SAFETY CERTIFICATE FOR STOCHASTIC FRACTIONAL-ORDER DYNAMICAL SYSTEMS

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ABSTRACT

Certifying the safety of stochastic systems with long-range memory is challenging because future risk can depend on the entire state history. Existing safety certificates commonly assume Markovian dynamics, and their safety and persistent-feasibility guarantees do not carry over when infinite histories are unavailable to an online safety certificate. In this paper, we study how to quantify risk and ensure safety when only limited history information is available to the online certificate. First, we propose a Bellman-like backward recursion that constructs lower bounds on the value and action-value functions, and hence on long-term risk, using only a finite-memory representation. Second, we derive constraints on control actions that guarantee long-term safety while ensuring persistent feasibility. By combining offline-computed lower bounds with these action constraints, we obtain an efficient online safety certificate whose certified action set remains nonempty at every decision step. Then, we apply the framework to stochastic fractional-order dynamical systems, whose long-range memory makes existing Markov-based risk evaluation and safety certification techniques inapplicable. Finally, we use numerical simulations to demonstrate the effectiveness of the proposed framework and compare with existing techniques.

1 INTRODUCTION

Many dynamical systems, including Lithium-ion battery (Manivannan & Vigneswar, 2025), Viscoelastic structures (Alotta et al., 2018), hydrology (Hurst, 1951), and network traffic (Leland et al., 1993), exhibit long-range memory. In these systems, future safety can depend on the entire state history, so identical current states can entail different risks depending on their pasts. Retaining and processing the complete history, however, becomes increasingly demanding as the system evolves. Limited onboard memory and computational resources may therefore restrict an online safety certificate to a finite history window or a compact summary. *How can we certify safety in real time in stochastic systems with long-range memory using limited information while accounting for the influence of discarded history?*

Existing safety certificates commonly assume Markov dynamics (Hsu et al., 2024; Wabersich & Zeilinger, 2021; Clark, 2020; Santoyo et al., 2019). These techniques usually exploit the Markovian nature to relate the current state and action to future safety. For systems with long-range memory, truncating the history generally does not preserve this property because the retained information may be insufficient to determine the successor distribution. Consequently, applying a Markov-based certificate directly to truncated dynamics does not preserve its guarantees for long-term safety and persistent feasibility for the original system. Our simulations further reveal practical limitations of existing methods. Ignoring discarded history can lead to conservative or unsafe decisions, while using full history can increase control intervention and online computation. These limitations motivate a certificate that accounts for discarded memory while remaining efficient online.

Motivated by this gap, we study safety certification for stochastic systems with long-range memory when the certificate has access to only a finite state history. First, we propose a Bellman-like backward recursion that computes lower bounds on the value and action-value functions while accounting for every full history compatible with the retained information. Building on these bounds, we derive online action constraints that guarantee the desired safety specification with persistent feasibility: every controller that selects only certified actions maintains risk within the prescribed

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tolerance, while the certified action set remains nonempty at every subsequent decision step. We then specialize the framework to stochastic discrete-time fractional-order dynamical (FoD) systems, whose risk quantification and safety certification are computationally prohibitive with infinite state history (Monje et al., 2010; Gomoyunov, 2019). To enable certification using limited history, we introduce a finite-memory surrogate and decompose its deviation from the true system into truncation error and stochastic deviation, as illustrated in Figure 2. We derive conditional bounds on both components and use them to construct the one-step lower-bound operators required for trajectory-safety certification. Finally, we demonstrate the effectiveness and merits of the proposed framework through numerical simulations.

Contributions. In short, the main contributions of this paper are: (i) We develop a safety certification framework for stochastic non-Markovian systems that uses finite certificate states and certified one-step lower bounds to account for all full histories consistent with the retained information. (ii) We construct Bellman-like lower value and action-value functions and derive an online certified action set that guarantees the prescribed safety requirement while remaining nonempty at every decision step, thereby ensuring persistent feasibility. (iii) We specialize the framework to stochastic FoD systems by introducing a finite-memory surrogate, decomposing its deviation from the true system into truncation error and stochastic deviation, and deriving probabilistic bounds on both components to construct a certificate for trajectory safety. Beyond long-range-memory systems, the proposed lower-bound recursion and safety-filter construction also apply to settings in which the exact transition law or risk is unavailable, providing a general approach for certification under model uncertainty.

2 Related Work

Our work connects online stochastic safety certificates with safety analysis for history-dependent systems, using fractional-order dynamics as a concrete example.

Online and stochastic safety certificates. For Markovian systems, control barrier functions and modular safety filters modify nominal control inputs to enforce online constraints (Ames et al., 2019; Hsu et al., 2024; Wabersich & Zeilinger, 2021). Stochastic barriers and probabilistic predictive certification provide safety guarantees under uncertainty (Clark, 2020; Santoyo et al., 2019; Cosner et al., 2023; Wabersich et al., 2021). Predictive barrier constructions also address feasibility and recovery of the safety layer (Wabersich & Zeilinger, 2023). Reachability and value-function approaches connect risk evaluation to barrier-style action constraints (Choi et al., 2021; Tan et al., 2023). Barrier functionals and methods for deterministic and stochastic delay systems address history dependence by constructing barrier certificates or functionals over the delayed state history (Prajna & Jadbabaie, 2005; Kiss et al., 2023; Pan et al., 2024). Our framework extends this perspective to certifiers whose retained statistic need not be Markov, using conditional lower bounds to preserve both safety and persistent feasibility.

Safety for systems with memory. Barrier certificates based on state-history functionals enable safety verification for time-delay systems (Prajna & Jadbabaie, 2005), and control barrier functionals support online safety filtering with state-history feedback (Kiss et al., 2023). Razumikhin and Krasovskii approaches combine safety with stabilization for nonlinear delay systems (Ren et al., 2022), while stochastic barrier-Krasovskii functionals address safe stabilization under both noise and state delays (Pan et al., 2024). We consider the additional restriction that the certifier retains only a recursively updated history summary. Our conditional one-step bounds remain valid for every full history compatible with this summary, accounting for the influence of discarded information.

Fractional-order dynamics and control. FoD systems capture long-range dependence (Monje et al., 2010; Podlubny, 1994), which has been documented in brain activity (Lundstrom et al., 2008), heart-rate variability (Ivanov et al., 1999), and large language models (Alabdulmohsin & Steiner, 2025). Existing control approaches include short-memory approximations (Deng, 2007), history-dependent dynamic programming (Gomoyunov, 2019), and finite-dimensional approximations with error bounds for recursively feasible model predictive control (Sopasakis & Sarimveis, 2016). Caputo barrier functions provide safety conditions for classes of fractional-order systems (Zhu et al., 2025). Our stochastic discrete-time FoD specialization combines a finite-memory surrogate with conditional truncation bounds and stochastic deviation bounds derived using the averaged moment generating function (AMGF) approach of Jafarpour et al. (2024).

3 PROBLEM FORMULATION

We formulate the problem for controlled stochastic systems that admit certified conditional lower bounds, including systems with history-dependent dynamics.

3.1 SYSTEM, DESIGN REQUIREMENT, AND CERTIFICATE

Consider a finite horizon $\mathcal{T} = \{0, \dots, T\}$, a controlled process $X_t \in \mathcal{X}$, and actions $U_t \in \mathcal{U}$, where $\mathcal{X}$ is a measurable state space and $\mathcal{U}$ is finite and nonempty. Let Y_t denote the full history available before choosing U_t, and set $\mathcal{I}_t = \sigma(Y_t)$. For fully observed systems, $Y_t = (X_{0:t}, U_{0:t-1})$ for $t \geq 1$. The general history-dependent dynamics take the form

$$X_{t+1} = F_t(Y_t, U_t, W_t), \qquad t = 0, \dots, T-1, \tag{1}$$

where F_t is a measurable state transition function and W_t is a disturbance. Its conditional distribution may depend on the history and current action. We denote by $\mu_t(u \mid Y_t)$ the actual online control policy and by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ its induced probability and expectation, respectively. This policy may depend on the full history and need not coincide with the reference policy π used to compute the certificate. For each $t \in \mathcal{T}$, let $\mathcal{Z}_t$ be a measurable certificate-state space, and let κ_t be a measurable map from the available histories at time t into $\mathcal{Z}_t$. The statistic $\Upsilon_t = \kappa_t(Y_t) \in \mathcal{Z}_t$ available to the certificate is called the certificate state.

For the fixed horizon T, prescribe a measurable terminal history payoff $\Theta_T(Y_T) \in [0, 1]$. We require a measurable $G : \mathcal{Z}_T \to [0, 1]$ satisfying $\Theta_T(h) = G(\kappa_T(h))$ for every compatible terminal history h. The design requirement is to ensure

$$\mathbb{E}_\mu[G(\Upsilon_T) \mid \Upsilon_0] \geq \eta, \qquad \eta \in [0, 1]. \tag{2}$$

Thus, G is a single terminal map for the chosen horizon. Indicator-valued payoffs express event-probability requirements, including the following two examples.

Trajectory safety. For trajectory safety, let $\phi : \mathcal{X} \to \mathbb{R}$ be measurable and define the safe set $\mathcal{C} = \{x \in \mathcal{X} : \phi(x) \geq 0\}$. Include in Υ_t the absorbing safety flag, updated for every $t < T$:

$$C_0 = \mathbf{1}\{X_0 \in \mathcal{C}\}, \qquad C_{t+1} = C_t \mathbf{1}\{X_{t+1} \in \mathcal{C}\}. \tag{3}$$

Then, $G(\Upsilon_T) = C_T = \mathbf{1}\{X_k \in \mathcal{C},\ 0 \leq k \leq T\}$, so (2) becomes the requirement $\mathbb{P}_\mu(X_k \in \mathcal{C} \text{ for all } 0 \leq k \leq T \mid \Upsilon_0) \geq \eta$. Per-step constraints $\mathbb{P}_\mu(X_k \in \mathcal{C}) \geq \eta$ would instead allow the violation probabilities of different steps to accumulate (Santoyo et al., 2019).

Finite-time target reachability. For a measurable target $\mathcal{D} \subseteq \mathcal{X}$, replace the safety flag by the absorbing hit flag, updated for every $t < T$:

$$B_0 = \mathbf{1}\{X_0 \in \mathcal{D}\}, \qquad B_{t+1} = \max\{B_t, \mathbf{1}\{X_{t+1} \in \mathcal{D}\}\}.$$

Include B_t in Υ_t and set $G(\Upsilon_T) = B_T$. Then (2) requires $\mathbb{P}_\mu(\exists k \in \{0, \dots, T\} : X_k \in \mathcal{D} \mid \Upsilon_0) \geq \eta$. This means reaching the target by time T, including at time zero. Keeping both flags and using $G(\Upsilon_T) = C_T B_T$ expresses reaching the target by T while remaining safe throughout the horizon.

Certificate requirements. We seek sets of certified actions $\Gamma_t(\Upsilon_t) \subseteq \mathcal{U}$ such that every policy supported on these sets satisfies (2) whenever the initial-value condition in Section 4 holds. These sets must be *persistently feasible*: they remain nonempty at every decision time whenever all preceding actions were certified. The certificate evaluates the model using only (t, Υ_t, u), while the nominal controller and performance cost may depend on the full history. The certificate state must admit a known recursive update $\Upsilon_{t+1} = \Phi_t(\Upsilon_t, U_t, O_{t+1})$, where O_{t+1} denotes the newly available observation. For fully observed systems, we may take $O_{t+1} = X_{t+1}$.

3.2 CERTIFIED LOWER BOUNDS

The information retained in Υ_t may be insufficient to reconstruct the full transition law. We therefore use lower bounds that hold uniformly over the histories represented by the same statistic. For each $t < T$ and $\upsilon \in \mathcal{Z}_t$, let $\mathcal{A}_t(\upsilon) \subseteq \mathcal{U}$ be a known nonempty candidate set with measurable membership. Both the reference policy and the online filter are restricted to these candidates, with $\Gamma_t(\upsilon) \subseteq \mathcal{A}_t(\upsilon)$. Compatible histories below are those attainable under policies supported on these candidate sets.

Assumption 1 (Certified one-step lower operator)**.** The candidate operators $\mathcal{L}_t^u$ map measurable $[0,1]$-valued functions to measurable $[0,1]$-valued functions on their candidate domains. For every $t < T$, measurable $v : \mathcal{Z}_{t+1} \to [0,1]$, compatible history h with $\kappa_t(h) = v$, and $u \in \mathcal{A}_t(v)$,

$$\mathbb{E}[v(\Upsilon_{t+1}) \mid Y_t = h, U_t = u] \geq (\mathcal{L}_t^u v)(v). \tag{4}$$

The expectation uses the controlled transition law for action u, including actions assigned zero probability by a particular policy. Since the zero operator $(\mathcal{L}_t^u v)(v) \equiv 0$ satisfies Assumption 1, valid lower bounds alone do not ensure a useful certificate. The prescribed probability guarantee additionally requires the initial-value condition in Section 4.

4 General certificate framework

Given the one-step lower operators in Assumption 1, we construct certificate functions by backward recursion and use them to constrain online actions. A concrete construction of these operators is provided in Section 4.3.

4.1 Offline certificate construction

Fix a measurable reference policy $\pi_t(u \mid v)$ supported on $\mathcal{A}_t(v)$. Initialize the terminal certificate and recurse backward for $t = T-1, \ldots, 0$:

$$\underline{V}_T^\pi(v) = G(v), \tag{5}$$

$$\underline{Q}_t^\pi(v,u) = (\mathcal{L}_t^u \underline{V}_{t+1}^\pi)(v), \qquad u \in \mathcal{A}_t(v), \tag{6}$$

$$\underline{V}_t^\pi(v) = \sum\nolimits_{u \in \mathcal{A}_t(v)} \pi_t(u \mid v) \underline{Q}_t^\pi(v,u). \tag{7}$$

Equation (5) initializes the terminal certificate with the prescribed payoff G. Equation (6) computes the lower action value by applying the one-step lower operator to the next-step certificate. Equation (7) computes the current lower value by averaging the lower action values under the reference policy π. The resulting bounds hold for every compatible full history.

Algorithm 1 Offline backward computation of value lower bounds

Require: Horizon T; terminal map G; candidate sets $\mathcal{A}_t$; reference policy π supported on $\mathcal{A}_t$; lower operators $\mathcal{L}_t^u$
Ensure: Functions $\underline{V}_t^\pi$ and $\underline{Q}_t^\pi$
1: Initialize $\underline{V}_T^\pi(v) = G(v)$ for all $v \in \mathcal{Z}_T$
2: **for** $t = T-1, T-2, \ldots, 0$ **do**
3: Compute $\underline{Q}_t^\pi(v,u)$ for all $v \in \mathcal{Z}_t$ and $u \in \mathcal{A}_t(v)$
4: Compute $\underline{V}_t^\pi(v)$ for all $v \in \mathcal{Z}_t$
5: **end for**
6: **return** $\{\underline{V}_t^\pi\}_{t=0}^T$ and $\{\underline{Q}_t^\pi\}_{t=0}^{T-1}$

Algorithm 2 Online certificate filter

Require: Initial statistic Υ_0; functions $\underline{V}_t^\pi, \underline{Q}_t^\pi$; candidate sets $\mathcal{A}_t$; cost J_t; update Φ_t
Ensure: Action sequence selected from $\Gamma_t(\Upsilon_t)$
1: **for** $t = 0, 1, \ldots, T-1$ **do**
2: Form $\Gamma_t(\Upsilon_t)$ using (8)
3: Choose $U_t \in \arg\min_{u \in \Gamma_t(\Upsilon_t)} J_t(Y_t, u)$
4: Apply U_t and obtain the next observations O_{t+1}
5: $\Upsilon_{t+1} \leftarrow \Phi_t(\Upsilon_t, U_t, O_{t+1})$
6: **end for**

4.2 Online filtering and guarantees

The online filter restricts action selection to

$$\Gamma_t(v) = \left\{ u \in \mathcal{A}_t(v) : \underline{Q}_t^\pi(v,u) \geq \underline{V}_t^\pi(v) \right\}. \tag{8}$$

The following theorem establishes persistent feasibility and the terminal guarantee for every policy selecting actions from these sets.

Theorem 1 (Terminal guarantee and persistent feasibility)**.** *Under Assumption 1, let π be supported on $\mathcal{A}_t(\upsilon)$ and let the certificate functions be defined by* (5)–(7)*. Any online policy μ satisfying $\sum_{u\in\Gamma_t(\Upsilon_t)} \mu_t(u \mid Y_t) = 1$ for every $t < T$ obeys*

$$\mathbb{E}_\mu[G(\Upsilon_T) \mid \Upsilon_0] \geq \underline{V}_0^\pi(\Upsilon_0). \tag{9}$$

Consequently, the initial-value condition $\underline{V}_0^\pi(\Upsilon_0) \geq \eta$ implies (2)*. Moreover, $\Gamma_t(\upsilon)$ is nonempty at every $t < T$ and certificate state $\upsilon \in \mathcal{Z}_t$.*

Nonemptiness follows because $\underline{V}_t^\pi(\upsilon)$ is a convex combination of the candidate action values $\underline{Q}_t^\pi(\upsilon, u)$. Under certified actions, Assumption 1 makes $\{\underline{V}_t^\pi(\Upsilon_t)\}_{t=0}^T$ a submartingale with respect to the full-history filtration, yielding the terminal bound. The full proof is given in Appendix B.1.

To select an action, let $J_t(Y_t, u)$ be a measurable, finite-valued performance cost, such as deviation from a nominal action $N_t(Y_t)$. The filter chooses

$$U_t \in \arg\min_{u\in\Gamma_t(\Upsilon_t)} J_t(Y_t, u). \tag{10}$$

Any measurable selection from these minimizers defines an online policy covered by Theorem 1. The offline recursion and online filter are summarized in Algorithms 1 and 2.

4.3 Constructing lower bounds

The following constructions apply to candidate actions $u \in \mathcal{A}_t(\upsilon)$. Suppose a nonempty ambiguity set $\mathfrak{P}_t(\upsilon, u)$ of probability measures on $\mathcal{Z}_{t+1}$ contains the controlled successor law of Υ_{t+1} under action u for every compatible history h with $\kappa_t(h) = \upsilon$. Then

$$(\mathcal{L}_t^u v)(\upsilon) = \inf_{P\in\mathfrak{P}_t(\upsilon,u)} \int v(\upsilon')\, P(d\upsilon')$$

provides a lower bound on the conditional expectation.

Alternatively, a lower operator can be constructed from successor sets with certified conditional coverage. Let $D_t \subseteq [0, 1)$ be a nonempty set of confidence parameters. For each $\upsilon \in \mathcal{Z}_t$, $u \in \mathcal{A}_t(\upsilon)$, and $\delta \in D_t$, suppose a nonempty measurable set $\mathcal{R}_t(\upsilon, u; \delta) \subseteq \mathcal{Z}_{t+1}$ satisfies

$$\mathbb{P}(\Upsilon_{t+1} \in \mathcal{R}_t(\upsilon, u; \delta) \mid Y_t = h, U_t = u) \geq 1 - \delta$$

for every compatible history h with $\kappa_t(h) = \upsilon$. Define

$$(\mathcal{L}_t^u v)(\upsilon) = \sup_{\delta\in D_t} (1 - \delta) \inf_{\upsilon'\in\mathcal{R}_t(\upsilon,u;\delta)} v(\upsilon'). \tag{11}$$

For each δ, nonnegativity of v gives $\mathbb{E}[v(\Upsilon_{t+1}) \mid Y_t = h, U_t = u] \geq (1 - \delta) \inf_{\upsilon'\in\mathcal{R}_t(\upsilon,u;\delta)} v(\upsilon')$. Taking the supremum over δ establishes (4). Both constructions are $[0, 1]$-valued when v is. We require the resulting maps to be measurable on their candidate domains, as stipulated in Assumption 1.

The successor-set construction extends to multiple confidence parameters when the resulting set has conditional coverage at least $1 - \sum_i \delta_i$, with $\sum_i \delta_i < 1$. The operator then uses the factor $1 - \sum_i \delta_i$ and takes the supremum over admissible parameter vectors. Section 5 constructs such successor sets for stochastic FoD systems using bounds on memory truncation and stochastic deviations.

5 Specialization to Fractional-Order Dynamical Systems

We specialize the general certificate framework to FoD systems. We first introduce the system and its finite-memory approximation, and then specify the trajectory-safety requirement and certificate state. Deviation bounds connect the approximation to the true system and provide the one-step lower operator.

5.1 System Model

The Grünwald–Letnikov difference operator (Grünwald, 1867; Letnikov, 1868) is a discrete counterpart of the fractional-order derivative:

$$\Delta^{\alpha} x_t := \sum\nolimits_{j=0}^{t} \psi(\alpha, j) x_{t-j}, \tag{12}$$

where $x_t \in \mathbb{R}^n$, $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]^\top \in (0,1)^n$ represents the fractional order, and $\psi(\alpha, j) \in \mathbb{R}^{n\times n}$ is a diagonal matrix defined as $\psi(\alpha, j) := \operatorname{diag}\big(\psi(\alpha_1, j), \psi(\alpha_2, j), \ldots, \psi(\alpha_n, j)\big)$ with $\psi(\alpha_i, j) := \frac{\Gamma(j-\alpha_i)}{\Gamma(-\alpha_i)\Gamma(j+1)}$ for $i = 1, 2, \ldots, n$. Here, $\Gamma(\cdot)$ denotes the gamma function.

We consider a stochastic discrete-time fractional-order dynamical system in the form of

$$\Delta^{\alpha} X_{t+1} = f(X_t) + g(X_t) U_t + W_t, \tag{13}$$

where $X_t \in \mathbb{R}^n$ denotes the state, $U_t \in \mathcal{U} \subseteq \mathbb{R}^m$ denotes the control input, and $W_t \in \mathbb{R}^n$ denotes an independent and identically distributed (i.i.d.) Gaussian noise sequence with zero mean and covariance Σ_w. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ specify the system dynamics. Using the Grünwald–Letnikov difference operator (12), we can rewrite the FoD system (13) as follows

$$X_{t+1} = f(X_t) + g(X_t) U_t - \sum\nolimits_{j=1}^{t+1} \psi(\alpha, j) X_{t+1-j} + W_t. \tag{14}$$

Equation (14) instantiates (1): the update map F_t contains the fractional-memory sum and an additive Gaussian disturbance. The noise W_t is independent of the pre-action history and of the current policy randomization. Write $U_{\max} = \max_{u\in\mathcal{U}} \|u\|$ for the finite action set introduced above. Let $S_t = [X_0^\top, X_1^\top, \ldots, X_t^\top]^\top$ denote the complete state history up to time t. We make the following assumption on the dynamics.

Assumption 2. The functions f and g in (13) are globally Lipschitz with constants $L_f, L_g > 0$, respectively, and bounded at the origin.

5.2 Finite-Memory Approximation

Fix a memory length $M \geq 1$. The full history S_t grows with time, so we retain only M states for prediction. The M-step stochastic truncated model is

$$X_{t+1}^{(M)} = f\left(X_t^{(M)}\right) + g\left(X_t^{(M)}\right) u_t - \sum\nolimits_{j=1}^{M} \psi(\alpha, j) X_{t+1-j}^{(M)} + W_t. \tag{15}$$

This stochastic truncated fractional-order dynamical system can be viewed as a noisy version of the following deterministic system

$$x_{t+1}^{(M)} = f\Big(x_t^{(M)}\Big) + g\Big(x_t^{(M)}\Big) u_t - \sum\nolimits_{j=1}^{M} \psi(\alpha, j)\, x_{t+1-j}^{(M)}. \tag{16}$$

The surrogate-state window introduced below is sufficient for the future evolution of the truncated stochastic model. States with negative indices are set to zero, consistently with the finite-start convolution.

Trajectory safety and certificate state. Trajectory safety is the problem studied for FoD systems in this paper. Using the safe set and trajectory-safety objective of Section 3.1 with $\mathcal{X} = \mathbb{R}^n$, the FoD requirement for $X_0 \in \mathcal{C}$ is

$$\mathbb{P}_{\mu}(X_t \in \mathcal{C},\ \forall t \in \{1, 2, \ldots, T\} \mid X_0) \geq \eta. \tag{17}$$

For the same memory length M, κ_t extracts the last M observed states (with zero padding) and the recursively updated flag: $\Upsilon_t = (\bar{S}_t^{(M)}, C_t)$, where $\bar{S}_t^{(M)} = [X_t^\top, \ldots, X_{t-M+1}^\top]^\top$. With $O_{t+1} = X_{t+1}$ and $s = [(s^1)^\top, \ldots, (s^M)^\top]^\top$, the update is

$$\Phi_t((s,c), u, x') = \Big(\big[(x')^\top, (s^1)^\top, \ldots, (s^{M-1})^\top\big]^\top, c\mathbf{1}\{x' \in \mathcal{C}\}\Big).$$

It prepends the new observation, drops the oldest state, and updates the flag.

Construction overview. The trajectory-safety flag is the absorbing flag in (3), with $C_0 = 1$ for the prescribed safe initial state. We next bound the memory truncation and stochastic deviations separately, then combine them into a conditional successor tube. Section 5.4 substitutes that tube into the general backward recursion.

5.3 SEPARATION STRATEGY

The trajectories of the stochastic FoD system (14) are generated by both deterministic inputs and stochastic disturbances. Due to the decaying influence of past states inherent to fractional-order dynamics, the contribution of remote history decreases over time. At a restart time $t_0 < T$, we condition on $X_{0:t_0} = x_{0:t_0}$ and initialize both truncated models with the last M observed states. All three systems (14)–(16) use a common input sequence fixed at t_0, and the true and truncated stochastic systems share the same future noise. The deviation from the deterministic surrogate is bounded by the sum of truncation error and stochastic deviation, as illustrated in Figure 2:

$$\max_{t_0\le t\le T}\|X_t - x_t^{(M)}\| \le \underbrace{\max_{t_0\le t\le T}\|X_t - X_t^{(M)}\|}_{\text{truncation error}} + \underbrace{\max_{t_0\le t\le T}\|X_t^{(M)} - x_t^{(M)}\|}_{\text{stochastic deviation}}.$$

To bound the truncation error and stochastic deviation, define the augmented state $S_t^{(M)} := \big[(X_t^{(M)})^\top, \dots, (X_{t-M+1}^{(M)})^\top\big]^\top \in \mathbb{R}^{Mn}$, and let $s_t^{(M)}$ denote the corresponding augmentation of $x_t^{(M)}$. The truncated stochastic system (15) has the Markovian representation $S_{t+1}^{(M)} = F_M(S_t^{(M)}, u_t) + G_M W_t$. Its deterministic counterpart satisfies $s_{t+1}^{(M)} = F_M(s_t^{(M)}, u_t)$. Here, $F_M : \mathbb{R}^{Mn} \times \mathbb{R}^m \to \mathbb{R}^{Mn}$ is defined by $F_M(S,u) = \big[(S^0)^\top, (S^1)^\top, \dots, (S^{M-1})^\top\big]^\top$, where S^j denotes the j-th block of S for $j = 1, \dots, M$, and $S^0 = f(S^1) + g(S^1)u - \sum_{j=1}^{M}\psi(\alpha, j)S^j$. The noise enters through $G_M = [I_n\ 0\ \cdots\ 0]^\top \in \mathbb{R}^{Mn\times n}$. Projection onto the first block gives $\|X_t^{(M)} - x_t^{(M)}\| \le \|S_t^{(M)} - s_t^{(M)}\|$. The block structure and Lipschitz continuity of f and g yield a Lipschitz bound for F_M that controls the propagation of both deviation terms; see Lemma 3 for the statement and Appendix B.2 for the proof.

Probabilistic bound for truncation error. We couple the true and truncated stochastic systems using the same noise, so their deviation is driven only by the discarded memory tail. The polynomial decay of the fractional coefficients, together with a uniform bound on the true trajectory, yields the truncation error bound.

Lemma 1 (Probabilistic truncation bound)**.** *Under Assumption 2, fix $t_0 \in \{0, \dots, T\}$ with $M \le t_0$ and condition on $X_{0:t_0} = x_{0:t_0}$. Initialize the M-step stochastic truncated system* (15) *with the true last M states and drive it with the same future inputs and noises as the true FoD system* (14)*. Then, for any $\delta_1 \in (0,1)$,*

$$\mathbb{P}\left(\max_{t_0\le t\le T}\|X_t - X_t^{(M)}\| \le r_{\text{tail}}\big(T, M, \delta_1\big)\,\middle|\, X_{0:t_0} = x_{0:t_0}\right) \ge 1 - \delta_1, \tag{18}$$

where $r_{\text{tail}}\big(T, M, \delta_1\big) \propto M^{-\alpha_{\min}}\sqrt{\log\frac{T-t_0}{\delta_1}}$, up to system- and history-dependent factors.

The proof is given in Appendix B.5. Next, we bound the stochastic deviation between the truncated stochastic system (15) and the corresponding truncated deterministic system (16), and show that it scales as $\mathcal{O}\big(\sqrt{\log(1/\delta)}\big)$ under Assumption 2.

Probabilistic bound for stochastic deviation. We use the AMGF to control the stochastic deviation (Jafarpour et al., 2024). The resulting normalized AMGF process is a nonnegative supermartingale, and Doob's maximal inequality yields a uniform bound over $[t, T]$.

Lemma 2 (Probabilistic stochastic deviation bound)**.** *Under Assumption 2, fix $t_0 \in \{0, \dots, T\}$ and condition on $X_{0:t_0} = x_{0:t_0}$. Let $\{X_t^{(M)}\}_{t=t_0}^T$ and $\{x_t^{(M)}\}_{t=t_0}^T$ be the stochastic and deterministic M-truncated trajectories driven by the same future input sequence and initialized with the same augmented state at time t_0. Then, for any $\delta_2 \in (0,1)$,*

$$\mathbb{P}\left(\max_{t_0\le t\le T}\|X_t^{(M)} - x_t^{(M)}\| \le r_{\text{noise}}(T, \delta_2)\,\middle|\, X_{0:t_0} = x_{0:t_0}\right) \ge 1 - \delta_2, \tag{19}$$

where $r_{\text{noise}}(T, \delta_2) = \mathcal{O}\Big(\Gamma_{t_0,T}\sqrt{Mn + \log(\frac{1}{\delta_2})}\Big)$. Here $\Gamma_{t_0,T}$ captures the horizon-dependent propagation of stochastic disturbances.

The proof is given in Appendix B.6. Applying the union bound to (18) and (19) yields a joint trajectory deviation bound, which we formalize in Theorem 2.

Theorem 2 (Joint trajectory deviation bound)**.** *Under Assumption 2, fix $t_0 \in \{0, \dots, T\}$ with $M \le t_0$ and condition on $X_{0:t_0} = x_{0:t_0}$. Let $\{X_t\}_{t=t_0}^T$ be the true FoD trajectory and $\{x_t^{(M)}\}_{t=t_0}^T$ its deterministic M-truncated counterpart, driven by the same future input sequence. Initialize the truncated models at t_0 with the true last M states, and couple the stochastic truncated model with the true system using the same future noise. Then, for any $\delta_1, \delta_2 \in (0,1)$ with $\delta_1 + \delta_2 < 1$,*

$$\mathbb{P}\left(\max_{t_0 \le t \le T} \|X_t - x_t^{(M)}\| \le r_{\mathrm{tail}}(T, M, \delta_1) + r_{\mathrm{noise}}(T, \delta_2) \,\middle|\, X_{0:t_0} = x_{0:t_0} \right) \ge 1 - \delta_1 - \delta_2.$$

5.4 Safety Certificate for FoD Systems

We use the conditional deviation bound in Theorem 2 to construct successor sets for the one-step lower operator and specialize the certificate framework of Section 4 to FoD systems.

Certificate state and conditional initialization. We use the certificate state $\zeta_t = (\bar{S}_t^{(M)}, C_t)$, where $\bar{S}_t^{(M)}$ is the observed true-state window, distinct from the surrogate window $S_t^{(M)}$. The certificate state evolves as $\zeta_{t+1} = \Phi_t(\zeta_t, U_t, X_{t+1})$, where Φ_t is defined in Section 5.2. The terminal certificate is

$$\underline{V}_T^\pi(\zeta) = c, \qquad \zeta = (s, c). \tag{20}$$

At time t, consider $\zeta = (s, 1)$ and fix $u \in \mathcal{A}_t(\zeta)$. Initialize both truncated models with the observed window at time t, and use this same action over $[t, t+1]$. Let $\widehat{x}^{(M)}(t, s, u)$ denote the resulting deterministic successor. For $t \ge M$, restart the joint bound over this one-step interval and define

$$r(t, \zeta, u; \delta_1, \delta_2) = r_{\mathrm{tail},t}(\zeta, u; \delta_1) + r_{\mathrm{noise},t}(\zeta, u; \delta_2).$$

For use in a table indexed only by ζ, the radii must be certified upper envelopes of the corresponding one-step bounds over all full histories compatible with $\zeta = (s, 1)$. A history-dependent radius from Theorem 2 can be used by taking such an envelope. For $t < M$, no history is discarded, so use $r_{\mathrm{tail},t} = 0$ and the stochastic-deviation bound.

Successor sets and backward computation. For $\zeta = (s, 1)$ define

$$\mathcal{R}_t(\zeta, u; \delta_1, \delta_2) = \{\Phi_t(\zeta, u, x') : \|x' - \widehat{x}^{(M)}(t, s, u)\| \le r(t, \zeta, u; \delta_1, \delta_2)\}.$$

For $\zeta = (s, 0)$, set $\mathcal{R}_t(\zeta, u; \delta_1, \delta_2) = \{\Phi_t(\zeta, u, x') : x' \in \mathbb{R}^n\}$, which contains every possible successor and preserves the zero flag. We require the supplied sets to satisfy, for every compatible history and candidate action $u \in \mathcal{A}_t(\zeta)$,

$$\mathbb{P}(\zeta_{t+1} \in \mathcal{R}_t(\zeta, u; \delta_1, \delta_2) \mid Y_t, U_t = u) \ge 1 - \delta_1 - \delta_2. \tag{21}$$

For a nonempty confidence-parameter set $\Delta_t \subset \{(\delta_1, \delta_2) : \delta_i \in (0,1), \delta_1 + \delta_2 < 1\}$, substitution into (11) gives, for $u \in \mathcal{A}_t(\zeta)$,

$$\begin{aligned} \underline{Q}_t^\pi(\zeta, u) &= \sup_{(\delta_1, \delta_2) \in \Delta_t} (1 - \delta_1 - \delta_2) \inf_{\zeta' \in \mathcal{R}_t(\zeta, u; \delta_1, \delta_2)} \underline{V}_{t+1}^\pi(\zeta'), \\ \underline{V}_t^\pi(\zeta) &= \mathbb{E}_{U \sim \pi_t(\cdot|\zeta)}[\underline{Q}_t^\pi(\zeta, U)]. \end{aligned} \tag{22}$$

For $c = 0$, the absorbing flag gives $\underline{Q}_t^\pi = \underline{V}_t^\pi = 0$. The online policy selects $U_t \in \mathcal{A}_t(\zeta_t)$ satisfying

$$\underline{Q}_t^\pi(\zeta_t, U_t) \ge \underline{V}_t^\pi(\zeta_t). \tag{23}$$

By Theorem 1 with $\Upsilon_t = \zeta_t$ and $G(\zeta_T) = C_T$, we have the following corollary:

Corollary 1 (FoD safety and persistent feasibility)**.** *Suppose the supplied successor sets satisfy* (21) *on the nonempty candidate sets and π is supported on these sets. If $\underline{V}_0^\pi(\zeta_0) \ge \eta$, every online policy μ selecting $U_t \in \mathcal{A}_t(\zeta_t)$ and satisfying* (23) *for every $t < T$ guarantees* (17)*. Moreover, the certified-action set is nonempty at every subsequent certificate state.*

FoD implementation. Algorithms 3 and 4 in Appendix A give the backward bound computation and online filter for $\zeta = (s, c)$. They specialize Algorithms 1 and 2 using the supplied successor sets and the update Φ_t. A nominal controller based on the observed window is obtained by choosing $J_t(Y_t, u) = J(N(\bar{S}_t^{(M)}), u)$ in (10).

6 NUMERIC SIMULATIONS

We evaluated the proposed method through two sets of experiments on fractional-order systems with linear and nonlinear dynamics. Both systems have a five-dimensional state $X_t \in \mathbb{R}^5$ and a scalar control input $u_t \in \mathbb{R}$. For each experiment, we simulated 500 trajectories over a horizon of $T = 20$. We compare our framework with the myopically verifiable probabilistic certificate method (MCLS) (Wang et al., 2026), the Jensen-enhanced discrete-time control barrier function controller (JED) (Cosner et al., 2023), and probabilistic safety barrier certificates (PrSBC) (Luo et al., 2020). These methods were originally developed for Markovian systems. We adapt the baselines to fractional dynamics by using the observed history $H_t = (X_0, \ldots, X_t)$ as an augmented Markov state. For comparison with our method, which uses a memory window of $M = 5$, we also evaluate MCLS and PrSBC with the same window, omitting older memory terms from their predictions. All methods used the same nominal controller and disturbance realizations. The safe set is defined as $\mathcal{C} = \{x : \|x\|_\infty \leq x_{\max}\}$, where $x_{\max} > 0$ is a prescribed state bound. Safety probability at time t is estimated as the proportion of simulated trajectories that remain within $\mathcal{C}$ up to time t. Our method achieves higher safety probabilities than the baselines in both experiments. Detailed experimental settings are provided in Appendix C.

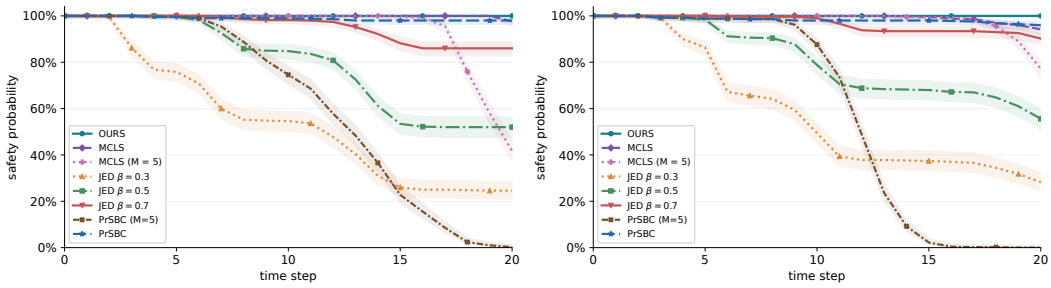

Figure 1: Safety probabilities for linear (left) and nonlinear (right) fractional-order systems. Shaded regions indicate 95% confidence intervals.

Linear fractional-order dynamics. We evaluate the proposed safety certificate framework on a five-dimensional fractional-order linear time-invariant (LTI) system (44) with a scalar control input. As shown in Figure 1, all 500 trajectories under our method remain safe using a memory window of $M = 5$. MCLS and PrSBC with full history achieve empirical safety rates of 97.6% and 98.0%, respectively, both above the 95% safety probability threshold. Their lower safety rates therefore do not indicate a failure to meet this requirement. In contrast, restricting these baselines to the same memory window as our method reduces their empirical safety rates to 41.8% and 0.2%, respectively, both below the threshold. This comparison illustrates the importance of accounting for discarded memory when assessing safety using a limited history window. To assess the tradeoff between safety and control performance, we measure intervention by the squared deviation from the nominal input, averaged over all trajectories and decision times. Our method incurs a cost of 62.16, compared with 71.11 for MCLS and 35.01 for PrSBC. Thus, our method achieves higher empirical safety with less intervention than MCLS, but requires greater intervention than PrSBC. For online computation, the measured median decision time is approximately 0.050 ms for our method, 1.110 ms for MCLS, and 0.076 ms for PrSBC; JED with $\beta = 0.7$ is faster at 0.027 ms.

Nonlinear fractional-order dynamics. We further evaluate our method on the nonlinear fractional-order system (45). Figure 1 shows that all 500 trajectories under our method remain safe. PrSBC with full history achieves an empirical safety rate of 96.0%, above the prescribed threshold $\eta = 0.95$, whereas MCLS achieves 94.2%, slightly below the threshold. Restricting the baselines to the same memory window as our method reduces their empirical safety rates to 77.2% for MCLS and 0% for PrSBC, both below the target. In terms of control performance, our method incurs an average intervention cost of 66.15, compared with 65.51 for MCLS and 38.86 for PrSBC with full history. Thus, our method achieves higher empirical safety with an intervention cost close to that of MCLS, while requiring greater intervention than PrSBC. The median online decision time is 0.100 ms for our method, compared with 49.460 ms for MCLS and 0.212 ms for PrSBC. JED with $\beta = 0.7$ is faster at 0.043 ms, but its empirical safety rate of 90.2% falls below the target.

AI USE STATEMENT

We used generative AI tools to improve the clarity and language of the manuscript, assist in drafting and checking mathematical proofs, support experimental code development, and identify relevant literature. We take full responsibility for the accuracy and integrity of the final manuscript, including its mathematical claims, proofs, code, and references.

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A ALGORITHMS

The following algorithms implement the FoD certificate with the observed window and absorbing safety flag as the certificate state.

Algorithm 3 FoD backward computation of value lower bounds

Require:
Horizon T; certificate-state domains $\{\mathcal{Z}_t\}_{t=0}^{T}$ with $\zeta = (s, c) \in \mathbb{R}^{nM} \times \{0, 1\}$; candidate sets $\mathcal{A}_t(\zeta) \subseteq \mathcal{U}$; reference policy π supported on these sets; confidence-parameter sets $\{\Delta_t\}_{t=0}^{T-1}$; successor sets $\mathcal{R}_t(\zeta, u; \delta_1, \delta_2)$.

Ensure: Functions $\{\underline{V}_t^{\pi}\}_{t=0}^{T}$ and $\{\underline{Q}_t^{\pi}\}_{t=0}^{T-1}$ representing the certificate functions.

$\underline{V}_T^{\pi}(\zeta) \leftarrow c$ for every $\zeta = (s, c) \in \mathcal{Z}_T$
for $t = T-1, T-2, \ldots, 0$ **do**
  for all $\zeta \in \mathcal{Z}_t$ and $u \in \mathcal{A}_t(\zeta)$ **do**
    $\underline{Q}_t^{\pi}(\zeta, u) \leftarrow \sup_{(\delta_1,\delta_2)\in\Delta_t} (1 - \delta_1 - \delta_2) \inf_{\zeta' \in \mathcal{R}_t(\zeta,u;\delta_1,\delta_2)} \underline{V}_{t+1}^{\pi}(\zeta')$
  end for
  for all $\zeta \in \mathcal{Z}_t$ **do**
    $\underline{V}_t^{\pi}(\zeta) \leftarrow \mathbb{E}_{U\sim\pi_t(\cdot|\zeta)}[\underline{Q}_t^{\pi}(\zeta, U)]$
  end for
end for

Algorithm 4 Online dynamically consistent safety filter

Require:
Horizon T; candidate sets $\mathcal{A}_t$; functions $\underline{Q}_t^{\pi}, \underline{V}_t^{\pi}$; nominal controller N; finite deviation cost J; initial state $X_0 \in \mathcal{C}$.

Ensure: Action sequence selected from $\Gamma_t(\zeta_t)$.

Initialize $\bar{S}_0^{(M)}$ with X_0 and zero padding, and set $\zeta_0 = (\bar{S}_0^{(M)}, 1)$
for $t = 0, 1, \ldots, T-1$ **do**
  $\Gamma_t(\zeta_t) \leftarrow \{u \in \mathcal{A}_t(\zeta_t) : \underline{Q}_t^{\pi}(\zeta_t, u) \geq \underline{V}_t^{\pi}(\zeta_t)\}$
  Choose $u_t \in \arg\min_{u\in\Gamma_t(\zeta_t)} J(N(\bar{S}_t^{(M)}), u)$
  Apply u_t to the true system and observe X_{t+1}
  $\zeta_{t+1} \leftarrow \Phi_t(\zeta_t, u_t, X_{t+1})$
end for

Limitations. The framework requires certified conditional one-step lower bounds, which may be difficult to construct for complex systems under limited history information. Conservative bounds and the choice of reference policy may prevent certification of an otherwise achievable safety level. Offline certificate construction can also become computationally expensive as the retained-state dimension, number of candidate actions, and horizon increase.

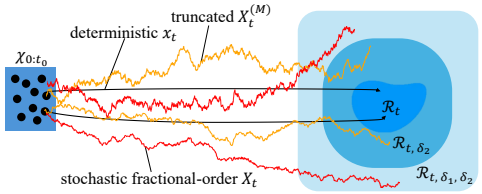

Figure 2: An illustration of separation strategy.

B THEOREMS AND PROOFS

B.1 PROOF OF THEOREM 1

Proof. First, G is measurable and $[0, 1]$-valued. If $\underline{V}_{t+1}^{\pi}$ has these properties, so does each candidate $\underline{Q}_t^{\pi}$ by Assumption 1; the finite reference-policy average has the same properties. Backward induction therefore justifies applying the assumption at every step.

The reference policy is supported on the finite nonempty candidate set, hence

$$\underline{V}_t^{\pi}(v) = \sum_{u\in\mathcal{A}_t(v)} \pi_t(u \mid v)\underline{Q}_t^{\pi}(v, u) \leq \max_{u\in\mathcal{A}_t(v)} \underline{Q}_t^{\pi}(v, u).$$

A maximizing action belongs to $\Gamma_t(\upsilon)$, proving nonemptiness at every time and certificate state. In particular this holds at subsequent states reached by the online filter.

Now fix an actual online policy μ supported on $\Gamma_t(\Upsilon_t)$. For a realized history h, let $\upsilon = \kappa_t(h)$ and $\mathcal{S}_t^\mu(h) = \{u \in \mathcal{U} : \mu_t(u \mid h) > 0\}$, we have $\mathcal{S}_t^\mu(h) \subseteq \Gamma_t(\upsilon) \subseteq \mathcal{A}_t(\upsilon)$. For every $u \in \mathcal{S}_t^\mu(h)$, Assumption 1, (6), and (8) give

$$\mathbb{E}_\mu[\underline{V}_{t+1}^\pi(\Upsilon_{t+1}) \mid Y_t = h, U_t = u] \geq \underline{Q}_t^\pi(\upsilon, u) \geq \underline{V}_t^\pi(\upsilon).$$

Averaging only over the actual policy's supported actions yields

$$\begin{aligned}\mathbb{E}_\mu[\underline{V}_{t+1}^\pi(\Upsilon_{t+1}) \mid Y_t = h] &= \sum_{u \in \mathcal{S}_t^\mu(h)} \mu_t(u \mid h) \mathbb{E}_\mu[\underline{V}_{t+1}^\pi(\Upsilon_{t+1}) \mid Y_t = h, U_t = u] \\ &\geq \sum_{u \in \mathcal{S}_t^\mu(h)} \mu_t(u \mid h) \underline{Q}_t^\pi(\upsilon, u) \geq \underline{V}_t^\pi(\upsilon).\end{aligned}$$

Since $\mathcal{I}_t = \sigma(Y_t)$, the process $\underline{V}_t^\pi(\Upsilon_t)$ is a bounded submartingale under μ. Moreover $\sigma(\Upsilon_0) \subseteq \mathcal{I}_t$, so the tower property gives

$$\begin{aligned}\mathbb{E}_\mu[\underline{V}_{t+1}^\pi(\Upsilon_{t+1}) \mid \Upsilon_0] &= \mathbb{E}_\mu[\mathbb{E}_\mu[\underline{V}_{t+1}^\pi(\Upsilon_{t+1}) \mid \mathcal{I}_t] \mid \Upsilon_0] \\ &\geq \mathbb{E}_\mu[\underline{V}_t^\pi(\Upsilon_t) \mid \Upsilon_0].\end{aligned}$$

Iterating from $t = 0$ to $T - 1$ and using $\underline{V}_T^\pi = G$ gives (9). □

B.2 Proof of Lemma 3

Lemma 3. *Under Assumption 2, there exists c_t such that for all S, s along the trajectories, the following holds*

$$\|F_M(S, u_t) - F_M(s, u_t)\| \leq e^{c_t} \|S - s\|.$$

Proof. Write the augmented states in n-dimensional blocks:

$$S = \begin{bmatrix} S(1) \\ S(2) \\ \vdots \\ S(M) \end{bmatrix}, \qquad s = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(M) \end{bmatrix}, \qquad \Delta_j := S(j) - s(j) \in \mathbb{R}^n.$$

By the definition of F_M, we have

$$F_M(S, u_t) - F_M(s, u_t) = \begin{bmatrix} \Delta_{\text{top}} \\ \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{M-1} \end{bmatrix},$$

where

$$\Delta_{\text{top}} = \big(f(S(1)) - f(s(1))\big) + \big(g(S(1)) - g(s(1))\big)u_t - \sum_{j=1}^{M} \psi(\alpha, j)\, \Delta_j.$$

By Assumption 2,

$$\|f(S(1)) - f(s(1))\| \leq L_f \|\Delta_1\|, \qquad \|g(S(1)) - g(s(1))\| \leq L_g \|\Delta_1\|.$$

Therefore,

$$\begin{aligned}\|\Delta_{\text{top}}\| &\leq \|f(S(1)) - f(s(1))\| + \|(g(S(1)) - g(s(1)))u_t\| + \left\| \sum_{j=1}^{M} \psi(\alpha, j) \Delta_j \right\| \\ &\leq L_f \|\Delta_1\| + L_g \|u_t\| \, \|\Delta_1\| + \sum_{j=1}^{M} \|\psi(\alpha, j)\| \, \|\Delta_j\| \\ &= \big(L_f + L_g \|u_t\| + |\psi(\alpha, 1)|\big) \|\Delta_1\| + \sum_{j=2}^{M} \|\psi(\alpha, j)\| \, \|\Delta_j\|.\end{aligned}$$

Let

$$w_1 := L_f + L_g\|u_t\| + \|\psi(\alpha, 1)\|, \qquad w_j := \|\psi(\alpha, j)\| \ \ (j = 2, \dots, M).$$

By Cauchy–Schwarz,

$$\|\Delta_{\text{top}}\| \le \sqrt{\sum_{j=1}^{M} w_j^2}\ \sqrt{\sum_{j=1}^{M} \|\Delta_j\|^2} = L_{\text{top}}(t)\, \|S - s\|,$$

where

$$L_{\text{top}}(t) := \sqrt{\big(L_f + L_g\|u_t\| + \|\psi(\alpha, 1)\|\big)^2 + \sum_{j=2}^{M} \|\psi(\alpha, j)\|^2}.$$

Finally, using the block form of $F_M(S, u_t) - F_M(s, u_t)$,

$$\begin{aligned}\|F_M(S, u_t) - F_M(s, u_t)\|^2 &= \|\Delta_{\text{top}}\|^2 + \sum_{j=1}^{M-1} \|\Delta_j\|^2 \\ &\le \|\Delta_{\text{top}}\|^2 + \sum_{j=1}^{M} \|\Delta_j\|^2 \\ &\le \big(L_{\text{top}}(t)^2 + 1\big)\|S - s\|^2.\end{aligned}$$

Taking square roots gives

$$\|F_M(S, u_t) - F_M(s, u_t)\| \le \sqrt{1 + L_{\text{top}}(t)^2}\ \|S - s\|.$$

Define

$$e^{c_t} := \sqrt{1 + L_{\text{top}}(t)^2}\ \to\ c_t = \tfrac{1}{2}\log\big(1 + L_{\text{top}}(t)^2\big),$$

which yields Lemma 3. □

B.3 Proof of Lemma 6

Lemma 4. *Let $X \in \mathbb{R}^n$ be a sub-Gaussian random vector with variance proxy σ^2. Then for any $\delta \in (0, 1)$ and any $\varepsilon \in (0, 1)$,*

$$\|X\| \le \sigma\sqrt{\varepsilon_1 n + \varepsilon_2 \log(1/\delta)}$$

holds with probability at least $1 - \delta$, where

$$\varepsilon_1 = \frac{2\log(1 + 2/\varepsilon)}{(1 - \varepsilon)^2}, \qquad \varepsilon_2 = \frac{2}{(1 - \varepsilon)^2}.$$

Lemma 5. *Under Assumption 2, for any initial condition x_0 and time horizon T, the following holds with probability at least $1 - \delta$*

$$\max_{0 \le t \le T} \|X_t\|_2\ \le\ H_T^{\max}\, \|x_0\|_2 + H_T^{\Sigma}\, b + H_T^{\Sigma}\, \sigma_w \sqrt{\varepsilon_1\, n\ +\ \varepsilon_2\, \log\frac{T}{\delta}}.$$

Here $b := b_f + U_{\max} b_g$, $H_T^{\max} := \max_{0 \le t \le T} H_t$, and $H_T^{\Sigma} := \sum_{k=0}^{T-1} H_k$, where $\{H_t\}_{t \ge 0}$ is defined recursively by

$$H_0 = 1, \qquad H_{t+1} = \sum_{k=0}^{t} \beta_k\, H_{t-k}, \quad t \ge 0.$$

The coefficients $\{\beta_k\}_{k \ge 0}$ are given by

$$\beta_0 := a + \|\psi(\alpha, 1)\|_2, \qquad \beta_k := \|\psi(\alpha, k + 1)\|_2, \quad k \ge 1,$$

with $a := L_f + U_{\max} L_g$.

Proof. We first prove Lemma 5. We want to find an explicit function $R(\delta)$ such that

$$\mathbb{P}\left(\max_{0\le t\le T}\|X_t\|_2 \le R(\delta)\right) \ge 1-\delta.$$

By Assumption 2, the following holds for any x,

$$\|f(x)\|_2 \le \|f(x)-f(0)\|_2 + \|f(0)\|_2 \le L_f\|x\|_2 + b_f,$$
$$\|g(x)\|_2 \le \|g(x)-g(0)\|_2 + \|g(0)\|_2 \le L_g\|x\|_2 + b_g.$$

Thus,

$$\|g(X_t)u_t\|_2 \le \|g(X_t)\|_2\,\|u_t\|_2 \le (L_g\|X_t\|_2 + b_g)\,U_{max}.$$

Let

$$a := L_f + U_{max}L_g, \qquad b := b_f + U_{max}b_g.$$

Then

$$\|f(X_t) + g(X_t)u_t\|_2 \le a\|X_t\|_2 + b.$$

Taking the ℓ_2-norm and applying the triangle inequality to the stochastic fractional-order system (14), we have

$$\|X_{t+1}\|_2 \le \|f(X_t)+g(X_t)u_t\|_2 + \left\|\sum_{j=1}^{t+1}\psi_j X_{t+1-j}\right\|_2 + \|W_t\|_2,$$

where $\psi_j := \psi(\alpha, j)$. Let $s_t := \|X_t\|_2$ and we have

$$s_{t+1} \le as_t + b + \sum_{j=1}^{t+1}\|\psi_j\|_2\, s_{t+1-j} + \|W_t\|_2.$$

Separate the $j=1$ term:

$$s_{t+1} \le \big(a+\|\psi_1\|_2\big)s_t + \sum_{j=2}^{t+1}\|\psi_j\|_2\, s_{t+1-j} + b + \|W_t\|_2.$$

Define the nonnegative coefficients

$$\beta_0 := a + \|\psi_1\|_2, \qquad \beta_k := \|\psi_{k+1}\|_2, \quad k \ge 1.$$

Then

$$s_{t+1} \le \sum_{k=0}^{t}\beta_k\, s_{t-k} + b + \|W_t\|_2. \tag{24}$$

Define H_t by

$$H_0 = 1, \qquad H_{t+1} = \sum_{k=0}^{t}\beta_k\, H_{t-k}, \quad t \ge 0. \tag{25}$$

Because all $\beta_k \ge 0$, we have $H_t \ge 0$ for all t.
We claim that for all $t \ge 0$,

$$s_t \;\le\; H_t s_0 \;+\; \sum_{k=0}^{t-1} H_{t-1-k}\,\big(b + \|W_k\|_2\big). \tag{26}$$

We prove (26) by induction as follows.
Base case $t = 0$**.** The right-hand side equals $H_0 s_0 = s_0$, so equality holds.
Inductive step. Assume (26) holds for all indices $\le t$. Using (24), we have

$$s_{t+1} \;\le\; \sum_{k=0}^{t}\beta_k s_{t-k} \;+\; b \;+\; \|W_t\|_2.$$

Apply the induction hypothesis (26) to each s_{t-k}

$$s_{t-k} \;\le\; H_{t-k}s_0 \;+\; \sum_{r=0}^{t-k-1} H_{t-k-1-r}\,\big(b + \|W_r\|_2\big).$$

Substituting it into (24), we obtain

$$s_{t+1} \le \Big(\sum_{k=0}^{t} \beta_k H_{t-k}\Big) s_0 + \sum_{k=0}^{t} \beta_k \sum_{r=0}^{t-k-1} H_{t-k-1-r} \big(b + \|W_r\|_2\big) + b + \|W_t\|_2. \tag{27}$$

By definition (25) $\sum_{k=0}^{t} \beta_k H_{t-k} = H_{t+1}$, we can rewrite the double sum in (27) as

$$\sum_{k=0}^{t} \beta_k \sum_{r=0}^{t-k-1} H_{t-k-1-r} \big(b + \|W_r\|_2\big) = \sum_{r=0}^{t-1} H_{t-r} \big(b + \|W_r\|_2\big).$$

Therefore,

$$s_{t+1} \;\le\; H_{t+1} s_0 \;+\; \sum_{r=0}^{t-1} H_{t-r} \big(b + \|W_r\|_2\big) \;+\; b + \|W_t\|_2 = H_{t+1} s_0 + \sum_{r=0}^{t} H_{t-r} \big(b + \|W_r\|_2\big),$$

which is exactly (26) with $t \mapsto t+1$.

Define

$$H_T^{\max} \;:=\; \max_{0 \le t \le T} H_t, \qquad H_T^{\Sigma} \;:=\; \sum_{k=0}^{T-1} H_k.$$

From (26), for any $0 \le t \le T$, we have

$$\begin{aligned} s_t &\le H_t s_0 + \sum_{k=0}^{t-1} H_{t-1-k} \big(b + \|W_k\|_2\big) \\ &\le H_T^{\max} s_0 + H_T^{\Sigma} b + H_T^{\Sigma} \max_{k \le T-1} \|W_k\|_2. \end{aligned}$$

Thus,

$$\max_{t \le T} \|X_t\|_2 \;\le\; H_T^{\max} \|X_0\|_2 \;+\; H_T^{\Sigma} b \;+\; H_T^{\Sigma} \max_{0 \le k \le T-1} \|W_k\|_2.$$

Let $n := \dim(W_t)$ and $\sigma_w^2 := \lambda_{\max}(\Sigma_w)$. Using Lemma 4, we have for any fixed $\epsilon \in (0,1)$,

$$\mathbb{P}\Big(\|W\|_2 \le \sigma_w \sqrt{\varepsilon_1(\epsilon)\, n + \varepsilon_2(\epsilon)\, \log(1/\eta)} \Big) \;\ge\; 1 - \eta,$$

where

$$\varepsilon_1(\epsilon) \;=\; \frac{2 \log(1 + 2/\epsilon)}{(1-\epsilon)^2}, \qquad \varepsilon_2(\epsilon) \;=\; \frac{2}{(1-\epsilon)^2}.$$

Apply this bound to each W_k with $\eta = \delta/T$, and use a union bound over $k = 0, \dots, T-1$

$$\mathbb{P}\left(\max_{0 \le k \le T-1} \|W_k\|_2 \le \sigma_w \sqrt{\varepsilon_1(\epsilon)\, n + \varepsilon_2(\epsilon)\, \log \frac{T}{\delta}} \right) \;\ge\; 1 - \delta.$$

Thus with probability at least $1 - \delta$,

$$\max_{t \le T} \|X_t\|_2 \;\le\; \underbrace{H_T^{\max} \|X_0\|_2 + H_T^{\Sigma} b + H_T^{\Sigma} \sigma_w \sqrt{\varepsilon_1(\epsilon)\, n + \varepsilon_2(\epsilon)\, \log \frac{T}{\delta}}}_{R(\delta)}.$$

□

Now let us generalize the result by fixing time t_0 and conditioning on the realized history $x_{0:t_0}$. Define the shifted sequence and noises for the future window as follows

$$\tilde{s}_\ell := s_{t_0+\ell}, \qquad \tilde{W}_\ell := W_{t_0+\ell}, \qquad T - t_0 \ge \ell \ge 0.$$

Now apply (24) at time $t = t_0 + \ell$:

$$\tilde{s}_{\ell+1} = s_{t_0+\ell+1} \le \sum_{k=0}^{t_0+\ell} \beta_k \, s_{t_0+\ell-k} \;+\; b \;+\; \|\tilde{W}_\ell\|.$$

The index inside $s_{t_0+\ell-k}$ is $\geq t_0$ iff $k \leq \ell$. So split:

$$\sum_{k=0}^{t_0+\ell} \beta_k \, s_{t_0+\ell-k} = \underbrace{\sum_{k=0}^{\ell} \beta_k \, \tilde{s}_{\ell-k}}_{\text{depends on future states}} + \underbrace{\sum_{k=\ell+1}^{t_0+\ell} \beta_k \, s_{t_0+\ell-k}}_{\text{history}}.$$

Define the history forcing term

$$h_\ell := \sum_{k=\ell+1}^{t_0+\ell} \beta_k \, s_{t_0+\ell-k}. \tag{28}$$

So the restarted recursion becomes

$$\tilde{s}_{\ell+1} \leq \sum_{k=0}^{\ell} \beta_k \, \tilde{s}_{\ell-k} \; + \; \underbrace{(b + h_\ell)}_{\tilde{b}_\ell} \; + \|\tilde{W}_\ell\|.$$

Define the same impulse response H but now on the restarted timeline

$$\tilde{H}_0 = 1, \qquad \tilde{H}_{\ell+1} = \sum_{k=0}^{\ell} \beta_k \, \tilde{H}_{\ell-k},$$

we claim for all $T - t_0 \geq \ell \geq 0$,

$$\tilde{s}_\ell \leq \tilde{H}_\ell \, \tilde{s}_0 + \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r} \Big(\tilde{b}_r + \|\tilde{W}_r\| \Big),$$

where

$$\tilde{s}_0 = s_{t_0} = \|x_{t_0}\|, \qquad \tilde{b}_r = b + h_r, \qquad h_r = \sum_{j=0}^{t_0-1} \beta_{t_0+r-j} \, s_j.$$

We prove (B.3) by induction as follows.

Base $\ell = 0$**.** RHS is $\tilde{H}_0 \tilde{s}_0 = \tilde{s}_0$. Equality holds.

Inductive step. Assume (B.3) holds for all indices $\leq \ell$. By (B.3) and induction hypothesis (B.3), we have

$$\tilde{s}_{\ell+1} \leq \Big(\sum_{k=0}^{\ell} \beta_k \, \tilde{H}_{\ell-k} \Big) \tilde{s}_0 + \sum_{k=0}^{\ell} \beta_k \sum_{r=0}^{\ell-k-1} \tilde{H}_{\ell-k-1-r} \big(\tilde{b}_r + \|\tilde{W}_r\| \big) + \tilde{b}_\ell + \|\tilde{W}_\ell\|.$$

Since

$$\sum_{k=0}^{\ell} \beta_k \sum_{r=0}^{\ell-k-1} \tilde{H}_{\ell-k-1-r} (\tilde{b}_r + \|\tilde{W}_r\|) = \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-r} \big(\tilde{b}_r + \|\tilde{W}_r\| \big),$$

then

$$\tilde{s}_{\ell+1} \leq \tilde{H}_{\ell+1} \tilde{s}_0 + \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-r} \big(\tilde{b}_r + \|\tilde{W}_r\| \big) + \tilde{b}_\ell + \|\tilde{W}_\ell\| = \tilde{H}_{\ell+1} \tilde{s}_0 + \sum_{r=0}^{\ell} \tilde{H}_{\ell-r} \big(\tilde{b}_r + \|\tilde{W}_r\| \big),$$

which is exactly (B.3) with $\ell \mapsto \ell + 1$.

Lemma 6. *Under Assumption 2, fix a time $t_0 \in \{0, 1, \ldots, T\}$ and suppose the past states $x_0, \ldots, x_{t_0}$ are given (i.e., we condition on $X_{0:t_0} = x_{0:t_0}$). Let $L := T - t_0$ and define the coefficients $\beta_0 := a + \|\psi(\alpha, 1)\|_2$, $\beta_k := \|\psi(\alpha, k+1)\|_2$, $k \geq 1$, $a := L_f + U_{\max} L_g$, and the sequence $\{\tilde{H}_\ell\}_{\ell \geq 0}$ by $\tilde{H}_0 = 1$, $\tilde{H}_{\ell+1} = \sum_{k=0}^{\ell} \beta_k \, \tilde{H}_{\ell-k}$. Let $\tilde{H}_L^{\max} := \max_{0 \leq \ell \leq L} \tilde{H}_\ell$, $\tilde{H}_L^{\Sigma} := \sum_{k=0}^{L-1} \tilde{H}_k$, $b := b_f + U_{\max} b_g$, and define $h_r := \sum_{j=0}^{t_0-1} \beta_{t_0+r-j} \, \|x_j\|_2$. Then for any $\delta \in (0, 1)$, with conditional probability at least $1 - \delta$ given $X_{0:t_0} = x_{0:t_0}$,*

$$\max_{t_0 \leq t \leq T} \|X_t\|_2 \leq \tilde{H}_L^{\max} \|x_{t_0}\|_2 + \max_{0 \leq \ell \leq L} \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r} h_r + b \tilde{H}_L^{\Sigma} + \tilde{H}_L^{\Sigma} \sigma_w \sqrt{\varepsilon_1 n + \varepsilon_2 \log \frac{L}{\delta}}, \tag{29}$$

where $\sigma_w = \sqrt{\lambda_{max}(\Sigma_w)}$.

Proof. Define

$$\tilde{H}_L^{\max} := \max_{0\le \ell \le L} \tilde{H}_\ell, \qquad \tilde{H}_L^{\Sigma} := \sum_{k=0}^{L-1} \tilde{H}_k,$$

where $L = T - t_0$. For any $\ell \le L$,

$$\begin{aligned}
\tilde{s}_\ell &\le \tilde{H}_\ell \tilde{s}_0 + \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r}\, \tilde{b}_r + \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r}\, \|\tilde{W}_r\|_2 \\
&\le \tilde{H}_L^{\max} \|x_{t_0}\|_2 + \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r}\, \tilde{b}_r + \tilde{H}_L^{\Sigma} \max_{0\le r\le L-1} \|\tilde{W}_r\| \\
&\le \tilde{H}_L^{\max} \|x_{t_0}\|_2 + b \sum_{k=0}^{\ell-1} \tilde{H}_k + \max_{0\le \ell\le L} \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r} h_r + \tilde{H}_L^{\Sigma} \max_{0\le r\le L-1} \|\tilde{W}_r\|.
\end{aligned}$$

Thus

$$\max_{t_0\le\tau\le T} \|X_\tau\|_2 \le \tilde{H}_L^{\max} \|x_{t_0}\|_2 + b\, \tilde{H}_L^{\Sigma} + \max_{0\le\ell\le L} \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r} h_r + \tilde{H}_L^{\Sigma} \max_{t_0\le k\le T-1} \|W_k\|_2.$$

Apply Lemma 4 to each W_k with $\eta = \delta/L$ and union bound over $k = t_0, \dots, T-1$:

$$\mathbb{P}\left(\max_{t_0\le k\le T-1} \|W_k\|_2 \le \sigma_w \sqrt{\epsilon_1(\epsilon)\, n + \epsilon_2(\epsilon)\, \log\frac{L}{\delta}} \right) \ge 1 - \delta.$$

Then with probability at least $1-\delta$ conditioning on $X_{0:t_0} = x_{0:t_0}$

$$\begin{aligned}
\max_{t_0\le\tau\le T} \|X_\tau\|_2 \le\ & \tilde{H}_L^{\max} \|x_{t_0}\|_2 + b\, \tilde{H}_L^{\Sigma} \\
&+ \max_{0\le\ell\le L} \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r} \left(\sum_{j=0}^{t_0-1} \beta_{t_0+r-j}\, \|x_j\|_2 \right) + \tilde{H}_L^{\Sigma}\, \sigma_w \sqrt{\epsilon_1(\epsilon)\, n + \epsilon_2(\epsilon)\, \log\frac{L}{\delta}}
\end{aligned}$$

□

B.4 Proof of Lemma 8

Lemma 7. *On any event under which the state trajectory is uniformly bounded, i.e., $\|x_t\| \le X_{\max}$ for all $0 \le t \le T$, the truncation error satisfies, for all $0 \le t \le T$,*

$$\|x_t - x_t^{(M)}\| \ \le\ \frac{X_{\max} C_{\max}}{\alpha_{\min}}\, M^{-\alpha_{\min}} \sum_{s=0}^{t-1} r_s.$$

Here $X_{\max} := \max_{0\le t\le T} \|x_t\|$, $\alpha_{\min} := \min_{1\le i\le n} \alpha_i$, and $C_{\max} := \max_{1\le i\le n} \frac{e^{\frac{1}{12(1-\alpha_i)}} \sqrt{2}\, (1-\alpha_i)^{-(\alpha_i+\frac{3}{2})}}{|\Gamma(-\alpha_i)|}$. The sequence $\{r_t\}_{t\ge 0}$ is defined by

$$r_0 = 1, \qquad r_t = \sum_{j=1}^{M} k_j\, r_{t-j}, \quad t \ge 1,$$

where $k_1 := L_f + L_g U_{max} + \max_{1\le i\le n} |\psi(\alpha_i, 1)|$ and $k_j := \max_{1\le i\le n} |\psi(\alpha_i, j)|$ with $j = 2, \dots, M$.

Proof. We first analyze the following term

$$\begin{aligned}
\sum_{j=1}^{\infty} \psi(\alpha, j) &= \sum_{j=1}^{\infty} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} \\
&= \frac{1}{\Gamma(-\alpha)} \sum_{j=1}^{\infty} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)}
\end{aligned}$$

By Bohr–Mollerup Theorem, there is a positive constant C such that

$$Cx^{x-1/2}e^{-x} \le \Gamma(x) \le Cx^{x-1/2}e^{-x}e^{1/(12x)}.$$

So we have the following equalities

$$\frac{C(j-\alpha)^{j-\alpha-\frac{1}{2}}e^{\alpha-j}e^{-\frac{1}{12(j+1)}}}{C(j+1)^{j+\frac{1}{2}}e^{-j-1}} \le \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} \le \frac{C(j-\alpha)^{j-\alpha-\frac{1}{2}}e^{\alpha-j}e^{\frac{1}{12(j-\alpha)}}}{C(j+1)^{j+\frac{1}{2}}e^{-j-1}}$$

$$(\frac{j-\alpha}{j+1})^{j+1}\frac{e^{-\frac{1}{12(j+1)}}e^{\alpha+1}}{(j+1)^{\frac{1}{2}}(j-\alpha)^{\alpha+\frac{1}{2}}}\frac{j+1}{j-\alpha} \le \frac{\Gamma(j-\alpha)}{\Gamma(j+1)}$$

$$\le (\frac{j-\alpha}{j+1})^{j+1}\frac{e^{\frac{1}{12(j-\alpha)}}e^{\alpha+1}}{(j+1)^{\frac{1}{2}}(j-\alpha)^{\alpha+\frac{1}{2}}}\frac{j+1}{j-\alpha}$$

$$(1-\frac{\alpha+1}{j+1})^{j+1}\frac{e^{-\frac{1}{12(j+1)}}e^{\alpha+1}(j+1)^{\frac{1}{2}}}{(j-\alpha)^{\alpha+\frac{3}{2}}} \le \frac{\Gamma(j-\alpha)}{\Gamma(j+1)}$$

$$\le (1-\frac{\alpha+1}{j+1})^{j+1}\frac{e^{\frac{1}{12(j-\alpha)}}e^{\alpha+1}(j+1)^{\frac{1}{2}}}{(j-\alpha)^{\alpha+\frac{3}{2}}}$$

Taking the limit of both side ($j \to \infty$), we have the following

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} \sim j^{-\alpha-1}.$$

By the convergence result, we have

$$\sum_{j=M+1}^{\infty}|\psi(\alpha,j)| \le \sum_{j=M+1}^{\infty}C_\alpha j^{-\alpha-1}$$

$$= C_\alpha \sum_{j=M+1}^{\infty} j^{-(\alpha+1)}$$

$$= \frac{C_\alpha}{\alpha}M^{-\alpha},$$

where $C_\alpha = \frac{e^{\frac{1}{12(1-\alpha)}}\sqrt{2}\,(1-\alpha)^{-(\alpha+\frac{3}{2})}}{|\Gamma(-\alpha)|}$. We now give the derivation of the constant C_α in the scalar case. We start from the bound

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} \le (1-\frac{\alpha+1}{j+1})^{j+1}\frac{e^{\frac{1}{12(j-\alpha)}}e^{\alpha+1}(j+1)^{\frac{1}{2}}}{(j-\alpha)^{\alpha+\frac{3}{2}}}.$$

For $0 < \beta < 1$ we have $\ln(1-\beta) \le -\beta$, hence

$$(1-\beta)^{j+1} = \exp\big((j+1)\ln(1-\beta)\big) \le \exp\big(-(j+1)\beta\big).$$

Take $\beta = \dfrac{\alpha+1}{j+1} \in (0,1)$. Then

$$\left(1-\frac{\alpha+1}{j+1}\right)^{j+1} \le \exp\left(-(j+1)\frac{\alpha+1}{j+1}\right) = e^{-(\alpha+1)}.$$

Multiplying by $e^{\alpha+1}$ we get

$$\left(1-\frac{\alpha+1}{j+1}\right)^{j+1}e^{\alpha+1} \le 1.$$

So it simplifies to

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} \le e^{\frac{1}{12(j-\alpha)}}\,\frac{(j+1)^{1/2}}{(j-\alpha)^{\alpha+3/2}}.$$

Since $j > \alpha$, we have $j-\alpha \ge 1-\alpha$, hence

$$\frac{1}{12(j-\alpha)} \le \frac{1}{12(1-\alpha)}.$$

Thus

$$e^{\frac{1}{12(j-\alpha)}} \leq e^{\frac{1}{12(1-\alpha)}}.$$

Consider

$$\frac{(j+1)^{1/2}}{(j-\alpha)^{\alpha+\frac{3}{2}}}.$$

Write everything in terms of j:

$$\frac{(j+1)^{1/2}}{(j-\alpha)^{\alpha+\frac{3}{2}}} = j^{-\alpha-1}\left(1+\frac{1}{j}\right)^{1/2}\left(1-\frac{\alpha}{j}\right)^{-(\alpha+\frac{3}{2})}.$$

(Just factor $j^{1/2}$ from $j+1$ and $j^{\alpha+3/2}$ from $j-\alpha$.) So it becomes

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} \leq e^{\frac{1}{12(j-\alpha)}}\, j^{-(1+\alpha)}\left(1+\frac{1}{j}\right)^{1/2}\left(1-\frac{\alpha}{j}\right)^{-(\alpha+\frac{3}{2})}.$$

For all $j \geq 1$,

$$1+\frac{1}{j} \leq 2 \qquad \Longrightarrow \qquad \left(1+\frac{1}{j}\right)^{1/2} \leq \sqrt{2}.$$

For $j \geq 1$ and $0 < \alpha < 1$,

$$1-\frac{\alpha}{j} \geq 1-\alpha.$$

The function $t \mapsto t^{-(\alpha+\frac{3}{2})}$ is decreasing on $(0,\infty)$, so the maximum occurs at the smallest t. Hence,

$$\left(1-\frac{\alpha}{j}\right)^{-(\alpha+\frac{3}{2})} \leq (1-\alpha)^{-(\alpha+\frac{3}{2})}.$$

Then we obtain

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} \leq e^{\frac{1}{12(1-\alpha)}}\sqrt{2}\,(1-\alpha)^{-(\alpha+\frac{3}{2})}\, j^{-(1+\alpha)}.$$

Thus we can choose the explicit constant

$$C_{\text{ratio}}(\alpha) = e^{\frac{1}{12(1-\alpha)}}\sqrt{2}\,(1-\alpha)^{-(\alpha+\frac{3}{2})}.$$

This depends only on α and works for all $j \geq 1$. Since

$$|\psi(\alpha,j)| \leq \frac{C_{\text{ratio}}(\alpha)}{|\Gamma(-\alpha)|}\, j^{-(1+\alpha)},$$

we can take

$$C_\alpha = \frac{e^{\frac{1}{12(1-\alpha)}}\sqrt{2}\,(1-\alpha)^{-(\alpha+\frac{3}{2})}}{|\Gamma(-\alpha)|}$$

Let the truncation error be

$$e_{t+1}^{(M)} := x_{t+1} - x_{t+1}^{(M)}.$$

Note there is no randomness in the truncation error since we use the same noise sequence in (14) and (15). By Assumption 2, we have

$$\begin{aligned}\|e_{t+1}^{(M)}\| &\leq L_f\|e_t^{(M)}\| + L_g\|u_t\|\|e_t^{(M)}\| + \sum_{j=1}^{M}\|\psi(\alpha,j)\|\,\|e_{t+1-j}^{(M)}\| + \left\|\sum_{j=M+1}^{t+1}\psi(\alpha,j)x_{t+1-j}\right\| \\ &\leq L_f\|e_t^{(M)}\| + L_g\|u_t\|\|e_t^{(M)}\| + \sum_{j=1}^{M}\|\psi(\alpha,j)\|\,\|e_{t+1-j}^{(M)}\| + \sum_{j=M+1}^{t+1}\|\psi(\alpha,j)\|\,X_{max}.\end{aligned} \tag{30}$$

Define

$$\alpha_{\min} := \min_i \alpha_i, \qquad C_{\max} := \max_i C_{\alpha_i},$$

where each scalar constant C_{α_i} is from the scalar bound

$$|\psi(\alpha_i, j)| \leq C_{\alpha_i}\, j^{-(1+\alpha_i)}.$$

Because $\psi(\alpha, j)$ is diagonal,

$$\|\psi(\alpha, j)\| = \max_{1\leq i\leq n} |\psi(\alpha_i, j)| \leq \max_i C_{\alpha_i}\, j^{-(1+\alpha_i)}.$$

Since $\alpha_i \geq \alpha_{\min}$, we have

$$j^{-(1+\alpha_i)} \leq j^{-(1+\alpha_{\min})},$$

so

$$\|\psi(\alpha, j)\| \leq C_{\max}\, j^{-(1+\alpha_{\min})}.$$

Now, for the tail sum:

$$\sum_{j=M+1}^{\infty} \|\psi(\alpha, j)\| \leq C_{\max} \sum_{j=M+1}^{\infty} j^{-(1+\alpha_{\min})}.$$

Thus we have

$$C_{\max} \sum_{j=M+1}^{\infty} j^{-(1+\alpha_{\min})} \leq \frac{C_{\max}}{\alpha_{\min}}\, M^{-\alpha_{\min}}.$$

For the last term, we have

$$\left\| \sum_{j=M+1}^{t+1} \psi(\alpha, j) x_{t+1-j} \right\| \leq \sum_{j=M+1}^{t+1} \|\psi(\alpha, j)\|\, \|x_{t+1-j}\|.$$

Since $\|x_k\| \leq X_{\max}$:

$$\left\| \sum_{j=M+1}^{t+1} \psi(\alpha, j) x_{t+1-j} \right\| \leq X_{\max} \sum_{j=M+1}^{\infty} \|\psi(\alpha, j)\|.$$

Then

$$\left\| \sum_{j=M+1}^{t+1} \psi(\alpha, j) x_{t+1-j} \right\| \leq X_{\max} \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}}.$$

Let us denote

$$a_t := \|e_t^{(M)}\|, \qquad b_j := \|\psi(\alpha, j)\| = \max_i |\psi(\alpha_i, j)| \geq 0, \qquad A_t := L_f + L_g \|u_t\|.$$

Then the inequality (30) becomes

$$a_{t+1} \;\leq\; A_t\, a_t \;+\; \sum_{j=1}^{M} b_j\, a_{t+1-j} \;+\; X_{\max} \sum_{j=M+1}^{t+1} b_j.$$

If we upper-bound $A_t \leq A := L_f + L_g U_{\max}$, then

$$a_{t+1} \;\leq\; A a_t + \sum_{j=1}^{M} b_j\, a_{t+1-j} + c_M, \qquad c_M := \frac{X_{\max} K_{\max}}{\alpha_{\min}}\, M^{-\alpha_{\min}}.$$

Combine the a_t terms by defining

$$k_1 := A + b_1, \qquad k_j := b_j \quad (j = 2, \ldots, M).$$

Then

$$a_{t+1} \;\leq\; \sum_{j=1}^{M} k_j\, a_{t+1-j} + c_M.$$

Define r_t by

$$r_0 = 1, \qquad r_t = \sum_{j=1}^{M} k_j \, r_{t-j}, \quad (t \geq 1).$$

Then

$$a_t \;\leq\; r_t \, a_0 + c_M \sum_{s=0}^{t-1} r_s, \qquad t \geq 1. \tag{31}$$

We now proof (31) by induction. Define

$$d_0 := a_0, \qquad d_t := r_t a_0 + c_M \sum_{s=0}^{t-1} r_s, \quad (t \geq 1).$$

We first show d_t satisfies the same recursion but with equality

$$d_t = \sum_{j=1}^{M} k_j \, d_{t-j} + c_M.$$

Start with the right-hand side:

$$\begin{aligned}\sum_{j=1}^{M} k_j d_{t-j} + c_M &= \sum_{j=1}^{M} k_j \left(r_{t-j} a_0 + c_M \sum_{s=0}^{t-j-1} r_s \right) + c_M \\ &= a_0 \sum_{j=1}^{M} k_j r_{t-j} + c_M \sum_{j=1}^{M} k_j \sum_{s=0}^{t-j-1} r_s + c_M.\end{aligned}$$

By definition of r_t,

$$\sum_{j=1}^{M} k_j r_{t-j} = r_t,$$

so

$$a_0 \sum_{j=1}^{M} k_j r_{t-j} = r_t a_0.$$

Define

$$R_t := \sum_{s=0}^{t-1} r_s \quad (t \geq 1), \qquad R_0 := 0.$$

Then $\sum_{s=0}^{t-j-1} r_s = R_{t-j}$, so the double sum is $\sum_{j=1}^{M} k_j R_{t-j}$. Now we prove

$$R_t = 1 + \sum_{j=1}^{M} k_j R_{t-j}.$$

By definition,

$$R_t = \sum_{s=0}^{t-1} r_s = r_0 + \sum_{s=1}^{t-1} r_s = 1 + \sum_{s=1}^{t-1} r_s.$$

For each $s \geq 1$, use the resolvent recursion

$$r_s = \sum_{j=1}^{M} k_j r_{s-j}.$$

Therefore,

$$\begin{aligned}\sum_{s=1}^{t-1} r_s &= \sum_{s=1}^{t-1} \sum_{j=1}^{M} k_j r_{s-j} \\ &= \sum_{j=1}^{M} k_j \sum_{s=j}^{t-1} r_{s-j}.\end{aligned}$$

Recognizing that $\sum_{s=j}^{t-1} r_{s-j} = R_{t-j}$, we obtain

$$\sum_{s=1}^{t-1} r_s = \sum_{j=1}^{M} k_j R_{t-j}.$$

Substituting back yields

$$R_t = 1 + \sum_{j=1}^{M} k_j R_{t-j},$$

which completes the proof.
Thus

$$\sum_{j=1}^{M} k_j d_{t-j} + c_M = r_t a_0 + c_M \left(\sum_{j=1}^{M} k_j R_{t-j} + 1 \right) = r_t a_0 + c_M R_t.$$

But $R_t = \sum_{s=0}^{t-1} r_s$, so this is exactly

$$r_t a_0 + c_M \sum_{s=0}^{t-1} r_s = d_t.$$

Thus we have shown

$$d_t = \sum_{j=1}^{M} k_j d_{t-j} + c_M.$$

Then we prove $a_t \le d_t$ by induction.
Base case: $t = 0$: $a_0 = d_0$.
Inductive step: Assume $a_s \le d_s$ for all $s \le t-1$. Then

$$a_t \le \sum_{j=1}^{M} k_j a_{t-j} + c_M \le \sum_{j=1}^{M} k_j d_{t-j} + c_M = d_t.$$

Therefore $a_t \le d_t$ for all t. Finally, substituting the definition of d_t gives (31). Then (7) follows since $a_0 = 0$. □

Lemma 8. *Fix $t_0 \in \{0, 1, \dots, T\}$ and assume $M \le t_0$. Conditioning on the history $x_{0:t_0}$, consider any event on which the true trajectory is uniformly bounded over $[t_0, T]$, i.e., $\|x_t\| \le X_{\text{fut}},\ \forall\, t \in \{t_0, t_0+1, \dots, T\}$. Then, for every $t \in \{t_0, t_0+1, \dots, T\}$, the truncation error satisfies*

$$\|x_t - x_t^{(M)}\| \ \le\ \Big(X_{\text{fut}} \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}} + X_{\text{past}} \sum_{j=M+1}^{T} \|\psi(\alpha, j)\| \Big) \sum_{s=0}^{t-t_0-1} r_s,$$

where $\alpha_{\min} := \min_{1\le i\le n} \alpha_i$, $X_{\text{past}} := \max_{0\le k\le t_0} \|x_k\|$, *and* $C_{\max} := \max_{1\le i\le n} \frac{e^{\frac{1}{12(1-\alpha_i)}} \sqrt{2}\,(1-\alpha_i)^{-(\alpha_i+\frac{3}{2})}}{|\Gamma(-\alpha_i)|}$. *The sequence $\{r_t\}_{t\ge 0}$ is defined recursively by $r_0 = 1$, and $r_t = \sum_{j=1}^{M} k_j\, r_{t-j}$, $t \ge 1$, with $k_1 := L_f + L_g U_{\max} + \max_{1\le i\le n} |\psi(\alpha_i, 1)|$ and $k_j := \max_{1\le i\le n} |\psi(\alpha_i, j)|$, $j = 2, \dots, M$.*

Proof. Now let us generalize the result by fixing time t_0 and conditioning on the realized history $x_{0:t_0}$. Then

$$\sum_{j=M+1}^{t+1} \|\psi(\alpha, j)\|\, \|x_{t+1-j}\| = \sum_{j\in S_{\text{fut}}(t)} \|\psi(\alpha, j)\|\, \|x_{t+1-j}\| + \sum_{j\in S_{\text{past}}(t)} \|\psi(\alpha, j)\|\, \|x_{t+1-j}\|,$$

where

$$S_{\text{fut}}(t) = \{\, M+1, \dots, \min(t+1,\, t-t_0)\,\}, \qquad S_{\text{past}}(t) = \{\, \max(M+1,\, t-t_0+1), \dots, t+1\,\}.$$

Since $x_{0:t_0}$ are known, define $X_{\text{past}} := \max_{0\le k\le t_0} \|x_k\|$. Then for all $j \in S_{\text{past}}(t)$, we have $t+1-j \le t_0$, so $\|x_{t+1-j}\| \le X_{\text{past}}$. Hence

$$\sum_{j\in S_{\text{past}}(t)} \|\psi(\alpha,j)\|\,\|x_{t+1-j}\| \le X_{\text{past}} \sum_{j=\max(M+1,\,t-t_0+1)}^{t+1} \|\psi(\alpha,j)\|.$$

Assume on the event $\|x_k\| \le X_{\text{fut}}$ for $\forall k \in \{t_0,\dots,T\}$, then for $j \in S_{\text{fut}}(t)$, $\|x_{t+1-j}\| \le X_{\text{fut}}$, giving

$$\sum_{j\in S_{\text{fut}}(t)} \|\psi(\alpha,j)\|\,\|x_{t+1-j}\| \le X_{\text{fut}} \sum_{j=M+1}^{\min(t+1,\,t-t_0)} \|\psi(\alpha,j)\|.$$

Thus

$$\begin{aligned}\left\|\sum_{j=M+1}^{t+1} \psi(\alpha,j)\,x_{t+1-j}\right\| &\le X_{\text{fut}} \sum_{j=M+1}^{\min(t+1,\,t-t_0)} \|\psi(\alpha,j)\| + X_{\text{past}} \sum_{j=\max(M+1,\,t-t_0+1)}^{t+1} \|\psi(\alpha,j)\| \\ &\le X_{\text{fut}} \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}} + X_{\text{past}} \sum_{j=\max(M+1,\,t-t_0+1)}^{T} \|\psi(\alpha,j)\| \\ &\le X_{\text{fut}} \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}} + X_{\text{past}} \sum_{j=M+1}^{T} \|\psi(\alpha,j)\|.\end{aligned}$$

Replace c_M by $c_M^{T,t_0} := X_{\text{fut}} \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}} + X_{\text{past}} \sum_{j=M+1}^{T} \|\psi(\alpha,j)\|$, then for all $T \ge t \ge t_0$,

$$a_{t+1} \le A a_t + \sum_{j=1}^{M} b_j\, a_{t+1-j} + c_M^{T,t_0}.$$

So (31) becomes

$$a_t \;\le\; r_{t-t_0}\, a_{t_0} + c_M^{T,t_0} \sum_{s=0}^{t-t_0-1} r_s, \qquad t \ge t_0.$$

□

B.5 Proof of Lemma 1

We first state the full version of the probabilistic truncation bound in Lemma 1 and then provide its proof.

Lemma 9 (Explicit probabilistic truncation bound)**.** *Consider the stochastic FoD system* (14) *and its M-step stochastic truncated counterpart* (15) *under Assumption 2. Fix $t_0 \in \{0,\dots,T\}$ with $M \le t_0$, and condition on $X_{0:t_0} = x_{0:t_0}$. Let $\{X_t\}_{t=t_0}^{T}$ and $\{X_t^{(M)}\}_{t=t_0}^{T}$ be driven by the same future input sequence and the same noise sequence, with the truncated system initialized at t_0 using the true last M states.*

For any $\delta_1 \in (0,1)$, with conditional probability at least $1-\delta_1$,

$$\max_{t_0\le t\le T} \|X_t - X_t^{(M)}\| \le r_{\text{tail}}\big(T, M, \delta_1\big),$$

where

$$\begin{aligned} r_{\text{tail}}\big(T, M, \delta_1 \mid x_{0:t_0}\big) := &\sum_{k=t_0}^{T-1} \left(\prod_{i=k+1}^{T-1} a_i\right) \\ &\times \left[R(\delta_1 \mid x_{0:t_0}) \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}} + X_{\text{past}} \sum_{j=M+1}^{T} \|\psi(\alpha,j)\|\right].\end{aligned}$$

Here $a_i = e^{c_i}$, with c_i defined in Lemma 3, and $R(\delta \mid x_{0:t_0})$ denotes the right-hand side of (29)*.*

Proof. Define the tail term:

$$\eta_{t+1}^{(M)} := \sum_{j=M+1}^{t+1} \psi_j \, X_{t+1-j}.$$

Then we can rewrite the stochastic fractional-order dynamical system (14) as follows

$$X_{t+1} = f(X_t) + g(X_t)u_t - \sum_{j=1}^{M} \psi_j \, X_{t+1-j} + W_t - \eta_{t+1}^{(M)}.$$

In augmented form (same F_M and G_M), this becomes

$$S_{t+1} = F_M(S_t, u_t) + G_M W_t + d_{t+1}^{(M)},$$

where

$$d_{t+1}^{(M)} := \begin{bmatrix} -\eta_{t+1}^{(M)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Define

$$v_t^{\text{tail}} := S_t - S_t^{(M)}.$$

Subtract the truncated stochastic dynamics from the true dynamics

$$\begin{aligned} v_{t+1}^{\text{tail}} &= \big(F_M(S_t, u_t) - F_M(S_t^{(M)}, u_t)\big) + \big(G_M W_t - G_M W_t\big) + d_{t+1}^{(M)} \\ &= F_M(S_t, u_t) - F_M(S_t^{(M)}, u_t) + d_{t+1}^{(M)}. \end{aligned} \tag{32}$$

By Lemma 3, we have

$$\|v_{t+1}^{\text{tail}}\| \le a_t \|v_t^{\text{tail}}\| + \|d_{t+1}^{(M)}\| = a_t \|v_t^{\text{tail}}\| + \|\eta_{t+1}^{(M)}\|,$$

where $a_t = e^{c_t}$. Unroll (32) with $v_0^{\text{tail}} = 0$

$$\|v_t^{\text{tail}}\| \le \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i \right) \|\eta_{k+1}^{(M)}\|. \tag{33}$$

On any event where the true states are bounded

$$\mathcal{G}_t := \bigcap_{s=0}^{t} \{\|x_s\| \le R_s\},$$

we have

$$\|\eta_{k+1}^{(M)}\| \le R_{max} \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}},$$

where $R_{\max} := \max_{0 \le s \le t} R_s$. Substitute (B.5) into (33), we have on $\mathcal{G}_t$,

$$\|S_t - S_t^{(M)}\| \le r_{\text{tail}}(t, M) := \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i \right) R_{max} \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}}.$$

Then using $\|X_t - X_t^{(M)}\| \le \|S_t - S_t^{(M)}\|$ and Lemma 5, we have

$$\|X_t - X_t^{(M)}\| \le \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i \right) R(\delta) \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}},$$

holds with probability at least $1 - \delta$, where $R(\delta) = H_T^{\max} \|x_0\|_2 + H_T^{\Sigma} b + H_T^{\Sigma} \sigma_w \sqrt{\varepsilon_1 \, n + \varepsilon_2 \, \log \frac{T}{\delta}}$. Now let us generalize the result by fixing time t_0 and conditioning on the realized history $x_{0:t_0}$. Then for any $t_0 \le t \le T$,

$$\|v_t^{\text{tail}}\| \le \sum_{k=t_0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i \right) \|\eta_{k+1}^{(M)}\|.$$

On the event $\mathcal{G}_{t_0:T} := \bigcap_{s=t_0}^{T} \{\|x_s\| \le R_s\}$, we have

$$\|\eta_{k+1}^{(M)}\| \le c_M^{T,t_0} = X_{\text{fut}} \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}} + X_{\text{past}} \sum_{j=M+1}^{T} \|\psi(\alpha, j)\|, \qquad \forall k \in \{t_0, \cdots, T-1\}.$$

Thus for any $t_0 \le t \le T$,

$$\|X_t - X_t^{(M)}\| \le \sum_{k=t_0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i \right) \left(R(\delta \mid x_{0:t_0}) \frac{C_{\max}}{\alpha_{\min}} M^{-\alpha_{\min}} + X_{\text{past}} \sum_{j=M+1}^{T} \|\psi(\alpha, j)\| \right)$$

holds with probability at least $1-\delta$, where $R(\delta \mid x_{0:t_0}) = \tilde{H}_L^{\max} \|x_{t_0}\|_2 + b\,\tilde{H}_L^{\Sigma} + \max_{0\le\ell\le L} \sum_{r=0}^{\ell-1} \tilde{H}_{\ell-1-r} h_r + \tilde{H}_L^{\Sigma}\, \sigma_w \sqrt{\varepsilon_1\, n + \varepsilon_2\, \log \frac{T-t_0}{\delta}}$.

Then, on the high-probability event $\mathcal{G}_{t_0:T}$ under which the state trajectory is bounded, the bound in (B.5) holds uniformly over $t \in \{t_0, \ldots, T\}$. Taking the maximum over the time horizon yields, on $\mathcal{G}_{t_0:T}$,

$$\max_{t_0 \le t \le T} \|X_t - X_t^{(M)}\| \le r_{\text{tail}}\big(T, M, \delta_1\big),$$

where $r_{\text{tail}}\big(T, M, \delta_1\big)$ denotes the right-hand side of (B.5) evaluated at $t = T$. Consequently,

$$\mathbb{P}\Big(\max_{t_0 \le t \le T} \|X_t - X_t^{(M)}\| \le r_{\text{tail}}\big(T, M, \delta_1\big) \Big) \ge 1 - \delta.$$

□

B.6 Proof of Lemma 2

The stochastic deviation compares the truncated stochastic system (15) with its truncated deterministic counterpart (16) for a fixed memory length M. To prove Lemma 2, we first introduce the averaged moment generating function (AMGF) and its concentration property.

Definition 1. Given $\lambda \in \mathbb{R}$, the Averaged Moment Generating Function $\Phi_{n,\lambda} : \mathbb{R}^n \to \mathbb{R}$ is defined as $\mathbb{E}_X[\Phi_{n,\lambda}(X)] := \mathbb{E}_X\, \mathbb{E}_{\ell \sim \text{Unif}(\mathbb{S}^{n-1})} \big[\exp\big(\lambda \langle \ell, X \rangle\big)\big]$.

Lemma 10 (Jafarpour et al. (2024))**.** *If a random variable $X \in \mathbb{R}^n$ satisfies*

$$\mathbb{E}_X[\Phi_{n,\lambda}(X)] \le \exp\left(\frac{\lambda^2 \sigma^2}{2}\right), \qquad \forall\, \lambda \in \mathbb{R},$$

with some $\sigma \in \mathbb{R}$, then for any $\delta > 0$, and any $\epsilon \in (0,1)$,

$$\|X\| \le \sigma \sqrt{\varepsilon_1\, n + \varepsilon_2\, \log(1/\delta)} \tag{34}$$

holds with probability at least $1-\delta$, where $\varepsilon_1 = \frac{2\log\left(1+\frac{2}{\varepsilon}\right)}{(1-\varepsilon)^2}$ and $\varepsilon_2 = \frac{2}{(1-\varepsilon)^2}$.

Lemma 11. *Consider the stochastic truncated fractional-order dynamical system* (15) *and the deterministic truncated fractional-order dynamical system* (16) *under Assumption 2. Fix a time $t_0 \in \{0, 1, \ldots, T\}$ and condition on the history $X_{0:t_0} = x_{0:t_0}$. Let $X_t^{(M)}$ be a trajectory of* (15) *and $x_t^{(M)}$ be an associated trajectory of* (16) *driven by the same input sequence $\{u_t\}_{t=t_0}^{T-1}$, and assume they share the same truncated augmented initial condition at time t_0. Then for any $t \in \{t_0+1, \ldots, T\}$, $\delta \in (0,1)$, and $\varepsilon \in (0,1)$, the following holds with probability at least $1-\delta$ given $X_{0:t_0} = x_{0:t_0}$,*

$$\|X_t^{(M)} - x_t^{(M)}\| \;\le\; \sqrt{\Psi_{t_0 \to t}(\varepsilon_1 d + \varepsilon_2 \log(1/\delta))},$$

where $\Psi_{t_0 \to t} := \sigma_w^2 \sum_{k=t_0}^{t-1} \exp\Big(2 \sum_{i=k+1}^{t-1} c_i\Big)$.

Proof. We have

$$S_{t+1}^{(M)} = F_M(S_t^{(M)}, u_t) + G_M W_t, \qquad s_{t+1}^{(M)} = F_M(s_t^{(M)}, u_t),$$

so the deviation

$$v_t := S_t^{(M)} - s_t^{(M)}$$

satisfies

$$v_{t+1} = \underbrace{F_M(S_t^{(M)}, u_t) - F_M(s_t^{(M)}, u_t)}_{:=\beta_t} + G_M W_t.$$

Here the augmented dimension is $d := Mn$. Define

$$\Phi_{d,\lambda}(x) := \mathbb{E}_{\ell \sim \mathbb{S}^{d-1}}\big[\exp(\lambda \langle \ell, x \rangle)\big]. \tag{35}$$

Define

$$h_t(\lambda) := \mathbb{E}[\Phi_{d,\lambda}(v_t)]\,.$$

Our goal becomes to prove an inequality of the form

$$h_t(\lambda) \le \exp\left(\frac{\lambda^2 \Sigma_t^2}{2}\right), \qquad \forall\, \lambda \in \mathbb{R}. \tag{36}$$

Condition on the sigma-field $\mathcal{F}_t$ generated by $\{W_0, \ldots, W_{t-1}\}$. Then β_t is $\mathcal{F}_t$-measurable and W_t is independent. From (35) we have

$$\Phi_{d,\lambda}(v_{t+1}) = \mathbb{E}_\ell\big[\exp\big(\lambda \langle \ell, \beta_t + G_M W_t \rangle\big)\big].$$

Take conditional expectation over W_t

$$\mathbb{E}_{w_t}[\Phi_{d,\lambda}(v_{t+1}) \mid \mathcal{F}_t] = \mathbb{E}_\ell\Big[\exp\big(\lambda \langle \ell, \beta_t \rangle\big)\ \underbrace{\mathbb{E}_{w_t}[\exp\big(\lambda \langle \ell, G_M W_t \rangle\big)]}_{\text{Gaussian MGF}}\Big].$$

Since $W_t \sim \mathcal{N}(0, \Sigma_w)$,

$$\mathbb{E}_{W_t}[\exp\big(\lambda \langle \ell, G_M W_t \rangle\big)] = \exp\left(\frac{\lambda^2}{2}\, \ell^\top G_M \Sigma_w G_M^\top \ell\right) \le \exp\left(\frac{\lambda^2}{2}\sigma^2\right),$$

where

$$\sigma^2 := \lambda_{\max}\big(G_M \Sigma_w G_M^\top\big)\,.$$

By the definition of G_M

$$\sigma^2 = \lambda_{\max}(\Sigma_w).$$

Therefore,

$$\mathbb{E}[\Phi_{d,\lambda}(v_{t+1}) \mid \mathcal{F}_t] \le \exp\left(\frac{\lambda^2 \sigma^2}{2}\right) \Phi_{d,\lambda}(\beta_t). \tag{37}$$

By Lemma 3, we have

$$\|\beta_t\| \le e^{c_t}\, \|v_t\|.$$

Then we have

$$\Phi_{d,\lambda}(\beta_t) \le \Phi_{d,\lambda}(e^{c_t} v_t) = \Phi_{d,\lambda e^{c_t}}(v_t). \tag{38}$$

Substitute (38) into (37), then take total expectation over $\mathcal{F}_t$

$$h_{t+1}(\lambda) \le \exp\left(\frac{\lambda^2 \sigma^2}{2}\right)\, h_t(\lambda e^{c_t}).$$

We want to use this recursion to prove (36) with an explicit Σ_t^2. Let $a_t := e^{c_t}$. Then we have

$$h_{t+1}(\lambda) \le \exp\left(\frac{\lambda^2 \sigma^2}{2}\right)\, h_t(\lambda a_t).$$

Since $v_0 = 0$, then

$$h_0(\lambda) = \Phi_{d,\lambda}(0) = 1.$$

For $t = 0$:

$$h_1(\lambda) \le \exp\left(\frac{\lambda^2 \sigma^2}{2}\right) h_0(\lambda a_0) = \exp\left(\frac{\lambda^2 \sigma^2}{2}\right).$$

For $t = 1$:

$$\begin{aligned} h_2(\lambda) &\le \exp\left(\frac{\lambda^2\sigma^2}{2}\right) h_1(\lambda a_1) \\ &\le \exp\left(\frac{\lambda^2\sigma^2}{2}\right) \exp\left(\frac{(\lambda a_1)^2\sigma^2}{2}\right) h_0(\lambda a_1 a_0) \\ &= \exp\left(\frac{\lambda^2\sigma^2}{2}\left(1 + a_1^2\right)\right). \end{aligned}$$

For $t = 2$:

$$\begin{aligned} h_3(\lambda) &\le \exp\left(\frac{\lambda^2\sigma^2}{2}\right) h_2(\lambda a_2) \\ &\le \exp\left(\frac{\lambda^2\sigma^2}{2}\right) \exp\left(\frac{(\lambda a_2)^2\sigma^2}{2}\left(1 + a_1^2\right)\right) \\ &= \exp\left(\frac{\lambda^2\sigma^2}{2}\left(1 + a_2^2 + a_2^2 a_1^2\right)\right). \end{aligned}$$

By time t, the exponent contains a sum of squares of products

$$1,\ a_{t-1}^2,\ (a_{t-1}a_{t-2})^2,\ \ldots,\ (a_{t-1}a_{t-2}\cdots a_0)^2.$$

We claim

$$h_t(\lambda) \le \exp\left(\frac{\lambda^2\sigma^2}{2} \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i\right)^2\right) h_0\left(\lambda \prod_{i=0}^{t-1} a_i\right). \tag{39}$$

Here we assume an empty product equals 1: when $k = t-1$, $\prod_{i=t}^{t-1} a_i = 1$.
Base case $t = 1$:

$$h_1(\lambda) \le \exp\left(\frac{\lambda^2\sigma^2}{2}\right) h_0(\lambda a_0),$$

which matches (39) since the sum has only the $k = 0$ term and $\prod_{i=1}^{0} a_i = 1$.
Inductive step: assume (39) holds for t. Then for $t + 1$,

$$h_{t+1}(\lambda) \le \exp\left(\frac{\lambda^2\sigma^2}{2}\right) h_t(\lambda a_t)$$

by (B.6). Apply the induction hypothesis (39) to $h_t(\lambda a_t)$ (replace λ by λa_t):

$$h_t(\lambda a_t) \le \exp\left(\frac{(\lambda a_t)^2\sigma^2}{2} \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i\right)^2\right) h_0\left(\lambda a_t \prod_{i=0}^{t-1} a_i\right).$$

Multiply by the front factor $\exp(\lambda^2\sigma^2/2)$:

$$\begin{aligned} h_{t+1}(\lambda) &\le \exp\left(\frac{\lambda^2\sigma^2}{2}\right) \exp\left(\frac{\lambda^2\sigma^2}{2} a_t^2 \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i\right)^2\right) h_0\left(\lambda \prod_{i=0}^{t} a_i\right) \\ &= \exp\left(\frac{\lambda^2\sigma^2}{2} \sum_{k=0}^{t} \left(\prod_{i=k+1}^{t} a_i\right)^2\right) h_0\left(\lambda \prod_{i=0}^{t} a_i\right), \end{aligned}$$

which is exactly (39) with t replaced by $t + 1$.
If $S_0 = s_0$, then $v_0 = 0 \Rightarrow h_0(\cdot) = 1$. So (39) simplifies to

$$h_t(\lambda) \le \exp\left(\frac{\lambda^2\sigma^2}{2} \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i\right)^2\right).$$

Now define

$$\Sigma_t^2 := \sigma^2 \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} a_i \right)^2 .$$

Then we have

$$h_t(\lambda) \le \exp\left(\frac{\lambda^2 \Sigma_t^2}{2} \right).$$

By $a_i = e^{c_i}$,

$$\prod_{i=k+1}^{t-1} a_i = \exp\left(\sum_{i=k+1}^{t-1} c_i \right).$$

Thus

$$\Sigma_t^2 = \sigma^2 \sum_{k=0}^{t-1} \exp\left(2 \sum_{i=k+1}^{t-1} c_i \right).$$

Then using $\|X_t - X_t^{(M)}\| \le \|S_t - S_t^{(M)}\|$ and Lemma 10, we have for any $t > 0$, $\delta \in (0,1)$, and $\varepsilon \in (0,1)$,

$$\|X_t^{(M)} - x_t^{(M)}\| \le \sqrt{\Psi_t(\varepsilon_1 d + \varepsilon_2 \log(1/\delta))},$$

holds with probability at least $1-\delta$, where $\Psi_t = \sigma^2 \sum_{k=0}^{t-1} \exp\Big(2 \sum_{i=k+1}^{t-1} c_i\Big)$.
Now let us generalize the result by fixing time t_0 and conditioning on the realized history $x_{0:t_0}$. We have

$$S_{t+1}^{(M)} = F_M(S_t^{(M)}, u_t) + G_M W_t, \qquad s_{t+1}^{(M)} = F_M(s_t^{(M)}, u_t), \qquad t \ge t_0,$$

and define $v_t := S_t^{(M)} - s_t^{(M)}$. Assume the two truncated systems share the same augmented initial condition at time t_0, so $v_{t_0} = 0$. Replace the filtration $\mathcal{F}_t = \sigma(W_0, \dots, W_{t-1})$ by the shifted filtration

$$\mathcal{F}_t^{(t_0)} := \sigma(W_{t_0}, W_{t_0+1}, \dots, W_{t-1}), \qquad t \ge t_0,$$

so that $\beta_t := F_M(S_t^{(M)}, u_t) - F_M(s_t^{(M)}, u_t)$ is $\mathcal{F}_t^{(t_0)}$-measurable and W_t is independent. All subsequent steps remain identical, except that the induction is initialized at $t = t_0$ rather than $t = 0$. Thus for any $t \in \{t_0+1, \dots, T\}$, $\delta \in (0,1)$, and $\varepsilon \in (0,1)$, the following holds with conditional probability at least $1-\delta$ given $X_{0:t_0} = x_{0:t_0}$:

$$\|X_t^{(M)} - x_t^{(M)}\| \le \sqrt{\Psi_{t_0 \to t}(\varepsilon_1 d + \varepsilon_2 \log(1/\delta))},$$

where $\Psi_{t_0 \to t} = \sigma^2 \sum_{k=t_0}^{t-1} \exp\Big(2 \sum_{i=k+1}^{t-1} c_i\Big)$, □

We have established a high-probability bound for the stochastic deviation at a fixed time t. To obtain a bound that is uniform over the horizon T, we introduce the auxiliary process $Z_t^{(t_0)} := \exp\Big(-\frac{1}{2} \sum_{k=t_0}^{t-1} \lambda_{k+1}^2 \sigma_w^2\Big)\ \Phi_{d,\lambda_t}(v_t)$, where $t \ge t_0$ and the parameters evolve according to $\lambda_{t+1} = \lambda_t / a_t$ with $a_t := e^{c_t}$.

Lemma 12. *The process $\{Z_t^{(t_0)}\}_{t \ge t_0}$ is a nonnegative supermartingale (with respect to the filtration generated by $\{W_{t_0}, \dots, W_{t-1}\}$).*

Proof. Let

$$\Psi_t^{(\lambda)} := \sum_{k=0}^{t-1} \lambda_{k+1}^2 \sigma_k^2, \qquad \Psi_0^{(\lambda)} = 0.$$

Then

$$\Psi_{t+1}^{(\lambda)} = \Psi_t^{(\lambda)} + \lambda_{t+1}^2 \sigma_t^2.$$

Nonnegativity: $\exp(\cdot) \geq 0$ and $\Phi_{d,\lambda_t}(\cdot) \geq 0$, so $Z_t \geq 0$. Also note $v_0 = 0 \Rightarrow \Phi_{d,\lambda_0}(v_0) = 1 \Rightarrow Z_0 = 1$.

$$\begin{aligned}
\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] &= \mathbb{E}\left[\exp\left(-\frac{1}{2}\Psi_{t+1}^{(\lambda)}\right)\Phi_{d,\lambda_{t+1}}(v_{t+1}) \;\middle|\; \mathcal{F}_t\right] \\
&= \exp\left(-\frac{1}{2}\Psi_{t+1}^{(\lambda)}\right)\mathbb{E}\big[\Phi_{d,\lambda_{t+1}}(v_{t+1}) \mid \mathcal{F}_t\big] \\
&\leq \exp\left(-\frac{1}{2}\Psi_{t+1}^{(\lambda)}\right)\exp\left(\frac{\lambda_{t+1}^2}{2}\sigma_t^2\right)\Phi_{d,\lambda_t}(v_t) \qquad (\text{by } \lambda_{t+1} = \frac{\lambda_t}{a_t}) \\
&= \exp\left(-\frac{1}{2}\big(\Psi_t^{(\lambda)} + \lambda_{t+1}^2\sigma_t^2\big)\right)\exp\left(\frac{\lambda_{t+1}^2}{2}\sigma_t^2\right)\Phi_{d,\lambda_t}(v_t) \qquad (\text{by (B.6)}) \\
&= \exp\left(-\frac{1}{2}\Psi_t^{(\lambda)}\right)\Phi_{d,\lambda_t}(v_t) = Z_t.
\end{aligned}$$

So

$$\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] \leq Z_t,$$

Z_t is a supermartingale.

Fix t_0 and define the shifted filtration $\mathcal{F}_t^{(t_0)} := \sigma(W_{t_0}, W_{t_0+1}, \ldots, W_{t-1})$, where $t \geq t_0$. Let

$$\Psi_t^{(\lambda,t_0)} := \sum_{k=t_0}^{t-1} \lambda_{k+1}^2\sigma_k^2, \qquad \Psi_{t_0}^{(\lambda,t_0)} = 0.$$

Then for all $t \geq t_0$,

$$\Psi_{t+1}^{(\lambda,t_0)} = \Psi_t^{(\lambda,t_0)} + \lambda_{t+1}^2\sigma_t^2. \tag{40}$$

Nonnegativity. Since $\exp(\cdot) \geq 0$ and $\Phi_{d,\lambda_t}(\cdot) \geq 0$, we have $Z_t^{(t_0)} \geq 0$ for all $t \geq t_0$. Moreover, if the truncated stochastic and deterministic systems share the same augmented initial condition at time t_0, then $v_{t_0} = 0$, hence $\Phi_{d,\lambda_{t_0}}(v_{t_0}) = 1$ and therefore $Z_{t_0}^{(t_0)} = 1$.

Now for any $t \geq t_0$,

$$\begin{aligned}
\mathbb{E}\Big[Z_{t+1}^{(t_0)} \mid \mathcal{F}_t^{(t_0)}\Big] &= \mathbb{E}\left[\exp\left(-\frac{1}{2}\Psi_{t+1}^{(\lambda,t_0)}\right)\Phi_{d,\lambda_{t+1}}(v_{t+1}) \;\middle|\; \mathcal{F}_t^{(t_0)}\right] \\
&= \exp\left(-\frac{1}{2}\Psi_{t+1}^{(\lambda,t_0)}\right)\mathbb{E}\Big[\Phi_{d,\lambda_{t+1}}(v_{t+1}) \mid \mathcal{F}_t^{(t_0)}\Big] \\
&\leq \exp\left(-\frac{1}{2}\Psi_{t+1}^{(\lambda,t_0)}\right)\exp\left(\frac{\lambda_{t+1}^2}{2}\sigma_t^2\right)\Phi_{d,\lambda_t}(v_t) \qquad (\text{by } \lambda_{t+1} = \lambda_t/a_t) \\
&= \exp\left(-\frac{1}{2}\big(\Psi_t^{(\lambda,t_0)} + \lambda_{t+1}^2\sigma_t^2\big)\right)\exp\left(\frac{\lambda_{t+1}^2}{2}\sigma_t^2\right)\Phi_{d,\lambda_t}(v_t) \qquad (\text{by (40)}) \\
&= \exp\left(-\frac{1}{2}\Psi_t^{(\lambda,t_0)}\right)\Phi_{d,\lambda_t}(v_t) = Z_t^{(t_0)}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\Big[Z_{t+1}^{(t_0)} \mid \mathcal{F}_t^{(t_0)}\Big] \leq Z_t^{(t_0)}, \qquad t \geq t_0,$$

so $\{Z_t^{(t_0)}\}_{t\geq t_0}$ is a supermartingale. Since it is also nonnegative, it is a nonnegative supermartingale. □

Consequently, for any $\delta \in (0,1)$, Doob's maximal inequality for nonnegative supermartingales yields

$$\mathbb{P}\left(\max_{t_0\leq t\leq T} Z_t^{(t_0)} \geq \frac{1}{\delta}\middle| X_{0:t_0} = x_{0:t_0}\right) \leq \delta. \tag{41}$$

Combining (41) with the AMGF bound (34) yields the following uniform bound over the horizon $[t_0, T]$. With conditional probability at least $1-\delta$,

$$\max_{t_0\le t\le T}\|X_t^{(M)}-x_t^{(M)}\| \le \underbrace{\frac{1}{1-\varepsilon}\max_{t_0\le t\le T}\left(\frac{\Psi_{t_0\to t}}{2}\,a_t\lambda_T+\frac{C(\delta)}{a_t\lambda_T}\right)}_{r_{\mathrm{noise}}(T,\delta)},$$

where $C(\delta) := d\log\!\big(1+\frac{2}{\varepsilon}\big)+\log\!\big(\frac{1}{\delta}\big)$. To tighten the bound, we choose λ_T by solving $\lambda_T^\star \in \arg\min_{\lambda_T>0}\ \max_{t_0\le t\le T}\left(\frac{\Psi_{t_0\to t}}{2}\,a_t\lambda_T+\frac{C(\delta)}{a_t\lambda_T}\right)$. We next state the explicit stochastic deviation bound and then provide its proof.

Lemma 13 (Explicit probabilistic stochastic deviation bound)**.** *Consider the stochastic truncated FoD system* (15) *and its deterministic counterpart* (16) *under Assumption 2. Fix $t_0 \in \{0,\dots,T\}$ and condition on $X_{0:t_0} = x_{0:t_0}$. Let $\{X_t^{(M)}\}_{t=t_0}^T$ and $\{x_t^{(M)}\}_{t=t_0}^T$ be driven by the same future input sequence and initialized with the same augmented state at time t_0. Fix $\varepsilon \in (0,1)$. Then, for any $\delta_2 \in (0,1)$, with conditional probability at least $1-\delta_2$,*

$$\max_{t_0\le t\le T}\|X_t^{(M)}-x_t^{(M)}\| \le r_{\mathrm{noise}}(T,\delta_2),$$

where

$$r_{\mathrm{noise}}(T,\delta_2) := \frac{1}{1-\varepsilon}\inf_{\lambda_T>0}\max_{t_0\le t\le T}\left[\frac{\Psi_{t_0\to t}}{2}A_t\lambda_T+\frac{C(\delta_2)}{A_t\lambda_T}\right],$$

with

$$\Psi_{t_0\to t} := \sigma_w^2\sum_{k=t_0}^{t-1}\exp\!\left(2\sum_{i=k+1}^{t-1}c_i\right),$$
$$A_t := \prod_{i=t}^{T-1}e^{c_i} = \exp\!\left(\sum_{i=t}^{T-1}c_i\right),$$
$$C(\delta_2) := Mn\log\!\left(1+\frac{2}{\varepsilon}\right)+\log\!\left(\frac{1}{\delta_2}\right).$$

Here c_i is the propagation coefficient from Lemma 3.

Proof. If we pick a terminal time t and set $\lambda_t = \lambda$, then for $k < t$

$$\lambda_{k+1} = \lambda\prod_{i=k+1}^{t-1}a_i.$$

Substituting it into $\Psi_t^{(\lambda)}$ gives

$$\Psi_t^{(\lambda)} = \lambda^2\sum_{k=0}^{t-1}\sigma_k^2\left(\prod_{i=k+1}^{t-1}a_i\right)^2 = \lambda^2\Psi_t,$$

where Ψ_t is exactly in Lemma 11 with $t_0 = 0$.
For any $\delta \in (0,1)$, Doob's maximal inequality for nonnegative supermartingales gives

$$\mathbb{P}\!\left(\max_{0\le t\le T}Z_t \ge \frac{1}{\delta}\right) \le \delta\,\mathbb{E}[Z_0] = \delta.$$

Equivalently, define the good event

$$\Omega := \left\{\max_{t\le T}Z_t \le \frac{1}{\delta}\right\},$$

then

$$\mathbb{P}(\Omega) \ge 1-\delta.$$

On Ω, for every $0 \le t \le T$,

$$Z_t \le \frac{1}{\delta} \quad \Longrightarrow \quad \Phi_{d,\lambda_t}(v_t) \le \frac{1}{\delta} \exp\left(\frac{1}{2}\Psi_t^{(\lambda)}\right).$$

Then on Ω, for every $0 \le t \le T$,

$$\Phi_{d,\lambda_t}(v_t) \le \frac{1}{\delta} \exp\left(\frac{\lambda_t^2}{2}\Psi_t\right). \tag{42}$$

For a given λ_T, on the Doob good event we have

$$\|v_t\| \le \frac{1}{1-\varepsilon}\left(\frac{\Psi_t}{2}\lambda_t + \frac{C(\delta)}{\lambda_t}\right), \qquad C(\delta) := d\log\left(1+\frac{2}{\varepsilon}\right) + \log(1/\delta). \tag{43}$$

We now prove (43). Set $u := x/\|x\| \in \mathbb{S}^{d-1}$. Consider the Euclidean cap

$$\mathcal{C}_\varepsilon(u) := \big\{\ell \in \mathbb{S}^{d-1} : \|\ell - u\| \le \varepsilon\big\}.$$

For any $\ell \in \mathcal{C}_\varepsilon(u)$,

$$\langle \ell, u\rangle = 1 - \frac{\|\ell-u\|^2}{2} \ge 1 - \frac{\varepsilon^2}{2} \ge 1-\varepsilon \qquad \big(\text{since } \varepsilon \in (0,1) \Rightarrow \varepsilon^2/2 \le \varepsilon\big).$$

Therefore

$$\langle \ell, x\rangle = \|x\|\,\langle \ell, u\rangle \ge (1-\varepsilon)\,\|x\|.$$

So on the cap,

$$\exp\big(\lambda\langle \ell, x\rangle\big) \ge \exp\big(\lambda(1-\varepsilon)\|x\|\big).$$

By definition of Φ,

$$\Phi_{d,\lambda}(x) = \mathbb{E}_\ell\big[\exp\big(\lambda\langle \ell, x\rangle\big)\big] \ge \mathbb{E}_\ell\big[\exp\big(\lambda\langle \ell, x\rangle\big)\,\mathbf{1}_{\{\ell\in\mathcal{C}_\varepsilon(u)\}}\big] \ge \mathbb{P}(\ell \in \mathcal{C}_\varepsilon(u))\,\exp\big(\lambda(1-\varepsilon)\|x\|\big).$$

Let

$$p_\varepsilon := \mathbb{P}(\ell \in \mathcal{C}_\varepsilon(u)).$$

So we have the deterministic lower bound

$$\Phi_{d,\lambda}(x) \ge p_\varepsilon\,\exp\big(\lambda(1-\varepsilon)\|x\|\big).$$

Using the fact that $\mathbb{S}^{d-1}$ admits an ε-net in Euclidean distance of size at most

$$N_\varepsilon \le (1+2/\varepsilon)^d,$$

we have that $\mathbb{S}^{d-1}$ can be covered by N_ε Euclidean caps of radius ε. All such caps have the same measure p_ε. Since the union of the N_ε caps covers the whole sphere,

$$1 = \mathbb{P}(\ell \in \mathbb{S}^{d-1}) \le \sum_{i=1}^{N_\varepsilon} \mathbb{P}(\ell \in \mathcal{C}_\varepsilon(u_i)) = N_\varepsilon p_\varepsilon.$$

Hence

$$p_\varepsilon \ge \frac{1}{N_\varepsilon} \ge (1+2/\varepsilon)^{-d}.$$

We have

$$\Phi_{d,\lambda}(x) \ge (1+2/\varepsilon)^{-d}\,\exp\big(\lambda(1-\varepsilon)\|x\|\big).$$

Combining (42), we have

$$\frac{1}{\delta}\exp\left(\frac{\lambda_t^2}{2}\Psi_t\right) \ge (1+2/\varepsilon)^{-d}\,\exp\big(\lambda_t(1-\varepsilon)\|v_t\|\big).$$

Taking logs

$$(1-\varepsilon)\,\|v_t\| \le \frac{\lambda_t}{2}\,\Psi_t + \frac{d\log(1+2/\varepsilon) + \log(1/\delta)}{\lambda_t}.$$

So

$$\|v_t\| \le \frac{1}{1-\varepsilon}\left(\frac{\Psi_t}{2}\lambda_t + \frac{C(\delta)}{\lambda_t}\right), \qquad C(\delta) := d\log\left(1+\frac{2}{\varepsilon}\right) + \log(1/\delta).$$

Substitute $\lambda_t = A_t \lambda_T$, where $A_t = \prod_{i=t}^{T-1} a_i$,

$$\|v_t\| \le \frac{1}{1-\varepsilon}\left(\frac{\Psi_t}{2} A_t \lambda_T + \frac{C(\delta)}{A_t \lambda_T}\right).$$

Then the max-over-time bound is

$$\max_{t \le T} \|v_t\| \le \frac{1}{1-\varepsilon} \max_{t \le T}\left(\frac{\Psi_t}{2} A_t \lambda_T + \frac{C(\delta)}{A_t \lambda_T}\right).$$

So if we want the best bound, we choose

$$\lambda_T^\star \in \arg\min_{\lambda_T > 0} \max_{t \le T}\left(\frac{\Psi_t}{2} A_t \lambda_T + \frac{C(\delta)}{A_t \lambda_T}\right).$$

Now let us generalize the result by fixing time t_0 and conditioning on the realized history $x_{0:t_0}$. The proof is identical to the $t_0 = 0$ case after a change of time origin. In particular, we (i) replace the filtration $\mathcal{F}_t = \sigma(W_0, \dots, W_{t-1})$ by $\mathcal{F}_t^{(t_0)} = \sigma(W_{t_0}, \dots, W_{t-1})$, where $t \ge t_0$, (ii) replace the auxiliary process by the shifted version $\{Z_t^{(t_0)}\}_{t \ge t_0}$, and (iii) shift all time-indexed sums/maxima from $[0, T]$ to $[t_0, T]$.

Applying Doob's maximal inequality to the nonnegative supermartingale $\{Z_t^{(t_0)}\}$ over the horizon $[t_0, T]$ and combining with the same geometric cap argument as in the $t_0 = 0$ proof yields that, with probability at least $1 - \delta$ given $X_{0:t_0} = x_{0:t_0}$,

$$\max_{t_0 \le t \le T} \|v_t\| \le \frac{1}{1-\varepsilon} \max_{t_0 \le t \le T}\left(\frac{\Psi_{t_0 \to t}}{2} A_t \lambda_T + \frac{C(\delta)}{A_t \lambda_T}\right).$$

Finally, an optimal choice of λ_T is

$$\lambda_T^\star \in \arg\min_{\lambda_T > 0} \max_{t_0 \le t \le T}\left(\frac{\Psi_{t_0 \to t}}{2} A_t \lambda_T + \frac{C(\delta)}{A_t \lambda_T}\right).$$

□

C Experiment Details

We provide a detailed description of the experimental settings used in Section 6. Both the linear and nonlinear experiments consider five-dimensional systems with scalar control inputs drawn from the discrete action set $\mathcal{U} = \{1, 2, \dots, 10\}$. We use a common fractional order $\alpha = 0.3$, a horizon of $T = 20$, and the safe set $\mathcal{S} = [-3, 3]^5$. All trajectories start from $x_0 = -0.2\mathbf{1}_5$, and the additive disturbances are i.i.d. as $W_t \sim \mathcal{N}(0, 0.01 I_5)$. Our method and MCLS use the same baseline policy, which selects $u^\pi = 1$ with probability 0.9 and $u^\pi = 10$ with probability 0.1, and assigns zero probability to all other actions. All methods use the constant nominal input $u_\star = 10$, with quadratic cost $\|u_t - u_\star\|_2^2$. Our method and the finite-memory variants of MCLS and PrSBC use a memory window of $M = 5$, while the simulated systems retain the complete state history. Each method is evaluated on 500 trajectories, using the same disturbance realizations across methods and across the two dynamics settings. At each time t, we estimate the safety probability as the fraction of trajectories that remain in $\mathcal{S}$ at every step from 0 through t, and report 95% confidence intervals.

Linear fractional-order dynamics. We consider the following fractional-order LTI system

$$X_{t+1} = AX_t + BU_t - \sum\nolimits_{j=1}^{t+1} \psi(\alpha, j) X_{t+1-j} + W_t. \tag{44}$$

with

$$A = \begin{bmatrix} 0.04 & 0.06 & 0 & -0.02 & 0.01 \\ -0.03 & 0.05 & 0.05 & 0 & 0.02 \\ 0.02 & -0.04 & 0.03 & 0.06 & 0 \\ 0 & 0.02 & -0.05 & 0.04 & 0.04 \\ 0.05 & 0 & 0.02 & -0.03 & 0.03 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.24 \\ 0.16 \\ -0.18 \\ 0.12 \\ -0.22 \end{bmatrix}.$$

For our certificate, each coordinate of the history summary is discretized into three uniform intervals on $[-3, 3]$. Both MCLS variants use 4096 Monte Carlo rollouts per candidate action, extending

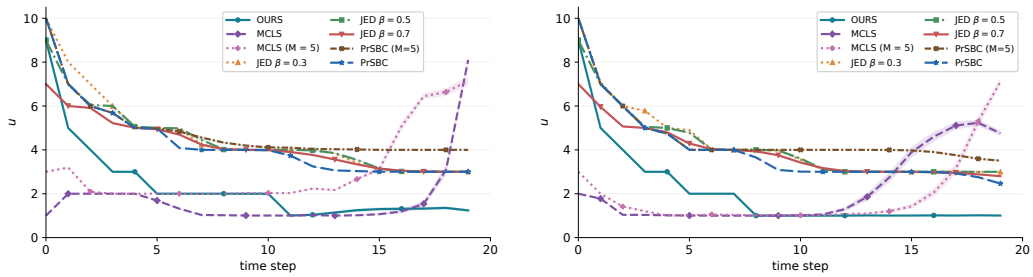

Figure 3: Control inputs for linear (left) and nonlinear (right) fractional-order systems. Curves show the control inputs averaged over all 500 trajectories at each time step with a horizon of $T = 20$. Shaded regions indicate 95% confidence intervals.

to terminal time T. JED uses $\beta \in \{0.3, 0.5, 0.7\}$ as the coefficient multiplying the current barrier value in its Discrete-Time Control Barrier Function (DTCBF) constraint, corresponding to α in the original paper. Both PrSBC variants use $\rho = 0.3$ in the barrier condition and set the allowed violation probability per step to $\delta = 0.1$. The $M = 5$ variants of MCLS and PrSBC truncate older fractional memory in their prediction models. Figure 3 (left) reports mean control inputs over all 500 trajectories, including inputs applied after the first safety violation. Our method gradually reduces its mean input toward the lower end of the action set. Both MCLS variants exhibit a late increase, whereas the JED and PrSBC curves generally decrease and settle in the range $u \in [3, 4]$.

Nonlinear fractional-order dynamics. We use the following nonlinear dynamics

$$X_{t+1} = AX_t + c\tanh(X_t) + BU_t - \sum_{j=1}^{t+1} \psi(\alpha, j) X_{t+1-j} + W_t. \tag{45}$$

with $c = 0.1$ and the same matrices A and B as in the linear experiment. The controller parameters and grid resolution remain unchanged, while the certificate and baseline predictions are recomputed for the nonlinear dynamics. Figure 3 (right) shows that our method reduces its mean input toward $u = 1$ and maintains approximately this value for the remainder of the horizon. Both MCLS variants exhibit a late increase, ending near $u = 4.8$ and $u = 7.1$ for MCLS and MCLS ($M = 5$), respectively. The JED curves decrease toward $u = 3$, while PrSBC and PrSBC ($M = 5$) end near $u = 2.5$ and $u = 3.5$, respectively.