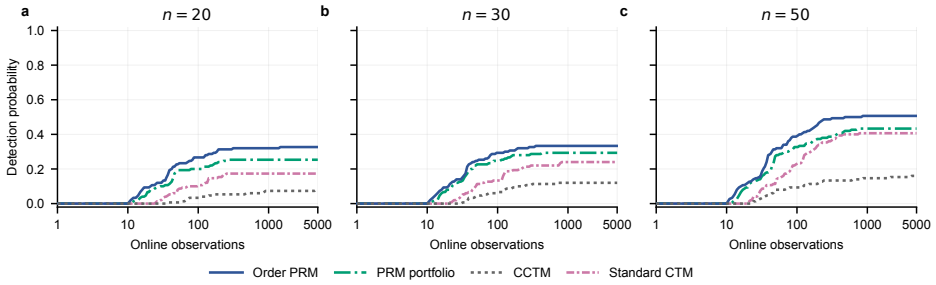


Figure 5: Detection of CIFAR-10-C corruptions at severity 1 for $n = 20, 30, 50$.

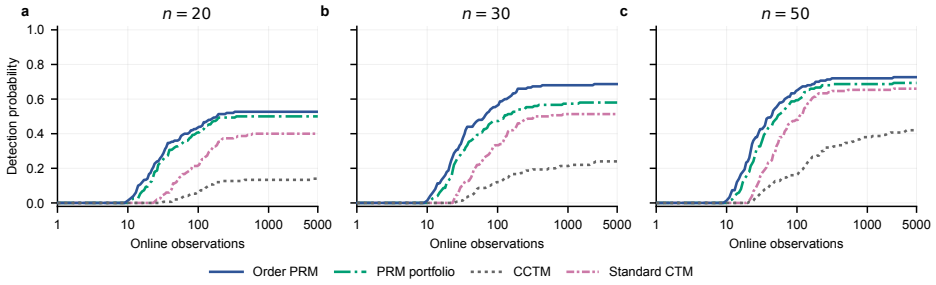


Figure 6: Detection of CIFAR-10-C corruptions at severity 2 for $n = 20, 30, 50$.

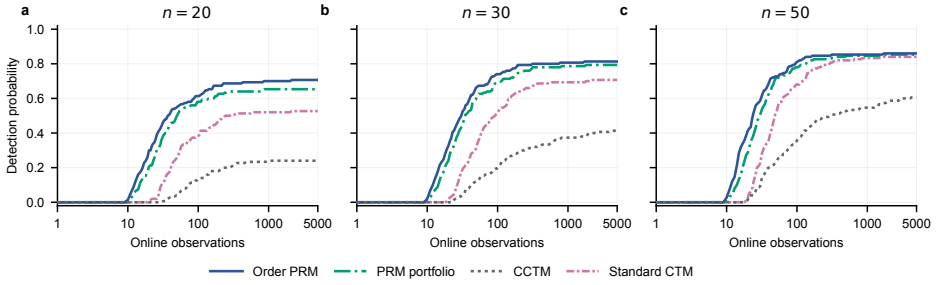


Figure 7: Detection of CIFAR-10-C corruptions at severity 3 for $n = 20, 30, 50$.

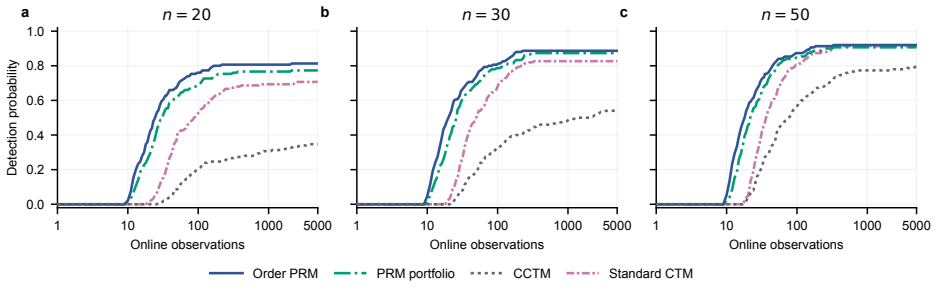


Figure 8: Detection of CIFAR-10-C corruptions at severity 4 for $n = 20, 30, 50$.

C.6 CIFAR-10-C WITH LARGER CALIBRATION SAMPLES

We also evaluate larger calibration samples, $n \in \{100, 500, 1000\}$. Figure 9 shows the detection curves, and Table 3 reports the time to 80% detection. Order PRM and the portfolio remain faster than both baselines by this measure, although their advantage over CCTM narrows as the calibration size increases.

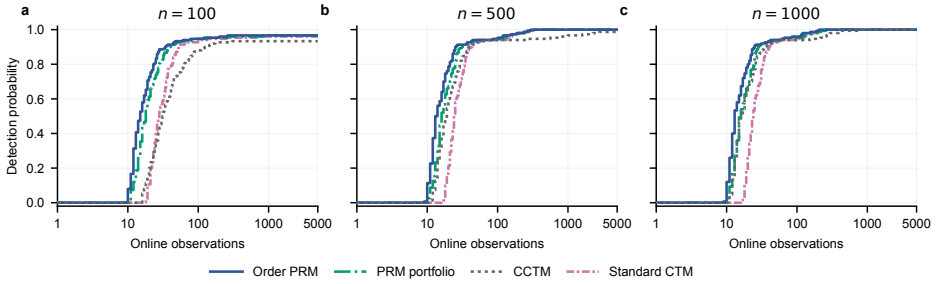


Figure 9: Detection of CIFAR-10-C corruptions at severity 5 for $n = 100, 500, 1000$. Each curve aggregates 15 corruption types and 10 random index splits.

Table 3: Time to 80% detection on CIFAR-10-C at severity 5 with larger calibration samples. Smaller is better.

n	Order PRM	PRM portfolio	CCTM	Standard CTM
100	**23**	28	63	43
500	**21**	25	29	34
1000	**20**	25	26	33