CMQNET: CONTINUOUS AND MONOTONE QUANTILE NETWORKS

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ABSTRACT

Deep quantile regression has emerged as a powerful paradigm for characterizing conditional distributions and quantifying uncertainty. However, existing neural network approaches are often limited to estimating quantiles at discrete grids, resulting in discontinuity with respect to the quantile level and failing to structurally guarantee the non-crossing property. To address these challenges, we propose the Continuous and Monotone Quantile Network (CMQNet). Leveraging the fundamental theorem of calculus for absolutely continuous functions, CMQNet models the conditional quantile function as the summation of a baseline term and the integral of a non-negative derivative with respect to the quantile index. By modeling this derivative using a neural network with strictly positive activation functions, our framework naturally ensures both monotonicity and continuity by construction, thereby eliminating quantile crossing and admitting a smooth conditional density. Furthermore, we introduce an efficient training strategy based on stochastic quantile sampling, allowing the model to learn the global quantile process beyond fixed grids. Extensive experiments on simulated datasets and real-world benchmarks demonstrate that CMQNet significantly outperforms state-of-the-art methods in terms of estimation accuracy and stability. The code can be found at https://anonymous.4open.science/r/CMQNet-C420/.

1 INTRODUCTION

Quantile regression (QR) (Koenker & Bassett Jr, 1978; Koenker, 2005) provides a powerful framework for characterizing conditional distributions beyond the conditional mean, and has found wide applications in uncertainty quantification, risk management, time-series forecasting (Meinshausen, 2006; Jensen et al., 2024), and more recently, conformal prediction and algorithmic fairness (Romano et al., 2019; Liu et al., 2022). In contrast to point estimation, quantile-based methods offer a more comprehensive description of predictive uncertainty and are particularly appealing in data-driven decision-making problems.

With the rapid development of deep learning, a growing body of literature has incorporated neural networks into quantile regression to enhance modeling flexibility and scalability. Representative approaches include deep quantile regression (DQR) (Padilla et al., 2022; Shen et al., 2021), its non-crossing variant DQR* (Padilla et al., 2022), penalized deep quantile regression (DQRP) (Shen et al., 2024), and more recent architectures such as NQ-Net (Shen et al., 2025). These methods have demonstrated strong empirical performance across diverse applications.

Despite their success, existing neural quantile regression methods suffer from several fundamental limitations. First, many approaches model quantiles only at a finite and pre-specified set of quantile levels, resulting in estimates that are discontinuous in the quantile index τ and incapable of producing quantiles outside the training grid. Second, ensuring the non-crossing property of estimated quantile functions remains challenging. Methods such as DQR do not enforce monotonicity in τ, while DQR* and NQ-Net rely on discrete constructions that guarantee non-crossing only for a fixed sequence of quantiles. Penalization-based approaches such as DQRP encourage monotonicity by adding derivative-based penalties, but their performance is sensitive to tuning parameters and may still exhibit quantile crossing in practice. Finally, most existing methods do not provide a continuous and well-defined conditional density implied by the estimated quantile function.

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Table 1 provides a high-level comparison of recent neural quantile regression methods in terms of three key methodological properties: continuity with respect to the quantile index τ, the non-crossing guarantee, and the existence of an implied conditional density. As shown in the table, most existing approaches satisfy at most one or two of these properties. In particular, methods based on discrete quantile grids are inherently discontinuous in τ, while penalization-based approaches may fail to strictly enforce monotonicity and are sensitive to tuning parameters. To the best of our knowledge, none of the existing methods simultaneously achieve continuity in τ, non-crossing quantile estimation, and a well-defined conditional density within a unified neural framework.

Table 1: A comparison of recent deep quantile regression methods in terms of continuity in τ, non-crossing guarantees, and density existence. The question mark **?** denotes no strict guarantee.

Method	Continuous in τ	Non-cross	Density
DQR	✗	✗	✗
DQR*	✗	✓	✗
DQRP	✓	**?**	**?**
NQ-Net	✗	✓	✗
CMQNet	✓	✓	✓

To address these challenges, we propose *CMQNet*, a continuous and monotone quantile network that models the entire conditional quantile function as a function of both covariates and the quantile index τ. Our key insight is to leverage the fundamental theorem of calculus for absolutely continuous functions, which enables a quantile function that is continuous in τ to be expressed as the integral of its derivative with respect to τ. By parameterizing this derivative using neural networks with strictly positive activation functions, such as the softplus or exponential function, we naturally enforce monotonicity and obtain a quantile function that is continuous in τ by construction.

Moreover, we introduce an efficient training strategy that randomly samples multiple quantile levels for each observation during training. This stochastic sampling scheme provides flexible control over computational cost and estimation accuracy, and allows the model to learn the global structure of the quantile function without relying on a fixed quantile grid. As a result, CMQNet yields non-crossing and continuous quantile estimates.

Our main contributions can be summarized as follows:

- We propose a continuous neural quantile regression framework that represents the conditional median function as a baseline plus the integral of a strictly positive derivative network, guaranteeing monotonicity and continuity in the quantile index.
- We develop a randomized quantile sampling strategy that improves computational efficiency and estimation accuracy.
- We demonstrate through extensive experiments that CMQNet outperforms existing neural quantile regression approaches in terms of accuracy, stability, and distributional consistency.

2 Background

2.1 Quantile Regression

Quantile regression, originally introduced by Koenker & Bassett Jr (1978), provides a comprehensive view of the relationship between a target scalar variable $Y \in \mathbb{R}$ and a set of covariates $\mathbf{X} \in \mathbb{R}^p$. Unlike ordinary least squares (OLS) regression, which estimates the conditional mean, quantile regression (QR) estimates conditional quantiles and thereby offers a richer description of the entire conditional distribution. This distributional perspective enables QR to capture tail behavior, asymmetry, and heterogeneous effects that are invisible to mean-based approaches, while maintaining robustness to heavy-tailed errors and outliers.

Consider a standard additive noise model formulation: $Y \mid \mathbf{X} = g(\mathbf{X}) + \epsilon$, where $g(\mathbf{X})$ is an unknown regression function and ϵ is an error term independent of $\mathbf{X}$, with cumulative distribution

function (CDF) F_ϵ. The conditional quantile function of Y at level $\tau \in (0,1)$ can be derived as:

$$Q_Y(\tau \mid \mathbf{X}) = \inf\{y : \mathbb{P}(Y \leq y \mid \mathbf{X}) \geq \tau\} = \inf\{y : F_Y(y \mid \mathbf{X}) \geq \tau\} = g(\mathbf{X}) + F_\epsilon^{-1}(\tau). \quad (1)$$

To estimate the conditional quantile $Q_Y(\tau \mid \mathbf{X})$, one typically minimizes the expected check loss (or pinball loss), defined as $\rho_\tau(u) = u(\tau - \mathbb{I}(u < 0))$. In classical nonparametric quantile regression, the regression function $g(\mathbf{X})$ is often modeled using smoothing splines methods (de Boor, 2001; Bondell et al., 2010), or more recent extensions based on vector-valued reproducing kernel Hilbert spaces (Takeuchi et al., 2006; Sangnier et al., 2016).

2.2 Deep Learning Methods and the Non-Crossing Constraints

With the advent of deep learning, Deep Quantile Regression (DQR) has emerged as a powerful tool for modeling complex non-linear dependencies. By replacing linear estimators with neural networks, DQR can approximate highly flexible quantile functions and has shown theoretical minimax optimality Shen et al. (2021); Padilla et al. (2022). Recent works have further extended these frameworks to simulation-based inference and complex distributional learning Jia (2024).

However, a fundamental challenge in DQR is the “quantile crossing problem" He (1997). When multiple quantiles are estimated independently or even via a shared-representation multi-output network, there is no structural guarantee that the monotonicity property of the CDF is preserved. That is, for $\tau_1 < \tau_2$, the model may predict $\hat{Q}(\tau_1 \mid \mathbf{X}) > \hat{Q}(\tau_2 \mid \mathbf{X})$, which violates basic probability axioms and leads to invalid distribution estimates.

The issue of quantile crossing is not unique to deep learning and has been extensively studied in the statistical literature. Various strategies have been developed to enforce monotonicity in traditional settings, including stepwise estimation (Liu & Wu, 2009), kernel-based constraints (Liu & Wu, 2011), and optimization-based approaches such as non-crossing convex quantile regression (Dai et al., 2023) and Fused LASSO (Szendrei et al., 2025). More recently, these simultaneous estimation techniques have been extended to functional data analysis (Hu et al., 2024).

In the context of neural networks, several approaches have been proposed to mitigate this issue. Some methods employ a “crossing penalty" in the loss function to discourage non-monotonicity Moon et al. (2021); Shin et al. (2024), though this acts as a soft constraint and does not strictly guarantee validity. Other approaches involve post-processing steps, such as sorting the predicted quantiles (rearrangement) Chernozhukov et al. (2010); Dette & Volgushev (2008), which lacks end-to-end differentiability.

To achieve structural non-crossing guarantees, monotonicity-constrained architectures have gained attention. Cannon (2018) proposed a monotone composite quantile regression neural network for environmental applications. Recent advances aim to enforce the non-decreasing property by design through positive weight constraints or specific structural formulations Bondell et al. (2010); Shen et al. (2024; 2025); Brando et al. (2022). In particular, Brando et al. (2022) focus on enforcing non-crossing via a Chebyshev-based analytic construction, but their framework does not provide direct density estimation through neural network structures.

3 Methodology

3.1 Problem Formulation

Consider a regression setting with a target variable $Y \in \mathbb{R}$ and a feature vector $\mathbf{X} \in \mathbb{R}^p$. The relationship between Y and $\mathbf{X}$ is generally formulated as $Y \mid \mathbf{X} = g(\mathbf{X}, \epsilon)$, where ϵ denotes a random noise term. The conditional quantile function of Y given $\mathbf{X}$ at quantile level $\tau \in (0,1)$ is defined as

$$Q_Y(\tau \mid \mathbf{X}) = \inf\{y : \mathbb{P}(Y \leq y \mid \mathbf{X}) \geq \tau\} = \inf\{y : F_Y(y \mid \mathbf{X}) \geq \tau\} \triangleq h(\mathbf{X}, \tau), \quad (2)$$

where $F_Y(\cdot \mid \mathbf{X})$ denotes the conditional cumulative distribution function. To obtain a quantile function that is continuous in τ, we construct $h(\mathbf{X}, \tau)$ through an integral representation based on its derivative with respect to τ.

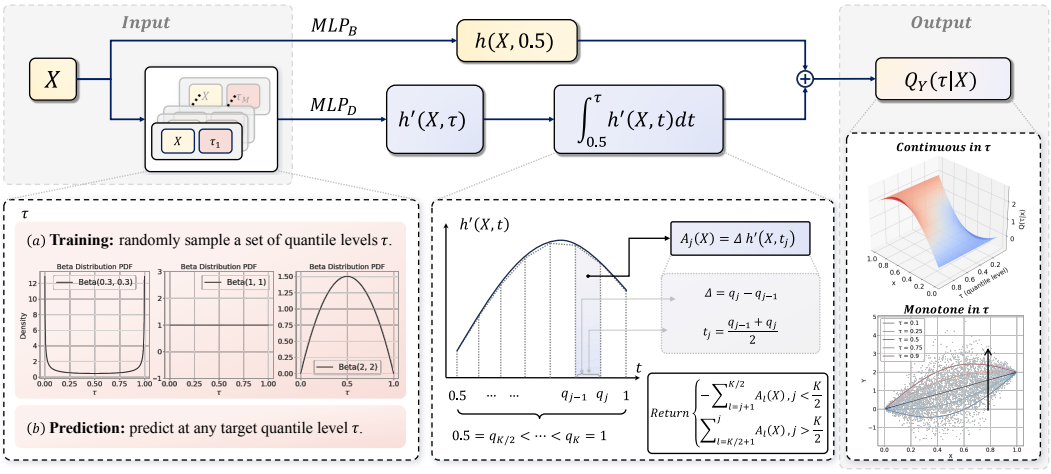


Figure 1: The overall architecture of our CMQNet model. The input features $\mathbf{X}$ are processed by a baseline network to estimate the conditional median, and by a derivative network that takes $(\mathbf{X}, \tau)$ as input to model quantile-specific variations. To guarantee non-crossing quantile curves, the derivative network is constrained to produce non-negative outputs. The final quantile prediction at level τ is obtained by numerically integrating these outputs from a reference level and adding them to the baseline. During training, quantile levels are randomly sampled, enabling CMQNet to learn a continuous and monotonic conditional quantile function over τ.

Integral representation. We assume that for any given $\mathbf{X}$, the mapping $\tau \mapsto h(\mathbf{X}, \tau)$ is absolutely continuous on $[0, 1]$. Under this mild regularity condition, $h(\mathbf{X}, \tau)$ is differentiable almost everywhere and admits an integral form, formalized by the following Lemma 1.

Lemma 1 (Fundamental theorem of calculus for absolutely continuous functions)**.** *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function. Then f is differentiable almost everywhere on $[a, b]$, its derivative $f' \in L^1([a, b])$, and for any $t, t_0 \in [a, b]$, $f(t) = f(t_0) + \int_{t_0}^{t} f'(s)\, ds$.*

Applying Lemma 1 to $f(\tau) = h(\mathbf{X}, \tau)$ with anchor point $\tau_0 = 0.5$ yields

$$h(\mathbf{X}, \tau) = h(\mathbf{X}, 0.5) + \int_{0.5}^{\tau} h'(\mathbf{X}, t)\, dt, \tag{3}$$

where $h'(\mathbf{X}, t) = \partial h(\mathbf{X}, t)/\partial t$ denotes the partial derivative with respect to the quantile level and exists almost everywhere. Eq. (3) decomposes the quantile function into a baseline term (the conditional median) and an accumulated derivative term. Importantly, it provides a sufficient condition for non-crossing across τ: if $h'(\mathbf{X}, t) \geq 0$ a.e., then $\tau \mapsto h(\mathbf{X}, \tau)$ is non-decreasing.

Neural parameterization. We model the two components in Eq. (3) with two networks:

- **Baseline network** MLP_B**:** Models the median $h(\mathbf{X}, 0.5)$. It maps $\mathbf{X} \in \mathbb{R}^p$ to a scalar output $\mathbb{R}$.
- **Derivative network** MLP_D**:** Models the quantile derivative $h'(\mathbf{X}, \tau)$ with the concatenated input $(\mathbf{X}, \tau)$ and a scalar output. To enforce $h'(\mathbf{X}, \tau) \geq 0$, we apply a strictly positive activation (e.g., Softplus or Exponential) at the output layer. The choice of the activation function is treated as a tunable design component of the network, and a comparative study of different choices is provided in Appendix 9.3.

This assumption is satisfied by standard MLP parameterizations with locally Lipschitz activations, as stated in the following Lemma 2.

Lemma 2 (Absolute continuity of MLPs in τ)**.** *Let $h(x, \tau) : \mathcal{X} \times [0, 1] \to \mathbb{R}$ be represented by an MLP with input (x, τ), where all layers are fully connected and all activation functions are locally Lipschitz continuous (e.g., ReLU, Softplus, ELU,* $\tanh$*, Sigmoid, Exponential). Then for any fixed $x \in \mathcal{X}$, the mapping $\tau \mapsto h(x, \tau)$ is absolutely continuous on* $[0, 1]$.

Numerical approximation. In practice, we approximate the integral in Eq. (3) by a fixed-grid cumulative-sum scheme. Specifically, we use an evenly spaced grid $0 = q_0 < q_1 < \cdots < q_K = 1, q_j = j/K$, where K is chosen to be even so that $q_{K/2} = 0.5$. For each interval $[q_{j-1}, q_j]$, we approximate its contribution by the positive area

$$A_j(\mathbf{X}) = (q_j - q_{j-1})\mathrm{MLP}_D\left(\mathbf{X}, \frac{q_{j-1} + q_j}{2}\right), \qquad j = 1, \ldots, K.$$

Since $\mathrm{MLP}_D(\mathbf{X}, \tau) > 0$, we have $A_j(\mathbf{X}) > 0$ for all j.

The estimated conditional quantile values on the grid are then obtained by accumulating these positive areas from the reference level 0.5:

$$\hat{h}(\mathbf{X}, q_j) = \begin{cases} \mathrm{MLP}_B(\mathbf{X}) - \sum_{\ell=j+1}^{K/2} A_\ell(\mathbf{X}), & j < K/2, \\ \mathrm{MLP}_B(\mathbf{X}), & j = K/2, \\ \mathrm{MLP}_B(\mathbf{X}) + \sum_{\ell=K/2+1}^{j} A_\ell(\mathbf{X}), & j > K/2. \end{cases} \tag{4}$$

For a quantile level τ between two adjacent grid points, we use linear interpolation between $\hat{h}(\mathbf{X}, q_{j-1})$ and $\hat{h}(\mathbf{X}, q_j)$. Because each interval area $A_j(\mathbf{X})$ is positive, the grid values are ordered, and the linear interpolation preserves both continuity and monotonicity in τ. An overview of our CMQNet architecture is provided in Figure 1.

3.2 Estimation

To estimate the network parameters Θ, we adopt the standard check (pinball) loss. For a quantile level τ and prediction error $e = Y - \hat{Y}$, the loss is defined as $\rho_\tau(e) = \max\{\tau e, (\tau - 1)e\}$.

During training, we employ a stochastic optimization strategy to learn the global quantile process. For each training sample $(\mathbf{X}_i, Y_i)$ in a mini-batch of size N, we randomly sample M quantile levels $\{\tau_{i,1}, \ldots, \tau_{i,M}\}$ independently from a specified distribution over $(0, 1)$ and minimize

$$\mathcal{L}(\Theta) = \frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} \rho_{\tau_{i,j}}\left(Y_i - \hat{h}(\mathbf{X}_i, \tau_{i,j})\right). \tag{5}$$

Minimizing this objective ensures the model accurately approximates the conditional quantile function across the continuous domain of τ. The design choices regarding the quantile sampling distribution and the number of sampled quantile levels M are discussed in Section 3.3. The detailed training procedure of the proposed method is provided in Algorithm 1 (Appendix 7). We state our main theoretical results on the bounds for the excess risk and estimation error of our estimator in Appendix 6.

3.3 Design of the Quantile Level Sampling Strategy

In this subsection, we introduce the impact of the quantile level sampling strategy on the performance of the proposed method CMQNet. In particular, we investigate (i) the choice of the sampling distribution for the quantile level τ, and (ii) the effect of the number of sampled quantile levels M.

3.3.1 Choice of the quantile-level distribution

Our framework explicitly treats the quantile level τ as an input to the neural network and allows τ to be randomly sampled during training. This design enables flexible and adaptive coverage of the joint input space (X, τ) and provides a principled way to control the effective amount of training information allocated to different quantile levels. In particular, when improved accuracy at extreme quantiles is desired, one may sample τ from distributions that place more mass near the boundaries, such as $\mathrm{Beta}(0.3, 0.3)$ or $\mathrm{Beta}(0.5, 0.5)$. These distributions emphasize tail regions while still maintaining coverage over the entire quantile range. When no particular preference is given to tail quantiles, or when balanced performance across quantile levels is desired, more uniform sampling schemes can be adopted, including $\mathrm{Beta}(1, 1)$ (uniform), $\mathrm{Beta}(2, 2)$, or truncated normal distributions centered at 0.5. These choices allocate more training mass to central quantiles while still preserving continuity in τ. This flexibility distinguishes our approach from traditional methods that rely on a fixed and deterministic set of quantile levels. The practical impact of the τ sampling distribution is quantitatively demonstrated by the ablation results in Section 4.3.1.

3.3.2 EFFECT OF THE NUMBER OF SAMPLED QUANTILE LEVELS

Existing quantile regression methods typically require pre-specifying both the number of quantile levels M and their exact locations, resulting in a fixed and discrete quantile grid throughout training. In contrast, our method allows τ to be randomly sampled at each iteration, so that even with a moderate value of M, the effective set of quantile levels encountered during training is substantially richer than a fixed grid of size M. Consequently, the model can leverage a larger and more diverse set of training pairs (X, τ) under the same data-generating setting, leading to improved estimation efficiency. Moreover, the estimation performance is relatively insensitive to the specific choice of M. This robustness is empirically examined in the ablation studies in Section 4.3.2 and Appendix 9.2.

3.4 THEORETICAL GUARANTEES

We summarize the main theoretical guarantees here and defer the full statements and proofs to Appendix 6. Let $h^*(x,\tau) = Q_Y(\tau \mid X = x)$ denote the true conditional quantile function, $m^*(x) = h^*(x, 0.5)$, and $g^*(x,\tau) = \partial_\tau h^*(x,\tau)$. For a fixed $\tau_0 \in (0, 1/2)$, define the interior quantile region $\mathcal{T} = [\tau_0, 1-\tau_0]$ and the enlarged region $\mathcal{T}^+ = [\tau_0/2, 1-\tau_0/2]$. Under Hölder smoothness of h^* on $\mathcal{X} \times \mathcal{T}^+$, together with $0 < g_{\min} \le g^* \le g_{\max} < \infty$, CMQNet represents the quantile process as

$$h(x,\tau) = f_B(x) + \int_{0.5}^{\tau} f_D(x,t)\, dt, \qquad f_D(x,t) > 0.$$

Hence the ideal continuous model is strictly increasing in τ. Moreover, the implemented fixed-grid construction, based on positive midpoint areas and linear interpolation, preserves monotonicity for every network parameter value.

Let

$$a := \frac{s-1}{p+1}, \qquad \theta := \min\{s-2, 1\}.$$

The implemented fixed-grid CMQNet retains the approximation rate of the continuous model. Specifically, for every $N \ge 2$ and every admissible even grid size K, there exists $h^{\text{grid}}_{N,K} \in \mathcal{F}^{\text{grid}}_{\text{CMQ},K}$, generated by networks with $O(\log N)$ depth and $O(N \log N)$ non-zero parameters, such that

$$\sup_{(x,\tau)\in\mathcal{X}\times\mathcal{T}} \left| h^{\text{grid}}_{N,K}(x,\tau) - h^*(x,\tau) \right| \le C_{\text{approx}} \left\{ N^{-a} + K^{-(1+\theta)} \right\}.$$

Choosing $K \asymp N^{a/(1+\theta)}$ therefore yields an approximation error of order N^{-a}.

For statistical estimation, let $\tau_{ij} \overset{\text{i.i.d.}}{\sim} \pi$, where π is any sampling distribution on $(0,1)$ independent of (X,Y) with $\pi(\mathcal{T}) > 0$, and let $\widehat{h}_{n,\pi,K}$ be the empirical pinball-risk minimizer over the fixed-grid CMQNet class.

Theorem 1 (Finite-sample interior guarantee)**.** *Under the regularity conditions stated in Appendix 6, for any comparator $h^\circ \in \mathcal{F}^{\text{grid}}_{\text{CMQ},K}$ satisfying*

$$\epsilon_{\mathcal{T}}(h^\circ) := \|h^\circ - h^*\|_{L^\infty(\mathcal{X}\times\mathcal{T})} \le \delta_0,$$

we have

$$E\left[\int_{\mathcal{T}} \|\widehat{h}_{n,\pi,K}(\cdot,\tau) - h^*(\cdot,\tau)\|^2_{L^2(P_X)}\, \pi(d\tau) \right] \le \frac{1}{c_{\mathcal{T}}} \left[\frac{\epsilon_{\mathcal{T}}(h^\circ)^2}{2 g_{\min}} + \pi(\mathcal{T}^c)\big(B_{\text{CMQ}} + E|Y|\big) + C_{\text{stat}} \sqrt{\frac{\mathfrak{C}_n}{n}} \right],$$

where $\mathfrak{C}_n = (S+1)\{L_{\max}\log(W_{\max}+1) + \log(en)\}$ measures the network complexity.

The theorem separates the estimation error into approximation, tail, and stochastic components. In particular, the result allows full-support sampling distributions such as $\text{Beta}(0.3, 0.3)$. In this case, $\pi(\mathcal{T}^c)$ is generally positive, so without additional regularity near the endpoints the guarantee is stated on the interior region $\mathcal{T}$, rather than as a global convergence rate over $(0,1)$.

4 EXPERIMENTS

In this section, we conduct several experiments to evaluate the performance of the proposed CMQNet. Our empirical study is organized into three parts. First, we conduct extensive simulation experiments covering a wide range of data-generating mechanisms with varying degrees of nonlinearity, smoothness, and heteroscedasticity to evaluate CMQNet's accuracy across the full quantile range. Additional experiments assessing the smoothness and accuracy of the induced conditional density under asymmetric and heavy-tailed errors are reported in Appendix 7.6. Second, we evaluate CMQNet on several real-world benchmark datasets and compare it with state-of-the-art methods in terms of predictive accuracy and quantile crossing behavior (see, in Appendix 8). Finally, we conduct detailed ablation studies to investigate the effects of key design choices, including the sampling distribution of quantile levels and the number of sampled quantiles during training.

4.1 EXPERIMENTAL SETUP

For simulated experiments, we follow the protocol of NQ-Net: we generate independent train/validation/test sets (512/512/512 for small-sample settings and 5120/512/512 for large-sample settings) and train all methods with Adam for up to 1000 epochs using early stopping on the validation loss. DQR, DQR*, and NQ-Net estimate quantiles on a fixed grid of 19 levels $(0.05, 0.10, \ldots, 0.95)$, while DQRP and CMQNet model the quantile process over $\tau \in (0, 1)$. In particular, CMQNet samples quantile levels from a Beta distribution during training to enable continuous estimation across τ. In simulations, we report MAE/MSE against the ground-truth quantiles on the 19-level evaluation grid, and for real-world datasets we report the averaged test loss and quantile crossing rate under 10-fold cross-validation. Full implementation details, data splitting, hyperparameters, and evaluation protocols are provided in Appendix 7.

4.2 SIMULATED DATA EXPERIMENTS

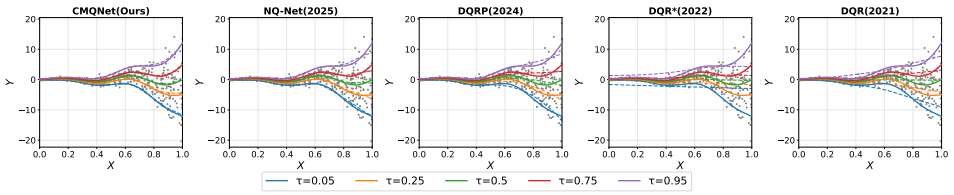
In this section, we evaluate the proposed CMQNet on nonparametric quantile regression problems using simulated data, and compare it with four popular deep quantile regression methods. All competing approaches aim to estimate conditional quantile functions, but differ substantially in how monotonicity across quantile levels and continuity with respect to the quantile index are enforced. We organize the simulation study into two parts: *univariate* settings with a single covariate (Section 4.2.1) and *multivariate* settings with p-dimensional covariates (Section 4.2.2). The complete simulation results are provided in Appendix 11.

4.2.1 UNIVARIATE MODELS

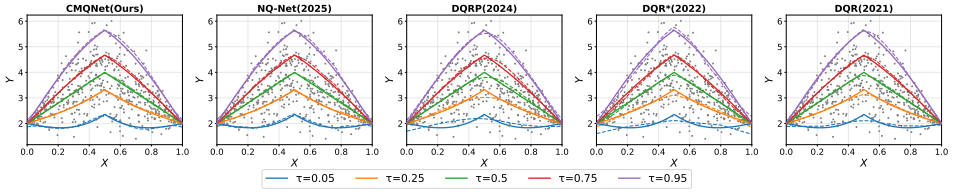
In the univariate simulation study, we draw covariates independently as $X \sim \mathrm{Unif}(0, 1)$. To assess performance across a variety of structural settings, we consider three representative data-generating mechanisms: a **Linear** homoscedastic baseline, a smooth nonlinear **Wave** model with heteroscedasticity, and a nonsmooth **Angle**-shaped model with heteroscedasticity. The explicit definitions of the conditional quantile functions are deferred to Appendix 7.5.

To further capture diverse noise characteristics, we consider three representative distributions for the error term: the normal, log-normal, and exponential distributions. The normal distribution represents a symmetric, light-tailed noise structure commonly assumed in classical quantile regression. In contrast, the log-normal and exponential distributions introduce asymmetric and skewed error structures. Including both symmetric and asymmetric cases enables us to assess the robustness of our method under increasingly challenging and distributionally heterogeneous scenarios.

Figure 2 displays the scatter plot of the simulated data together with the estimated conditional quantile curves of the five methods at five quantile levels $\tau \in \{0.05, 0.25, 0.50, 0.75, 0.95\}$ for the wave and angle models. The results show that CMQNet achieves accurate quantile estimation under both settings. By contrast, the competing methods exhibit noticeable degradation in estimation accuracy near the boundary quantiles, whereas CMQNet maintains stable and accurate performance even at extreme quantile levels. In terms of MAE and MSE, CMQNet achieves excellent performance across the considered settings and, in most cases, outperforms the competing methods. An overall summary is reported in Table 10, with detailed results provided in Tables 11–28 in Appendix 11.



(a) Wave model (smooth nonlinear)



(b) Angle model (nonsmooth nonlinear)

Figure 2: Scatter plot of the simulated data together with conditional quantile curves at $\tau \in \{0.05, 0.25, 0.50, 0.75, 0.95\}$ under the univariate setting. Solid lines denote the ground-truth quantile functions, and dashed lines denote the estimates produced by each competing method (each panel corresponds to one method). Curves of different colors correspond to different quantile levels.

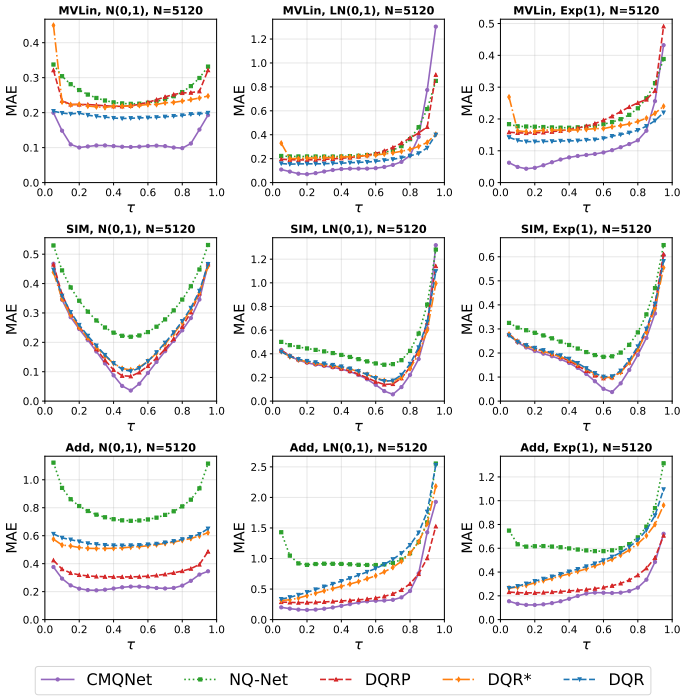


Figure 3: Comparison of MAE across quantile levels $\tau \in [0.05, 0.95]$ for five quantile regression methods under normal, log-normal, and exponential error distributions. Results are averaged over 100 runs with $N = 5120$ training samples.

4.2.2 Multivariate Models

We further evaluate CMQNet in multivariate settings and compare it with other methods. In all experiments, the covariate vector $\mathbf{X} = (X_1, \ldots, X_p)^\top$ is generated i.i.d. from $\mathrm{Unif}([0,1]^p)$, and the error term follows the same three distributions used in the univariate study (normal, log-normal, and exponential). To cover a range of structural complexities, we consider three representative multivariate data-generating mechanisms: a multivariate linear (**MVLin**) model that serves as a

baseline and extends the univariate linear setting to higher dimensions; a single-index (**SIM**) nonlinear model that introduces smooth nonlinear effects through a low-dimensional projection and allows for heteroscedasticity across quantile levels; and an additive (**Add**) nonlinear model that further combines smooth and non-smooth components with multiplicative heteroscedastic effects, yielding a flexible and challenging multivariate setting. The explicit definitions of these models, including coefficient choices, are provided in Appendix 7.5.

Figure 3 visualizes the comparison of MAE across quantile levels $\tau \in [0.05, 0.95]$ for five methods under normal, log-normal, and exponential error distributions. All results are averaged over 100 simulation runs with $N = 5120$ training samples. As shown in the figure, our CMQNet method demonstrates superior estimation accuracy across a wide range of quantile levels, different model settings and error distributions, outperforming the competing methods in most cases. An overall summary of the numerical results is reported in Table 10, with more detailed results provided in Tables 29–46 in Appendix 11.

4.3 Ablation Study

4.3.1 Effect of the τ Sampling Distribution.

We investigate the impact of the τ sampling distribution on the performance of CMQNet. Since estimation near boundary quantiles is typically less stable due to fewer effective observations in the tails, we consider five Beta distributions, $\mathrm{Beta}(0.3, 0.3)$, $\mathrm{Beta}(0.5, 0.5)$, $\mathrm{Beta}(1, 1)$, $\mathrm{Beta}(2, 2)$, and $\mathrm{Beta}(1, 0.5)$. These distributions respectively emphasize extreme quantiles, provide near-uniform coverage, concentrate more mass around central quantiles, or introduce asymmetric emphasis toward one side of the quantile range.

We evaluate performance using two complementary criteria: (i) the test loss computed on a fixed grid of 19 quantile levels $\tau \in \{0.05, 0.10, \ldots, 0.95\}$ (Table 4a in Appendix 9.1), and (ii) tail-focused errors evaluated at extreme quantiles $\tau < 0.05$ and $\tau > 0.95$ (Table 4b in Appendix 9.1). The test losses in Table 4a are broadly comparable across sampling schemes, suggesting that CMQNet is robust when estimating central and moderately extreme quantiles. By comparison, the tail-focused results in Table 4b show that emphasizing boundary quantiles via $\mathrm{Beta}(0.3, 0.3)$ yields the best tail performance, indicating that allocating more sampling mass to extreme τ values improves estimation accuracy in the tails.

4.3.2 Effect of the Number of Sampled Quantile Levels and Positive Activations.

We further study the effect of the number of sampled quantile levels M and the choice of positive activation function used to enforce monotonicity. As shown in Appendix 9.2, CMQNet is relatively insensitive to M, with test losses remaining nearly unchanged once a moderate number of quantile levels is sampled. Appendix 9.3 demonstrates that CMQNet is robust to the choice of positive activation functions and we adopt Exponential in our main experiments for numerical stability and conditional density estimation. In addition, Appendix 10 reports the average runtime of the five competing methods under varying numbers of sampled quantile levels M, providing a detailed comparison of their computational efficiency.

5 Conclusion

In this paper, we propose **Continuous and Monotone Quantile Networks (CMQNet)** for learning conditional quantile functions as continuous and non-crossing processes over τ. By modeling the quantile function as a baseline plus the integral of a strictly nonnegative derivative network, CMQNet guarantees monotonicity by construction and yields a smooth implied conditional density. Extensive experiments on both simulated and real datasets demonstrate consistent improvements in accuracy, stability, and distributional consistency over state-of-the-art deep quantile regression methods. One limitation of CMQNet is that its performance, particularly at extreme quantile levels, may depend on the availability of sufficient tail observations and the choice of quantile-level sampling strategy. Future work may further investigate adaptive sampling schemes to improve robustness in extreme quantile estimation.

AI USE STATEMENT

Generative AI tools were used only for language polishing. They were not used to develop the research ideas, methodology, theoretical results or proofs, experimental design, code, data analysis, or interpretation of results. All scientific contributions, implementations, experiments, and conclusions were produced by the authors. All AI-assisted edits were reviewed and revised by the authors as appropriate, and the authors take full responsibility for the final content of the paper.

REPRODUCIBILITY STATEMENT

We have made extensive efforts to ensure the reproducibility of our results. An anonymous implementation of CMQNet is provided at `https://anonymous.4open.science/r/CMQNet-C420/`. The appendix provides complete descriptions of the model architecture, training procedure, numerical integration scheme, data preprocessing and splitting strategies, hyperparameter settings, and evaluation protocols. Detailed definitions of all simulation settings and additional experimental results are also reported in the appendix. The real-world benchmark datasets used in our experiments are publicly available.

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6 APPENDIX A: THEORETICAL ANALYSIS

In this appendix we provide the theoretical justification for the CMQNet framework proposed in the main text.

6.1 SETUP AND NOTATION

Let $(X, Y) \in \mathcal{X} \times \mathbb{R}$, where $\mathcal{X} = [0,1]^p$. For every $\tau \in (0,1)$ define the true conditional quantile function by

$$h^*(x,\tau) = Q_Y(\tau \mid X = x) = \inf\{y : F_{Y|X}(y \mid x) \geq \tau\}. \tag{6}$$

Fix $0 < \tau_0 < \frac{1}{2}$ and set

$$\mathcal{T} = [\tau_0, 1-\tau_0], \qquad \mathcal{T}^+ = \left[\tfrac{\tau_0}{2}, 1-\tfrac{\tau_0}{2}\right], \qquad \mathcal{D} = \mathcal{X} \times \mathcal{T}, \qquad \mathcal{D}^+ = \mathcal{X} \times \mathcal{T}^+.$$

On $\mathcal{D}^+$ define

$$m^*(x) = h^*(x, 0.5), \qquad g^*(x,\tau) = \partial_\tau h^*(x,\tau).$$

The finite-sample L^2 guarantees below are stated on $\mathcal{T}$. $\mathcal{T}^+$ serves only as a technical buffer for the fixed-grid argument. Let π be a Borel probability measure on $(0,1)$, independent of (X,Y), with $\pi(\mathcal{T}) > 0$. The sampling law π may be continuous, discrete, or mixed, and may have full support on $(0,1)$. When it admits a Lebesgue density, we denote the density by w.

Definition 1 (Hölder function class)**.** *Let $\mathcal{R} \subset \mathbb{R}^r$ be a compact rectangle. For $s = m + \gamma > 0$ with $m \in \mathbb{Z}_{\geq 0}$ and $\gamma \in (0,1]$, let $\mathcal{H}^s(\mathcal{R}; C_H)$ be the set of functions f satisfying*

$$\max_{|\alpha| \leq m} \|D^\alpha f\|_\infty + \max_{|\alpha| = m} \sup_{z \neq z'} \frac{|D^\alpha f(z) - D^\alpha f(z')|}{\|z - z'\|^\gamma} \leq C_H.$$

Allowing $\gamma = 1$ includes integer values of s. When no confusion arises we write $\mathcal{H}^s(\mathcal{R})$.

Definition 2 (Sparse ReLU network class)**.** *For input dimension r, let $\mathcal{N}_r(L, W, S, B_W)$ denote the class of functions represented by ReLU feedforward networks with at most L hidden layers, at most W neurons per hidden layer, at most S non-zero weights and biases, and all parameters bounded in absolute value by a fixed constant $B_W \geq 1$. Throughout, B_W is fixed independently of n and of the architecture.*

For fixed constants $B_0 > 0$ and $A > 0$ define

$$\chi_{B_0}(u) = \mathrm{clip}(u, -B_0, B_0), \qquad \psi_A(u) = \exp\{\mathrm{clip}(u, -A, A)\}.$$

The map χ_{B_0} is 1-Lipschitz, ψ_A is globally e^A-Lipschitz, and $e^{-A} \leq \psi_A(u) \leq e^A$. Define

$$\mathcal{N}_r^{B_0}(L, W, S, B_W) = \{\chi_{B_0} \circ q : q \in \mathcal{N}_r(L, W, S, B_W)\},$$
$$\mathcal{N}_{A,r}^{+}(L, W, S, B_W) = \{\psi_A \circ q : q \in \mathcal{N}_r(L, W, S, B_W)\}.$$

Definition 3 (Continuous CMQNet function class)**.** *Let $f_B \in \mathcal{N}_p^{B_0}(L_B, W_B, S_B, B_W)$ and $f_D \in \mathcal{N}_{A,p+1}^{+}(L_D, W_D, S_D, B_W)$. The ideal continuous CMQNet generated by (f_B, f_D) is*

$$h^{\mathrm{cont}}_{f_B, f_D}(x,\tau) = f_B(x) + \int_{0.5}^{\tau} f_D(x,t)\, dt, \qquad \tau \in [0,1], \tag{7}$$

and the corresponding class is denoted $\mathcal{F}^{\mathrm{cont}}_{\mathrm{CMQ}}$.

Definition 4 (Fixed-grid numerical CMQNet class)**.** *Fix an even integer $K \geq 2$, let $q_j = j/K$ for $j = 0, \ldots, K$, so that $q_{K/2} = 0.5$, and write $\bar{q}_j = (q_{j-1} + q_j)/2$. For the same pair (f_B, f_D) define the positive midpoint areas*

$$\alpha_j(x) = \frac{1}{K} f_D(x, \bar{q}_j), \qquad j = 1, \ldots, K.$$

At the grid points set

$$h^{\mathrm{grid}}_{f_B,f_D,K}(x,q_j)=\begin{cases} f_B(x)-\displaystyle\sum_{\ell=j+1}^{K/2}\alpha_\ell(x), & j<K/2,\\ f_B(x), & j=K/2,\\ f_B(x)+\displaystyle\sum_{\ell=K/2+1}^{j}\alpha_\ell(x), & j>K/2, \end{cases} \tag{8}$$

and extend to every $\tau\in[0,1]$ by linear interpolation between adjacent grid points. The implemented class is

$$\mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}=\big\{h^{\mathrm{grid}}_{f_B,f_D,K}:(f_B,f_D)\ \textit{as in Definition 3}\big\}=\mathcal{I}_K\big(\mathcal{F}^{\mathrm{cont}}_{\mathrm{CMQ}}\big),$$

where $\mathcal{I}_K$ denotes the midpoint cumulative-sum and linear-interpolation operator acting on the generating pair.

Proposition 1 (Structural properties)**.** *For every $h^{\mathrm{cont}}_{f_B,f_D}$ and every fixed x, the map $\tau\mapsto h^{\mathrm{cont}}_{f_B,f_D}(x,\tau)$ is absolutely continuous with $\partial_\tau h^{\mathrm{cont}}_{f_B,f_D}(x,\tau)=f_D(x,\tau)\geq e^{-A}$, hence strictly increasing, and $\|h^{\mathrm{cont}}_{f_B,f_D}\|_\infty\leq B_0+\frac{1}{2}e^A$. For every $h^{\mathrm{grid}}_{f_B,f_D,K}$ the map $\tau\mapsto h^{\mathrm{grid}}_{f_B,f_D,K}(x,\tau)$ is continuous, piecewise affine, absolutely continuous and strictly increasing, and*

$$\big\|h^{\mathrm{grid}}_{f_B,f_D,K}\big\|_\infty\leq B_0+\tfrac{1}{2}e^A=:B_{\mathrm{CMQ}}. \tag{9}$$

The bound is independent of K and of the architecture.

Proof. For the continuous class, absolute continuity follows from Eq. (7) and the boundedness and continuity of f_D, and the fundamental theorem of calculus gives the derivative identity; $f_D\geq e^{-A}>0$ gives strict monotonicity, and $|h^{\mathrm{cont}}|\leq|f_B|+|\tau-0.5|\,\|f_D\|_\infty\leq B_0+\frac{1}{2}e^A$. For the fixed-grid class, $h^{\mathrm{grid}}(x,q_j)-h^{\mathrm{grid}}(x,q_{j-1})=\alpha_j(x)\geq K^{-1}e^{-A}>0$, so the grid values are strictly increasing; linear interpolation preserves strict monotonicity and yields a continuous piecewise affine, hence absolutely continuous, function. The accumulated path from 0.5 to either endpoint contains at most $K/2$ areas, each bounded by $K^{-1}e^A$, which gives Eq. (9). □

Remark 1 (Induced density of the continuous model)**.** *For the ideal continuous CMQNet,*

$$\partial_\tau h^{\mathrm{cont}}_{f_B,f_D}(x,\tau)=f_D(x,\tau)>0.$$

Consequently, for $\tau\in(0,1)$ and at points in the interior of the support of the conditional distribution induced by $h^{\mathrm{cont}}_{f_B,f_D}(x,\cdot)$, the usual quantile–density identity gives

$$f_{h^{\mathrm{cont}}}\big(h^{\mathrm{cont}}_{f_B,f_D}(x,\tau)\mid x\big)=\{f_D(x,\tau)\}^{-1}.$$

Thus the density associated with the ideal continuous model is determined directly by the positive derivative network; it is not obtained by interpolating or differencing the fixed-grid quantile estimates. The results below concern quantile estimation and do not, by themselves, establish density-estimation consistency.

Remark 2 (Existence and measurability of the sieve minimizer)**.** *For a fixed sparsity pattern the parameter set is the compact cube $[-B_W,B_W]^S$, the map from parameters to the fixed-grid predictor is continuous in the supremum norm, and the empirical risk in Eq. (20) is continuous in the predictor. Since there are finitely many admissible sparsity patterns, the infimum in Eq. (22) is attained; a measurable empirical minimizer can be chosen by a standard measurable-selection argument. If an approximate minimizer $\widetilde{h}$ is used instead and satisfies*

$$\widehat{\mathcal{L}}_{n,\pi,K}(\widetilde{h})\leq\inf_{h\in\mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}}\widehat{\mathcal{L}}_{n,\pi,K}(h)+\epsilon_n,$$

then the right-hand sides of the oracle inequalities below acquire an additional additive term ϵ_n. In particular, their stated stochastic order is unchanged whenever $\epsilon_n=O\{\sqrt{\mathfrak{C}_n/n}\}$, with $\mathfrak{C}_n$ defined in Eq. (26).

6.2 Structural Properties of CMQNet

6.2.1 Mathematical Foundations

The validity of Eq. (3) follows from the classical characterization of absolutely continuous functions: an absolutely continuous function can be reconstructed from its derivative by integration. This result provides the mathematical foundation for the integral representation of the conditional quantile process used by CMQNet.

6.2.2 Absolute Continuity of Neural Networks in the Quantile Index

We next verify that standard neural networks satisfy the absolute continuity requirement needed in Lemma 1.

Proof of Lemma 2. Fix x and argue by induction over the finitely many layers, the induction hypothesis being that every coordinate of the layer output is absolutely continuous as a function of $\tau \in [0, 1]$. The pre-activation of the first affine layer is affine in τ, hence Lipschitz, hence absolutely continuous in each coordinate. Suppose the vector-valued output $v(\tau) \in \mathbb{R}^k$ of a given layer is absolutely continuous coordinatewise. Absolute continuity implies continuity, so $v([0, 1])$ is a compact subset of $\mathbb{R}^k$. Every locally Lipschitz activation is Lipschitz on a neighbourhood of that compact set; composing an absolutely continuous vector-valued map with a map that is Lipschitz on a compact set containing its range preserves absolute continuity coordinatewise. Affine transformations are globally Lipschitz and therefore also preserve it. Since a finite linear combination of absolutely continuous functions is absolutely continuous, the induction closes, and $\tau \mapsto h(x, \tau)$ is absolutely continuous. □

Remark 3. *The result holds for common locally Lipschitz activations, including ReLU, Softplus, ELU,* $\tanh$*, sigmoid, and exponential activations. No differentiability of the activation at every point is required.*

6.2.3 Structural Properties of the Numerical Approximation in Eq. (4)

We further justify that the implemented fixed-grid predictor is absolutely continuous in the quantile index τ for any fixed covariate $\mathbf{X}$.

Proposition 2 (Absolute continuity and monotonicity of the fixed-grid predictor)**.** *Fix any* $\mathbf{X}$*. Under the approximation in Eq. (4), together with linear interpolation between adjacent grid points, the mapping* $\tau \mapsto \hat{h}(\mathbf{X}, \tau)$ *is absolutely continuous. If every interval area is strictly positive, then this mapping is strictly increasing.*

Proof. Fix $\mathbf{X}$, and let $q_0 < q_1 < \cdots < q_K$ be the fixed grid used in Eq. (4). Let $\hat{h}(\mathbf{X}, q_j)$ denote the grid-level estimates obtained by cumulative summation of the interval areas $\alpha_j(\mathbf{X})$. For $\tau \in [q_{j-1}, q_j]$,

$$\hat{h}(\mathbf{X}, \tau) = \frac{q_j - \tau}{q_j - q_{j-1}} \hat{h}(\mathbf{X}, q_{j-1}) + \frac{\tau - q_{j-1}}{q_j - q_{j-1}} \hat{h}(\mathbf{X}, q_j),$$

which is affine in τ on each grid interval and continuous at all grid points. Hence $\tau \mapsto \hat{h}(\mathbf{X}, \tau)$ is continuous and piecewise affine on a finite partition, and therefore absolutely continuous. Moreover $\hat{h}(\mathbf{X}, q_j) - \hat{h}(\mathbf{X}, q_{j-1}) = \alpha_j(\mathbf{X})$, so if $\alpha_j(\mathbf{X}) > 0$ for every j the grid values are strictly increasing. Linear interpolation between two strictly ordered adjacent values has positive slope, so strict monotonicity is preserved on the whole quantile interval. □

6.3 Assumptions and Basic Consequences

Assumption 1 (Hölder smoothness)**.** *For some* $s \geq 2$*, written* $s = m + \gamma$ *with* $m \in \mathbb{Z}_{\geq 0}$ *and* $\gamma \in (0, 1]$ *as in Definition 1,* $h^* \in \mathcal{H}^s(\mathcal{D}^+; C_H)$.

Assumption 2 (Strict monotonicity)**.** *There exist constants* $0 < g_{\min} \leq g_{\max} < \infty$ *such that*

$$g_{\min} \leq g^*(x, \tau) \leq g_{\max}, \qquad (x, \tau) \in \mathcal{D}^+.$$

Assumption 3 (Moment and interior boundedness)**.** $E|Y| < \infty$*, and for some finite* B*,* $\|h^*\|_{L^\infty(\mathcal{D}^+)} \leq B$.

Remark 4. *Assumption 1 implies $g^* \in \mathcal{H}^{s-1}(\mathcal{D}^+)$ and, since $s \geq 2$, g^* is Lipschitz in τ on $\mathcal{T}^+$. Assumption 2 guarantees strict monotonicity of the target and permits a stable positive-network approximation. Assumption 3 serves two purposes: the moment condition $E|Y| < \infty$ ensures that the population pinball risks used below are finite, while the interior bound $\|h^*\|_{L^\infty(\mathcal{D}^+)} \leq B$ makes the baseline clipping inactive for $B_0 > B$ and provides a uniform interior bound needed in the self-calibration argument.*

To simplify the approximation bounds below, define

$$\theta := \min\{s-2,\ 1\} \in [0,1], \qquad a := \frac{s-1}{p+1}. \tag{10}$$

Here a is the approximation exponent associated with the derivative function $g^*(x,\tau)$, which has smoothness $s-1$ and input dimension $p+1$, while θ determines the order of the midpoint quadrature error. Choose the clipping levels such that

$$B_0 > B, \qquad A > \max\{-\log g_{\min},\ \log g_{\max}\}. \tag{11}$$

Since $|m^*(x)| \leq B$ and $g_{\min} \leq g^*(x,\tau) \leq g_{\max}$ on $\mathcal{D}^+$, these choices ensure that the clipping operations do not alter the target functions:

$$\chi_{B_0}\{m^*(x)\} = m^*(x), \qquad \psi_A\{\log g^*(x,\tau)\} = g^*(x,\tau), \qquad (x,\tau) \in \mathcal{D}^+.$$

Thus, clipping is used only to control the range of the neural-network class and does not introduce additional approximation bias at the target.

Lemma 3 (Distributional regularity implied by the derivative bounds)**.** *Let Assumptions 1–2 hold and set*

$$\delta_0 := \frac{g_{\min}\tau_0}{4}. \tag{12}$$

Then, for every $x \in \mathcal{X}$ and $\tau \in \mathcal{T}$,

$$F_{Y|X}\big(h^*(x,\tau) \mid x\big) = \tau,$$

and for all $u_1 \leq u_2$ with $|u_i - h^*(x,\tau)| \leq \delta_0$, $i = 1,2$,

$$\frac{u_2 - u_1}{g_{\max}} \ \leq\ F_{Y|X}(u_2 \mid x) - F_{Y|X}(u_1 \mid x) \ \leq\ \frac{u_2 - u_1}{g_{\min}}. \tag{13}$$

Proof. Fix x. On $\mathcal{T}^+$, the map $t \mapsto h^*(x,t)$ is continuously differentiable and strictly increasing, because its derivative satisfies $g_{\min} \leq \partial_t h^*(x,t) \leq g_{\max}$. Hence it is one-to-one on $\mathcal{T}^+$ and admits a continuous inverse on its image $\mathcal{U}_x := h^*(x,\mathcal{T}^+)$. By the standard duality between a continuous strictly increasing quantile function and its distribution function, $F_{Y|X}(h^*(x,t) \mid x) = t$ for every t in the interior of $\mathcal{T}^+$.

Let $\tau \in \mathcal{T}$. Since $\mathcal{T} = [\tau_0, 1-\tau_0]$ and $\mathcal{T}^+ = [\tau_0/2, 1-\tau_0/2]$, the point τ is at least $\tau_0/2$ away from both endpoints of $\mathcal{T}^+$. Therefore, using $g^*(x,t) \geq g_{\min}$,

$$h^*(x,\tau) - h^*(x,\tau_0/2) = \int_{\tau_0/2}^{\tau} g^*(x,t)\,dt \geq g_{\min}\frac{\tau_0}{2},$$

and similarly,

$$h^*(x,1-\tau_0/2) - h^*(x,\tau) \geq g_{\min}\frac{\tau_0}{2}.$$

Since $\delta_0 = g_{\min}\tau_0/4$, both quantities are larger than δ_0. Hence

$$[h^*(x,\tau) - \delta_0,\ h^*(x,\tau) + \delta_0] \subset \mathcal{U}_x.$$

Now let $u_1 \leq u_2$ belong to this interval. Since $t \mapsto h^*(x,t)$ is strictly increasing on $\mathcal{T}^+$ and hence invertible on $\mathcal{U}_x$, there exist unique $t_1, t_2 \in \mathcal{T}^+$ such that

$$u_i = h^*(x,t_i), \qquad i = 1,2.$$

Moreover, $u_1 \leq u_2$ implies $t_1 \leq t_2$. By the quantile identity,

$$F_{Y|X}(u_i \mid x) = t_i, \qquad i = 1,2.$$

Therefore,

$$u_2 - u_1 = h^*(x, t_2) - h^*(x, t_1) = \int_{t_1}^{t_2} g^*(x, t)\, dt.$$

Using $g_{\min} \le g^*(x, t) \le g_{\max}$ gives

$$g_{\min}(t_2 - t_1) \le u_2 - u_1 \le g_{\max}(t_2 - t_1).$$

Since

$$t_2 - t_1 = F_{Y|X}(u_2 \mid x) - F_{Y|X}(u_1 \mid x),$$

rearranging yields Eq. (13). □

6.4 Approximation Theory

Lemma 4 (Positive-network approximation). *Let $\beta > 0$ and let $g \in \mathcal{H}^\beta(\mathcal{D}^+)$ satisfy $g_{\min} \le g \le g_{\max}$ on $\mathcal{D}^+$. Under Eq. (11), for every integer $N \ge 2$ there exists $f_D \in \mathcal{N}^+_{A,p+1}(L_D, W_D, S_D, B_W)$ with*

$$L_D = O(\log N), \qquad W_D = O(N), \qquad S_D = O(N \log N),$$

such that

$$\sup_{z \in \mathcal{D}^+} |f_D(z) - g(z)| \le C_D\, N^{-\beta/(p+1)}, \tag{14}$$

where C_D depends only on the fixed smoothness and boundedness constants, the fixed clipping level A, and τ_0. Moreover $e^{-A} \le f_D \le e^A$ everywhere.

Proof. Let $\phi := \log g$. Since $g \in \mathcal{H}^\beta(\mathcal{D}^+)$ and $g_{\min} \le g \le g_{\max}$ on $\mathcal{D}^+$, the function $\log(\cdot)$ is smooth with bounded derivatives on $[g_{\min}, g_{\max}]$. Hence, the standard composition rule for Hölder functions implies that $\phi \in \mathcal{H}^\beta(\mathcal{D}^+)$, with Hölder norm bounded by a constant depending only on $g_{\min}$, $g_{\max}$, β, p, and the Hölder norm of g. After an affine rescaling of $\mathcal{D}^+$ to the unit cube, standard sparse ReLU approximation results for Hölder functions (Schmidt-Hieber, 2020) imply that, for every $N \ge 2$, there exists a ReLU network q_N with

$$L_D = O(\log N), \qquad W_D = O(N), \qquad S_D = O(N \log N),$$

and uniformly bounded parameters such that

$$\|q_N - \phi\|_{L^\infty(\mathcal{D}^+)} \le C N^{-\beta/(p+1)}.$$

Define $f_D := \psi_A \circ q_N$. By Eq. (11), $\phi(x, \tau) \in (-A, A)$ on $\mathcal{D}^+$, and therefore

$$\psi_A\{\phi(x, \tau)\} = \exp\{\phi(x, \tau)\} = g(x, \tau).$$

Moreover, ψ_A is globally e^A-Lipschitz. Hence

$$\begin{aligned} \|f_D - g\|_{L^\infty(\mathcal{D}^+)} &= \|\psi_A(q_N) - \psi_A(\phi)\|_{L^\infty(\mathcal{D}^+)} \\ &\le e^A \|q_N - \phi\|_{L^\infty(\mathcal{D}^+)} \\ &\le C_D N^{-\beta/(p+1)}. \end{aligned}$$

Finally, by the definition of ψ_A,

$$e^{-A} \le f_D(x, \tau) \le e^A \qquad \text{for all } (x, \tau) \in \mathcal{D}^+.$$

This proves the claim. □

Theorem 2 (Approximation by the continuous CMQNet class). *Under Assumptions 1–3, for every integer $N \ge 2$ there exists $h_N^{\text{cont}} \in \mathcal{F}_{\text{CMQ}}^{\text{cont}}$ generated by networks with*

$$L_B, L_D = O(\log N), \qquad W_B, W_D = O(N), \qquad S_B, S_D = O(N \log N),$$

such that

$$\left\| h_N^{\text{cont}} - h^* \right\|_{L^\infty(\mathcal{D})} \le C_{\text{cont}}\, N^{-a}. \tag{15}$$

Proof. Since $m^* \in \mathcal{H}^s([0,1]^p)$, a sparse ReLU Hölder approximation result followed by the non-expansive clipping χ_{B_0} yields $f_{B,N}$ with $\|f_{B,N} - m^*\|_\infty \le C_B N^{-s/p}$. Also $g^* \in \mathcal{H}^{s-1}(\mathcal{D}^+)$ with $g_{\min} \le g^* \le g_{\max}$, so Lemma 4 with $\beta = s-1$ gives $f_{D,N}$ with $\|f_{D,N} - g^*\|_{L^\infty(\mathcal{D}^+)} \le C_D N^{-a}$. Setting $h_N^{\mathrm{cont}}(x,\tau) = f_{B,N}(x) + \int_{0.5}^{\tau} f_{D,N}(x,t)\,dt$ and noting that for $\tau \in \mathcal{T}$ the whole integration path lies in $\mathcal{T}^+$,

$$|h_N^{\mathrm{cont}}(x,\tau) - h^*(x,\tau)| \le C_B N^{-s/p} + \Big(\tfrac{1}{2} - \tau_0\Big) C_D N^{-a}.$$

Since $s/p \ge (s-1)/(p+1) = a$, Eq. (15) follows. □

Lemma 5 (Interior quadrature error of the target fixed-grid area sum)**.** *Let Assumption 1 hold and let $\theta = \min\{s-2, 1\}$ as in Eq. (10). Consider the uniform grid $q_j = j/K$ with K even and $K^{-1} \le \tau_0/2$. For $x \in \mathcal{X}$ define $H_K^*(x,\cdot)$ at those grid points q_j lying on the accumulation path between 0.5 and $\mathcal{T}$ by*

$$H_K^*(x, q_j) = m^*(x) + \mathrm{sgn}(q_j - 0.5) \sum_{\ell \text{ between } K/2 \text{ and } j} \frac{1}{K} g^*(x, \bar{q}_\ell),$$

and extend it to $\mathcal{T}$ by linear interpolation between adjacent such grid points. Then

$$\sup_{(x,\tau)\in\mathcal{D}} \big|H_K^*(x,\tau) - h^*(x,\tau)\big| \le C_R\, K^{-(1+\theta)}, \tag{16}$$

where C_R depends only on C_H and τ_0.

Proof. Write $\Delta = K^{-1}$ and fix x; all τ-arguments below lie in $\mathcal{T}^+$. If $\theta = 0$, then g^* is C_H-Lipschitz in τ and, on a grid interval with midpoint $\bar{q}_j$,

$$\Big| \int_{q_{j-1}}^{q_j} g^*(x,t)\,dt - \Delta g^*(x, \bar{q}_j) \Big| \le \int_{q_{j-1}}^{q_j} C_H |t - \bar{q}_j|\,dt = \frac{C_H \Delta^2}{4}.$$

If $\theta > 0$, then $\partial_\tau^2 h^*$ exists and is θ-Hölder in τ on $\mathcal{T}^+$ with constant at most C_H. Using $g^*(x,t) - g^*(x,\bar{q}_j) - \partial_\tau g^*(x,\bar{q}_j)(t-\bar{q}_j) = \int_{\bar{q}_j}^t \{\partial_\tau g^*(x,v) - \partial_\tau g^*(x,\bar{q}_j)\}dv$ and the fact that the linear term integrates to zero over the interval by symmetry of the midpoint rule,

$$\Big| \int_{q_{j-1}}^{q_j} g^*(x,t)\,dt - \Delta g^*(x, \bar{q}_j) \Big| \le \int_{q_{j-1}}^{q_j} C_H |t - \bar{q}_j|^{1+\theta}\,dt \le C_H\, \Delta^{2+\theta}.$$

Both cases are covered by the bound $C_H \Delta^{2+\theta}$ up to a fixed constant. At most $K/2$ intervals are accumulated between 0.5 and a point of $\mathcal{T}$, so the grid-point error is at most $\frac{1}{2} C_H \Delta^{1+\theta}$.

Since $s \ge 2$, $g^* = \partial_\tau h^*$ is Lipschitz in τ on $\mathcal{T}^+$. Therefore, the linear interpolation error of h^* on one grid interval is at most $C_H \Delta^2/8$. Since $\theta \le 1$ we have $\Delta^2 \le \Delta^{1+\theta}$, and adding the two contributions gives Eq. (16). □

Theorem 3 (Approximation by the implemented fixed-grid class)**.** *Under Assumptions 1–3, for every integer $N \ge 2$ and every even K with $K^{-1} \le \tau_0/2$ there exists $h_{N,K}^{\mathrm{grid}} \in \mathcal{F}_{\mathrm{CMQ},K}^{\mathrm{grid}}$ generated by networks with $L_B, L_D = O(\log N)$, $W_B, W_D = O(N)$, $S_B, S_D = O(N \log N)$, such that*

$$\big\|h_{N,K}^{\mathrm{grid}} - h^*\big\|_{L^\infty(\mathcal{D})} \le C_{\mathrm{grid}} \big\{N^{-a} + K^{-(1+\theta)}\big\}. \tag{17}$$

In particular, taking

$$K = 2\Big\lceil \max\Big\{\tfrac{1}{\tau_0},\ \tfrac{1}{2} N^{a/(1+\theta)}\Big\}\Big\rceil \tag{18}$$

gives

$$\big\|h_{N,K}^{\mathrm{grid}} - h^*\big\|_{L^\infty(\mathcal{D})} \le C_{\mathrm{approx}}\, N^{-a}. \tag{19}$$

Proof. Use the networks $f_{B,N}$ and $f_{D,N}$ from Theorem 2, and let H_K^* be as in Lemma 5. At any grid point on the accumulation path, the difference between the network cumulative sum and the target cumulative sum is bounded by

$$\|f_{B,N} - m^*\|_\infty + |q_j - 0.5|\,\|f_{D,N} - g^*\|_{L^\infty(\mathcal{D}^+)}.$$

Both $h^{\mathrm{grid}}_{N,K}$ and H^*_K are linearly interpolated between adjacent grid points using the same interpolation weights. Therefore, the same bound extends to every $\tau \in \mathcal{T}$. Hence,

$$\left|h^{\mathrm{grid}}_{N,K}(x,\tau) - H^*_K(x,\tau)\right| \le C_B N^{-s/p} + \left(\tfrac{1}{2} - \tau_0 + \tfrac{1}{K}\right) C_D N^{-a} \lesssim N^{-a}.$$

Here, $K^{-1} \le \tau_0/2$ ensures that every derivative evaluation needed for a point in $\mathcal{T}$ lies within $\mathcal{T}^+$. By the triangle inequality, combining this bound with Eq. (16) yields Eq. (17). Finally, choosing K according to Eq. (18) gives $K^{-(1+\theta)} \lesssim N^{-a}$, and hence Eq. (19) follows. □

6.5 Estimation under a General Sampling Distribution

The implemented estimator belongs to $\mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}$. Training quantile indices are drawn from π defined in section 6.1. With J draws attached to each observation, define

$$\widehat{\mathcal{L}}_{n,\pi,K}(h) = \frac{1}{nJ}\sum_{i=1}^{n}\sum_{j=1}^{J} \rho_{\tau_{ij}}\{Y_i - h(X_i,\tau_{ij})\}, \qquad \tau_{ij} \overset{\text{i.i.d.}}{\sim} \pi, \tag{20}$$

$$\mathcal{L}_\pi(h) = E_{X,Y} E_{\tau\sim\pi}\big[\rho_\tau\{Y - h(X,\tau)\}\big], \tag{21}$$

and let

$$\widehat{h}_{n,\pi,K} \in \arg\min_{h\in\mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}} \widehat{\mathcal{L}}_{n,\pi,K}(h). \tag{22}$$

Write $L_h(x,y,\tau) = \rho_\tau\{y - h(x,\tau)\}$ and define the population excess risk

$$\mathcal{E}_\pi(h) := \mathcal{L}_\pi(h) - \mathcal{L}_\pi(h^*).$$

Lemma 6 (Well-posedness and a uniform excess-risk bound)**.** *Let $E|Y| < \infty$. Then $\mathcal{L}_\pi(h^*) \le E|Y| < \infty$ and, for every h with $\|h\|_\infty \le B_{\mathrm{CMQ}}$,*

$$0 \;\le\; \mathcal{E}_\pi(h) \;\le\; B_{\mathrm{CMQ}} + E|Y|. \tag{23}$$

Moreover, for any Borel $\mathcal{S} \subseteq (0,1)$,

$$\mathcal{R}_{\pi,\mathcal{S}}(h) := \int_{\mathcal{S}} E_{X,Y}\big[L_h - L_{h^*}\big]\pi(d\tau) \;\le\; \pi(\mathcal{S})\big(B_{\mathrm{CMQ}} + E|Y|\big), \qquad \mathcal{R}_{\pi,\mathcal{S}}(h) \ge 0. \tag{24}$$

Proof. Because $h^*(x,\tau)$ is a conditional τ-quantile,

$$E[L_h(X,Y,\tau) - L_{h^*}(X,Y,\tau) \mid X,\tau] \ge 0.$$

Moreover, since $0 \le \rho_\tau(u) \le |u|$,

$$\mathcal{L}_\pi(h^*) \le \mathcal{L}_\pi(0) \le E|Y| < \infty.$$

For every h satisfying $\|h\|_\infty \le B_{\mathrm{CMQ}}$,

$$\begin{aligned} 0 \le \mathcal{E}_\pi(h) &\le \mathcal{L}_\pi(h) \\ &\le E|Y - h(X,\tau)| \\ &\le E|Y| + B_{\mathrm{CMQ}}. \end{aligned}$$

This proves Eq. (23). Restricting the same argument to any Borel set $\mathcal{S} \subseteq (0,1)$ gives

$$0 \le \mathcal{R}_{\pi,\mathcal{S}}(h) \le \pi(\mathcal{S})\big(E|Y| + B_{\mathrm{CMQ}}\big),$$

which is Eq. (24). □

Lemma 7 (Metric entropy of the fixed-grid class)**.** *Let $S = S_B + S_D$, $L_{\max} = \max(L_B, L_D)$, $W_{\max} = \max(W_B, W_D)$. For every $0 < \varepsilon \le 2B_{\mathrm{CMQ}}$,*

$$\log \mathcal{N}\big(\varepsilon, \mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}, \|\cdot\|_\infty\big) \le C_{\mathrm{ent}}(S+1)\Big[\log\Big(\frac{C_{\mathrm{ent}}}{\varepsilon}\Big) + L_{\max}\log(W_{\max}+1)\Big]. \tag{25}$$

Here C_{ent} depends only on the fixed constants p, B_W, B_0, and A, and is independent of K and n.

Proof. A covering-number bound for sparse bounded-parameter ReLU networks (see, e.g., Lemma 5 of Schmidt-Hieber (2020) together with the correction in Schmidt-Hieber & Vu (2024)) gives, up to fixed constants,

$$\log \mathcal{N}\big(\eta, \mathcal{N}_r(L, W, S, B_W), \|\cdot\|_\infty\big) \lesssim (S+1)\big[\log(1/\eta) + L\log(W+1)\big].$$

The output maps χ_{B_0} and ψ_A are 1- and e^A-Lipschitz respectively, so covers of the pre-activation classes at radius η and ηe^{-A} induce covers of the baseline and derivative classes at radius η. If h, h' are generated by (f_B, f_D) and (f'_B, f'_D), the cumulative-sum construction and linear interpolation give

$$\|h - h'\|_\infty \le \|f_B - f'_B\|_\infty + \tfrac{1}{2}\|f_D - f'_D\|_\infty,$$

so the generating map is Lipschitz with a constant independent of K. Combining the two covers at radius $\varepsilon/2$ each proves Eq. (25). The range $\varepsilon \le 2B_{\mathrm{CMQ}}$ is the relevant one because the class has diameter at most $2B_{\mathrm{CMQ}}$ in $\|\cdot\|_\infty$; for larger ε a single ball suffices. □

Define the complexity functional

$$\mathfrak{C}_n := (S+1)\big[L_{\max}\log(W_{\max}+1) + \log(en)\big]. \tag{26}$$

Theorem 4 (Oracle inequality under a general sampling distribution)**.** *Under Assumptions 1–3, for any $h^\circ \in \mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}$ and any fixed $J \ge 1$,*

$$E\big[\mathcal{E}_\pi(\widehat{h}_{n,\pi,K})\big] \le \mathcal{E}_\pi(h^\circ) + C_{\mathrm{stat}}\sqrt{\frac{\mathfrak{C}_n}{n}}, \tag{27}$$

where C_{stat} depends only on the fixed constants in the assumptions and the sieve, and is independent of n, K, and J.

Proof. For any measurable function $\ell(x, y, \tau)$, define the population and empirical averaging operators by

$$P_\pi \ell := E_{X,Y} E_{\tau\sim\pi}\big[\ell(X, Y, \tau)\big], \qquad P_{n,J}\ell := \frac{1}{nJ}\sum_{i=1}^{n}\sum_{j=1}^{J}\ell(X_i, Y_i, \tau_{ij}).$$

In particular,

$$\mathcal{L}_\pi(h) = P_\pi L_h, \qquad \widehat{\mathcal{L}}_{n,\pi,K}(h) = P_{n,J}L_h.$$

By empirical risk minimization,

$$P_{n,J}L_{\widehat{h}_{n,\pi,K}} \le P_{n,J}L_{h^\circ}.$$

Therefore,

$$\begin{aligned}\mathcal{E}_\pi(\widehat{h}_{n,\pi,K}) - \mathcal{E}_\pi(h^\circ) &= P_\pi\big(L_{\widehat{h}_{n,\pi,K}} - L_{h^\circ}\big)\\ &\le (P_\pi - P_{n,J})\big(L_{\widehat{h}_{n,\pi,K}} - L_{h^\circ}\big)\\ &\le \sup_{h\in\mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}} \big|(P_\pi - P_{n,J})(L_h - L_{h^\circ})\big|.\end{aligned}$$

Hence,

$$\mathcal{E}_\pi(\widehat{h}_{n,\pi,K}) \le \mathcal{E}_\pi(h^\circ) + \sup_{h\in\mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}} \big|(P_\pi - P_{n,J})(L_h - L_{h^\circ})\big|.$$

The J quantile draws corresponding to the same observation share the common pair (X_i, Y_i) and therefore should not be treated as J independent observations. Instead, define the cluster

$$Z_i := (X_i, Y_i, \tau_{i1}, \dots, \tau_{iJ}), \qquad i = 1, \dots, n.$$

Then $Z_1, \dots, Z_n$ are i.i.d. For each $h \in \mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}$, define the cluster-level loss difference

$$\overline{G}_h(Z_i) := \frac{1}{J}\sum_{j=1}^{J}\big[L_h(X_i, Y_i, \tau_{ij}) - L_{h^\circ}(X_i, Y_i, \tau_{ij})\big].$$

By construction,

$$P_{n,J}(L_h - L_{h^\circ}) = \frac{1}{n}\sum_{i=1}^{n} \overline{G}_h(Z_i),$$

while

$$P_\pi(L_h - L_{h^\circ}) = E\big[\overline{G}_h(Z_i)\big].$$

Thus the preceding supremum is an empirical process indexed by h based on the n independent clusters $Z_1, \dots, Z_n$. Since the pinball loss is 1-Lipschitz in its prediction argument,

$$|\overline{G}_h(Z_i)| \le \|h - h^\circ\|_\infty \le 2B_{\mathrm{CMQ}},$$

and, for any h, h',

$$|\overline{G}_h(Z_i) - \overline{G}_{h'}(Z_i)| \le \|h - h'\|_\infty.$$

Hence the cluster-level function class has a uniformly bounded envelope and inherits the covering-number bound in Eq. (25). Applying symmetrization followed by the Dudley entropy integral, with the entropy integral truncated at $\varepsilon = 1/n$, gives

$$E \sup_{h \in \mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}} |(P_\pi - P_{n,J})(L_h - L_{h^\circ})| \le C_{\mathrm{stat}} \sqrt{\frac{\mathfrak{C}_n}{n}},$$

where $\mathfrak{C}_n$ is defined in Eq. (26). Combining the preceding inequalities yields the claimed oracle bound. □

Lemma 8 (Interior self-calibration)**.** *Let Assumptions 1–3 hold, let δ_0 be as in Eq. (12), and set*

$$D_0 := B + B_{\mathrm{CMQ}}, \qquad \kappa_0 := \min\Big\{1, \frac{\delta_0}{D_0}\Big\}, \qquad c_{\mathcal{T}} := \frac{\kappa_0}{2g_{\max}}.$$

Then every $h \in \mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}$ satisfies

$$\mathcal{E}_\pi(h) \;\ge\; c_{\mathcal{T}} \int_{\mathcal{T}} \big\|h(\cdot,\tau) - h^*(\cdot,\tau)\big\|^2_{L^2(P_X)}\, \pi(d\tau). \tag{28}$$

If in addition $\|h - h^*\|_{L^\infty(\mathcal{D})} \le \delta_0$, then

$$\int_{\mathcal{T}} E_{X,Y}\big[L_h - L_{h^*}\big]\pi(d\tau) \;\le\; \frac{1}{2g_{\min}} \big\|h - h^*\big\|^2_{L^\infty(\mathcal{D})}\, \pi(\mathcal{T}). \tag{29}$$

Proof. Fix $x \in \mathcal{X}$ and $\tau \in \mathcal{T}$, and define

$$c^* := h^*(x,\tau), \qquad c := h(x,\tau), \qquad d := |c - c^*|.$$

By Eq. (9) and Assumption 3,

$$|c| \le B_{\mathrm{CMQ}}, \qquad |c^*| \le B,$$

and hence

$$d \le B_{\mathrm{CMQ}} + B = D_0.$$

Define the conditional excess pinball risk at (x,τ) by

$$\Delta_{x,\tau}(c, c^*) := E[\rho_\tau(Y - c) - \rho_\tau(Y - c^*) \mid X = x]\,.$$

Since $c^* = h^*(x,\tau)$ is the conditional τ-quantile, Lemma 3 gives

$$F_{Y|X}(c^* \mid x) = \tau,$$

and Eq. (13) holds on $[c^* - \delta_0, c^* + \delta_0]$. By Knight's identity,

$$\Delta_{x,\tau}(c, c^*) = \int_{c^*}^{c} \big\{F_{Y|X}(t \mid x) - \tau\big\}\, dt.$$

We first establish the lower bound. It is enough to consider $c > c^*$, since the case $c < c^*$ is analogous. If $d \le \delta_0$, then for every $t \in [c^*, c]$, Eq. (13) implies

$$F_{Y|X}(t \mid x) - \tau \ge \frac{t - c^*}{g_{\max}}.$$

Therefore,

$$\Delta_{x,\tau}(c, c^*) \geq \frac{1}{g_{\max}} \int_{c^*}^{c} (t - c^*)\, dt = \frac{d^2}{2g_{\max}} \geq c_{\mathcal{T}} d^2,$$

where the last inequality follows from $c_{\mathcal{T}} \leq (2g_{\max})^{-1}$. Now suppose that $d > \delta_0$. Applying the preceding local bound on $[c^*, c^* + \delta_0]$ gives

$$\int_{c^*}^{c^*+\delta_0} \left\{ F_{Y|X}(t \mid x) - \tau \right\} dt \geq \frac{\delta_0^2}{2g_{\max}}.$$

Moreover, by monotonicity of the conditional distribution function, for every $t \geq c^* + \delta_0$,

$$F_{Y|X}(t \mid x) - \tau \geq \frac{\delta_0}{g_{\max}}.$$

Hence,

$$\begin{aligned} \Delta_{x,\tau}(c, c^*) &\geq \frac{\delta_0^2}{2g_{\max}} + \frac{\delta_0}{g_{\max}}(d - \delta_0) = \frac{\delta_0}{g_{\max}} \left(d - \frac{\delta_0}{2} \right) \\ &\geq \frac{\delta_0 d}{2g_{\max}} \geq \frac{\delta_0}{2g_{\max} D_0} d^2 \geq c_{\mathcal{T}} d^2. \end{aligned}$$

where we used $d > \delta_0$ and $d \leq D_0$. Thus, for every $(x, \tau) \in \mathcal{D}$,

$$\Delta_{x,\tau}(c, c^*) \geq c_{\mathcal{T}} |h(x, \tau) - h^*(x, \tau)|^2.$$

Integrating over $P_X \otimes \pi$ on $\mathcal{T}$, and using the nonnegativity of the conditional excess pinball risk for $\tau \notin \mathcal{T}$, yields Eq. (28). For the upper bound, suppose additionally that

$$|h(x, \tau) - h^*(x, \tau)| \leq \delta_0.$$

Using the upper inequality in Eq. (13) gives

$$\Delta_{x,\tau}(c, c^*) \leq \frac{d^2}{2g_{\min}}.$$

Consequently, if $\|h - h^*\|_{L^\infty(\mathcal{D})} \leq \delta_0$, then

$$\int_{\mathcal{T}} E[L_h(X, Y, \tau) - L_{h^*}(X, Y, \tau)]\, \pi(d\tau) \leq \frac{1}{2g_{\min}} \|h - h^*\|^2_{L^\infty(\mathcal{D})} \pi(\mathcal{T}),$$

which proves Eq. (29). □

Theorem 5 (Interior oracle inequality with an explicit approximation term)**.** *Let Assumptions 1–3 hold and let $h^\circ \in \mathcal{F}^{\mathrm{grid}}_{\mathrm{CMQ},K}$ satisfy $\epsilon_{\mathcal{T}}(h^\circ) := \|h^\circ - h^*\|_{L^\infty(\mathcal{D})} \leq \delta_0$. Then*

$$\begin{aligned} &E\left[\int_{\mathcal{T}} \left\| \widehat{h}_{n,\pi,K}(\cdot, \tau) - h^*(\cdot, \tau) \right\|^2_{L^2(P_X)} \pi(d\tau) \right] \\ &\qquad \leq \frac{1}{c_{\mathcal{T}}} \left[\underbrace{\frac{\epsilon_{\mathcal{T}}(h^\circ)^2}{2g_{\min}}}_{\text{interior approximation}} + \underbrace{\pi(\mathcal{T}^c)\big(B_{\mathrm{CMQ}} + E|Y|\big)}_{\text{tail cost}} + \underbrace{C_{\mathrm{stat}} \sqrt{\frac{\mathfrak{C}_n}{n}}}_{\text{stochastic}} \right], \end{aligned} \tag{30}$$

where $\mathcal{T}^c = (0, 1) \setminus \mathcal{T}$. In particular, choosing $h^\circ = h^{\mathrm{grid}}_{N,K}$ from Theorem 3 with K as in Eq. (18) and N large enough that $C_{\mathrm{approx}} N^{-a} \leq \delta_0$, the first term is $O(N^{-2a})$. If moreover π admits a density w on $\mathcal{T}$ with $w_{\mathcal{T},\min} := \operatorname{ess\,inf}_{\tau \in \mathcal{T}} w(\tau) > 0$, the same bound divided by $w_{\mathcal{T},\min}$ holds for the Lebesgue-weighted quantity $E \int_{\mathcal{T}} \|\widehat{h}_{n,\pi,K}(\cdot, \tau) - h^*(\cdot, \tau)\|^2_{L^2(P_X)}\, d\tau$.

Proof. Lemma 8 applied to $\widehat{h}_{n,\pi,K}$ gives

$$c_{\mathcal{T}}\, E\left[\int_{\mathcal{T}} \|\widehat{h}_{n,\pi,K} - h^*\|^2_{L^2(P_X)} \pi(d\tau) \right] \leq E\big[\mathcal{E}_\pi(\widehat{h}_{n,\pi,K})\big].$$

By Theorem 4,

$$E\big[\mathcal{E}_\pi(\widehat{h}_{n,\pi,K})\big] \le \mathcal{E}_\pi(h^\circ) + C_{\mathrm{stat}}\sqrt{\frac{\mathfrak{C}_n}{n}}.$$

Decompose the excess risk of h° into its contributions over $\mathcal{T}$ and $\mathcal{T}^c$. On $\mathcal{T}$, Eq. (29) applies because $\epsilon_{\mathcal{T}}(h^\circ) \le \delta_0$, whereas the contribution over $\mathcal{T}^c$ is bounded by Eq. (24) with $\mathcal{S} = \mathcal{T}^c$. Using $\pi(\mathcal{T}) \le 1$ then yields Eq. (30). The Lebesgue statement follows from

$$\int_{\mathcal{T}} \|u(\cdot,\tau)\|^2_{L^2(P_X)}\, d\tau \le w^{-1}_{\mathcal{T},\min} \int_{\mathcal{T}} \|u(\cdot,\tau)\|^2_{L^2(P_X)}\, \pi(d\tau).$$

□

Remark 5 (Role of the tail term)**.** *For a full-support sampling distribution such as* $\mathrm{Beta}(0.3, 0.3)$*, one generally has* $\pi(\mathcal{T}^c) > 0$*, so the oracle bound retains an explicit contribution from quantile levels outside* $\mathcal{T}$*. Without additional regularity near the endpoints, this term need not vanish. Accordingly, the result is stated as an interior oracle inequality rather than a global convergence rate.*

7 Appendix B: Implementation Details

All experiments are implemented in PyTorch (Paszke et al., 2019) and conducted on a single NVIDIA A100 GPU with 80GB memory. The ADAM optimizer (Kingma & Ba, 2017) is used across all experiments. We use a fixed random seed (e.g., 2025) for reproducibility unless otherwise specified. The specific training procedure can be found in Algorithm 1.

7.1 Detailed Description of Competing Methods

In this section, we provide detailed descriptions of the proposed method and the competing approaches considered in our experiments:

- **Deep Quantile Regression (DQR).** DQR (Padilla et al., 2022; Shen et al., 2021) serves as a baseline deep quantile regression method. It estimates conditional quantile curves independently at each quantile level using ReLU neural networks, without imposing any non-crossing constraints. As a result, estimated quantile curves may intersect.
- **Deep Quantile Regression with Non-Crossing Constraints (DQR*).** DQR* (Padilla et al., 2022) extends DQR to jointly estimate multiple quantiles under non-crossing constraints. Given a finite set of quantile levels, DQR* estimates the conditional quantile function at the smallest quantile level and models the gaps between successive quantiles using a non-negative transformation, $\log\{1 + \exp(\cdot)\}$, applied to the outputs of ReLU networks, thereby enforcing monotonicity across quantiles.
- **Deep Quantile Regression Process (DQRP).** DQRP (Shen et al., 2024) models the entire quantile process nonparametrically using Rectified Quadratic Unit (ReQU) networks. Monotonicity with respect to the quantile level is encouraged through a penalty on the non-positivity of the partial derivative of the estimated quantile function with respect to τ. Following Shen et al. (2024), we implement DQRP using the tuning parameter $\lambda = \log(n)$ and sample $\xi \sim \mathrm{Unif}(0, 1)$ during training.
- **Non-Crossing Quantile Network (NQ-Net).** NQ-Net (Shen et al., 2025) estimates a finite collection of quantile curves by decomposing them into a mean quantile function and a sequence of non-negative gap functions. The gaps are enforced to be positive via the activation function $\mathrm{ELU} + 1$, ensuring non-crossing quantiles. Both the mean network and the gap network share the same architecture and are trained in parallel.
- **Continuous and Monotone Quantile Network (CMQNet).** The proposed CMQNet models the conditional quantile function as a continuous function of the quantile level τ. Leveraging the fundamental theorem of calculus, CMQNet represents the quantile function as the sum of a baseline network and the integral of a derivative network with respect to τ. Monotonicity is guaranteed by constraining the derivative network to be non-negative. Importantly, continuity in τ is achieved by explicitly including τ as an input to the derivative network, which enables the valid application of the calculus-based representation and distinguishes CMQNet from existing discretely indexed approaches.

Algorithm 1 Training Procedure for the Proposed Method

Require: Training data $\mathcal{D} = \{(\mathbf{X}_i, Y_i)\}_{i=1}^{N_{\text{train}}}$, batch size N, sampling size M, number of fixed-grid intervals K

Initialize neural network parameters Θ for MLP_B and MLP_D
Construct a fixed grid $0 = q_0 < q_1 < \cdots < q_K = 1$ containing 0.5, and let j_0 satisfy $q_{j_0} = 0.5$
while not converged **do**
  Sample a mini-batch $\{(\mathbf{X}_i, Y_i)\}_{i=1}^{N}$ from $\mathcal{D}$
  Sample quantile levels $\tau_{i,j} \sim \text{Beta}(0.3, 0.3)$, $j = 1, \ldots, M$
  for $i = 1$ to N **do**
    Compute baseline:

$$\hat{h}_i^{\text{base}} = \text{MLP}_B(\mathbf{X}_i)$$

    for $\ell = 1$ to K **do**
      Compute the positive interval area:

$$A_{i,\ell} = (q_\ell - q_{\ell-1})\text{MLP}_D\left(\mathbf{X}_i, \frac{q_{\ell-1} + q_\ell}{2}\right)$$

    end for
    Obtain grid-level quantile estimates by cumulative summation:

$$\hat{h}_i(q_m) = \begin{cases} \hat{h}_i^{\text{base}} - \sum_{\ell=m+1}^{j_0} A_{i,\ell}, & m < j_0, \\ \hat{h}_i^{\text{base}}, & m = j_0, \\ \hat{h}_i^{\text{base}} + \sum_{\ell=j_0+1}^{m} A_{i,\ell}, & m > j_0. \end{cases}$$

    for $j = 1$ to M **do**
      Find m such that $\tau_{i,j} \in [q_{m-1}, q_m]$
      Estimate the quantile by linear interpolation:

$$\hat{y}_{i,j} = \frac{q_m - \tau_{i,j}}{q_m - q_{m-1}}\hat{h}_i(q_{m-1}) + \frac{\tau_{i,j} - q_{m-1}}{q_m - q_{m-1}}\hat{h}_i(q_m)$$

    end for
  end for
  Compute $\mathcal{L}(\Theta)$ using Eq. (5)
  Update parameters $\Theta \leftarrow \Theta - \eta\nabla_\Theta\mathcal{L}(\Theta)$
end while

7.2 Data Splitting

For the simulated datasets, we follow the experimental protocol of NQ-Net (Shen et al., 2025) to ensure a fair comparison. Specifically, for small-sample settings, we generate datasets consisting of 512 training samples, 512 validation samples, and 512 testing samples. For large-sample settings, the sizes are set to 5120 for training, 512 for validation, and 512 for testing.

For real-world datasets, where the ground truth conditional quantiles are unknown, we cannot directly compute MAE or MSE. Instead, we evaluate performance using the average check loss via 10-fold cross-validation. In each fold, the data is partitioned into 10 subsets; one subset is reserved for testing, while the remaining nine subsets are used for model development. Within these nine subsets, we further perform a random split, allocating 90% of the data for training and 10% for validation (used for early stopping). The final reported results are the averages over the 10 folds.

7.3 Hyperparameter Settings

We determine the optimal hyperparameters for our CMQNet through a grid search on the validation set. The search space for the hyperparameters is defined as follows:

- **Learning rate:** $\{10^{-2}, 10^{-3}, 10^{-4}\}$;

- **Batch size:** $\{128, 256, 512\}$;
- **Dropout rate:** $\{0, 0.1\}$;
- **Hidden layer architecture (MLP$_B$ size):** $\{[16], [32], [64]\}$;
- **Hidden layer architecture (MLP$_D$ size):** $\{[128, 128, 128], [128, 256, 128], [256, 256, 256]\}$.

The number of grid intervals K controls the numerical accuracy–efficiency trade-off of the fixed-grid approximation in Eq. (4). We find that CMQNet is relatively insensitive to the choice of K once it exceeds a moderate value. Unless otherwise stated, we fix $K = 16$ throughout the paper.

The models are trained for a maximum of 1000 epochs (consistent with the baselines), utilizing an early stopping mechanism with a patience of 5 epochs (i.e., training is terminated if the validation loss does not improve for 5 consecutive epochs). The optimization is performed using the Adam optimizer.

7.4 Quantile Level Generation Strategy

The handling of the quantile level τ differs between the baseline methods and our proposed CMQNet:

- **Baseline Methods:** For methods such as DQR, DQR*, and NQ-Net, the quantile levels are typically pre-specified. Given a number of quantiles K, these methods automatically generate an equi-spaced grid in $[0, 1]$. For DQRP, a single τ is randomly generated for each training step.
- **CMQNet (Ours):** We employ a flexible randomized sampling strategy. Given a target number of τ's (denoted as M), we sample these levels from a candidate Beta distribution (e.g., Beta$(1, 1)$ or Beta$(0.3, 0.3)$). For each training sample $\mathbf{X}$, we construct pairs $(\mathbf{X}, \tau_1), \ldots, (\mathbf{X}, \tau_M)$ as inputs to the derivative network. This allows the model to learn the global quantile structure continuously rather than at fixed points.

7.5 Detailed Data-Generating Processes for Simulations

- **Linear model (baseline):**

$$Q_Y(\tau \mid X = x) = 2x + F^{-1}(\tau),$$

  where $F(\cdot)$ is the cumulative distribution function (CDF) of the error term. This setting serves as a baseline where the relationship between X and Y is linear and homoscedastic, and thus all competing methods are expected to perform well.

- **Wave model (smooth nonlinear):**

$$Q_Y(\tau \mid X = x) = 2x \sin(4\pi x) + \exp(4x - 2)\, F^{-1}(\tau).$$

  This model introduces a smooth but highly nonlinear dependence on X. The multiplicative structure produces heteroscedastic noise whose magnitude varies with x, making estimation more challenging than in the linear case.

- **Angle model (nonsmooth nonlinear):**

$$Q_Y(\tau \mid X = x) = 4(1 - |x - 0.5|) + |\sin(\pi x)|\, F^{-1}(\tau).$$

  This setting features a continuous but non-differentiable conditional mean structure, together with heteroscedasticity driven by the amplitude modulation through $|\sin(\pi x)|$. It is therefore a stringent test of the method's ability to recover sharp features and adapt to non-smooth dynamics.

- **Multivariate Linear Model:**

$$Q_Y(\tau \mid \mathbf{X} = \mathbf{x}) = 2\, A^\top \mathbf{x} + F^{-1}(\tau),$$

  where $A \in \mathbb{R}^p$ is the coefficient vector. This setting provides a multivariate extension of the linear baseline.

- **Single-Index Nonlinear Model:**

$$Q_Y(\tau \mid \mathbf{X} = \mathbf{x}) = \exp\big(A^\top \mathbf{x}\big) + \big|\sin\big(\pi B^\top \mathbf{x}\big)\big|\, F^{-1}(\tau),$$

  where $A, B \in \mathbb{R}^p$ are the coefficient vectors. This model incorporates smooth nonlinear effects and heteroscedasticity.

- **Additive Nonlinear Model:**

$$\begin{aligned} Q_Y(\tau \mid \mathbf{X} = \mathbf{x}) = \; & 3x_1 + 4(x_2 - 0.5)^2 + 2\sin(\pi x_3) \\ & -5|x_4 - 0.5| + \exp\big\{0.1(B^\top \mathbf{x} - 0.5)\big\} F^{-1}(\tau), \end{aligned}$$

  This setting combines smooth, non-smooth, and multiplicative heteroscedastic components, yielding the most challenging multivariate scenario.

To evaluate the robustness of the proposed method under various noise structures, we consider three distinct error distributions for the random noise term ϵ across all simulated models (i.e., Linear, Wave, Angle, Multivariate Linear, Single-Index Nonlinear, and Additive Nonlinear models). Let $F^{-1}(\cdot)$ denote the inverse cumulative distribution function (quantile function) of the error distribution. The specific settings are as follows:

- **Normal Distribution:** The error term follows a standard normal distribution, $\epsilon \sim \mathcal{N}(0, 1)$, representing a symmetric noise structure. In this case, $F^{-1}(\tau) = \Phi^{-1}(\tau)$, where Φ is the standard normal cumulative distribution function.
- **Log-normal Distribution:** To introduce asymmetry, we generate the error term from a log-normal distribution. Specifically, let $Z \sim \text{LogNormal}(0, 1)$. The error term is defined as the centered variable $\epsilon = Z - \mathbb{E}[Z] = Z - e^{1/2}$, ensuring that $\mathbb{E}[\epsilon] = 0$. The corresponding quantile function is $F^{-1}(\tau) = \exp(\Phi^{-1}(\tau)) - e^{1/2}$.
- **Exponential Distribution:** We also consider a skewed error distribution derived from an exponential distribution. Let $Z \sim \text{Exp}(1)$. The error term is defined as $\epsilon = Z - \mathbb{E}[Z] = Z - 1$, which also satisfies $\mathbb{E}[\epsilon] = 0$. The quantile function for this centered distribution is $F^{-1}(\tau) = -\ln(1 - \tau) - 1$.

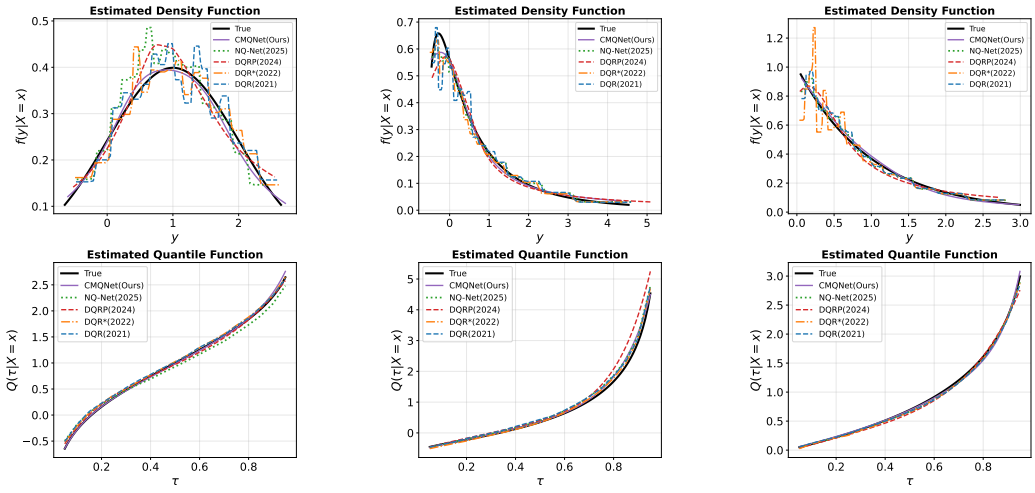
For each simulation setting, we generate training data $\{(X_i^{\text{train}}, Y_i^{\text{train}})\}_{i=1}^{N_1}$ from the joint distribution of (X, Y). In addition, an independent validation dataset of size N_2 is generated and used exclusively for early stopping during training. For all competing methods, we use a batch size of 128 or 256, depending on the sample size, and set the maximum number of training epochs to 1000.

To evaluate generalization performance, we further generate an independent test dataset $\{(X_t^{\text{test}}, Y_t^{\text{test}})\}_{t=1}^{N_3}$ from the same data-generating distribution. Let $Q_Y(\tau \mid X)$ denote the true conditional quantile function of Y given X at quantile level τ. For methods that estimate quantiles at discrete levels, we denote by $\widehat{f}_{\tau_k}$ the estimated conditional quantile function at level τ_k, $k = 1, \ldots, 19$. Performance is evaluated using the mean absolute error (MAE) and mean squared error (MSE) between the estimated conditional quantile functions $\widehat{f}_\tau$ and the true quantile function $Q_Y(\tau \mid X)$ at each quantile level $\tau \in \{0.05, 0.10, \ldots, 0.90, 0.95\}$. In addition to quantile-specific errors, we report overall MAE and MSE averaged across all quantile levels. All reported results correspond to the mean and standard deviation of these error measures over $R = 100$ independent simulation replications.

7.6 Conditional Density Estimation Results

Implied conditional density. CMQNet provides a direct estimate of the conditional density without additional modeling. Under mild regularity conditions, the quantile derivative satisfies $\frac{\partial}{\partial \tau} Q_Y(\tau \mid \mathbf{X}) = 1/f_{Y|\mathbf{X}}(Q_Y(\tau \mid \mathbf{X}) \mid \mathbf{X})$, where $f_{Y|\mathbf{X}}(\cdot \mid \mathbf{X})$ denotes the conditional probability density function. Since $\text{MLP}_D(\mathbf{X}, \tau)$ is constrained to be positive and is used to model the quantile derivative, we obtain the conditional density at $y = \hat{h}(\mathbf{X}, \tau)$ as $\hat{f}_{Y|\mathbf{X}}(y \mid \mathbf{X}) = 1/\text{MLP}_D(\mathbf{X}, \tau)$.

Figure 4 presents the estimated conditional density and quantile functions at $x = 0.5$ under the linear model, comparing five methods across symmetric (normal) and asymmetric (log-normal and exponential) error distributions. Although all competing methods produce accurate conditional



(a) Linear model, normal distribution

(b) Linear model, log-normal distribution

(c) Linear model, exponential distribution

Figure 4: Estimated conditional density functions (top row) and conditional quantile functions (bottom row) under the linear model at $x = 0.5$. Panels correspond to normal, log-normal, and exponential errors. The black curves represent the ground truth. Empirical densities for NQ-Net, DQR*, and DQR are computed using finite differences of adjacent quantile estimates.

quantile estimates that closely match the ground truth across quantile levels, substantial differences arise in the estimation of the implied conditional densities. Except for CMQNet and DQRP, the remaining methods tend to produce irregular and non-smooth density estimates due to the lack of continuity in their estimated quantile functions with respect to τ. Although DQRP yields smooth density estimates by modeling the quantile process continuously, its accuracy is consistently lower than that of CMQNet. In comparison, CMQNet reliably recovers density functions that closely match the true densities under both symmetric and asymmetric error distributions, demonstrating its strong robustness to distributional asymmetry.

7.7 Evaluation Metrics

Simulated Data. Since the true conditional quantile function $Q_Y(\tau \mid \mathbf{X})$ is known in simulation studies, we evaluate the model performance using the Mean Absolute Error (MAE) and Mean Squared Error (MSE) between the estimated and true quantiles. These are calculated over a fixed grid of quantile levels $\mathcal{T}_{\text{eval}} = \{0.05, 0.10, \ldots, 0.95\}$:

$$\text{MAE} = \frac{1}{|\mathcal{T}_{\text{eval}}| \cdot N_{\text{test}}} \sum_{\tau \in \mathcal{T}_{\text{eval}}} \sum_{i=1}^{N_{\text{test}}} \left| \hat{Q}_Y(\tau \mid \mathbf{X}_i) - Q_Y(\tau \mid \mathbf{X}_i) \right|, \tag{31}$$

$$\text{MSE} = \frac{1}{|\mathcal{T}_{\text{eval}}| \cdot N_{\text{test}}} \sum_{\tau \in \mathcal{T}_{\text{eval}}} \sum_{i=1}^{N_{\text{test}}} \left(\hat{Q}_Y(\tau \mid \mathbf{X}_i) - Q_Y(\tau \mid \mathbf{X}_i) \right)^2. \tag{32}$$

Real Data. For real-world datasets, model performance is evaluated using the test loss. In the main real-data experiments, we report the test loss computed on a fixed grid of 19 quantile levels $\mathcal{T}_{\text{eval}} = \{0.05, 0.10, \ldots, 0.95\}$. which enables a standardized and fair comparison with baseline methods that are defined on discrete quantile grids. All real-data results presented in the main text (e.g., Table 3) are based on this fixed-grid evaluation.

Ablation Study. In the ablation study on the τ sampling strategy, except for the fixed-grid evaluation, we additionally consider tail-focused evaluation criteria to assess performance at extreme quantiles. Specifically, we compute test losses over 50 densely sampled quantile levels in each tail, with $\tau \in [0.001, 0.05)$ for the lower tail and $\tau \in (0.95, 0.999]$ for the upper tail. This complementary

evaluation allows us to more precisely examine the impact of different τ sampling distributions on tail estimation accuracy.

8 APPENDIX C: REAL DATA EXPERIMENTS

To evaluate the performance of the methods on real data, we consider four benchmark datasets (Liu et al., 2022): Law School (LAW), Community and Crime (CRIME), MEPS 2016 (MEPS), and Government Salary (GOV). A detailed description of these datasets is provided below and summarized in Table 2.

Community and Crime (CRIME). The Community and Crime dataset consists of socio-economic characteristics, law enforcement indicators, and crime statistics for 1,994 U.S. communities. The prediction target is the number of violent crimes per 100,000 population, with race serving as the sensitive attribute. Following standard practice, missing values are imputed using feature-wise means. The race attribute is constructed by selecting the dominant race category among four racial proportions.

Government Salary (GOV). The Government Salary dataset, available through the "fairadapt" R package, is derived from the 2018 American Community Survey conducted by the U.S. Census Bureau. It contains salary information for over 200,000 government employees. The annual salary is treated as the response variable, and race, categorized into seven groups, is considered the sensitive attribute.

Law School (LAW). The Law School dataset contains 27,472 observations and aims to predict students' grade point average (GPA) based on academic and demographic information. Gender (male vs. female) is treated as the sensitive attribute. This dataset has been extensively used in fairness studies involving educational outcomes.

MEPS 2016 (MEPS). The Medical Expenditure Panel Survey (MEPS) 2016 dataset includes detailed information on individuals' health status, medical utilization, and socio-demographic characteristics. The task is to predict a health care utilization score using $p = 41$ features, including age, marital status, race, poverty level, health condition, and insurance type. The dataset contains 15,656 observations, and race (nonwhite vs. white) is used as the sensitive attribute.

Pre-processing. To ensure stable and reliable model fitting, we apply standard preprocessing procedures prior to training. These include mean imputation for missing values and feature normalization. All cleaned and processed datasets are made publicly available in our GitHub repository to facilitate reproducibility and future research.

Table 2: Detailed dataset descriptions.

Dataset	Datasete Size	Number of features
CRIME	1994	97
GOV	204309	15
LAW	27472	9
MEPS	15656	41

Following common practice, we report the test loss computed by averaging the loss over a fixed grid of quantile levels, and the crossing rate, defined as the proportion of test instances for which the estimated quantile curves violate monotonicity across quantile levels.

Table 3 summarizes the results. CMQNet achieves the lowest test loss on CRIME and LAW, and remains competitive on GOV and MEPS where the best losses are nearly tied. In terms of monotonicity, CMQNet, NQ-Net, and DQR* guarantee non-crossing quantiles (zero crossing rate) across all datasets; DQRP exhibits a small but nonzero crossing rate on some datasets, whereas DQR shows substantial crossing on CRIME and MEPS.

Table 3: Real-data comparison on four benchmark datasets. We report the averaged test loss and the quantile crossing rate, with the standard deviation shown in parentheses. The best results for test loss are highlighted in **bold**.

Dataset	Metric	CMQNet (Ours)	NQ-Net (2025)	DQRP (2024)	DQR* (2022)	DQR (2021)
CRIME	Test Loss	**0.0022(0.0000)**	0.0024(0.0000)	0.0025(0.0000)	0.0029(0.0000)	0.0027(0.0001)
	Crossing Rate	0.0000(0.0000)	0.0000(0.0000)	0.0000(0.0000)	0.0000(0.0000)	0.3077(0.0184)
GOV	Test Loss	0.0100(0.0003)	0.0100(0.0003)	**0.0099(0.0002)**	**0.0099(0.0003)**	**0.0099(0.0003)**
	Crossing Rate	0.0000(0.0000)	0.0000(0.0000)	0.0001(0.0000)	0.0000(0.0000)	0.0043(0.0004)
LAW	Test Loss	**0.0297(0.0068)**	0.0305(0.0065)	0.0298(0.0069)	0.0301(0.0066)	0.0301(0.0072)
	Crossing Rate	0.0000(0.0000)	0.0000(0.0000)	0.0000(0.0000)	0.0000(0.0000)	0.0021(0.0003)
MEPS	Test Loss	0.0059(0.0004)	0.0067(0.0004)	**0.0058(0.0005)**	0.0071(0.0005)	0.0066(0.0005)
	Crossing Rate	0.0000(0.0000)	0.0000(0.0000)	0.0002(0.0001)	0.0000(0.0000)	0.1186(0.0127)

9 Appendix D: Additional Ablation Study Results

9.1 Effect of the τ sampling distribution

Table 4: Ablation on the τ sampling distribution.

(a) Test loss evaluated on a fixed grid $\tau \in \{0.05, 0.10, \ldots, 0.95\}$.

Beta(α, β)	Wave	SIM	LAW
Beta$(0.3, 0.3)$	0.5625 (0.0367)	0.2019 (0.0080)	0.0298 (0.0067)
Beta$(0.5, 0.5)$	0.5600 (0.0368)	0.2018 (0.0080)	0.0298 (0.0068)
Beta$(1, 1)$	0.5565 (0.0366)	0.2018 (0.0082)	0.0300 (0.0067)
Beta$(2, 2)$	0.5555 (0.0372)	0.2021 (0.0082)	0.0299 (0.0068)
Beta$(1, 0.5)$	0.5585 (0.0370)	0.2016 (0.0081)	0.0298 (0.0067)

(b) Tail-focused test losses evaluated on extreme quantiles ($\tau < 0.05$ and $\tau > 0.95$).

Beta(α, β)	Wave	SIM	LAW
Beta$(0.3, 0.3)$	0.1049 (0.0036)	0.0463 (0.0009)	0.0056 (0.0013)
Beta$(0.5, 0.5)$	0.1048 (0.0040)	0.0462 (0.0008)	0.0057 (0.0013)
Beta$(1, 1)$	0.1054 (0.0044)	0.0468 (0.0010)	0.0059 (0.0016)
Beta$(2, 2)$	0.1081 (0.0060)	0.0492 (0.0015)	0.0060 (0.0013)
Beta$(1, 0.5)$	0.1051 (0.0042)	0.0468 (0.0012)	0.0057 (0.0014)

9.2 Effect of the Number of Sampled τ (M).

Table 5 reports the ablation results on the effect of the number of sampled quantile levels M during training. We consider a wide range of values, from $M = 1$ to $M = 999$, under the univariate Wave model, the multivariate SIM model, and the real-data Law School (LAW) dataset. Here, we report the test loss evaluated on a fixed grid of quantile levels which providing a more controlled and comparable measure of estimation accuracy across different choices of M.

As shown in the table, increasing M leads to only marginal improvements in the test loss, and the overall performance remains stable once a moderate number of sampled quantile levels is used. In particular, the differences in loss between $M = 1$, $M = 19$, $M = 99$, and $M = 999$ are negligible across all three datasets. This indicates that the proposed CMQNet is not sensitive to the choice of M, and reliable estimation can be achieved even with a relatively small number of sampled quantile levels. Unlike existing methods that rely on a fixed quantile grid and may require careful tuning of M, CMQNet exhibits strong robustness to the number of sampled quantile levels, offering greater flexibility and practical convenience.

Table 5: Ablation study on the effect of the number of sampled quantile levels. We report the average loss of CMQNet under different numbers of sampled τ per observation across three datasets.

Number of τ	Wave	SIM	LAW
$M = 1$	0.5844 (0.0381)	0.2037 (0.0083)	0.0310 (0.0082)
$M = 9$	0.5658 (0.0363)	0.2022 (0.0080)	0.0299 (0.0068)
$M = 19$	0.5630 (0.0375)	0.2019 (0.0081)	0.0299 (0.0068)
$M = 99$	0.5625 (0.0367)	0.2019 (0.0080)	0.0298 (0.0067)
$M = 999$	0.5619 (0.0369)	0.2018 (0.0080)	0.0298 (0.0067)

9.3 Effect of Positive Activation Functions.

In CMQNet, the monotonicity of the conditional quantile function with respect to the quantile level τ is guaranteed by enforcing the non-negativity of the derivative network output. This is achieved by applying a strictly positive activation function at the final layer of the derivative network. While several activation functions can be used to impose non-negativity, their practical impact on estimation accuracy and stability may vary.

To investigate this effect, we conduct an ablation study comparing three commonly used activation functions: ReLU, Exponential (Exp), and Softplus. Table 6 reports the average loss and standard deviation across the Wave, SIM, and LAW datasets. Overall, the results indicate that CMQNet is relatively robust to the choice of positive activation function, as all three options yield comparable performance across different settings. Among them, the Exponential and Softplus activations tend to provide slightly more stable and accurate results, especially on the real LAW dataset. In addition, ReLU may output exact zeros, leading to vanishing derivatives that are undesirable for conditional density estimation, while Softplus introduces a mild bias due to its saturation near zero. Taking both numerical stability and density estimation into account, we therefore adopt the Exponential activation as the choice in our main experiments.

Table 6: Ablation study on the effect of different activation functions across datasets.

Activation Function	Wave	SIM	LAW
ReLU	0.5504 (0.0165)	0.2042 (0.0129)	0.0364 (0.0095)
Exp	0.5630 (0.0375)	0.2019 (0.0081)	0.0299 (0.0068)
Softplus	0.5511 (0.0166)	0.2016 (0.0035)	0.0299 (0.0069)

10 Appendix E: Computational Efficiency

Tables 7–9 report the average runtime (in seconds) and standard deviation of five quantile regression methods under different numbers of sampled quantile levels M. Results are shown for the Wave and SIM simulation models with Gaussian errors ($N = 5120$), as well as for the real LAW dataset.

The symbol "✗" indicates that a given method is not applicable or not evaluated under the corresponding setting. In particular, DQR, DQR*, and NQ-Net require a predefined finite set of quantile levels and therefore cannot be applied when $M = 1$. Conversely, DQRP is designed to model the quantile process by sampling a single quantile level at each iteration, and extending it to multiple τ values per observation would require substantial modifications to the training objective and optimization procedure. For this reason, we follow the original implementation and evaluate DQRP only under its default single-τ setting.

It is worth noting that, even for the same nominal number of sampled quantile levels M, the effective number of quantile evaluations differs substantially across methods. For competing approaches such as DQR, DQR*, and NQ-Net, the value of M corresponds to a fixed and finite set of predefined quantile levels that are shared across all training samples. Consequently, when $M = 99$, these methods only evaluate the model at exactly 99 quantile levels throughout training.

In contrast, CMQNet adopts a stochastic quantile sampling strategy, where for each training sample $\mathbf{X}_i$ and each mini-batch iteration, M quantile levels are independently drawn from a continuous distribution over $(0, 1)$. As a result, although the nominal value of M is the same, CMQNet effectively explores a much richer and continuously varying set of quantile levels over the course of training. This leads to a substantially larger total number of distinct τ evaluations compared to methods based on fixed quantile grids.

Therefore, the slightly higher runtime of CMQNet under the same M is a natural consequence of its broader coverage of the $(\mathbf{X}, \tau)$ space and should be interpreted as a trade-off for improved estimation accuracy and robustness, particularly near boundary quantile regions.

Table 7: Runtime (seconds) of five quantile regression methods under the Wave model with Gaussian errors ($N = 5120$) for different numbers of sampled quantile levels $M \in \{1, 9, 19, 99\}$.

M	CMQNet (Ours)	NQ-Net (2025)	DQRP (2024)	DQR* (2022)	DQR (2021)
M=1	23.15 ± 14.84	✗	57.90 ± 20.24	✗	✗
M=9	52.20 ± 5.87	38.65 ± 11.84	✗	63.40 ± 16.86	40.97 ± 9.48
M=19	64.99 ± 11.29	26.60 ± 0.22	✗	27.13 ± 0.58	24.28 ± 0.79
M=99	217.01 ± 28.64	65.53 ± 0.89	✗	65.92 ± 0.31	63.47 ± 3.78

Table 8: Runtime (seconds) of five quantile regression methods under the SIM model with Gaussian errors ($N = 5120$) for different numbers of sampled quantile levels $M \in \{1, 9, 19, 99\}$.

M	CMQNet (Ours)	NQ-Net (2025)	DQRP (2024)	DQR* (2022)	DQR (2021)
M=1	18.07 ± 3.07	✗	26.75 ± 3.54	✗	✗
M=9	20.18 ± 2.33	16.51 ± 3.82	✗	17.45 ± 4.23	13.10 ± 2.48
M=19	28.32 ± 5.33	24.13 ± 5.89	✗	26.11 ± 7.23	20.40 ± 5.28
M=99	90.28 ± 16.23	62.38 ± 8.63	✗	66.13 ± 1.04	65.75 ± 0.55

Table 9: Runtime (seconds) of five quantile regression methods on the LAW dataset for different numbers of sampled quantile levels $M \in \{1, 9, 19, 99\}$.

M	CMQNet (Ours)	NQ-Net (2025)	DQRP (2024)	DQR* (2022)	DQR (2021)
M=1	154.20 ± 43.53	✗	262.66 ± 57.88	✗	✗
M=9	189.08 ± 18.74	191.54 ± 41.87	✗	264.91 ± 51.33	154.26 ± 26.69
M=19	190.42 ± 21.27	232.79 ± 43.64	✗	254.21 ± 78.34	266.48 ± 72.56
M=99	367.82 ± 54.85	685.71 ± 159.98	✗	1014.50 ± 246.48	660.44 ± 101.39

11 APPENDIX F: COMPLETE SIMULATION RESULTS

Table 10 provides an overall summary of the simulation results across all datasets and settings, while the subsequent detailed tables (Tables 11–46) present the corresponding results for each dataset and distribution individually. Each entry in Table 10 is computed as the arithmetic mean of the 19 quantile-specific results ($\tau \in \{0.05, 0.10, \ldots, 0.95\}$) reported in its corresponding detailed table.

Specifically, the row labels "Linear", "Wave", "Angle", "MVLin", "SIM", and "Add" correspond to the Linear model, Wave model, Angle model, Multivariate linear model, Single-Index nonlinear model, and Additive nonlinear model, respectively. Regarding the noise distributions, N(0,1), LN(0,1), and Exp(1) denote the standard normal, log-normal, and exponential distributions. The values N=512 and N=5120 represent the sample sizes of the training sets. Furthermore, **red bold** text indicates the best performance (lowest error), while blue underlined text denotes the second-best result.

Table 10: Comparison of different methods on several simulated datasets in terms of MAE and MSE.

Data	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
Linear, N(0,1), N=512	**0.0854 (0.0183)**	**0.0129 (0.0055)**	0.0953 (0.0228)	0.0151 (0.0064)	0.1136 (0.0356)	0.0226 (0.0134)	0.0942 (0.0256)	0.0146 (0.0074)	0.0904 (0.0218)	0.0136 (0.0060)
Linear, N(0,1), N=5120	**0.0405 (0.0156)**	**0.0035 (0.0029)**	0.0570 (0.0197)	0.0053 (0.0035)	0.0810 (0.0343)	0.0125 (0.0109)	0.0842 (0.0436)	0.0113 (0.0108)	0.0652 (0.0227)	0.0070 (0.0049)
Linear, LN(0,1), N=512	0.1795 (0.0572)	0.1856 (0.1441)	0.1379 (0.0341)	**0.0520 (0.0330)**	0.1824 (0.0505)	0.0978 (0.0550)	0.1391 (0.0363)	0.0554 (0.0332)	**0.1374 (0.0341)**	0.0596 (0.0372)
Linear, LN(0,1), N=5120	0.1370 (0.0516)	0.1364 (0.1294)	**0.0701 (0.0239)**	**0.0106 (0.0074)**	0.1231 (0.0540)	0.0500 (0.0454)	0.0798 (0.0339)	0.0127 (0.0091)	0.0757 (0.0251)	0.0139 (0.0092)
Linear, Exp(1), N=512	**0.0812 (0.0184)**	0.0230 (0.0131)	0.0863 (0.0211)	0.0154 (0.0084)	0.1121 (0.0387)	0.0284 (0.0205)	0.0878 (0.0252)	0.0154 (0.0097)	0.0843 (0.0188)	**0.0149 (0.0074)**
Linear, Exp(1), N=5120	**0.0350 (0.0144)**	0.0055 (0.0070)	0.0523 (0.0173)	**0.0049 (0.0033)**	0.0734 (0.0325)	0.0128 (0.0119)	0.0585 (0.0292)	0.0062 (0.0056)	0.0533 (0.0149)	0.0054 (0.0031)
Wave, N(0,1), N=512	**0.6452 (0.0582)**	**0.7659 (0.1724)**	1.0162 (0.0855)	3.3440 (0.6895)	1.1911 (0.1276)	3.9556 (1.0207)	1.0518 (0.0893)	2.6531 (0.6085)	1.0535 (0.0807)	3.3633 (0.6006)
Wave, N(0,1), N=5120	**0.2843 (0.0481)**	**0.1907 (0.0650)**	1.0078 (0.0739)	3.2917 (0.5217)	1.0167 (0.0859)	3.4171 (0.6237)	1.0227 (0.0781)	3.2737 (0.5260)	1.0041 (0.0671)	3.2236 (0.4448)
Wave, LN(0,1), N=512	**0.9777 (0.1077)**	**2.6014 (0.9551)**	1.4549 (0.1377)	6.7212 (2.3201)	1.5714 (0.2139)	7.2928 (3.1298)	1.5545 (0.1405)	6.9786 (2.5235)	1.4907 (0.1574)	7.5354 (3.0068)
Wave, LN(0,1), N=5120	**0.6602 (0.1157)**	**1.5307 (0.5252)**	1.4157 (0.0812)	6.6078 (0.9641)	1.4494 (0.1743)	7.3648 (2.8949)	1.4367 (0.0822)	6.8410 (1.0322)	1.4157 (0.0823)	6.6248 (0.9974)
Wave, Exp(1), N=512	**0.7295 (0.0609)**	**1.0621 (0.2766)**	0.9223 (0.0832)	2.7361 (0.7275)	0.9850 (0.1178)	2.8529 (0.9686)	0.9816 (0.0951)	2.8419 (0.7855)	0.9364 (0.0795)	2.8332 (0.7450)
Wave, Exp(1), N=5120	**0.3609 (0.0428)**	**0.3413 (0.1071)**	0.9077 (0.0551)	2.6470 (0.3324)	0.9228 (0.0965)	2.8089 (0.7698)	0.9150 (0.0589)	2.7075 (0.4185)	0.9051 (0.0532)	2.6095 (0.3263)
Angle, N(0,1), N=512	0.3393 (0.0165)	0.2548 (0.0201)	0.2915 (0.0192)	0.1763 (0.0147)	0.3050 (0.0285)	0.1759 (0.0253)	0.2991 (0.0155)	**0.1700 (0.0137)**	**0.2897 (0.0173)**	0.1767 (0.0146)
Angle, N(0,1), N=5120	**0.0630 (0.0072)**	**0.0092 (0.0019)**	0.2741 (0.0226)	0.1702 (0.0173)	0.2854 (0.0218)	0.1662 (0.0171)	0.2735 (0.0201)	0.1680 (0.0159)	0.2747 (0.0191)	0.1705 (0.0168)
Angle, LN(0,1), N=512	**0.3422 (0.0422)**	**0.2728 (0.1144)**	0.4334 (0.0362)	0.4082 (0.0492)	0.4604 (0.0381)	0.4324 (0.0641)	0.4438 (0.0308)	0.4088 (0.0531)	0.4240 (0.0336)	0.3945 (0.0551)
Angle, LN(0,1), N=5120	**0.1179 (0.0266)**	**0.0673 (0.0539)**	0.3918 (0.0240)	0.3814 (0.0284)	0.4083 (0.0258)	0.3724 (0.0429)	0.3919 (0.0212)	0.3717 (0.0325)	0.3964 (0.0178)	0.3835 (0.0286)
Angle, Exp(1), N=512	0.2730 (0.0199)	**0.1492 (0.0192)**	0.2709 (0.0236)	0.1521 (0.0174)	0.2856 (0.0244)	0.1519 (0.0237)	0.2785 (0.0177)	0.1507 (0.0143)	**0.2692 (0.0156)**	0.1511 (0.0145)
Angle, Exp(1), N=5120	**0.0707 (0.0083)**	**0.0120 (0.0039)**	0.2390 (0.0158)	0.1403 (0.0104)	0.2527 (0.0174)	0.1390 (0.0145)	0.2380 (0.0162)	0.1345 (0.0088)	0.2489 (0.0165)	0.1442 (0.0128)
MVLin, N(0,1), N=512	**0.2244 (0.0242)**	**0.0911 (0.0178)**	0.4495 (0.0393)	0.3482 (0.0614)	0.4453 (0.0586)	0.3573 (0.0980)	0.5051 (0.0402)	0.6221 (0.1049)	0.3226 (0.0389)	0.1770 (0.0423)
MVLin, N(0,1), N=5120	**0.1187 (0.0114)**	**0.0272 (0.0059)**	0.2597 (0.0244)	0.1181 (0.0198)	0.2424 (0.0668)	0.1054 (0.0603)	0.2378 (0.0385)	0.1021 (0.0300)	0.1914 (0.0439)	0.0607 (0.0244)
MVLin, LN(0,1), N=512	0.4354 (0.0774)	0.6916 (0.2982)	0.5110 (0.0542)	0.5295 (0.1265)	0.6280 (0.0830)	0.8311 (0.2170)	0.5597 (0.0617)	0.7235 (0.1539)	**0.3421 (0.0454)**	**0.2197 (0.0568)**
MVLin, LN(0,1), N=5120	0.2287 (0.0517)	0.2288 (0.1394)	0.3043 (0.0302)	0.2072 (0.0419)	0.2892 (0.0493)	0.1818 (0.0480)	0.2493 (0.0435)	0.1117 (0.0320)	**0.1944 (0.0290)**	**0.0698 (0.0174)**
MVLin, Exp(1), N=512	**0.2117 (0.0279)**	**0.1134 (0.0337)**	0.3652 (0.0348)	0.2420 (0.0466)	0.4232 (0.0601)	0.3430 (0.0935)	0.4634 (0.0389)	0.5836 (0.1090)	0.2672 (0.0338)	0.1266 (0.0294)
MVLin, Exp(1), N=5120	**0.1129 (0.0169)**	**0.0373 (0.0128)**	0.2063 (0.0243)	0.0784 (0.0150)	0.2087 (0.0582)	0.0840 (0.0452)	0.1830 (0.0253)	0.0601 (0.0153)	0.1475 (0.0288)	0.0381 (0.0132)
SIM, N(0,1), N=512	**0.2501 (0.0189)**	**0.1235 (0.0154)**	0.4785 (0.0312)	0.3826 (0.0508)	0.2995 (0.0226)	0.1575 (0.0244)	0.2719 (0.0157)	0.1275 (0.0149)	0.2826 (0.0180)	0.1347 (0.0165)
SIM, N(0,1), N=5120	**0.2116 (0.0047)**	**0.0898 (0.0055)**	0.3274 (0.0168)	0.1839 (0.0185)	0.2279 (0.0102)	0.0960 (0.0082)	0.2353 (0.0119)	0.0971 (0.0083)	0.2395 (0.0120)	0.1021 (0.0092)
SIM, LN(0,1), N=512	**0.3963 (0.0349)**	0.5105 (0.1660)	0.6641 (0.0564)	0.8221 (0.1521)	0.4270 (0.0414)	0.3843 (0.0842)	0.3988 (0.0246)	**0.2896 (0.0360)**	0.4142 (0.0314)	0.3280 (0.0497)
SIM, LN(0,1), N=5120	**0.3266 (0.0156)**	0.2751 (0.0608)	0.4719 (0.0248)	0.4306 (0.0501)	0.3441 (0.0192)	0.2520 (0.0373)	0.3390 (0.0176)	**0.2125 (0.0182)**	0.3564 (0.0149)	0.2526 (0.0208)
SIM, Exp(1), N=512	**0.2191 (0.0114)**	0.1185 (0.0221)	0.4127 (0.0377)	0.3005 (0.0556)	0.2654 (0.0210)	0.1334 (0.0205)	0.2531 (0.0151)	**0.1130 (0.0133)**	0.2563 (0.0169)	0.1170 (0.0152)
SIM, Exp(1), N=5120	**0.1958 (0.0049)**	**0.0812 (0.0087)**	0.2846 (0.0166)	0.1448 (0.0160)	0.2165 (0.0142)	0.0904 (0.0124)	0.2129 (0.0093)	0.0818 (0.0064)	0.2197 (0.0085)	0.0898 (0.0064)
Add, N(0,1), N=512	0.6549 (0.0613)	**0.7222 (0.1288)**	1.0990 (0.0454)	1.9856 (0.1755)	**0.6524 (0.0643)**	0.7471 (0.1577)	0.9126 (0.0645)	1.3647 (0.1920)	0.9546 (0.0614)	1.5118 (0.1893)
Add, N(0,1), N=5120	**0.2522 (0.0227)**	**0.1082 (0.0209)**	0.8174 (0.0469)	1.2319 (0.1584)	0.3404 (0.0416)	0.2036 (0.0500)	0.5406 (0.0520)	0.5076 (0.1021)	0.5620 (0.0434)	0.5614 (0.0921)
Add, LN(0,1), N=512	1.0754 (0.1288)	**2.6369 (0.8370)**	1.5975 (0.1334)	7.2165 (3.3732)	**0.9733 (0.1592)**	3.9777 (7.9798)	1.3184 (0.1471)	4.2333 (1.6105)	1.5054 (0.1450)	5.3825 (1.9839)
Add, LN(0,1), N=5120	**0.4309 (0.0777)**	0.6987 (0.4245)	1.0975 (0.0895)	3.8739 (2.0253)	0.4622 (0.0948)	**0.6730 (0.5307)**	0.7699 (0.0902)	2.0190 (0.7744)	0.8805 (0.0846)	2.2751 (0.6536)
Add, Exp(1), N=512	0.6419 (0.0713)	**0.7878 (0.1881)**	1.0075 (0.0542)	2.1214 (0.5262)	**0.6260 (0.0773)**	0.7991 (0.2744)	0.8379 (0.0809)	1.2957 (0.2943)	0.8702 (0.0810)	1.3924 (0.3105)
Add, Exp(1), N=5120	**0.2369 (0.0256)**	**0.1316 (0.0451)**	0.6800 (0.0405)	1.0096 (0.2246)	0.3069 (0.0468)	0.1967 (0.0657)	0.4771 (0.0492)	0.4873 (0.1080)	0.5051 (0.0468)	0.5245 (0.1022)
1st Count	27	25	1	3	3	1	0	4	5	3
2nd Count	5	1	10	6	4	9	7	9	10	11

Across all the simulation settings, we observe three representative performance patterns.

1. In Table 28, 29, 30, 41, 42, 44, 45, 46, CMQNet achieves the best or second-best performance across all 19 evaluated quantile levels, demonstrating strong and uniform estimation accuracy over the entire quantile range.

2. In Table 17 - 27, CMQNet exhibits slightly inferior performance at several central quantile levels, while remaining clearly superior near the boundary quantiles. For consistency and robustness across experiments, we adopt a $\mathrm{Beta}(0.3, 0.3)$ distribution for sampling τ, which emphasizes extreme quantiles to improve tail estimation accuracy, while potentially placing slightly less emphasis on central quantiles in certain settings. In fact, CMQNet allows the sampling distribution of τ to be flexibly adapted to the underlying data characteristics. By selecting alternative sampling distributions (e.g., $\mathrm{Beta}(1, 1)$ or $\mathrm{Beta}(2, 2)$), one can readily improve estimation accuracy at central quantiles when desired, as demonstrated in the ablation study in Section 4.3.1.

3. In the remaining tables, CMQNet continues to deliver the best performance at the vast majority of quantile levels, while exhibiting slightly inferior accuracy at a small number of boundary quantiles in specific settings. This behavior is closely related to the choice of the τ sampling distribution used during training. Developing a more adaptive and data-driven strategy for selecting an optimal τ sampling distribution remains an interesting direction for future work.

Table 11: Linear, N(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.1638 (0.0724)	0.0406 (0.0318)	0.1331 (0.0512)	0.0281 (0.0190)	0.1681 (0.0809)	0.0475 (0.0411)	**0.1212 (0.0521)**	**0.0245 (0.0183)**	0.1279 (0.0552)	0.0259 (0.0215)
0.10	0.0968 (0.0426)	0.0151 (0.0131)	0.1153 (0.0421)	0.0214 (0.0134)	0.1301 (0.0641)	0.0288 (0.0266)	**0.0950 (0.0391)**	**0.0145 (0.0111)**	0.1100 (0.0455)	0.0194 (0.0156)
0.15	**0.0677 (0.0321)**	**0.0080 (0.0079)**	0.1033 (0.0321)	0.0171 (0.0096)	0.1077 (0.0502)	0.0191 (0.0167)	0.0828 (0.0328)	0.0109 (0.0080)	0.0957 (0.0355)	0.0147 (0.0099)
0.20	**0.0715 (0.0363)**	**0.0083 (0.0084)**	0.0961 (0.0338)	0.0151 (0.0092)	0.0941 (0.0411)	0.0143 (0.0118)	0.0800 (0.0322)	0.0103 (0.0071)	0.0878 (0.0341)	0.0125 (0.0082)
0.25	**0.0747 (0.0378)**	**0.0086 (0.0084)**	0.0914 (0.0317)	0.0136 (0.0084)	0.0862 (0.0365)	0.0121 (0.0095)	0.0807 (0.0325)	0.0103 (0.0075)	0.0818 (0.0305)	0.0108 (0.0071)
0.30	**0.0699 (0.0354)**	**0.0075 (0.0072)**	0.0895 (0.0298)	0.0129 (0.0075)	0.0823 (0.0334)	0.0110 (0.0083)	0.0797 (0.0316)	0.0101 (0.0071)	0.0814 (0.0319)	0.0109 (0.0074)
0.35	**0.0613 (0.0313)**	**0.0059 (0.0055)**	0.0854 (0.0295)	0.0119 (0.0074)	0.0806 (0.0323)	0.0105 (0.0078)	0.0792 (0.0299)	0.0100 (0.0068)	0.0763 (0.0316)	0.0100 (0.0073)
0.40	**0.0549 (0.0279)**	**0.0048 (0.0042)**	0.0842 (0.0289)	0.0115 (0.0069)	0.0809 (0.0335)	0.0106 (0.0080)	0.0784 (0.0327)	0.0101 (0.0077)	0.0768 (0.0331)	0.0101 (0.0080)
0.45	**0.0545 (0.0283)**	**0.0047 (0.0044)**	0.0821 (0.0294)	0.0112 (0.0070)	0.0833 (0.0367)	0.0111 (0.0090)	0.0791 (0.0314)	0.0102 (0.0082)	0.0778 (0.0325)	0.0101 (0.0072)
0.50	**0.0596 (0.0312)**	**0.0056 (0.0054)**	0.0812 (0.0275)	0.0109 (0.0067)	0.0878 (0.0408)	0.0122 (0.0107)	0.0822 (0.0350)	0.0111 (0.0090)	0.0757 (0.0323)	0.0098 (0.0074)
0.55	**0.0673 (0.0343)**	**0.0070 (0.0065)**	0.0824 (0.0280)	0.0110 (0.0070)	0.0931 (0.0457)	0.0137 (0.0129)	0.0853 (0.0330)	0.0117 (0.0081)	0.0772 (0.0312)	0.0099 (0.0073)
0.60	**0.0733 (0.0371)**	**0.0083 (0.0073)**	0.0864 (0.0294)	0.0120 (0.0073)	0.0981 (0.0506)	0.0154 (0.0153)	0.0900 (0.0334)	0.0127 (0.0086)	0.0806 (0.0317)	0.0104 (0.0074)
0.65	**0.0753 (0.0378)**	**0.0089 (0.0077)**	0.0860 (0.0286)	0.0118 (0.0068)	0.1034 (0.0550)	0.0172 (0.0177)	0.0930 (0.0345)	0.0134 (0.0086)	0.0848 (0.0312)	0.0112 (0.0077)
0.70	**0.0730 (0.0355)**	**0.0084 (0.0081)**	0.0863 (0.0305)	0.0121 (0.0073)	0.1097 (0.0595)	0.0195 (0.0204)	0.0977 (0.0350)	0.0146 (0.0089)	0.0841 (0.0284)	0.0109 (0.0072)
0.75	**0.0703 (0.0328)**	**0.0079 (0.0087)**	0.0886 (0.0327)	0.0129 (0.0082)	0.1186 (0.0655)	0.0229 (0.0240)	0.1017 (0.0384)	0.0159 (0.0103)	0.0831 (0.0311)	0.0108 (0.0077)
0.80	**0.0752 (0.0414)**	**0.0095 (0.0105)**	0.0931 (0.0357)	0.0142 (0.0097)	0.1316 (0.0736)	0.0283 (0.0295)	0.1031 (0.0432)	0.0168 (0.0128)	0.0877 (0.0393)	0.0123 (0.0106)
0.85	0.1054 (0.0521)	0.0167 (0.0145)	0.0982 (0.0368)	0.0157 (0.0102)	0.1476 (0.0835)	0.0358 (0.0369)	0.1083 (0.0472)	0.0188 (0.0155)	**0.0954 (0.0442)**	**0.0145 (0.0118)**
0.90	0.1499 (0.0618)	0.0311 (0.0218)	**0.1068 (0.0399)**	0.0186 (0.0118)	0.1638 (0.0898)	0.0437 (0.0435)	0.1159 (0.0537)	0.0220 (0.0190)	**0.1068 (0.0496)**	**0.0184 (0.0151)**
0.95	0.1576 (0.0736)	0.0376 (0.0310)	**0.1221 (0.0500)**	**0.0244 (0.0176)**	0.1916 (0.0911)	0.0558 (0.0501)	0.1366 (0.0598)	0.0299 (0.0236)	0.1263 (0.0569)	0.0250 (0.0209)
1st Count	14	14	2	1	0	0	2	2	2	2
2nd Count	1	1	1	2	0	0	5	6	12	12

Table 12: Linear, N(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.0903 (0.0622)	0.0132 (0.0166)	0.0742 (0.0397)	0.0089 (0.0084)	0.1193 (0.0712)	0.0252 (0.0311)	0.0853 (0.0411)	0.0111 (0.0097)	**0.0736 (0.0366)**	**0.0088 (0.0086)**
0.10	**0.0506 (0.0315)**	**0.0043 (0.0048)**	0.0671 (0.0353)	0.0072 (0.0071)	0.0926 (0.0598)	0.0152 (0.0198)	0.0539 (0.0260)	0.0046 (0.0042)	0.0631 (0.0307)	0.0064 (0.0057)
0.15	**0.0400 (0.0209)**	**0.0025 (0.0024)**	0.0610 (0.0315)	0.0060 (0.0059)	0.0709 (0.0435)	0.0085 (0.0102)	0.0583 (0.0328)	0.0054 (0.0057)	0.0583 (0.0277)	0.0053 (0.0046)
0.20	**0.0325 (0.0202)**	**0.0018 (0.0023)**	0.0570 (0.0300)	0.0053 (0.0055)	0.0604 (0.0322)	0.0058 (0.0057)	0.0596 (0.0302)	0.0053 (0.0050)	0.0525 (0.0259)	0.0044 (0.0039)
0.25	**0.0299 (0.0201)**	**0.0016 (0.0022)**	0.0542 (0.0286)	0.0048 (0.0050)	0.0558 (0.0268)	0.0049 (0.0042)	0.0629 (0.0340)	0.0060 (0.0060)	0.0524 (0.0243)	0.0044 (0.0035)
0.30	**0.0298 (0.0182)**	**0.0015 (0.0018)**	0.0514 (0.0278)	0.0044 (0.0047)	0.0539 (0.0249)	0.0045 (0.0038)	0.0656 (0.0374)	0.0065 (0.0068)	0.0533 (0.0237)	0.0044 (0.0036)
0.35	**0.0288 (0.0152)**	**0.0014 (0.0013)**	0.0491 (0.0266)	0.0040 (0.0044)	0.0526 (0.0252)	0.0044 (0.0039)	0.0695 (0.0408)	0.0073 (0.0079)	0.0536 (0.0251)	0.0044 (0.0037)
0.40	**0.0262 (0.0127)**	**0.0011 (0.0010)**	0.0477 (0.0258)	0.0038 (0.0042)	0.0522 (0.0266)	0.0045 (0.0043)	0.0712 (0.0425)	0.0077 (0.0082)	0.0543 (0.0275)	0.0046 (0.0044)
0.45	**0.0239 (0.0118)**	**0.0010 (0.0011)**	0.0466 (0.0247)	0.0036 (0.0040)	0.0531 (0.0294)	0.0048 (0.0051)	0.0761 (0.0449)	0.0087 (0.0093)	0.0571 (0.0294)	0.0051 (0.0053)
0.50	**0.0247 (0.0130)**	**0.0012 (0.0015)**	0.0469 (0.0243)	0.0036 (0.0040)	0.0559 (0.0330)	0.0055 (0.0062)	0.0786 (0.0462)	0.0092 (0.0097)	0.0577 (0.0333)	0.0054 (0.0066)
0.55	**0.0284 (0.0157)**	**0.0015 (0.0022)**	0.0468 (0.0232)	0.0035 (0.0035)	0.0602 (0.0369)	0.0064 (0.0078)	0.0831 (0.0476)	0.0102 (0.0104)	0.0583 (0.0326)	0.0055 (0.0062)
0.60	**0.0322 (0.0187)**	**0.0019 (0.0027)**	0.0477 (0.0235)	0.0037 (0.0035)	0.0654 (0.0413)	0.0075 (0.0096)	0.0879 (0.0509)	0.0114 (0.0115)	0.0631 (0.0344)	0.0062 (0.0061)
0.65	**0.0337 (0.0203)**	**0.0021 (0.0031)**	0.0499 (0.0240)	0.0040 (0.0037)	0.0711 (0.0460)	0.0089 (0.0116)	0.0922 (0.0536)	0.0125 (0.0128)	0.0658 (0.0346)	0.0067 (0.0070)
0.70	**0.0327 (0.0196)**	**0.0020 (0.0030)**	0.0519 (0.0248)	0.0043 (0.0038)	0.0779 (0.0505)	0.0105 (0.0139)	0.0966 (0.0560)	0.0136 (0.0142)	0.0670 (0.0364)	0.0071 (0.0075)
0.75	**0.0310 (0.0180)**	**0.0017 (0.0027)**	0.0556 (0.0262)	0.0049 (0.0041)	0.0870 (0.0565)	0.0129 (0.0169)	0.1012 (0.0591)	0.0150 (0.0157)	0.0691 (0.0383)	0.0075 (0.0082)
0.80	**0.0332 (0.0188)**	**0.0020 (0.0025)**	0.0604 (0.0283)	0.0057 (0.0047)	0.1008 (0.0657)	0.0170 (0.0217)	0.1052 (0.0627)	0.0164 (0.0175)	0.0751 (0.0428)	0.0090 (0.0104)
0.85	**0.0436 (0.0287)**	**0.0032 (0.0042)**	0.0657 (0.0314)	0.0067 (0.0056)	0.1205 (0.0786)	0.0237 (0.0294)	0.1096 (0.0666)	0.0180 (0.0195)	0.0797 (0.0471)	0.0103 (0.0118)
0.90	**0.0542 (0.0491)**	**0.0059 (0.0108)**	0.0713 (0.0347)	0.0078 (0.0067)	0.1403 (0.0913)	0.0317 (0.0389)	0.1166 (0.0737)	0.0208 (0.0231)	0.0880 (0.0507)	0.0123 (0.0135)
0.95	0.1046 (0.0636)	0.0165 (0.0181)	**0.0783 (0.0400)**	**0.0095 (0.0092)**	0.1493 (0.0948)	0.0359 (0.0463)	0.1259 (0.0773)	0.0240 (0.0260)	0.0970 (0.0566)	0.0152 (0.0163)
1st Count	17	17	1	1	0	0	0	0	1	1
2nd Count	0	0	14	14	0	0	2	1	4	5

Table 13: Linear, LN(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0337 (0.0143)**	**0.0016 (0.0012)**	0.0652 (0.0272)	0.0070 (0.0058)	0.0868 (0.0451)	0.0130 (0.0125)	0.0660 (0.0299)	0.0075 (0.0083)	0.0475 (0.0196)	0.0036 (0.0028)
0.10	**0.0252 (0.0110)**	**0.0010 (0.0007)**	0.0637 (0.0249)	0.0066 (0.0051)	0.0784 (0.0380)	0.0101 (0.0089)	0.0521 (0.0191)	0.0043 (0.0029)	0.0482 (0.0189)	0.0037 (0.0028)
0.15	**0.0275 (0.0123)**	**0.0012 (0.0010)**	0.0623 (0.0246)	0.0064 (0.0050)	0.0764 (0.0335)	0.0092 (0.0074)	0.0557 (0.0232)	0.0049 (0.0038)	0.0508 (0.0186)	0.0040 (0.0026)
0.20	**0.0370 (0.0186)**	**0.0021 (0.0019)**	0.0616 (0.0235)	0.0064 (0.0049)	0.0773 (0.0313)	0.0093 (0.0069)	0.0587 (0.0266)	0.0056 (0.0048)	0.0533 (0.0187)	0.0045 (0.0029)
0.25	**0.0459 (0.0238)**	**0.0032 (0.0029)**	0.0622 (0.0220)	0.0065 (0.0047)	0.0791 (0.0304)	0.0098 (0.0068)	0.0629 (0.0256)	0.0063 (0.0046)	0.0566 (0.0221)	0.0050 (0.0036)
0.30	**0.0511 (0.0269)**	**0.0039 (0.0036)**	0.0653 (0.0236)	0.0071 (0.0053)	0.0813 (0.0302)	0.0103 (0.0071)	0.0667 (0.0284)	0.0072 (0.0054)	0.0604 (0.0238)	0.0059 (0.0047)
0.35	**0.0523 (0.0280)**	**0.0042 (0.0039)**	0.0681 (0.0242)	0.0079 (0.0056)	0.0841 (0.0313)	0.0112 (0.0078)	0.0707 (0.0278)	0.0080 (0.0058)	0.0647 (0.0255)	0.0067 (0.0049)
0.40	**0.0511 (0.0265)**	**0.0040 (0.0038)**	0.0731 (0.0259)	0.0091 (0.0064)	0.0883 (0.0339)	0.0125 (0.0091)	0.0766 (0.0291)	0.0094 (0.0062)	0.0693 (0.0270)	0.0076 (0.0053)
0.45	**0.0506 (0.0242)**	**0.0040 (0.0035)**	0.0804 (0.0274)	0.0110 (0.0070)	0.0948 (0.0378)	0.0144 (0.0107)	0.0838 (0.0322)	0.0114 (0.0077)	0.0774 (0.0273)	0.0093 (0.0062)
0.50	**0.0538 (0.0268)**	**0.0046 (0.0041)**	0.0886 (0.0300)	0.0132 (0.0084)	0.1036 (0.0417)	0.0173 (0.0127)	0.0917 (0.0354)	0.0136 (0.0090)	0.0863 (0.0333)	0.0117 (0.0087)
0.55	**0.0644 (0.0349)**	**0.0067 (0.0063)**	0.0974 (0.0341)	0.0158 (0.0105)	0.1140 (0.0460)	0.0210 (0.0151)	0.0982 (0.0368)	0.0159 (0.0112)	0.0934 (0.0394)	0.0142 (0.0115)
0.60	**0.0830 (0.0443)**	**0.0109 (0.0102)**	0.1081 (0.0384)	0.0196 (0.0127)	0.1264 (0.0514)	0.0262 (0.0189)	0.1069 (0.0397)	0.0187 (0.0130)	0.1045 (0.0418)	0.0177 (0.0131)
0.65	**0.1047 (0.0541)**	**0.0169 (0.0154)**	0.1216 (0.0406)	0.0243 (0.0145)	0.1432 (0.0621)	0.0344 (0.0269)	0.1189 (0.0452)	0.0230 (0.0154)	0.1161 (0.0477)	0.0223 (0.0169)
0.70	**0.1217 (0.0643)**	**0.0234 (0.0216)**	0.1376 (0.0458)	0.0312 (0.0189)	0.1692 (0.0842)	0.0498 (0.0448)	0.1335 (0.0511)	0.0288 (0.0195)	0.1332 (0.0527)	0.0289 (0.0209)
0.75	**0.1317 (0.0704)**	**0.0290 (0.0270)**	0.1638 (0.0570)	0.0442 (0.0290)	0.2126 (0.1184)	0.0804 (0.0809)	0.1606 (0.0626)	0.0412 (0.0283)	0.1628 (0.0638)	0.0427 (0.0309)
0.80	**0.1672 (0.0719)**	**0.0461 (0.0356)**	0.1991 (0.0760)	0.0643 (0.0484)	0.2783 (0.1631)	0.1373 (0.1462)	0.1897 (0.0819)	0.0574 (0.0459)	0.1921 (0.0848)	0.0612 (0.0540)
0.85	0.3167 (0.1591)	0.1705 (0.1664)	0.2516 (0.1053)	0.1017 (0.0820)	0.3573 (0.2138)	0.2245 (0.2490)	**0.2478 (0.1085)**	**0.0956 (0.0772)**	0.2497 (0.1104)	0.1026 (0.0850)
0.90	0.7289 (0.3448)	0.8337 (0.7766)	**0.3399 (0.1646)**	0.1885 (0.1747)	0.4355 (0.2490)	0.3230 (0.3540)	0.3429 (0.1638)	**0.1872 (0.1636)**	0.3538 (0.1614)	0.2045 (0.1707)
0.95	1.2645 (0.5692)	2.3599 (1.8314)	**0.5101 (0.2831)**	**0.4168 (0.4059)**	0.7798 (0.3375)	0.8437 (0.6197)	0.5599 (0.2819)	0.5070 (0.4564)	0.5899 (0.2886)	0.5767 (0.5349)
1st Count	16	16	2	1	0	0	1	2	0	0
2nd Count	0	0	0	2	0	0	4	4	15	13

Table 14: Linear, LN(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0283 (0.0154)**	**0.0012 (0.0011)**	0.0526 (0.0353)	0.0047 (0.0067)	0.0500 (0.0265)	0.0043 (0.0041)	0.0469 (0.0314)	0.0037 (0.0046)	0.0438 (0.0309)	0.0033 (0.0050)
0.10	**0.0245 (0.0139)**	**0.0009 (0.0009)**	0.0483 (0.0298)	0.0038 (0.0049)	0.0462 (0.0240)	0.0036 (0.0034)	0.0460 (0.0292)	0.0033 (0.0039)	0.0418 (0.0272)	0.0029 (0.0035)
0.15	**0.0178 (0.0112)**	**0.0005 (0.0006)**	0.0450 (0.0259)	0.0033 (0.0038)	0.0441 (0.0232)	0.0033 (0.0031)	0.0462 (0.0300)	0.0034 (0.0042)	0.0406 (0.0233)	0.0026 (0.0030)
0.20	**0.0174 (0.0089)**	**0.0005 (0.0005)**	0.0423 (0.0222)	0.0028 (0.0030)	0.0445 (0.0236)	0.0033 (0.0033)	0.0473 (0.0310)	0.0036 (0.0043)	0.0417 (0.0231)	0.0027 (0.0028)
0.25	**0.0224 (0.0108)**	**0.0007 (0.0006)**	0.0407 (0.0200)	0.0026 (0.0025)	0.0460 (0.0249)	0.0036 (0.0037)	0.0498 (0.0333)	0.0040 (0.0049)	0.0420 (0.0240)	0.0028 (0.0033)
0.30	**0.0287 (0.0132)**	**0.0012 (0.0010)**	0.0397 (0.0186)	0.0025 (0.0023)	0.0480 (0.0264)	0.0039 (0.0042)	0.0532 (0.0335)	0.0044 (0.0051)	0.0437 (0.0228)	0.0030 (0.0033)
0.35	**0.0339 (0.0147)**	**0.0016 (0.0012)**	0.0402 (0.0185)	0.0026 (0.0023)	0.0505 (0.0279)	0.0044 (0.0049)	0.0549 (0.0341)	0.0047 (0.0054)	0.0439 (0.0252)	0.0031 (0.0037)
0.40	**0.0365 (0.0151)**	**0.0018 (0.0014)**	0.0415 (0.0198)	0.0028 (0.0026)	0.0536 (0.0299)	0.0050 (0.0058)	0.0585 (0.0351)	0.0053 (0.0058)	0.0466 (0.0272)	0.0036 (0.0041)
0.45	**0.0360 (0.0147)**	**0.0018 (0.0013)**	0.0446 (0.0224)	0.0032 (0.0030)	0.0578 (0.0331)	0.0059 (0.0072)	0.0621 (0.0374)	0.0060 (0.0067)	0.0487 (0.0276)	0.0038 (0.0042)
0.50	**0.0334 (0.0133)**	**0.0016 (0.0011)**	0.0484 (0.0238)	0.0038 (0.0034)	0.0638 (0.0373)	0.0073 (0.0090)	0.0661 (0.0389)	0.0068 (0.0074)	0.0543 (0.0308)	0.0047 (0.0050)
0.55	**0.0312 (0.0121)**	**0.0015 (0.0010)**	0.0539 (0.0262)	0.0046 (0.0040)	0.0721 (0.0429)	0.0093 (0.0117)	0.0707 (0.0414)	0.0078 (0.0083)	0.0580 (0.0277)	0.0051 (0.0044)
0.60	**0.0342 (0.0164)**	**0.0020 (0.0018)**	0.0608 (0.0302)	0.0058 (0.0050)	0.0825 (0.0496)	0.0122 (0.0153)	0.0750 (0.0436)	0.0088 (0.0092)	0.0611 (0.0302)	0.0058 (0.0053)
0.65	**0.0472 (0.0256)**	**0.0038 (0.0039)**	0.0702 (0.0359)	0.0078 (0.0069)	0.0950 (0.0574)	0.0160 (0.0202)	0.0813 (0.0462)	0.0103 (0.0102)	0.0677 (0.0330)	0.0072 (0.0064)
0.70	**0.0682 (0.0375)**	**0.0074 (0.0073)**	0.0771 (0.0390)	0.0095 (0.0086)	0.1107 (0.0667)	0.0215 (0.0270)	0.0889 (0.0478)	0.0122 (0.0111)	0.0737 (0.0358)	0.0087 (0.0075)
0.75	0.0905 (0.0504)	0.0126 (0.0123)	0.0875 (0.0460)	0.0124 (0.0127)	0.1337 (0.0806)	0.0309 (0.0380)	0.0971 (0.0524)	0.0147 (0.0132)	**0.0837 (0.0378)**	**0.0112 (0.0093)**
0.80	0.1098 (0.0641)	0.0194 (0.0218)	**0.0981 (0.0516)**	0.0158 (0.0159)	0.1766 (0.1071)	0.0518 (0.0616)	0.1077 (0.0566)	0.0180 (0.0158)	0.0983 (0.0459)	**0.0156 (0.0137)**
0.85	0.1653 (0.1290)	0.0518 (0.0944)	**0.1140 (0.0615)**	**0.0214 (0.0215)**	0.2609 (0.1632)	0.1091 (0.1248)	0.1196 (0.0609)	0.0224 (0.0198)	0.1243 (0.0542)	0.0244 (0.0212)
0.90	0.5059 (0.3597)	0.4175 (0.6116)	**0.1388 (0.0717)**	**0.0317 (0.0307)**	0.3991 (0.2503)	0.2491 (0.2741)	0.1480 (0.0714)	0.0345 (0.0309)	0.1691 (0.0709)	0.0448 (0.0336)
0.95	1.2715 (0.6151)	2.0630 (1.8131)	**0.1873 (0.0990)**	**0.0611 (0.0643)**	0.5030 (0.3020)	0.4063 (0.4229)	0.1975 (0.1056)	0.0666 (0.0710)	0.2560 (0.1149)	0.1089 (0.0905)
1st Count	14	14	4	3	0	0	0	0	1	2
2nd Count	0	0	9	10	0	0	3	3	7	7

Table 15: Linear, Exp(1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0196 (0.0096)**	**0.0006 (0.0005)**	0.0511 (0.0202)	0.0043 (0.0033)	0.0553 (0.0230)	0.0059 (0.0053)	0.0525 (0.0221)	0.0044 (0.0033)	0.0424 (0.0206)	0.0029 (0.0027)
0.10	**0.0162 (0.0077)**	**0.0004 (0.0004)**	0.0500 (0.0199)	0.0042 (0.0034)	0.0521 (0.0230)	0.0050 (0.0046)	0.0471 (0.0202)	0.0035 (0.0026)	0.0422 (0.0191)	0.0029 (0.0024)
0.15	**0.0234 (0.0121)**	**0.0009 (0.0008)**	0.0500 (0.0206)	0.0042 (0.0036)	0.0521 (0.0237)	0.0049 (0.0045)	0.0492 (0.0219)	0.0038 (0.0031)	0.0445 (0.0219)	0.0033 (0.0033)
0.20	**0.0313 (0.0155)**	**0.0014 (0.0012)**	0.0505 (0.0203)	0.0043 (0.0036)	0.0536 (0.0245)	0.0052 (0.0047)	0.0507 (0.0220)	0.0041 (0.0032)	0.0470 (0.0199)	0.0036 (0.0030)
0.25	**0.0362 (0.0180)**	**0.0019 (0.0016)**	0.0509 (0.0199)	0.0045 (0.0036)	0.0559 (0.0255)	0.0056 (0.0050)	0.0529 (0.0218)	0.0045 (0.0033)	0.0482 (0.0195)	0.0038 (0.0031)
0.30	**0.0381 (0.0192)**	**0.0022 (0.0019)**	0.0532 (0.0186)	0.0048 (0.0035)	0.0591 (0.0267)	0.0062 (0.0053)	0.0546 (0.0232)	0.0048 (0.0038)	0.0498 (0.0170)	0.0040 (0.0027)
0.35	**0.0379 (0.0187)**	**0.0022 (0.0019)**	0.0555 (0.0176)	0.0052 (0.0034)	0.0634 (0.0280)	0.0070 (0.0059)	0.0580 (0.0230)	0.0054 (0.0042)	0.0548 (0.0180)	0.0048 (0.0029)
0.40	**0.0368 (0.0175)**	**0.0021 (0.0019)**	0.0591 (0.0199)	0.0059 (0.0038)	0.0691 (0.0294)	0.0081 (0.0066)	0.0647 (0.0262)	0.0065 (0.0051)	0.0589 (0.0195)	0.0055 (0.0036)
0.45	**0.0366 (0.0182)**	**0.0022 (0.0021)**	0.0642 (0.0205)	0.0068 (0.0042)	0.0758 (0.0309)	0.0095 (0.0076)	0.0672 (0.0240)	0.0070 (0.0048)	0.0661 (0.0208)	0.0068 (0.0039)
0.50	**0.0395 (0.0220)**	**0.0026 (0.0027)**	0.0688 (0.0228)	0.0080 (0.0051)	0.0833 (0.0327)	0.0113 (0.0087)	0.0726 (0.0278)	0.0083 (0.0057)	0.0709 (0.0269)	0.0083 (0.0055)
0.55	**0.0474 (0.0269)**	**0.0038 (0.0037)**	0.0743 (0.0252)	0.0094 (0.0063)	0.0915 (0.0353)	0.0135 (0.0101)	0.0792 (0.0291)	0.0098 (0.0064)	0.0759 (0.0275)	0.0096 (0.0064)
0.60	**0.0595 (0.0318)**	**0.0058 (0.0053)**	0.0797 (0.0281)	0.0108 (0.0072)	0.1007 (0.0394)	0.0163 (0.0123)	0.0840 (0.0313)	0.0113 (0.0077)	0.0809 (0.0301)	0.0109 (0.0073)
0.65	**0.0725 (0.0370)**	**0.0083 (0.0073)**	0.0875 (0.0301)	0.0129 (0.0084)	0.1120 (0.0457)	0.0202 (0.0161)	0.0912 (0.0357)	0.0133 (0.0097)	0.0857 (0.0311)	0.0121 (0.0080)
0.70	**0.0829 (0.0418)**	**0.0108 (0.0095)**	0.0981 (0.0338)	0.0159 (0.0106)	0.1271 (0.0557)	0.0260 (0.0229)	0.0999 (0.0377)	0.0158 (0.0107)	0.0975 (0.0373)	0.0154 (0.0108)
0.75	**0.0901 (0.0429)**	**0.0129 (0.0111)**	0.1105 (0.0392)	0.0201 (0.0145)	0.1474 (0.0706)	0.0352 (0.0344)	0.1105 (0.0434)	0.0195 (0.0137)	0.1070 (0.0422)	0.0185 (0.0128)
0.80	**0.1001 (0.0404)**	**0.0158 (0.0122)**	0.1241 (0.0431)	0.0250 (0.0165)	0.1732 (0.0897)	0.0492 (0.0518)	0.1231 (0.0477)	0.0238 (0.0176)	0.1173 (0.0477)	0.0229 (0.0170)
0.85	**0.1332 (0.0614)**	0.0302 (0.0239)	0.1427 (0.0544)	0.0330 (0.0250)	0.2019 (0.1090)	0.0676 (0.0735)	0.1383 (0.0568)	0.0305 (0.0231)	0.1359 (0.0525)	**0.0296 (0.0204)**
0.90	0.2426 (0.1052)	0.0916 (0.0676)	0.1597 (0.0694)	0.0429 (0.0353)	0.2274 (0.1208)	0.0861 (0.0904)	0.1631 (0.0712)	0.0437 (0.0380)	**0.1593 (0.0713)**	**0.0427 (0.0358)**
0.95	0.3985 (0.1615)	0.2408 (0.1730)	**0.2088 (0.0909)**	**0.0708 (0.0613)**	0.3283 (0.1505)	0.1580 (0.1280)	0.2094 (0.0981)	0.0734 (0.0751)	0.2169 (0.0917)	0.0762 (0.0611)
1st Count	17	16	1	1	0	0	0	0	1	2
2nd Count	0	1	5	5	0	0	1	1	13	13

Table 16: Linear, Exp(1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0124 (0.0092)**	**0.0003 (0.0005)**	0.0463 (0.0262)	0.0033 (0.0034)	0.0394 (0.0213)	0.0028 (0.0028)	0.0381 (0.0216)	0.0023 (0.0023)	0.0348 (0.0211)	0.0020 (0.0024)
0.10	**0.0109 (0.0074)**	**0.0002 (0.0003)**	0.0438 (0.0240)	0.0029 (0.0029)	0.0374 (0.0205)	0.0025 (0.0024)	0.0379 (0.0226)	0.0022 (0.0022)	0.0340 (0.0198)	0.0018 (0.0020)
0.15	**0.0112 (0.0065)**	**0.0002 (0.0003)**	0.0405 (0.0221)	0.0025 (0.0024)	0.0373 (0.0207)	0.0025 (0.0024)	0.0380 (0.0223)	0.0022 (0.0022)	0.0343 (0.0189)	0.0018 (0.0020)
0.20	**0.0121 (0.0066)**	**0.0002 (0.0003)**	0.0372 (0.0202)	0.0022 (0.0021)	0.0380 (0.0211)	0.0025 (0.0026)	0.0391 (0.0232)	0.0023 (0.0023)	0.0352 (0.0182)	0.0019 (0.0019)
0.25	**0.0133 (0.0069)**	**0.0003 (0.0003)**	0.0350 (0.0188)	0.0020 (0.0018)	0.0390 (0.0216)	0.0027 (0.0028)	0.0395 (0.0231)	0.0024 (0.0024)	0.0348 (0.0162)	0.0019 (0.0016)
0.30	**0.0144 (0.0073)**	**0.0003 (0.0003)**	0.0340 (0.0173)	0.0019 (0.0016)	0.0403 (0.0223)	0.0029 (0.0031)	0.0416 (0.0241)	0.0026 (0.0027)	0.0355 (0.0177)	0.0020 (0.0020)
0.35	**0.0154 (0.0079)**	**0.0004 (0.0004)**	0.0340 (0.0165)	0.0019 (0.0016)	0.0419 (0.0234)	0.0031 (0.0035)	0.0435 (0.0250)	0.0029 (0.0030)	0.0363 (0.0171)	0.0020 (0.0018)
0.40	**0.0163 (0.0085)**	**0.0004 (0.0005)**	0.0349 (0.0165)	0.0020 (0.0017)	0.0440 (0.0249)	0.0035 (0.0041)	0.0454 (0.0275)	0.0032 (0.0035)	0.0372 (0.0198)	0.0022 (0.0023)
0.45	**0.0173 (0.0092)**	**0.0005 (0.0006)**	0.0364 (0.0170)	0.0021 (0.0020)	0.0469 (0.0273)	0.0040 (0.0049)	0.0475 (0.0284)	0.0035 (0.0040)	0.0397 (0.0214)	0.0026 (0.0027)
0.50	**0.0184 (0.0099)**	**0.0006 (0.0008)**	0.0386 (0.0186)	0.0024 (0.0024)	0.0509 (0.0303)	0.0048 (0.0059)	0.0501 (0.0305)	0.0040 (0.0045)	0.0400 (0.0207)	0.0026 (0.0025)
0.55	**0.0197 (0.0106)**	**0.0007 (0.0010)**	0.0419 (0.0208)	0.0029 (0.0029)	0.0560 (0.0337)	0.0058 (0.0070)	0.0531 (0.0323)	0.0045 (0.0050)	0.0427 (0.0188)	0.0028 (0.0023)
0.60	**0.0217 (0.0116)**	**0.0009 (0.0012)**	0.0463 (0.0233)	0.0035 (0.0036)	0.0619 (0.0368)	0.0069 (0.0082)	0.0568 (0.0343)	0.0051 (0.0056)	0.0457 (0.0194)	0.0032 (0.0025)
0.65	**0.0245 (0.0134)**	**0.0011 (0.0014)**	0.0521 (0.0259)	0.0043 (0.0044)	0.0685 (0.0395)	0.0083 (0.0094)	0.0617 (0.0369)	0.0060 (0.0066)	0.0518 (0.0225)	0.0042 (0.0034)
0.70	**0.0284 (0.0161)**	**0.0014 (0.0019)**	0.0582 (0.0279)	0.0052 (0.0051)	0.0763 (0.0422)	0.0100 (0.0108)	0.0671 (0.0405)	0.0072 (0.0079)	0.0584 (0.0250)	0.0052 (0.0040)
0.75	**0.0345 (0.0201)**	**0.0021 (0.0026)**	0.0656 (0.0315)	0.0065 (0.0060)	0.0880 (0.0487)	0.0130 (0.0137)	0.0734 (0.0448)	0.0086 (0.0097)	0.0652 (0.0288)	0.0064 (0.0050)
0.80	**0.0452 (0.0256)**	**0.0033 (0.0038)**	0.0737 (0.0356)	0.0082 (0.0074)	0.1084 (0.0637)	0.0195 (0.0209)	0.0812 (0.0482)	0.0105 (0.0114)	0.0732 (0.0357)	0.0084 (0.0072)
0.85	**0.0610 (0.0358)**	**0.0060 (0.0070)**	0.0817 (0.0389)	0.0101 (0.0087)	0.1419 (0.0850)	0.0322 (0.0355)	0.0878 (0.0520)	0.0124 (0.0134)	0.0845 (0.0400)	0.0112 (0.0097)
0.90	**0.0868 (0.0614)**	0.0129 (0.0217)	0.0910 (0.0442)	**0.0128 (0.0112)**	0.1759 (0.1060)	0.0489 (0.0553)	0.0991 (0.0557)	0.0156 (0.0159)	0.1005 (0.0428)	0.0154 (0.0113)
0.95	0.2023 (0.1620)	0.0728 (0.1112)	**0.1032 (0.0521)**	**0.0170 (0.0154)**	0.2024 (0.1289)	0.0672 (0.0829)	0.1098 (0.0613)	0.0196 (0.0189)	0.1280 (0.0596)	0.0260 (0.0202)
1st Count	18	17	1	2	0	0	0	0	0	0
2nd Count	0	1	9	8	0	0	1	1	9	10

Table 17: Wave, N(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.7067 (0.1251)**	**1.0495 (0.4953)**	2.2693 (0.2937)	12.2306 (3.9544)	2.4671 (0.4306)	13.0560 (6.0978)	1.6019 (0.3576)	4.2397 (2.2361)	2.3877 (0.2778)	11.2324 (3.3130)
0.10	**0.6962 (0.1114)**	**1.0148 (0.4185)**	1.8054 (0.2230)	7.8407 (2.4533)	2.0444 (0.3553)	9.0407 (4.1142)	1.5695 (0.2905)	4.1256 (1.8450)	1.8497 (0.2004)	7.0572 (2.0816)
0.15	**0.6652 (0.0965)**	**0.8678 (0.3017)**	1.4522 (0.1859)	5.1089 (1.6699)	1.6841 (0.3005)	6.1155 (2.7920)	1.4664 (0.2467)	3.5675 (1.4799)	1.4851 (0.1779)	4.6795 (1.5269)
0.20	**0.6404 (0.0719)**	**0.7610 (0.2054)**	1.1816 (0.1621)	3.4317 (1.2261)	1.3766 (0.2622)	4.0834 (1.9240)	1.2995 (0.2142)	2.8043 (1.0661)	1.2123 (0.1633)	3.2560 (1.1576)
0.25	**0.6231 (0.0509)**	**0.6959 (0.1345)**	0.9510 (0.1612)	2.2893 (0.9915)	1.1129 (0.2332)	2.6996 (1.3440)	1.1039 (0.2020)	2.0700 (0.8410)	0.9804 (0.1684)	2.2553 (0.9588)
0.30	**0.6123 (0.0440)**	**0.6521 (0.0995)**	0.7392 (0.1586)	1.4460 (0.7744)	0.8882 (0.2045)	1.7717 (0.9403)	0.8917 (0.1748)	1.3836 (0.5849)	0.7659 (0.1636)	1.4798 (0.7701)
0.35	0.6063 (0.0414)	**0.6220 (0.0869)**	**0.5557 (0.1539)**	0.8886 (0.5801)	0.7074 (0.1679)	1.1632 (0.6453)	0.7082 (0.1686)	0.9385 (0.5078)	0.5834 (0.1463)	0.9659 (0.5850)
0.40	0.6039 (0.0359)	0.6053 (0.0708)	**0.3996 (0.1320)**	**0.5101 (0.3862)**	0.5781 (0.1298)	0.7823 (0.4217)	0.5469 (0.1317)	0.5841 (0.3032)	0.4427 (0.1203)	0.6254 (0.4155)
0.45	0.6052 (0.0313)	0.6030 (0.0514)	**0.2903 (0.0999)**	**0.2937 (0.2547)**	0.5052 (0.1099)	0.5717 (0.2594)	0.4439 (0.0986)	0.3998 (0.2026)	0.3365 (0.0860)	0.3903 (0.2539)
0.50	0.6105 (0.0392)	0.6154 (0.0733)	**0.2417 (0.0778)**	**0.2013 (0.1749)**	0.4879 (0.1154)	0.5004 (0.1913)	0.4099 (0.0979)	0.3551 (0.1828)	0.2949 (0.0801)	0.2926 (0.1788)
0.55	0.6184 (0.0578)	0.6411 (0.1398)	**0.2795 (0.0919)**	**0.2311 (0.1889)**	0.5256 (0.1309)	0.5590 (0.2534)	0.4299 (0.1189)	0.4125 (0.2564)	0.3344 (0.1000)	0.3231 (0.2150)
0.60	0.6280 (0.0797)	0.6778 (0.2226)	**0.3845 (0.1220)**	**0.3904 (0.2966)**	0.6059 (0.1479)	0.7569 (0.3820)	0.5016 (0.1429)	0.5887 (0.3912)	0.4357 (0.1227)	0.4891 (0.3210)
0.65	0.6386 (0.1015)	0.7222 (0.3088)	**0.5356 (0.1429)**	**0.6988 (0.4383)**	0.7229 (0.1599)	1.1227 (0.5430)	0.6047 (0.1615)	0.8534 (0.5354)	0.5676 (0.1522)	0.7934 (0.4961)
0.70	**0.6498 (0.1207)**	**0.7702 (0.3879)**	0.7106 (0.1579)	1.1621 (0.6017)	0.8758 (0.1671)	1.7059 (0.7423)	0.7402 (0.1876)	1.2582 (0.7393)	0.7313 (0.1714)	1.2692 (0.6935)
0.75	**0.6612 (0.1350)**	**0.8176 (0.4489)**	0.9139 (0.1679)	1.8561 (0.8120)	1.0645 (0.1748)	2.5794 (1.0037)	0.9337 (0.2015)	1.9599 (0.9629)	0.9345 (0.1710)	2.0021 (0.8554)
0.80	**0.6723 (0.1415)**	**0.8596 (0.4799)**	1.1457 (0.1708)	2.8515 (1.0229)	1.2925 (0.1931)	3.8433 (1.3691)	1.1656 (0.2054)	2.9460 (1.2000)	1.1560 (0.1582)	2.9953 (0.9979)
0.85	**0.6815 (0.1371)**	**0.8880 (0.4719)**	1.4240 (0.1917)	4.3908 (1.4678)	1.5649 (0.2279)	5.6226 (1.9020)	1.4544 (0.2145)	4.4504 (1.6045)	1.4350 (0.1815)	4.6082 (1.3809)
0.90	**0.6829 (0.1236)**	**0.8823 (0.4309)**	1.7857 (0.2161)	6.9030 (2.0527)	1.8836 (0.2805)	8.0466 (2.6896)	1.8185 (0.2455)	6.8069 (2.3360)	1.7952 (0.2030)	7.2567 (1.9325)
0.95	**0.6570 (0.1093)**	**0.8068 (0.3573)**	2.2427 (0.2802)	10.8113 (3.3542)	2.2432 (0.3450)	11.1367 (3.8215)	2.2944 (0.3083)	10.6645 (3.4308)	2.2881 (0.2969)	11.9320 (3.5560)
1st Count	12	13	7	6	0	0	0	0	0	0
2nd Count	0	1	10	5	0	0	2	9	7	4

Table 18: Wave, N(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.2989 (0.1293)**	**0.2914 (0.2697)**	2.3280 (0.1942)	12.5483 (2.6169)	2.2966 (0.2524)	11.9823 (3.1789)	2.2674 (0.2216)	9.4604 (1.6231)	2.3009 (0.1919)	11.8323 (2.1299)
0.10	**0.2114 (0.0840)**	**0.1457 (0.1405)**	1.8068 (0.1524)	7.4400 (1.5030)	1.8382 (0.2060)	7.7684 (2.1124)	1.8775 (0.1603)	7.3819 (1.5236)	1.7865 (0.1605)	7.1615 (1.4051)
0.15	**0.1775 (0.0686)**	**0.0801 (0.0878)**	1.4576 (0.1267)	4.7850 (0.9921)	1.4665 (0.1695)	4.9337 (1.4074)	1.4729 (0.1309)	4.8798 (1.0584)	1.4442 (0.1411)	4.7174 (1.0131)
0.20	**0.1778 (0.0661)**	**0.0725 (0.0883)**	1.1827 (0.1084)	3.1367 (0.6809)	1.1635 (0.1411)	3.0904 (0.9396)	1.1786 (0.1127)	3.2856 (0.7460)	1.1686 (0.1241)	3.1167 (0.7236)
0.25	**0.1897 (0.0685)**	**0.0769 (0.0948)**	0.9481 (0.0966)	2.0168 (0.4733)	0.9104 (0.1191)	1.8896 (0.6278)	0.9311 (0.0981)	2.1283 (0.5433)	0.9388 (0.1128)	2.0421 (0.5272)
0.30	**0.2169 (0.0733)**	**0.0988 (0.0976)**	0.7395 (0.0904)	1.2379 (0.3455)	0.6923 (0.1023)	1.1026 (0.4171)	0.7107 (0.1067)	1.2904 (0.4301)	0.7292 (0.1068)	1.2630 (0.4036)
0.35	**0.2735 (0.0743)**	**0.1513 (0.0956)**	0.5495 (0.0812)	0.7012 (0.2364)	0.5005 (0.0888)	0.5932 (0.2696)	0.5287 (0.1029)	0.7399 (0.3107)	0.5439 (0.1023)	0.7426 (0.3061)
0.40	0.3780 (0.0642)	**0.2609 (0.0864)**	0.3701 (0.0733)	0.3360 (0.1482)	**0.3306 (0.0778)**	0.2797 (0.1649)	0.3511 (0.0937)	0.3533 (0.1950)	0.3664 (0.0888)	0.3726 (0.1961)
0.45	0.5084 (0.0380)	0.4339 (0.0579)	0.2084 (0.0596)	0.1231 (0.0775)	**0.1945 (0.0672)**	**0.1133 (0.0939)**	0.2206 (0.0718)	0.1487 (0.1098)	0.2070 (0.0640)	0.1454 (0.1010)
0.50	0.5828 (0.0220)	0.5556 (0.0350)	**0.1250 (0.0353)**	**0.0465 (0.0294)**	0.1501 (0.0612)	0.0673 (0.0653)	0.1790 (0.0613)	0.0852 (0.0657)	0.1430 (0.0438)	0.0698 (0.0514)
0.55	0.5062 (0.0480)	0.4155 (0.0809)	**0.2015 (0.0542)**	**0.1032 (0.0636)**	0.2317 (0.0647)	0.1329 (0.0922)	0.2442 (0.0957)	0.1584 (0.1319)	0.2151 (0.0467)	0.1169 (0.0558)
0.60	**0.3517 (0.0706)**	**0.2090 (0.0915)**	0.3538 (0.0711)	0.2963 (0.1342)	0.3724 (0.0697)	0.3179 (0.1513)	0.3868 (0.1175)	0.3722 (0.2514)	0.3574 (0.0649)	0.2974 (0.1146)
0.65	**0.2528 (0.0692)**	**0.1178 (0.0719)**	0.3357 (0.0804)	0.6562 (0.2309)	0.5402 (0.0805)	0.6497 (0.2410)	0.5628 (0.1342)	0.7507 (0.3795)	0.5293 (0.0749)	0.6253 (0.1824)
0.70	**0.2144 (0.0587)**	**0.0897 (0.0539)**	0.7294 (0.0912)	1.1990 (0.3539)	0.7316 (0.0963)	1.1812 (0.3796)	0.7529 (0.1467)	1.3245 (0.5554)	0.7242 (0.0814)	1.1496 (0.2697)
0.75	**0.1879 (0.0546)**	**0.0768 (0.0489)**	0.9392 (0.1030)	1.9746 (0.5078)	0.9503 (0.1184)	2.0018 (0.5963)	0.9630 (0.1626)	2.1348 (0.7798)	0.9356 (0.0944)	1.9117 (0.4058)
0.80	**0.1784 (0.0527)**	**0.0706 (0.0502)**	1.1697 (0.1140)	3.0395 (0.6776)	1.2068 (0.1471)	3.2562 (0.9356)	1.1949 (0.1748)	3.2568 (1.0294)	1.1670 (0.1069)	2.9591 (0.5751)
0.85	**0.1823 (0.0568)**	**0.0757 (0.0666)**	1.4419 (0.1255)	4.5993 (0.8783)	1.5139 (0.1834)	5.1709 (1.4639)	1.4708 (0.1887)	4.9030 (1.3546)	1.4446 (0.1146)	4.5377 (0.7697)
0.90	**0.2087 (0.0786)**	**0.1320 (0.1358)**	1.7811 (0.1431)	6.9770 (1.2032)	1.8865 (0.2287)	8.0724 (2.2806)	1.8132 (0.2133)	7.3996 (1.8630)	1.7823 (0.1318)	6.8635 (1.0679)
0.95	**0.3046 (0.1322)**	**0.2701 (0.2463)**	2.2807 (0.1732)	11.3252 (1.8558)	2.3414 (0.2835)	12.3207 (3.5205)	2.3242 (0.2489)	12.1466 (2.7960)	2.2938 (0.1667)	11.3230 (1.7595)
1st Count	15	16	2	2	2	1	0	0	0	0
2nd Count	0	0	4	2	4	6	2	1	9	10

Table 19: Wave, LN(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.6807 (0.0954)**	**0.9779 (0.4118)**	2.0211 (0.1219)	9.1177 (1.1848)	2.2958 (0.2507)	11.6269 (2.9917)	2.2692 (0.2187)	9.1574 (1.1764)	2.1053 (0.1324)	9.2808 (1.1071)
0.10	**0.6377 (0.0567)**	**0.7994 (0.2220)**	1.9049 (0.1181)	8.1245 (1.0936)	2.1105 (0.2298)	9.7426 (2.5154)	2.0216 (0.1491)	7.8102 (0.9652)	1.9425 (0.1168)	8.2307 (1.0045)
0.15	**0.6156 (0.0342)**	**0.6957 (0.1118)**	1.8023 (0.1163)	7.3059 (1.0313)	1.9467 (0.2169)	8.2289 (2.1860)	1.9314 (0.1512)	6.9080 (0.9788)	1.8297 (0.1109)	7.4480 (0.9593)
0.20	**0.6165 (0.0329)**	**0.6702 (0.0741)**	1.6982 (0.1171)	6.5165 (1.0055)	1.7992 (0.2102)	7.0017 (1.9571)	1.8309 (0.1501)	6.1696 (1.0522)	1.7186 (0.1051)	6.6302 (0.8955)
0.25	**0.6392 (0.0451)**	**0.7127 (0.0951)**	1.5877 (0.1130)	5.7221 (0.9443)	1.6634 (0.2083)	5.9881 (1.7939)	1.7134 (0.1519)	5.4311 (1.1022)	1.6097 (0.1048)	5.8482 (0.8970)
0.30	**0.6813 (0.0531)**	**0.8125 (0.1284)**	1.4765 (0.1132)	4.9854 (0.9019)	1.5349 (0.2099)	5.1287 (1.6699)	1.5645 (0.1455)	4.5477 (1.0115)	1.5051 (0.1089)	5.1517 (0.8713)
0.35	**0.7341 (0.0518)**	**0.9613 (0.1537)**	1.3659 (0.1193)	4.3201 (0.8842)	1.4089 (0.2140)	4.3755 (1.5645)	1.4064 (0.1485)	3.7138 (0.9589)	1.3895 (0.1130)	4.4210 (0.8524)
0.40	**0.7902 (0.0466)**	**1.1549 (0.1655)**	1.2432 (0.1261)	3.6363 (0.8570)	1.2800 (0.2196)	3.6894 (1.4607)	1.2614 (0.1743)	3.0795 (1.0352)	1.2684 (0.1198)	3.7278 (0.8268)
0.45	**0.8469 (0.0438)**	**1.3953 (0.1748)**	1.1049 (0.1348)	2.9302 (0.8280)	1.1426 (0.2253)	3.0415 (1.3460)	1.1647 (0.2007)	2.7471 (1.0721)	1.1307 (0.1322)	3.0236 (0.8138)
0.50	**0.9053 (0.0508)**	**1.6920 (0.2380)**	0.9494 (0.1433)	2.2367 (0.7958)	0.9936 (0.2240)	2.4162 (1.2091)	1.0317 (0.2224)	2.2729 (1.0595)	0.9785 (0.1529)	2.3227 (0.7967)
0.55	0.9671 (0.0743)	2.0662 (0.4324)	**0.7774 (0.1563)**	**1.5978 (0.7452)**	0.8340 (0.2062)	1.8165 (1.0366)	0.9265 (0.2336)	1.9692 (1.0569)	0.8082 (0.1666)	1.6719 (0.7319)
0.60	1.0352 (0.1154)	2.5552 (0.8068)	**0.5923 (0.1536)**	**1.0328 (0.6288)**	0.6771 (0.1670)	1.2799 (0.8162)	0.7781 (0.2368)	1.5326 (1.0321)	0.6207 (0.1569)	1.0752 (0.5747)
0.65	1.1128 (0.1734)	3.2115 (1.4046)	**0.4210 (0.1375)**	**0.6075 (0.4762)**	0.5640 (0.1276)	0.9074 (0.5764)	0.6316 (0.2365)	1.1402 (0.9955)	0.4518 (0.1293)	0.6254 (0.4216)
0.70	1.2045 (0.2477)	4.0874 (2.2320)	**0.3511 (0.1410)**	**0.4367 (0.4075)**	0.5715 (0.1801)	0.9177 (0.6899)	0.5590 (0.2571)	0.9143 (1.0568)	0.4123 (0.1403)	0.4776 (0.4029)
0.75	1.3130 (0.3344)	5.1858 (3.2086)	**0.5343 (0.2204)**	**0.8314 (0.8213)**	0.7978 (0.3138)	1.7427 (1.6574)	0.6578 (0.2741)	1.1152 (1.1723)	0.5987 (0.2263)	0.9046 (0.8678)
0.80	1.4374 (0.4212)	6.3024 (4.0989)	**0.9674 (0.3472)**	2.4864 (2.0902)	1.2559 (0.4869)	4.1795 (3.7167)	0.9906 (0.3421)	**2.3655 (1.9891)**	1.0012 (0.3673)	2.6074 (2.3248)
0.85	**1.5612 (0.4797)**	**6.7599 (4.4379)**	1.6569 (0.4927)	6.8214 (4.9584)	1.9575 (0.6957)	9.5903 (7.8076)	1.6882 (0.5376)	7.0630 (5.5595)	1.6374 (0.5359)	7.0908 (5.5929)
0.90	**1.5320 (0.4468)**	**5.2385 (3.4527)**	2.7480 (0.7022)	17.2963 (11.6960)	2.8996 (0.9702)	19.9771 (15.9585)	2.7271 (0.7623)	17.8733 (12.5423)	2.6223 (0.7609)	18.0449 (13.3952)
0.95	**1.2648 (0.3845)**	**4.1477 (3.4560)**	4.4406 (1.1209)	41.6963 (27.5186)	4.1229 (1.3096)	36.9133 (31.2125)	4.4809 (1.1506)	46.7821 (29.3575)	4.6924 (1.3191)	54.5905 (38.6933)
1st Count	13	13	6	5	0	0	0	1	0	0
2nd Count	0	0	10	5	1	1	1	8	7	5

Table 20: Wave, LN(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.1526 (0.1201)**	**0.0720 (0.1898)**	2.0068 (0.1097)	8.9409 (0.9556)	2.0527 (0.1972)	9.5689 (2.0220)	2.0745 (0.1757)	8.5369 (0.9809)	1.9948 (0.1031)	8.8058 (0.8811)
0.10	**0.1537 (0.1100)**	**0.0675 (0.1642)**	1.8870 (0.1019)	7.8790 (0.8167)	1.9220 (0.1820)	8.3528 (1.7510)	1.9330 (0.1102)	7.8943 (0.8776)	1.8771 (0.1006)	7.8053 (0.8044)
0.15	**0.1516 (0.1024)**	**0.0619 (0.1416)**	1.7818 (0.0956)	7.0206 (0.7179)	1.7989 (0.1697)	7.2811 (1.5308)	1.8287 (0.1047)	7.4176 (0.8877)	1.7703 (0.0925)	6.9612 (0.6869)
0.20	**0.1535 (0.0985)**	**0.0616 (0.1278)**	1.6765 (0.0887)	6.2139 (0.6220)	1.6810 (0.1596)	6.3310 (1.3480)	1.7228 (0.1041)	6.7544 (0.8584)	1.6678 (0.0899)	6.1989 (0.6346)
0.25	**0.1733 (0.0995)**	**0.0749 (0.1251)**	1.5684 (0.0841)	5.4368 (0.5581)	1.5656 (0.1509)	5.4740 (1.1913)	1.5974 (0.1158)	5.8712 (0.8974)	1.5580 (0.0842)	5.4298 (0.5722)
0.30	**0.2346 (0.1060)**	**0.1199 (0.1360)**	1.4552 (0.0813)	4.6829 (0.5089)	1.4493 (0.1433)	4.6832 (1.0538)	1.4583 (0.1229)	4.8959 (0.8573)	1.4471 (0.0825)	4.7108 (0.5345)
0.35	**0.3541 (0.1053)**	**0.2372 (0.1612)**	1.3322 (0.0760)	3.9279 (0.4429)	1.3282 (0.1361)	3.9337 (0.9270)	1.3305 (0.1232)	4.0541 (0.7985)	1.3259 (0.0813)	3.9772 (0.4898)
0.40	**0.5231 (0.0859)**	**0.5004 (0.1787)**	1.2005 (0.0753)	3.1972 (0.4117)	1.1981 (0.1291)	3.2092 (0.8049)	1.2126 (0.1400)	3.3860 (0.8052)	1.1959 (0.0761)	3.2704 (0.4267)
0.45	**0.7108 (0.0505)**	**0.9454 (0.1509)**	1.0547 (0.0742)	2.4860 (0.3684)	1.0555 (0.1226)	2.5063 (0.6862)	1.0602 (0.1435)	2.5800 (0.7492)	1.0513 (0.0732)	2.5672 (0.3729)
0.50	**0.8466 (0.0492)**	**1.3896 (0.1883)**	0.8932 (0.0749)	1.8050 (0.3206)	0.8975 (0.1173)	1.8346 (0.5703)	0.9193 (0.1629)	1.9821 (0.7755)	0.8896 (0.0742)	1.8892 (0.3350)
0.55	0.8828 (0.0705)	1.6057 (0.3287)	0.7125 (0.0783)	**1.1795 (0.2806)**	0.7211 (0.1149)	1.2153 (0.4573)	0.7489 (0.1492)	1.3429 (0.5650)	**0.7097 (0.0762)**	1.2634 (0.2858)
0.60	0.8391 (0.1198)	1.6223 (0.5865)	0.5028 (0.0854)	**0.6252 (0.2220)**	0.5233 (0.1166)	0.6833 (0.3427)	0.5447 (0.1718)	0.7770 (0.5128)	**0.4998 (0.0750)**	0.7038 (0.2132)
0.65	0.7741 (0.1738)	1.6214 (0.8754)	**0.2745 (0.0792)**	**0.2246 (0.1305)**	0.3203 (0.1010)	0.2952 (0.2111)	0.3393 (0.1527)	0.3486 (0.3639)	0.2785 (0.0614)	0.2960 (0.1292)
0.70	0.7519 (0.2198)	1.8683 (1.2078)	**0.1891 (0.0533)**	**0.1006 (0.0563)**	0.2365 (0.0803)	0.1559 (0.1197)	0.2416 (0.1043)	0.1767 (0.1690)	0.2156 (0.0422)	0.1505 (0.0597)
0.75	0.8607 (0.2458)	2.6765 (1.5957)	0.4495 (0.1130)	**0.4870 (0.2816)**	**0.4408 (0.1625)**	0.4938 (0.4717)	0.4593 (0.1820)	0.5485 (0.4613)	0.4583 (0.0953)	0.4956 (0.2431)
0.80	1.0778 (0.2725)	3.8540 (1.9578)	0.9046 (0.1397)	1.8305 (0.6558)	**0.8917 (0.2615)**	1.9114 (1.4245)	0.9048 (0.2429)	1.9108 (1.0619)	0.9140 (0.1301)	**1.7834 (0.5954)**
0.85	**1.2859 (0.3112)**	**4.5986 (2.1287)**	1.5854 (0.1850)	5.5297 (1.5377)	1.6309 (0.4282)	6.3287 (4.0616)	1.5661 (0.2785)	5.5433 (2.1450)	1.5963 (0.1679)	5.3511 (1.3578)
0.90	**1.2995 (0.3478)**	**3.7096 (1.8177)**	2.6758 (0.2675)	15.6601 (3.7729)	2.8778 (0.7345)	19.6751 (12.0349)	2.6341 (0.3334)	15.5898 (4.5613)	2.6646 (0.2793)	15.0443 (3.9360)
0.95	**1.3176 (0.2825)**	**3.9967 (1.5917)**	4.7485 (0.4848)	48.3211 (11.5576)	4.9480 (1.2652)	55.9972 (35.5540)	4.7217 (0.5032)	50.3695 (12.3446)	4.7844 (0.4940)	49.1668 (11.9083)
1st Count	13	13	2	5	2	0	0	0	2	1
2nd Count	0	0	4	7	0	4	3	1	12	7

Table 21: Wave, Exp(1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.6800 (0.0576)**	**0.9238 (0.1779)**	1.2996 (0.0901)	3.7795 (0.5478)	1.4813 (0.2097)	4.9781 (1.5893)	1.4675 (0.2138)	4.1380 (0.8215)	1.3492 (0.1058)	4.0172 (0.6185)
0.10	**0.6306 (0.0405)**	**0.7612 (0.1215)**	1.2184 (0.0920)	3.3249 (0.5358)	1.3439 (0.1838)	4.0581 (1.2901)	1.3417 (0.1295)	3.5477 (0.5872)	1.2381 (0.0964)	3.4510 (0.5861)
0.15	**0.6116 (0.0356)**	**0.6853 (0.0962)**	1.1442 (0.0932)	2.9506 (0.5172)	1.2203 (0.1668)	3.3281 (1.0799)	1.2525 (0.1288)	3.1180 (0.5628)	1.1562 (0.0965)	3.0473 (0.5556)
0.20	**0.6133 (0.0367)**	**0.6690 (0.0830)**	1.0627 (0.0970)	2.5620 (0.5153)	1.1066 (0.1567)	2.7390 (0.9260)	1.1631 (0.1375)	2.7346 (0.6274)	1.0695 (0.1003)	2.6387 (0.5425)
0.25	**0.6291 (0.0391)**	**0.6916 (0.0793)**	0.9778 (0.1003)	2.1868 (0.5133)	0.9992 (0.1518)	2.2541 (0.8070)	1.0648 (0.1408)	2.3382 (0.6356)	0.9841 (0.1019)	2.2754 (0.5237)
0.30	**0.6518 (0.0409)**	**0.7371 (0.0848)**	0.8866 (0.1063)	1.8235 (0.5166)	0.8952 (0.1506)	1.8456 (0.7095)	0.9427 (0.1264)	1.8410 (0.5156)	0.8930 (0.1054)	1.9169 (0.5073)
0.35	**0.6763 (0.0410)**	**0.7937 (0.0952)**	0.7918 (0.1070)	1.4847 (0.4766)	0.7942 (0.1477)	1.4938 (0.6248)	0.8268 (0.1279)	1.4511 (0.4812)	0.7946 (0.1139)	1.5631 (0.4922)
0.40	0.6993 (0.0401)	**0.8533 (0.1059)**	**0.6838 (0.1095)**	1.1389 (0.4319)	0.6965 (0.1406)	1.1863 (0.5461)	0.7084 (0.1293)	1.1039 (0.4576)	0.6876 (0.1182)	1.2290 (0.4409)
0.45	0.7190 (0.0398)	0.9109 (0.1163)	**0.5672 (0.1134)**	0.8212 (0.3856)	0.6023 (0.1292)	0.9177 (0.4675)	0.5886 (0.1256)	**0.7885 (0.3686)**	0.5700 (0.1219)	0.9046 (0.3792)
0.50	0.7354 (0.0415)	0.9647 (0.1316)	**0.4449 (0.1088)**	**0.5433 (0.3127)**	0.5119 (0.1178)	0.6897 (0.3830)	0.4879 (0.1218)	0.5824 (0.3115)	0.4532 (0.1108)	0.6237 (0.3132)
0.55	0.7488 (0.0473)	1.0153 (0.1614)	**0.3346 (0.0991)**	**0.3456 (0.2446)**	0.4322 (0.1135)	0.5142 (0.2950)	0.4052 (0.1244)	0.4391 (0.2879)	0.3497 (0.0907)	0.4114 (0.2453)
0.60	0.7600 (0.0591)	1.0660 (0.2157)	**0.2608 (0.0843)**	**0.2422 (0.1977)**	0.3795 (0.1242)	0.4177 (0.2447)	0.3526 (0.1262)	0.3714 (0.2792)	0.2847 (0.0707)	0.2878 (0.1723)
0.65	0.7698 (0.0779)	1.1231 (0.3006)	**0.2691 (0.1041)**	**0.2714 (0.2757)**	0.3936 (0.1470)	0.4481 (0.3368)	0.3474 (0.1366)	0.3688 (0.3534)	0.2968 (0.0984)	0.3076 (0.2687)
0.70	0.7798 (0.1040)	1.1959 (0.4215)	**0.3911 (0.1397)**	**0.4899 (0.4688)**	0.4997 (0.1838)	0.6879 (0.5925)	0.4537 (0.1704)	0.5926 (0.5451)	0.4184 (0.1407)	0.5058 (0.4179)
0.75	0.7929 (0.1365)	1.2985 (0.5870)	**0.5855 (0.1753)**	0.9543 (0.7171)	0.6970 (0.2274)	1.2724 (1.0015)	0.6119 (0.1960)	0.9608 (0.7605)	0.6053 (0.1787)	**0.9488 (0.7241)**
0.80	**0.8131 (0.1738)**	**1.4496 (0.8115)**	0.8632 (0.2245)	1.9039 (1.1675)	0.9872 (0.2774)	2.4165 (1.6366)	0.8830 (0.2484)	1.9182 (1.2125)	0.8849 (0.2344)	1.9050 (1.2114)
0.85	**0.8439 (0.2100)**	**1.6563 (1.0899)**	1.2650 (0.2760)	3.8655 (1.9942)	1.3710 (0.3498)	4.4421 (2.7080)	1.2823 (0.3012)	3.9137 (2.1236)	1.2712 (0.2971)	3.8254 (2.0751)
0.90	**0.8786 (0.2287)**	**1.8184 (1.2706)**	1.8281 (0.3411)	7.7400 (3.6151)	1.8524 (0.4573)	7.7750 (4.6540)	1.8199 (0.3628)	7.6149 (3.7852)	1.7998 (0.3471)	7.5029 (3.7467)
0.95	**0.8272 (0.2033)**	**1.5661 (0.9703)**	2.6485 (0.4979)	15.5577 (7.3914)	2.4502 (0.6048)	12.7398 (8.1195)	2.6506 (0.5358)	16.1733 (8.7712)	2.6856 (0.5613)	16.4703 (8.4315)
1st Count	11	12	8	5	0	0	0	1	0	1
2nd Count	0	0	9	9	1	1	0	3	9	6

Table 22: Wave, Exp(1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0926 (0.0295)**	**0.0169 (0.0131)**	1.3250 (0.0877)	3.9162 (0.5285)	1.3714 (0.1537)	4.2407 (1.0845)	1.3410 (0.1039)	3.7498 (0.5048)	1.3161 (0.0852)	3.8349 (0.4881)
0.10	**0.0960 (0.0282)**	**0.0184 (0.0117)**	1.2462 (0.0824)	3.4579 (0.4688)	1.2665 (0.1410)	3.5885 (0.9210)	1.2527 (0.0798)	3.5651 (0.5045)	1.2401 (0.0829)	3.4011 (0.4495)
0.15	**0.1059 (0.0260)**	**0.0226 (0.0113)**	1.1666 (0.0780)	3.0285 (0.4177)	1.1667 (0.1302)	3.0260 (0.7861)	1.1994 (0.0796)	3.3842 (0.5035)	1.1612 (0.0808)	2.9820 (0.4127)
0.20	**0.1196 (0.0252)**	**0.0295 (0.0119)**	1.0818 (0.0740)	2.6040 (0.3676)	1.0701 (0.1211)	2.5346 (0.6738)	1.0906 (0.0871)	2.7511 (0.5355)	1.0805 (0.0782)	2.5908 (0.3806)
0.25	**0.1405 (0.0291)**	**0.0416 (0.0147)**	0.9901 (0.0704)	2.1838 (0.3191)	0.9748 (0.1135)	2.0984 (0.5786)	0.9886 (0.0713)	2.2047 (0.3500)	0.9906 (0.0758)	2.1819 (0.3418)
0.30	**0.1835 (0.0425)**	**0.0675 (0.0269)**	0.8921 (0.0674)	1.7755 (0.2810)	0.8785 (0.1072)	1.7051 (0.4966)	0.8888 (0.0729)	1.7550 (0.3134)	0.8932 (0.0755)	1.7851 (0.3090)
0.35	**0.2791 (0.0553)**	**0.1356 (0.0556)**	0.7896 (0.0661)	1.3994 (0.2517)	0.7791 (0.1022)	1.3457 (0.4240)	0.7970 (0.0692)	1.4194 (0.2809)	0.7892 (0.0703)	1.4077 (0.2659)
0.40	**0.4335 (0.0502)**	**0.3113 (0.0796)**	0.6806 (0.0644)	1.0516 (0.2156)	0.6743 (0.0985)	1.0159 (0.3574)	0.6876 (0.0747)	1.0492 (0.2637)	0.6826 (0.0699)	1.0702 (0.2388)
0.45	0.6049 (0.0333)	**0.6190 (0.0676)**	**0.5613 (0.0635)**	0.7282 (0.1808)	0.5620 (0.0963)	0.7169 (0.2941)	0.5734 (0.0743)	0.7384 (0.2293)	0.5648 (0.0666)	0.7522 (0.1997)
0.50	0.7155 (0.0354)	0.8695 (0.0851)	**0.4300 (0.0621)**	**0.4424 (0.1402)**	0.4402 (0.0959)	0.4552 (0.2314)	0.4437 (0.0778)	0.4531 (0.1902)	0.4370 (0.0662)	0.4729 (0.1610)
0.55	0.6960 (0.0424)	0.8265 (0.1062)	**0.2875 (0.0597)**	**0.2149 (0.0975)**	0.3096 (0.0918)	0.2430 (0.1656)	0.3080 (0.0825)	0.2337 (0.1428)	0.2965 (0.0632)	0.2442 (0.1185)
0.60	0.5594 (0.0609)	0.5490 (0.1257)	**0.1561 (0.0495)**	**0.0754 (0.0560)**	0.1859 (0.0761)	0.1015 (0.0950)	0.1799 (0.0707)	0.0924 (0.0898)	0.1647 (0.0498)	0.0968 (0.0700)
0.65	0.4104 (0.0694)	0.3021 (0.1179)	**0.1397 (0.0517)**	**0.0555 (0.0483)**	0.1600 (0.0634)	0.0696 (0.0704)	0.1596 (0.0599)	0.0725 (0.0654)	0.1464 (0.0439)	0.0674 (0.0455)
0.70	0.3075 (0.0638)	**0.1720 (0.0874)**	0.3082 (0.0778)	0.2328 (0.1387)	**0.2915 (0.1063)**	0.2214 (0.2031)	0.3028 (0.1124)	0.2495 (0.2035)	0.2973 (0.0712)	0.2229 (0.1220)
0.75	**0.2748 (0.0645)**	**0.1516 (0.0951)**	0.5502 (0.1002)	0.6932 (0.2940)	0.5360 (0.1478)	0.7009 (0.4631)	0.5391 (0.1395)	0.7052 (0.4091)	0.5339 (0.0941)	0.6582 (0.2788)
0.80	**0.3175 (0.0937)**	**0.2627 (0.2347)**	0.8573 (0.1139)	1.6452 (0.5142)	0.8698 (0.1925)	1.7939 (0.9376)	0.8498 (0.1641)	1.6750 (0.7482)	0.8442 (0.1120)	1.5807 (0.5260)
0.85	**0.4291 (0.1177)**	**0.5465 (0.4282)**	1.2446 (0.1278)	3.4387 (0.8438)	1.3188 (0.2537)	4.0696 (1.8260)	1.2433 (0.1809)	3.5034 (1.1998)	1.2304 (0.1289)	3.2936 (0.8742)
0.90	**0.5419 (0.1553)**	**0.8442 (0.5851)**	1.8032 (0.1388)	7.1509 (1.2607)	1.9276 (0.3405)	8.5760 (3.5166)	1.7988 (0.2040)	7.2276 (1.9037)	1.7949 (0.1483)	6.9634 (1.4114)
0.95	**0.5490 (0.1992)**	**0.6979 (0.5411)**	2.7370 (0.1976)	16.1995 (2.5300)	2.7510 (0.4654)	16.8662 (6.7685)	2.7406 (0.2683)	16.6126 (3.5690)	2.7327 (0.1954)	15.9744 (2.6338)
1st Count	13	15	5	4	1	0	0	0	0	0
2nd Count	0	0	0	0	6	7	0	4	13	8

Table 23: Angle, N(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.2021 (0.0212)**	**0.0500 (0.0104)**	0.6288 (0.0641)	0.6041 (0.0857)	0.6865 (0.0820)	0.6432 (0.1201)	0.5960 (0.0503)	0.5122 (0.0821)	0.6374 (0.0525)	0.6269 (0.0670)
0.10	**0.1472 (0.0092)**	**0.0397 (0.0062)**	0.4940 (0.0500)	0.3771 (0.0529)	0.5056 (0.0516)	0.3620 (0.0670)	0.5152 (0.0377)	0.3725 (0.0469)	0.5005 (0.0418)	0.3867 (0.0429)
0.15	**0.2288 (0.0195)**	**0.1072 (0.0164)**	0.4043 (0.0427)	0.2512 (0.0376)	0.4175 (0.0425)	0.2435 (0.0475)	0.4264 (0.0329)	0.2528 (0.0328)	0.4079 (0.0347)	0.2546 (0.0294)
0.20	0.3387 (0.0258)	0.2034 (0.0252)	**0.3331 (0.0380)**	0.1685 (0.0276)	0.3547 (0.0396)	0.1720 (0.0374)	0.3523 (0.0315)	0.1710 (0.0258)	0.3332 (0.0302)	**0.1682 (0.0215)**
0.25	0.4404 (0.0285)	0.3158 (0.0332)	0.2733 (0.0359)	0.1111 (0.0213)	0.3002 (0.0382)	0.1206 (0.0298)	0.2860 (0.0311)	0.1121 (0.0201)	**0.2712 (0.0279)**	**0.1094 (0.0156)**
0.30	0.5330 (0.0302)	0.4398 (0.0408)	0.2203 (0.0336)	0.0703 (0.0162)	0.2486 (0.0374)	0.0814 (0.0235)	0.2301 (0.0305)	0.0714 (0.0154)	**0.2180 (0.0301)**	**0.0688 (0.0138)**
0.35	0.6190 (0.0317)	0.5740 (0.0481)	0.1717 (0.0310)	0.0412 (0.0117)	0.1992 (0.0374)	0.0525 (0.0187)	0.1770 (0.0288)	0.0420 (0.0111)	**0.1677 (0.0285)**	**0.0396 (0.0104)**
0.40	0.7006 (0.0330)	0.7182 (0.0553)	0.1259 (0.0278)	0.0217 (0.0080)	0.1532 (0.0376)	0.0327 (0.0147)	0.1292 (0.0262)	0.0226 (0.0074)	**0.1214 (0.0258)**	**0.0201 (0.0073)**
0.45	0.7794 (0.0342)	0.8731 (0.0626)	0.0865 (0.0250)	0.0108 (0.0056)	0.1139 (0.0372)	0.0209 (0.0118)	0.0879 (0.0222)	0.0115 (0.0053)	**0.0786 (0.0231)**	**0.0089 (0.0052)**
0.50	0.8397 (0.0339)	0.9924 (0.0634)	0.0659 (0.0220)	0.0077 (0.0048)	0.0975 (0.0365)	0.0164 (0.0112)	0.0682 (0.0201)	0.0082 (0.0047)	**0.0555 (0.0228)**	**0.0058 (0.0053)**
0.55	0.4474 (0.0401)	0.2498 (0.0477)	0.0907 (0.0235)	0.0117 (0.0057)	0.1104 (0.0417)	0.0188 (0.0131)	0.0928 (0.0250)	0.0123 (0.0061)	**0.0810 (0.0253)**	**0.0095 (0.0060)**
0.60	0.1655 (0.0392)	0.0372 (0.0172)	0.1285 (0.0272)	0.0231 (0.0085)	0.1382 (0.0471)	0.0280 (0.0171)	0.1340 (0.0283)	0.0241 (0.0093)	**0.1215 (0.0268)**	**0.0207 (0.0077)**
0.65	**0.0885 (0.0217)**	**0.0120 (0.0056)**	0.1714 (0.0310)	0.0430 (0.0125)	0.1726 (0.0512)	0.0442 (0.0226)	0.1793 (0.0297)	0.0441 (0.0129)	0.1665 (0.0291)	0.0402 (0.0114)
0.70	**0.0900 (0.0163)**	**0.0128 (0.0047)**	0.2190 (0.0344)	0.0726 (0.0172)	0.2112 (0.0552)	0.0679 (0.0298)	0.2294 (0.0329)	0.0738 (0.0179)	0.2165 (0.0328)	0.0699 (0.0161)
0.75	**0.1201 (0.0265)**	**0.0226 (0.0092)**	0.2725 (0.0369)	0.1151 (0.0220)	0.2547 (0.0604)	0.1011 (0.0395)	0.2847 (0.0356)	0.1154 (0.0231)	0.2691 (0.0333)	0.1106 (0.0201)
0.80	**0.1544 (0.0320)**	**0.0359 (0.0133)**	0.3332 (0.0394)	0.1743 (0.0279)	0.3057 (0.0684)	0.1476 (0.0530)	0.3463 (0.0401)	0.1729 (0.0301)	0.3328 (0.0378)	0.1703 (0.0282)
0.85	**0.1826 (0.0358)**	**0.0492 (0.0171)**	0.4037 (0.0441)	0.2578 (0.0362)	0.3716 (0.0823)	0.2176 (0.0736)	0.4169 (0.0444)	0.2531 (0.0389)	0.4045 (0.0408)	0.2539 (0.0370)
0.90	**0.1954 (0.0411)**	**0.0573 (0.0209)**	0.4929 (0.0505)	0.3837 (0.0483)	0.4731 (0.1056)	0.3399 (0.1089)	0.5032 (0.0497)	0.3731 (0.0512)	0.4956 (0.0453)	0.3832 (0.0489)
0.95	**0.1742 (0.0424)**	**0.0498 (0.0211)**	0.6230 (0.0630)	0.6055 (0.0737)	0.6808 (0.1421)	0.6316 (0.1842)	0.6288 (0.0584)	0.5844 (0.0771)	0.6258 (0.0551)	0.6104 (0.0759)
1st Count	10	10	1	0	0	0	0	0	8	9
2nd Count	0	0	11	9	5	7	1	2	2	1

Table 24: Angle, N(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0770 (0.0405)**	**0.0109 (0.0129)**	0.6014 (0.0695)	0.5985 (0.0877)	0.6236 (0.0521)	0.5850 (0.0771)	0.6109 (0.0355)	0.5784 (0.0558)	0.6099 (0.0439)	0.6084 (0.0646)
0.10	**0.0515 (0.0159)**	**0.0046 (0.0031)**	0.4748 (0.0595)	0.3708 (0.0610)	0.4906 (0.0417)	0.3642 (0.0535)	0.4832 (0.0312)	0.3656 (0.0380)	0.4798 (0.0405)	0.3734 (0.0471)
0.15	**0.0459 (0.0124)**	**0.0037 (0.0022)**	0.3885 (0.0535)	0.2458 (0.0456)	0.4131 (0.0416)	0.2532 (0.0440)	0.3902 (0.0277)	0.2409 (0.0269)	0.3905 (0.0383)	0.2451 (0.0362)
0.20	**0.0424 (0.0109)**	**0.0032 (0.0017)**	0.3195 (0.0491)	0.1641 (0.0350)	0.3467 (0.0429)	0.1743 (0.0369)	0.3160 (0.0250)	0.1591 (0.0198)	0.3230 (0.0345)	0.1648 (0.0265)
0.25	**0.0394 (0.0086)**	**0.0027 (0.0013)**	0.2597 (0.0457)	0.1068 (0.0267)	0.2840 (0.0434)	0.1150 (0.0301)	0.2533 (0.0234)	0.1024 (0.0147)	0.2594 (0.0324)	0.1058 (0.0200)
0.30	**0.0382 (0.0074)**	**0.0026 (0.0011)**	0.2054 (0.0425)	0.0659 (0.0202)	0.2267 (0.0431)	0.0723 (0.0238)	0.1978 (0.0220)	0.0621 (0.0108)	0.2030 (0.0302)	0.0641 (0.0148)
0.35	**0.0524 (0.0139)**	**0.0056 (0.0028)**	0.1559 (0.0386)	0.0372 (0.0145)	0.1749 (0.0429)	0.0426 (0.0183)	0.1474 (0.0206)	0.0341 (0.0077)	0.1526 (0.0300)	0.0356 (0.0110)
0.40	**0.0866 (0.0156)**	**0.0148 (0.0046)**	0.1103 (0.0348)	0.0181 (0.0096)	0.1278 (0.0424)	0.0230 (0.0136)	0.1009 (0.0190)	0.0157 (0.0049)	0.1052 (0.0274)	0.0164 (0.0074)
0.45	0.1240 (0.0113)	0.0297 (0.0049)	0.0691 (0.0302)	0.0071 (0.0056)	0.0880 (0.0408)	0.0119 (0.0097)	**0.0604 (0.0168)**	**0.0056 (0.0026)**	0.0644 (0.0249)	0.0060 (0.0043)
0.50	0.1404 (0.0085)	0.0397 (0.0046)	0.0468 (0.0253)	0.0035 (0.0034)	0.0705 (0.0372)	0.0083 (0.0079)	**0.0392 (0.0153)**	**0.0027 (0.0020)**	0.0420 (0.0211)	0.0028 (0.0023)
0.55	0.0969 (0.0135)	0.0186 (0.0060)	0.0690 (0.0278)	0.0069 (0.0051)	0.0880 (0.0378)	0.0117 (0.0096)	0.0682 (0.0237)	0.0069 (0.0043)	**0.0675 (0.0285)**	**0.0067 (0.0051)**
0.60	**0.0625 (0.0110)**	**0.0064 (0.0028)**	0.1086 (0.0318)	0.0177 (0.0085)	0.1232 (0.0420)	0.0222 (0.0140)	0.1118 (0.0267)	0.0184 (0.0075)	0.1113 (0.0368)	0.0185 (0.0102)
0.65	**0.0489 (0.0096)**	**0.0039 (0.0016)**	0.1537 (0.0342)	0.0367 (0.0126)	0.1645 (0.0454)	0.0403 (0.0197)	0.1582 (0.0293)	0.0380 (0.0112)	0.1552 (0.0374)	0.0375 (0.0141)
0.70	**0.0376 (0.0074)**	**0.0025 (0.0010)**	0.2025 (0.0360)	0.0652 (0.0172)	0.2090 (0.0481)	0.0669 (0.0267)	0.2072 (0.0325)	0.0670 (0.0159)	0.2079 (0.0429)	0.0680 (0.0214)
0.75	**0.0346 (0.0080)**	**0.0023 (0.0010)**	0.2565 (0.0390)	0.1060 (0.0229)	0.2569 (0.0505)	0.1032 (0.0350)	0.2610 (0.0365)	0.1080 (0.0218)	0.2619 (0.0470)	0.1092 (0.0290)
0.80	**0.0393 (0.0092)**	**0.0028 (0.0012)**	0.3171 (0.0426)	0.1633 (0.0301)	0.3090 (0.0530)	0.1520 (0.0451)	0.3207 (0.0406)	0.1652 (0.0292)	0.3206 (0.0493)	0.1657 (0.0375)
0.85	**0.0418 (0.0100)**	**0.0030 (0.0013)**	0.3876 (0.0463)	0.2455 (0.0391)	0.3692 (0.0573)	0.2199 (0.0588)	0.3899 (0.0451)	0.2467 (0.0390)	0.3899 (0.0504)	0.2470 (0.0463)
0.90	**0.0448 (0.0158)**	**0.0036 (0.0029)**	0.4760 (0.0518)	0.3719 (0.0521)	0.4503 (0.0679)	0.3274 (0.0808)	0.4763 (0.0509)	0.3717 (0.0532)	0.4753 (0.0537)	0.3700 (0.0607)
0.95	**0.0922 (0.0419)**	**0.0142 (0.0121)**	0.6063 (0.0611)	0.6036 (0.0746)	0.6064 (0.0984)	0.5642 (0.1298)	0.6044 (0.0605)	0.6028 (0.0802)	0.5993 (0.0605)	0.5942 (0.0878)
1st Count	16	16	0	0	0	0	2	2	1	1
2nd Count	0	0	7	4	3	6	6	8	3	2

Table 25: Angle, LN(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.1262 (0.0105)**	**0.0251 (0.0042)**	0.5353 (0.0452)	0.4676 (0.0524)	0.5560 (0.0525)	0.4555 (0.0782)	0.5346 (0.0271)	0.4303 (0.0429)	0.5359 (0.0404)	0.4591 (0.0571)
0.10	**0.1281 (0.0131)**	**0.0322 (0.0069)**	0.5160 (0.0464)	0.4281 (0.0496)	0.5305 (0.0501)	0.4140 (0.0716)	0.5269 (0.0315)	0.4206 (0.0414)	0.5153 (0.0355)	0.4213 (0.0466)
0.15	**0.1401 (0.0161)**	**0.0443 (0.0098)**	0.4924 (0.0458)	0.3870 (0.0460)	0.5092 (0.0495)	0.3800 (0.0677)	0.5027 (0.0324)	0.3814 (0.0388)	0.4932 (0.0360)	0.3835 (0.0436)
0.20	**0.1621 (0.0187)**	**0.0616 (0.0129)**	0.4674 (0.0451)	0.3469 (0.0427)	0.4878 (0.0499)	0.3468 (0.0647)	0.4736 (0.0332)	0.3388 (0.0367)	0.4685 (0.0344)	0.3446 (0.0388)
0.25	**0.1946 (0.0215)**	**0.0848 (0.0164)**	0.4416 (0.0441)	0.3079 (0.0391)	0.4645 (0.0509)	0.3127 (0.0621)	0.4440 (0.0352)	0.2978 (0.0353)	0.4396 (0.0339)	0.3024 (0.0353)
0.30	**0.2372 (0.0247)**	**0.1150 (0.0209)**	0.4118 (0.0438)	0.2672 (0.0361)	0.4389 (0.0525)	0.2772 (0.0598)	0.4126 (0.0369)	0.2575 (0.0338)	0.4115 (0.0344)	0.2633 (0.0329)
0.35	**0.2870 (0.0280)**	**0.1536 (0.0268)**	0.3789 (0.0437)	0.2256 (0.0329)	0.4107 (0.0548)	0.2408 (0.0576)	0.3787 (0.0383)	0.2173 (0.0314)	0.3794 (0.0347)	0.2226 (0.0300)
0.40	**0.3412 (0.0331)**	0.2020 (0.0352)	0.3441 (0.0431)	0.1847 (0.0294)	0.3797 (0.0579)	0.2039 (0.0556)	0.3415 (0.0386)	**0.1773 (0.0283)**	0.3432 (0.0331)	0.1811 (0.0259)
0.45	0.3975 (0.0404)	0.2592 (0.0469)	0.3065 (0.0420)	0.1449 (0.0257)	0.3459 (0.0617)	0.1672 (0.0536)	**0.2997 (0.0385)**	**0.1374 (0.0241)**	0.3045 (0.0356)	0.1412 (0.0245)
0.50	0.4448 (0.0465)	0.3085 (0.0562)	0.2646 (0.0412)	0.1062 (0.0221)	0.3094 (0.0664)	0.1319 (0.0516)	**0.2572 (0.0393)**	**0.1009 (0.0208)**	0.2623 (0.0361)	0.1034 (0.0203)
0.55	0.4497 (0.0412)	0.2975 (0.0481)	0.2204 (0.0411)	0.0717 (0.0189)	0.2707 (0.0719)	0.0995 (0.0497)	**0.2106 (0.0388)**	**0.0678 (0.0174)**	0.2157 (0.0353)	0.0679 (0.0162)
0.60	0.4039 (0.0536)	0.2315 (0.0576)	0.1709 (0.0390)	0.0422 (0.0159)	0.2304 (0.0796)	0.0722 (0.0485)	**0.1606 (0.0363)**	0.0406 (0.0143)	0.1633 (0.0375)	**0.0381 (0.0139)**
0.65	0.3296 (0.0781)	0.1554 (0.0661)	0.1240 (0.0382)	0.0238 (0.0153)	0.1907 (0.0886)	0.0534 (0.0493)	0.1243 (0.0323)	0.0261 (0.0146)	**0.1117 (0.0391)**	**0.0187 (0.0127)**
0.70	0.2354 (0.0926)	0.0863 (0.0610)	0.1161 (0.0461)	0.0247 (0.0200)	0.1663 (0.0929)	0.0472 (0.0532)	0.1327 (0.0441)	0.0312 (0.0228)	**0.0977 (0.0425)**	**0.0173 (0.0164)**
0.75	**0.1719 (0.0758)**	0.0512 (0.0467)	0.2096 (0.0604)	0.0598 (0.0316)	0.1921 (0.0896)	0.0610 (0.0613)	0.2293 (0.0580)	0.0720 (0.0376)	0.1892 (0.0574)	**0.0494 (0.0270)**
0.80	**0.2542 (0.0927)**	0.1138 (0.0834)	0.3424 (0.0723)	0.1562 (0.0506)	0.2751 (0.0952)	**0.1130 (0.0782)**	0.3759 (0.0767)	0.1793 (0.0695)	0.3247 (0.0744)	0.1459 (0.0527)
0.85	0.4856 (0.1481)	0.3922 (0.2493)	0.5453 (0.0915)	0.4038 (0.0964)	**0.4308 (0.1198)**	**0.2632 (0.1240)**	0.5857 (0.0994)	0.4356 (0.1311)	0.5253 (0.0988)	0.3888 (0.1057)
0.90	0.7833 (0.2237)	1.0104 (0.6492)	0.8610 (0.1350)	1.0268 (0.2162)	**0.7631 (0.1995)**	**0.7889 (0.3020)**	0.9117 (0.1358)	1.0692 (0.2586)	0.8363 (0.1512)	1.0059 (0.2424)
0.95	**0.9287 (0.3383)**	**1.5587 (1.2907)**	1.4856 (0.2451)	3.0796 (0.6206)	1.7965 (0.3736)	3.7865 (1.1513)	1.5297 (0.2082)	3.0865 (0.6324)	1.4396 (0.2479)	2.9403 (0.6780)
1st Count	11	8	0	0	2	3	4	4	2	4
2nd Count	2	2	4	2	1	2	3	6	9	7

Table 26: Angle, LN(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0375 (0.0116)**	**0.0024 (0.0014)**	0.5217 (0.0418)	0.4580 (0.0490)	0.5328 (0.0405)	0.4455 (0.0582)	0.5315 (0.0260)	0.4603 (0.0357)	0.5317 (0.0337)	0.4683 (0.0443)
0.10	**0.0324 (0.0088)**	**0.0018 (0.0009)**	0.4984 (0.0414)	0.4149 (0.0455)	0.5079 (0.0402)	0.4052 (0.0544)	0.5032 (0.0269)	0.4121 (0.0335)	0.5094 (0.0319)	0.4269 (0.0389)
0.15	**0.0295 (0.0074)**	**0.0016 (0.0008)**	0.4735 (0.0406)	0.3734 (0.0421)	0.4856 (0.0407)	0.3697 (0.0517)	0.4768 (0.0263)	0.3712 (0.0309)	0.4832 (0.0323)	0.3835 (0.0367)
0.20	**0.0296 (0.0074)**	**0.0016 (0.0008)**	0.4471 (0.0400)	0.3325 (0.0391)	0.4618 (0.0413)	0.3335 (0.0490)	0.4498 (0.0255)	0.3325 (0.0283)	0.4560 (0.0322)	0.3413 (0.0347)
0.25	**0.0329 (0.0087)**	**0.0019 (0.0011)**	0.4196 (0.0394)	0.2926 (0.0361)	0.4358 (0.0422)	0.2962 (0.0466)	0.4217 (0.0251)	0.2933 (0.0255)	0.4281 (0.0328)	0.3005 (0.0327)
0.30	**0.0421 (0.0101)**	**0.0032 (0.0017)**	0.3899 (0.0384)	0.2523 (0.0329)	0.4071 (0.0436)	0.2581 (0.0443)	0.3923 (0.0249)	0.2540 (0.0230)	0.3988 (0.0347)	0.2602 (0.0323)
0.35	**0.0595 (0.0103)**	**0.0063 (0.0023)**	0.3580 (0.0371)	0.2124 (0.0294)	0.3755 (0.0454)	0.2194 (0.0421)	0.3613 (0.0251)	0.2148 (0.0207)	0.3690 (0.0359)	0.2212 (0.0303)
0.40	**0.0821 (0.0088)**	**0.0118 (0.0027)**	0.3231 (0.0360)	0.1725 (0.0259)	0.3406 (0.0473)	0.1805 (0.0396)	0.3272 (0.0257)	0.1753 (0.0185)	0.3314 (0.0375)	0.1786 (0.0284)
0.45	**0.1052 (0.0070)**	**0.0195 (0.0031)**	0.2845 (0.0348)	0.1333 (0.0222)	0.3019 (0.0494)	0.1421 (0.0368)	0.2898 (0.0266)	0.1365 (0.0163)	0.2926 (0.0371)	0.1388 (0.0247)
0.50	**0.1224 (0.0063)**	**0.0267 (0.0034)**	0.2416 (0.0325)	0.0955 (0.0180)	0.2591 (0.0515)	0.1048 (0.0333)	0.2478 (0.0279)	0.0988 (0.0143)	0.2497 (0.0384)	0.1006 (0.0218)
0.55	**0.1251 (0.0087)**	**0.0279 (0.0047)**	0.1937 (0.0303)	0.0605 (0.0137)	0.2120 (0.0535)	0.0702 (0.0289)	0.2015 (0.0288)	0.0640 (0.0120)	0.2009 (0.0379)	0.0647 (0.0178)
0.60	**0.1119 (0.0146)**	**0.0218 (0.0073)**	0.1389 (0.0268)	0.0304 (0.0090)	0.1612 (0.0547)	0.0406 (0.0236)	0.1487 (0.0290)	0.0337 (0.0093)	0.1459 (0.0368)	0.0336 (0.0131)
0.65	0.0927 (0.0184)	0.0155 (0.0082)	**0.0806 (0.0212)**	**0.0096 (0.0043)**	0.1124 (0.0527)	0.0199 (0.0173)	0.0913 (0.0286)	0.0123 (0.0063)	0.0863 (0.0306)	0.0112 (0.0070)
0.70	0.0733 (0.0208)	0.0111 (0.0081)	**0.0569 (0.0294)**	**0.0053 (0.0052)**	0.0991 (0.0464)	0.0152 (0.0135)	0.0636 (0.0243)	0.0066 (0.0054)	0.0581 (0.0334)	0.0054 (0.0055)
0.75	**0.0679 (0.0211)**	**0.0103 (0.0076)**	0.1470 (0.0390)	0.0312 (0.0135)	0.1640 (0.0514)	0.0388 (0.0207)	0.1451 (0.0374)	0.0302 (0.0128)	0.1395 (0.0454)	0.0287 (0.0156)
0.80	**0.0895 (0.0276)**	**0.0145 (0.0087)**	0.2785 (0.0457)	0.1181 (0.0264)	0.2835 (0.0586)	0.1147 (0.0408)	0.2679 (0.0531)	0.1114 (0.0301)	0.2710 (0.0515)	0.1131 (0.0334)
0.85	**0.1179 (0.0382)**	**0.0226 (0.0176)**	0.4643 (0.0529)	0.3390 (0.0477)	0.4528 (0.0692)	0.2999 (0.0820)	0.4476 (0.0699)	0.3207 (0.0639)	0.4587 (0.0635)	0.3314 (0.0694)
0.90	**0.2397 (0.1506)**	**0.1245 (0.1941)**	0.7638 (0.0655)	0.9305 (0.0942)	0.7253 (0.0999)	0.7857 (0.1904)	0.7414 (0.0871)	0.8821 (0.1314)	0.7611 (0.0842)	0.9157 (0.1476)
0.95	**0.7483 (0.3747)**	**0.9535 (0.8508)**	1.3623 (0.1039)	2.9840 (0.2555)	1.4385 (0.2594)	2.9352 (0.7268)	1.3378 (0.1261)	2.8520 (0.3476)	1.3608 (0.1103)	2.9627 (0.3512)
1st Count	17	17	2	2	0	0	0	0	0	0
2nd Count	0	0	12	9	1	5	3	3	3	3

Table 27: Angle, Exp(1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.1869 (0.0121)**	**0.0712 (0.0086)**	0.3520 (0.0402)	0.2010 (0.0340)	0.3695 (0.0382)	0.1928 (0.0365)	0.3577 (0.0254)	0.1857 (0.0255)	0.3591 (0.0372)	0.2010 (0.0313)
0.10	**0.2080 (0.0136)**	**0.0879 (0.0102)**	0.3417 (0.0427)	0.1853 (0.0329)	0.3572 (0.0381)	0.1791 (0.0348)	0.3557 (0.0325)	0.1840 (0.0292)	0.3472 (0.0263)	0.1864 (0.0211)
0.15	**0.2336 (0.0149)**	**0.1085 (0.0119)**	0.3228 (0.0426)	0.1642 (0.0306)	0.3426 (0.0384)	0.1637 (0.0331)	0.3372 (0.0343)	0.1643 (0.0281)	0.3275 (0.0255)	0.1657 (0.0191)
0.20	**0.2635 (0.0162)**	**0.1335 (0.0137)**	0.3014 (0.0413)	0.1424 (0.0277)	0.3253 (0.0387)	0.1464 (0.0313)	0.3125 (0.0345)	0.1414 (0.0259)	0.3076 (0.0250)	0.1454 (0.0171)
0.25	0.2976 (0.0173)	0.1640 (0.0158)	**0.2786 (0.0404)**	0.1211 (0.0253)	0.3053 (0.0392)	0.1279 (0.0294)	0.2876 (0.0343)	**0.1199 (0.0237)**	0.2833 (0.0244)	0.1230 (0.0155)
0.30	0.3358 (0.0182)	0.2008 (0.0180)	**0.2534 (0.0395)**	0.0996 (0.0227)	0.2826 (0.0399)	0.1086 (0.0273)	0.2615 (0.0334)	**0.0992 (0.0209)**	0.2576 (0.0256)	0.1010 (0.0144)
0.35	0.3780 (0.0191)	0.2453 (0.0205)	**0.2260 (0.0384)**	**0.0786 (0.0199)**	0.2574 (0.0408)	0.0892 (0.0252)	0.2336 (0.0328)	0.0791 (0.0182)	0.2319 (0.0267)	0.0809 (0.0133)
0.40	0.4244 (0.0200)	0.2990 (0.0234)	**0.1977 (0.0363)**	**0.0592 (0.0165)**	0.2300 (0.0421)	0.0706 (0.0229)	0.2033 (0.0326)	0.0599 (0.0157)	0.2037 (0.0285)	0.0615 (0.0128)
0.45	0.4757 (0.0209)	0.3642 (0.0265)	**0.1678 (0.0348)**	**0.0418 (0.0136)**	0.2007 (0.0437)	0.0534 (0.0205)	0.1720 (0.0324)	0.0426 (0.0132)	0.1733 (0.0296)	0.0436 (0.0111)
0.50	0.5263 (0.0212)	0.4310 (0.0286)	**0.1355 (0.0328)**	**0.0267 (0.0106)**	0.1704 (0.0450)	0.0385 (0.0181)	0.1371 (0.0309)	0.0274 (0.0103)	0.1402 (0.0302)	0.0279 (0.0094)
0.55	0.3566 (0.0328)	0.1719 (0.0346)	**0.1028 (0.0296)**	**0.0153 (0.0080)**	0.1402 (0.0456)	0.0269 (0.0157)	0.1032 (0.0312)	0.0161 (0.0090)	0.1044 (0.0296)	**0.0153 (0.0075)**
0.60	0.1637 (0.0393)	0.0391 (0.0187)	0.0755 (0.0259)	0.0092 (0.0067)	0.1133 (0.0453)	0.0195 (0.0138)	0.0763 (0.0273)	0.0098 (0.0075)	**0.0728 (0.0263)**	**0.0081 (0.0061)**
0.65	0.0993 (0.0280)	0.0159 (0.0090)	0.0776 (0.0288)	0.0103 (0.0075)	0.1034 (0.0426)	0.0175 (0.0130)	0.0806 (0.0281)	0.0113 (0.0090)	**0.0667 (0.0273)**	**0.0076 (0.0065)**
0.70	**0.0933 (0.0201)**	**0.0141 (0.0066)**	0.1264 (0.0328)	0.0220 (0.0105)	0.1227 (0.0422)	0.0229 (0.0150)	0.1285 (0.0325)	0.0230 (0.0119)	0.1133 (0.0310)	0.0178 (0.0087)
0.75	**0.1209 (0.0347)**	**0.0246 (0.0141)**	0.1906 (0.0366)	0.0498 (0.0161)	0.1631 (0.0499)	0.0391 (0.0215)	0.1962 (0.0360)	0.0506 (0.0172)	0.1736 (0.0355)	0.0430 (0.0139)
0.80	**0.1727 (0.0499)**	**0.0491 (0.0271)**	0.2731 (0.0424)	0.1057 (0.0245)	0.2233 (0.0631)	0.0736 (0.0353)	0.2823 (0.0412)	0.1053 (0.0271)	0.2588 (0.0424)	0.0980 (0.0239)
0.85	**0.2400 (0.0645)**	**0.0918 (0.0499)**	0.3817 (0.0501)	0.2106 (0.0391)	0.3155 (0.0863)	0.1479 (0.0638)	0.3938 (0.0487)	0.2070 (0.0443)	0.3700 (0.0530)	0.2035 (0.0417)
0.90	**0.3028 (0.0920)**	**0.1495 (0.0956)**	0.5304 (0.0630)	0.4133 (0.0644)	0.4852 (0.1277)	0.3364 (0.1286)	0.5507 (0.0615)	0.4108 (0.0774)	0.5278 (0.0719)	0.4156 (0.0780)
0.95	**0.3086 (0.1304)**	**0.1740 (0.1618)**	0.8126 (0.1065)	0.9330 (0.1596)	0.9183 (0.1861)	1.0326 (0.3168)	0.8225 (0.0846)	0.9266 (0.1590)	0.7952 (0.1088)	0.9265 (0.1761)
1st Count	10	10	7	5	0	0	0	2	2	3
2nd Count	0	0	6	4	4	6	4	7	5	2

Table 28: Angle, Exp(1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0248 (0.0076)**	**0.0012 (0.0006)**	0.3259 (0.0327)	0.1850 (0.0242)	0.3467 (0.0324)	0.1869 (0.0285)	0.3297 (0.0198)	0.1787 (0.0165)	0.3473 (0.0497)	0.1985 (0.0395)
0.10	**0.0248 (0.0048)**	**0.0012 (0.0004)**	0.3160 (0.0311)	0.1714 (0.0209)	0.3348 (0.0323)	0.1732 (0.0270)	0.3115 (0.0241)	0.1608 (0.0176)	0.3306 (0.0417)	0.1813 (0.0313)
0.15	**0.0264 (0.0046)**	**0.0013 (0.0004)**	0.2990 (0.0294)	0.1530 (0.0185)	0.3196 (0.0326)	0.1569 (0.0257)	0.2922 (0.0248)	0.1422 (0.0168)	0.3110 (0.0437)	0.1607 (0.0307)
0.20	**0.0244 (0.0040)**	**0.0011 (0.0003)**	0.2792 (0.0280)	0.1334 (0.0164)	0.3008 (0.0327)	0.1383 (0.0239)	0.2728 (0.0246)	0.1248 (0.0158)	0.2919 (0.0390)	0.1408 (0.0259)
0.25	**0.0240 (0.0045)**	**0.0012 (0.0004)**	0.2572 (0.0264)	0.1132 (0.0144)	0.2783 (0.0326)	0.1181 (0.0218)	0.2515 (0.0255)	0.1065 (0.0153)	0.2696 (0.0382)	0.1198 (0.0237)
0.30	**0.0357 (0.0070)**	**0.0028 (0.0009)**	0.2338 (0.0252)	0.0934 (0.0127)	0.2529 (0.0328)	0.0976 (0.0200)	0.2295 (0.0262)	0.0882 (0.0142)	0.2424 (0.0357)	0.0972 (0.0203)
0.35	**0.0592 (0.0078)**	**0.0068 (0.0015)**	0.2085 (0.0235)	0.0738 (0.0108)	0.2246 (0.0334)	0.0773 (0.0181)	0.2050 (0.0280)	0.0699 (0.0131)	0.2141 (0.0356)	0.0754 (0.0181)
0.40	**0.0883 (0.0073)**	**0.0141 (0.0020)**	0.1807 (0.0211)	0.0552 (0.0088)	0.1938 (0.0337)	0.0579 (0.0160)	0.1771 (0.0275)	0.0520 (0.0111)	0.1877 (0.0401)	0.0577 (0.0171)
0.45	**0.1159 (0.0061)**	**0.0239 (0.0023)**	0.1498 (0.0195)	0.0378 (0.0069)	0.1607 (0.0335)	0.0402 (0.0135)	0.1462 (0.0273)	0.0354 (0.0093)	0.1541 (0.0353)	0.0387 (0.0131)
0.50	0.1317 (0.0050)	0.0309 (0.0024)	0.1168 (0.0191)	0.0227 (0.0052)	0.1260 (0.0326)	0.0249 (0.0107)	**0.1143 (0.0256)**	**0.0211 (0.0070)**	0.1247 (0.0368)	0.0245 (0.0116)
0.55	0.1240 (0.0055)	0.0269 (0.0026)	0.0819 (0.0187)	0.0108 (0.0036)	0.0905 (0.0303)	0.0129 (0.0077)	**0.0797 (0.0231)**	**0.0099 (0.0046)**	0.0882 (0.0289)	0.0118 (0.0066)
0.60	0.1007 (0.0089)	0.0157 (0.0028)	**0.0453 (0.0178)**	**0.0031 (0.0020)**	0.0575 (0.0257)	0.0054 (0.0049)	0.0459 (0.0165)	0.0032 (0.0020)	0.0600 (0.0278)	0.0052 (0.0038)
0.65	0.0847 (0.0115)	0.0102 (0.0025)	**0.0368 (0.0118)**	**0.0020 (0.0011)**	0.0508 (0.0276)	0.0043 (0.0041)	0.0456 (0.0159)	0.0031 (0.0017)	0.0545 (0.0215)	0.0040 (0.0027)
0.70	**0.0650 (0.0130)**	**0.0065 (0.0024)**	0.0855 (0.0183)	0.0106 (0.0033)	0.0945 (0.0280)	0.0126 (0.0072)	0.0935 (0.0236)	0.0126 (0.0051)	0.0951 (0.0408)	0.0130 (0.0099)
0.75	**0.0472 (0.0116)**	**0.0039 (0.0018)**	0.1498 (0.0242)	0.0347 (0.0073)	0.1536 (0.0276)	0.0349 (0.0120)	0.1576 (0.0272)	0.0373 (0.0101)	0.1548 (0.0423)	0.0361 (0.0170)
0.80	**0.0506 (0.0129)**	**0.0047 (0.0023)**	0.2320 (0.0329)	0.0855 (0.0147)	0.2268 (0.0259)	0.0790 (0.0187)	0.2374 (0.0352)	0.0871 (0.0180)	0.2371 (0.0555)	0.0882 (0.0327)
0.85	**0.0649 (0.0192)**	**0.0074 (0.0039)**	0.3314 (0.0337)	0.1821 (0.0224)	0.3227 (0.0281)	0.1631 (0.0305)	0.3357 (0.0446)	0.1819 (0.0314)	0.3437 (0.0587)	0.1907 (0.0494)
0.90	**0.0693 (0.0305)**	**0.0090 (0.0090)**	0.4761 (0.0385)	0.3853 (0.0372)	0.4667 (0.0420)	0.3437 (0.0594)	0.4752 (0.0471)	0.3745 (0.0499)	0.4898 (0.0655)	0.3941 (0.0860)
0.95	**0.1824 (0.0897)**	**0.0586 (0.0593)**	0.7360 (0.0674)	0.9125 (0.0857)	0.7999 (0.1107)	0.9135 (0.1832)	0.7219 (0.0566)	0.8663 (0.0943)	0.7324 (0.0779)	0.9023 (0.1521)
1st Count	15	15	2	2	0	0	2	2	0	0
2nd Count	0	0	5	4	3	3	11	12	0	0

Table 29: MVLin, N(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.2579 (0.0500)**	**0.1183 (0.0492)**	0.6029 (0.1130)	0.6036 (0.1993)	0.5756 (0.1142)	0.5478 (0.1797)	1.9594 (0.2910)	6.3325 (1.5796)	0.3759 (0.0857)	0.2348 (0.0979)
0.10	**0.2508 (0.0446)**	**0.1070 (0.0398)**	0.5246 (0.0891)	0.4627 (0.1404)	0.4657 (0.0717)	0.3909 (0.1157)	0.6614 (0.1019)	0.7673 (0.2617)	0.3602 (0.0705)	0.2145 (0.0751)
0.15	**0.2443 (0.0427)**	**0.0977 (0.0327)**	0.4771 (0.0749)	0.3848 (0.1121)	0.4349 (0.0578)	0.3461 (0.0984)	0.4839 (0.0742)	0.3705 (0.0961)	0.3461 (0.0655)	0.1991 (0.0673)
0.20	**0.2336 (0.0406)**	**0.0895 (0.0270)**	0.4450 (0.0651)	0.3364 (0.0943)	0.4236 (0.0553)	0.3276 (0.0931)	0.4515 (0.0767)	0.3217 (0.0951)	0.3340 (0.0604)	0.1870 (0.0628)
0.25	**0.2229 (0.0375)**	**0.0834 (0.0228)**	0.4235 (0.0579)	0.3052 (0.0815)	0.4180 (0.0554)	0.3175 (0.0905)	0.4225 (0.0703)	0.2863 (0.0844)	0.3254 (0.0534)	0.1776 (0.0535)
0.30	**0.2131 (0.0344)**	**0.0789 (0.0206)**	0.4084 (0.0533)	0.2842 (0.0728)	0.4143 (0.0557)	0.3105 (0.0883)	0.3992 (0.0574)	0.2596 (0.0674)	0.3188 (0.0496)	0.1714 (0.0495)
0.35	**0.2041 (0.0322)**	**0.0755 (0.0198)**	0.3980 (0.0492)	0.2697 (0.0662)	0.4117 (0.0560)	0.3056 (0.0864)	0.3842 (0.0469)	0.2437 (0.0552)	0.3102 (0.0433)	0.1630 (0.0431)
0.40	**0.1956 (0.0310)**	**0.0728 (0.0198)**	0.3917 (0.0462)	0.2606 (0.0613)	0.4101 (0.0564)	0.3025 (0.0852)	0.3801 (0.0468)	0.2398 (0.0549)	0.3051 (0.0423)	0.1588 (0.0410)
0.45	**0.1879 (0.0303)**	**0.0707 (0.0200)**	0.3886 (0.0438)	0.2558 (0.0572)	0.4099 (0.0573)	0.3017 (0.0853)	0.3763 (0.0421)	0.2359 (0.0489)	0.3027 (0.0421)	0.1566 (0.0399)
0.50	**0.1822 (0.0296)**	**0.0694 (0.0201)**	0.3889 (0.0433)	0.2555 (0.0556)	0.4112 (0.0593)	0.3034 (0.0874)	0.3768 (0.0415)	0.2362 (0.0493)	0.2999 (0.0426)	0.1546 (0.0403)
0.55	**0.1805 (0.0286)**	**0.0694 (0.0202)**	0.3913 (0.0416)	0.2577 (0.0545)	0.4142 (0.0623)	0.3077 (0.0920)	0.3760 (0.0423)	0.2372 (0.0493)	0.3014 (0.0426)	0.1553 (0.0410)
0.60	**0.1846 (0.0285)**	**0.0712 (0.0207)**	0.3989 (0.0414)	0.2666 (0.0546)	0.4188 (0.0665)	0.3148 (0.0991)	0.3801 (0.0492)	0.2420 (0.0588)	0.3015 (0.0447)	0.1555 (0.0434)
0.65	**0.1950 (0.0311)**	**0.0755 (0.0221)**	0.4053 (0.0429)	0.2746 (0.0579)	0.4246 (0.0716)	0.3245 (0.1085)	0.3878 (0.0535)	0.2505 (0.0649)	0.3029 (0.0474)	0.1574 (0.0468)
0.70	**0.2113 (0.0364)**	**0.0829 (0.0249)**	0.4168 (0.0472)	0.2898 (0.0650)	0.4315 (0.0767)	0.3364 (0.1195)	0.3940 (0.0607)	0.2585 (0.0720)	0.3066 (0.0506)	0.1613 (0.0512)
0.75	**0.2316 (0.0427)**	**0.0935 (0.0290)**	0.4322 (0.0526)	0.3102 (0.0730)	0.4391 (0.0811)	0.3501 (0.1307)	0.4066 (0.0681)	0.2739 (0.0839)	0.3111 (0.0541)	0.1660 (0.0548)
0.80	**0.2530 (0.0475)**	**0.1067 (0.0335)**	0.4553 (0.0636)	0.3449 (0.0906)	0.4476 (0.0838)	0.3657 (0.1404)	0.4181 (0.0699)	0.2875 (0.0869)	0.3153 (0.0576)	0.1705 (0.0584)
0.85	**0.2704 (0.0490)**	**0.1198 (0.0371)**	0.4851 (0.0738)	0.3954 (0.1099)	0.4592 (0.0836)	0.3852 (0.1461)	0.4346 (0.0822)	0.3096 (0.1064)	0.3224 (0.0583)	0.1773 (0.0594)
0.90	**0.2758 (0.0481)**	**0.1266 (0.0393)**	0.5305 (0.0935)	0.4793 (0.1627)	0.4837 (0.0833)	0.4196 (0.1469)	0.4454 (0.0845)	0.3240 (0.1114)	0.3352 (0.0662)	0.1905 (0.0700)
0.95	**0.2686 (0.0463)**	**0.1225 (0.0403)**	0.5759 (0.1060)	0.5793 (0.2285)	0.5678 (0.1079)	0.5317 (0.1730)	0.4589 (0.0929)	0.3437 (0.1272)	0.3551 (0.0638)	0.2113 (0.0699)
1st Count	19	19	0	0	0	0	0	0	0	0
2nd Count	0	0	0	0	0	0	0	0	19	19

Table 30: MVLin, N(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.1996 (0.0604)**	**0.0622 (0.0325)**	0.3376 (0.0464)	0.1980 (0.0485)	0.3214 (0.0898)	0.1589 (0.0759)	0.4496 (0.0814)	0.3772 (0.1600)	0.2043 (0.0440)	0.0685 (0.0253)
0.10	**0.1484 (0.0438)**	**0.0355 (0.0181)**	0.3041 (0.0403)	0.1573 (0.0364)	0.2343 (0.0545)	0.0966 (0.0426)	0.2306 (0.0474)	0.0896 (0.0341)	0.1988 (0.0503)	0.0652 (0.0286)
0.15	**0.1091 (0.0236)**	**0.0212 (0.0081)**	0.2813 (0.0372)	0.1339 (0.0315)	0.2235 (0.0611)	0.0901 (0.0493)	0.2225 (0.0445)	0.0840 (0.0332)	0.1963 (0.0534)	0.0637 (0.0306)
0.20	**0.1003 (0.0153)**	**0.0187 (0.0053)**	0.2646 (0.0347)	0.1184 (0.0279)	0.2234 (0.0694)	0.0902 (0.0572)	0.2237 (0.0421)	0.0844 (0.0299)	0.1986 (0.0528)	0.0643 (0.0298)
0.25	**0.1033 (0.0169)**	**0.0200 (0.0060)**	0.2517 (0.0332)	0.1074 (0.0254)	0.2231 (0.0738)	0.0901 (0.0616)	0.2189 (0.0427)	0.0813 (0.0294)	0.1928 (0.0503)	0.0610 (0.0276)
0.30	**0.1062 (0.0181)**	**0.0214 (0.0067)**	0.2420 (0.0316)	0.0995 (0.0232)	0.2217 (0.0751)	0.0891 (0.0633)	0.2167 (0.0433)	0.0796 (0.0310)	0.1893 (0.0492)	0.0588 (0.0266)
0.35	**0.1063 (0.0177)**	**0.0220 (0.0070)**	0.2347 (0.0308)	0.0939 (0.0219)	0.2196 (0.0747)	0.0878 (0.0632)	0.2147 (0.0396)	0.0781 (0.0268)	0.1873 (0.0470)	0.0577 (0.0247)
0.40	**0.1045 (0.0163)**	**0.0221 (0.0071)**	0.2293 (0.0297)	0.0899 (0.0207)	0.2182 (0.0733)	0.0867 (0.0621)	0.2185 (0.0430)	0.0806 (0.0306)	0.1845 (0.0468)	0.0563 (0.0251)
0.45	**0.1025 (0.0149)**	**0.0222 (0.0072)**	0.2264 (0.0296)	0.0879 (0.0204)	0.2180 (0.0720)	0.0864 (0.0609)	0.2185 (0.0429)	0.0808 (0.0302)	0.1832 (0.0461)	0.0556 (0.0248)
0.50	**0.1018 (0.0141)**	**0.0224 (0.0073)**	0.2252 (0.0293)	0.0872 (0.0201)	0.2198 (0.0717)	0.0874 (0.0601)	0.2191 (0.0421)	0.0812 (0.0297)	0.1839 (0.0450)	0.0558 (0.0241)
0.55	**0.1028 (0.0144)**	**0.0228 (0.0074)**	0.2257 (0.0280)	0.0877 (0.0195)	0.2237 (0.0728)	0.0899 (0.0606)	0.2207 (0.0435)	0.0826 (0.0318)	0.1850 (0.0467)	0.0566 (0.0254)
0.60	**0.1047 (0.0152)**	**0.0232 (0.0074)**	0.2283 (0.0279)	0.0898 (0.0197)	0.2296 (0.0756)	0.0940 (0.0627)	0.2234 (0.0426)	0.0846 (0.0306)	0.1856 (0.0469)	0.0570 (0.0251)
0.65	**0.1057 (0.0158)**	**0.0230 (0.0071)**	0.2323 (0.0273)	0.0931 (0.0198)	0.2371 (0.0798)	0.0995 (0.0667)	0.2234 (0.0438)	0.0848 (0.0322)	0.1865 (0.0502)	0.0577 (0.0275)
0.70	**0.1043 (0.0153)**	**0.0219 (0.0065)**	0.2388 (0.0281)	0.0985 (0.0212)	0.2450 (0.0850)	0.1060 (0.0724)	0.2281 (0.0422)	0.0882 (0.0308)	0.1882 (0.0520)	0.0589 (0.0290)
0.75	**0.1003 (0.0133)**	**0.0199 (0.0055)**	0.2469 (0.0285)	0.1054 (0.0223)	0.2520 (0.0900)	0.1127 (0.0790)	0.2293 (0.0433)	0.0896 (0.0319)	0.1899 (0.0513)	0.0599 (0.0277)
0.80	**0.0985 (0.0122)**	**0.0186 (0.0049)**	0.2588 (0.0299)	0.1158 (0.0247)	0.2563 (0.0938)	0.1182 (0.0855)	0.2332 (0.0428)	0.0927 (0.0318)	0.1919 (0.0526)	0.0612 (0.0292)
0.85	**0.1117 (0.0247)**	**0.0221 (0.0084)**	0.2758 (0.0327)	0.1316 (0.0285)	0.2572 (0.0945)	0.1218 (0.0902)	0.2372 (0.0429)	0.0960 (0.0321)	0.1953 (0.0515)	0.0630 (0.0286)
0.90	**0.1516 (0.0475)**	**0.0372 (0.0200)**	0.2991 (0.0378)	0.1556 (0.0367)	0.2611 (0.0934)	0.1266 (0.0935)	0.2419 (0.0433)	0.1001 (0.0331)	0.1965 (0.0574)	0.0647 (0.0339)
0.95	**0.1929 (0.0659)**	**0.0598 (0.0368)**	0.3317 (0.0465)	0.1934 (0.0560)	0.3215 (0.1183)	0.1709 (0.1213)	0.2475 (0.0437)	0.1045 (0.0344)	0.1992 (0.0560)	0.0671 (0.0342)
1st Count	19	19	0	0	0	0	0	0	0	0
2nd Count	0	0	0	0	0	0	0	0	19	19

Table 31: MVLin, LN(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.1743 (0.0384)**	**0.0732 (0.0612)**	0.4115 (0.0748)	0.4184 (0.4146)	0.4350 (0.0829)	0.3592 (0.1286)	2.0389 (0.3521)	6.6161 (1.8282)	0.2793 (0.0489)	0.1372 (0.0428)
0.10	**0.1662 (0.0340)**	**0.0628 (0.0491)**	0.4085 (0.0705)	0.3540 (0.2215)	0.4322 (0.0807)	0.3550 (0.1256)	0.5874 (0.0834)	0.6366 (0.1941)	0.2820 (0.0515)	0.1408 (0.0458)
0.15	**0.1620 (0.0313)**	**0.0555 (0.0382)**	0.4048 (0.0700)	0.3181 (0.1402)	0.4301 (0.0772)	0.3528 (0.1230)	0.4413 (0.0658)	0.3236 (0.0874)	0.2868 (0.0567)	0.1454 (0.0517)
0.20	**0.1623 (0.0295)**	**0.0518 (0.0288)**	0.4011 (0.0699)	0.2984 (0.1061)	0.4305 (0.0744)	0.3548 (0.1232)	0.4485 (0.0831)	0.3310 (0.1087)	0.2882 (0.0593)	0.1474 (0.0536)
0.25	**0.1670 (0.0287)**	**0.0518 (0.0214)**	0.3963 (0.0694)	0.2864 (0.0950)	0.4345 (0.0736)	0.3625 (0.1279)	0.4444 (0.0842)	0.3287 (0.1132)	0.2902 (0.0606)	0.1494 (0.0553)
0.30	**0.1757 (0.0296)**	**0.0555 (0.0174)**	0.3927 (0.0663)	0.2793 (0.0883)	0.4427 (0.0758)	0.3771 (0.1383)	0.4348 (0.0802)	0.3191 (0.1110)	0.2904 (0.0607)	0.1497 (0.0573)
0.35	**0.1878 (0.0318)**	**0.0630 (0.0178)**	0.3893 (0.0631)	0.2743 (0.0832)	0.4557 (0.0815)	0.3998 (0.1551)	0.4232 (0.0716)	0.3062 (0.1010)	0.2906 (0.0610)	0.1500 (0.0581)
0.40	**0.2026 (0.0348)**	**0.0744 (0.0213)**	0.3882 (0.0608)	0.2731 (0.0815)	0.4734 (0.0905)	0.4315 (0.1785)	0.4184 (0.0660)	0.3014 (0.0943)	0.2898 (0.0592)	0.1487 (0.0562)
0.45	**0.2197 (0.0382)**	**0.0902 (0.0264)**	0.3878 (0.0563)	0.2735 (0.0814)	0.4954 (0.1020)	0.4729 (0.2085)	0.4133 (0.0613)	0.2956 (0.0886)	0.2888 (0.0568)	0.1478 (0.0550)
0.50	**0.2394 (0.0421)**	**0.1113 (0.0335)**	0.3935 (0.0568)	0.2833 (0.0962)	0.5211 (0.1152)	0.5239 (0.2447)	0.4167 (0.0629)	0.3003 (0.0940)	0.2890 (0.0547)	0.1472 (0.0535)
0.55	**0.2634 (0.0479)**	**0.1400 (0.0453)**	0.3994 (0.0537)	0.2915 (0.1072)	0.5492 (0.1293)	0.5840 (0.2868)	0.4139 (0.0602)	0.2974 (0.0879)	0.2909 (0.0515)	0.1482 (0.0503)
0.60	**0.2959 (0.0566)**	0.1816 (0.0666)	0.4167 (0.0569)	0.3141 (0.1132)	0.5779 (0.1431)	0.6512 (0.3343)	0.4144 (0.0613)	0.2967 (0.0888)	0.2964 (0.0489)	**0.1522 (0.0473)**
0.65	0.3442 (0.0695)	0.2477 (0.1035)	0.4393 (0.0581)	0.3420 (0.1156)	0.6050 (0.1558)	0.7221 (0.3854)	0.4217 (0.0629)	0.3040 (0.0874)	**0.3065 (0.0491)**	**0.1604 (0.0478)**
0.70	0.4205 (0.0898)	0.3638 (0.1666)	0.4791 (0.0633)	0.3978 (0.1274)	0.6282 (0.1656)	0.7915 (0.4368)	0.4317 (0.0705)	0.3137 (0.0959)	**0.3206 (0.0514)**	**0.1732 (0.0498)**
0.75	0.5404 (0.1203)	0.5841 (0.2786)	0.5297 (0.0706)	0.4798 (0.1428)	0.6475 (0.1697)	0.8544 (0.4812)	0.4592 (0.0848)	0.3485 (0.1172)	**0.3437 (0.0548)**	**0.1969 (0.0548)**
0.80	0.7207 (0.1640)	1.0172 (0.4896)	0.6186 (0.0874)	0.6455 (0.1925)	0.6714 (0.1638)	0.9144 (0.5041)	0.4947 (0.0971)	0.3963 (0.1411)	**0.3772 (0.0675)**	**0.2353 (0.0758)**
0.85	0.9733 (0.2258)	1.8531 (0.8953)	0.7471 (0.1150)	0.9304 (0.2985)	0.7386 (0.1460)	1.0247 (0.4829)	0.5437 (0.1216)	0.4720 (0.1920)	**0.4308 (0.0946)**	**0.3026 (0.1213)**
0.90	1.2828 (0.3108)	3.2818 (1.5997)	0.9058 (0.1622)	1.3540 (0.4865)	0.9875 (0.1628)	1.5032 (0.4663)	0.6169 (0.1688)	0.6025 (0.3006)	**0.5176 (0.1339)**	**0.4372 (0.2071)**
0.95	1.5733 (0.3820)	4.7811 (2.3058)	1.1999 (0.2786)	2.2475 (0.8999)	1.9761 (0.3202)	4.7560 (1.2411)	0.7704 (0.2854)	0.9571 (0.6376)	**0.7410 (0.2112)**	**0.9039 (0.4877)**
1st Count	12	11	0	0	0	0	0	0	7	8
2nd Count	2	2	0	0	0	0	5	6	12	11

Table 32: MVLin, LN(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.1093 (0.0306)**	**0.0191 (0.0090)**	0.2226 (0.0367)	0.0928 (0.0349)	0.1952 (0.0628)	0.0712 (0.0470)	0.3293 (0.0749)	0.2321 (0.1239)	0.1594 (0.0498)	0.0435 (0.0251)
0.10	**0.0919 (0.0244)**	**0.0146 (0.0067)**	0.2184 (0.0376)	0.0866 (0.0277)	0.1914 (0.0621)	0.0685 (0.0451)	0.2012 (0.0510)	0.0687 (0.0321)	0.1540 (0.0397)	0.0409 (0.0185)
0.15	**0.0745 (0.0184)**	**0.0115 (0.0054)**	0.2185 (0.0405)	0.0860 (0.0296)	0.1896 (0.0606)	0.0670 (0.0427)	0.2008 (0.0489)	0.0681 (0.0305)	0.1563 (0.0364)	0.0417 (0.0166)
0.20	**0.0703 (0.0148)**	**0.0111 (0.0048)**	0.2184 (0.0415)	0.0858 (0.0305)	0.1897 (0.0590)	0.0666 (0.0406)	0.2052 (0.0485)	0.0708 (0.0302)	0.1570 (0.0369)	0.0422 (0.0171)
0.25	**0.0791 (0.0171)**	**0.0132 (0.0051)**	0.2181 (0.0406)	0.0855 (0.0302)	0.1912 (0.0576)	0.0673 (0.0392)	0.2056 (0.0524)	0.0718 (0.0335)	0.1568 (0.0383)	0.0424 (0.0180)
0.30	**0.0928 (0.0202)**	**0.0168 (0.0058)**	0.2177 (0.0389)	0.0851 (0.0295)	0.1941 (0.0564)	0.0691 (0.0385)	0.2090 (0.0530)	0.0737 (0.0333)	0.1575 (0.0348)	0.0425 (0.0161)
0.35	**0.1051 (0.0217)**	**0.0207 (0.0065)**	0.2178 (0.0367)	0.0851 (0.0284)	0.1981 (0.0553)	0.0718 (0.0382)	0.2094 (0.0562)	0.0744 (0.0372)	0.1608 (0.0365)	0.0441 (0.0170)
0.40	**0.1133 (0.0214)**	**0.0240 (0.0069)**	0.2186 (0.0339)	0.0856 (0.0271)	0.2033 (0.0542)	0.0755 (0.0384)	0.2146 (0.0525)	0.0773 (0.0335)	0.1620 (0.0356)	0.0447 (0.0169)
0.45	**0.1168 (0.0198)**	**0.0263 (0.0068)**	0.2210 (0.0318)	0.0873 (0.0263)	0.2097 (0.0532)	0.0803 (0.0390)	0.2154 (0.0546)	0.0783 (0.0359)	0.1662 (0.0393)	0.0471 (0.0196)
0.50	**0.1170 (0.0169)**	**0.0278 (0.0063)**	0.2243 (0.0292)	0.0897 (0.0250)	0.2180 (0.0529)	0.0867 (0.0402)	0.2209 (0.0472)	0.0815 (0.0307)	0.1690 (0.0397)	0.0484 (0.0200)
0.55	**0.1169 (0.0134)**	**0.0291 (0.0059)**	0.2304 (0.0282)	0.0939 (0.0244)	0.2289 (0.0541)	0.0955 (0.0427)	0.2255 (0.0550)	0.0853 (0.0380)	0.1717 (0.0382)	0.0499 (0.0200)
0.60	**0.1205 (0.0108)**	**0.0314 (0.0058)**	0.2384 (0.0271)	0.0998 (0.0238)	0.2440 (0.0581)	0.1079 (0.0475)	0.2306 (0.0462)	0.0887 (0.0312)	0.1775 (0.0375)	0.0530 (0.0198)
0.65	**0.1306 (0.0125)**	**0.0358 (0.0067)**	0.2510 (0.0271)	0.1096 (0.0244)	0.2650 (0.0656)	0.1260 (0.0566)	0.2392 (0.0490)	0.0954 (0.0344)	0.1856 (0.0403)	0.0578 (0.0222)
0.70	**0.1477 (0.0189)**	**0.0433 (0.0092)**	0.2710 (0.0285)	0.1265 (0.0266)	0.2935 (0.0765)	0.1526 (0.0717)	0.2488 (0.0501)	0.1030 (0.0369)	0.1942 (0.0428)	0.0634 (0.0244)
0.75	**0.1733 (0.0289)**	**0.0570 (0.0155)**	0.3045 (0.0322)	0.1593 (0.0339)	0.3299 (0.0894)	0.1908 (0.0933)	0.2615 (0.0520)	0.1134 (0.0397)	0.2083 (0.0463)	0.0728 (0.0287)
0.80	0.2266 (0.0574)	0.0953 (0.0435)	0.3629 (0.0412)	0.2284 (0.0533)	0.3725 (0.1017)	0.2427 (0.1210)	0.2783 (0.0572)	0.1283 (0.0462)	**0.2235 (0.0459)**	**0.0838 (0.0304)**
0.85	0.3814 (0.1405)	0.2612 (0.1771)	0.4629 (0.0573)	0.3784 (0.0989)	0.4145 (0.1103)	0.3045 (0.1519)	0.3019 (0.0670)	0.1511 (0.0587)	**0.2437 (0.0418)**	**0.0997 (0.0317)**
0.90	0.7742 (0.3121)	1.0042 (0.7370)	0.6148 (0.0851)	0.6621 (0.1903)	0.4655 (0.1010)	0.3792 (0.1686)	0.3356 (0.0784)	0.1868 (0.0751)	**0.2904 (0.0562)**	**0.1418 (0.0526)**
0.95	1.3040 (0.4963)	2.6038 (1.7310)	0.8501 (0.1313)	1.2095 (0.3385)	0.9015 (0.2571)	1.1307 (0.5118)	0.4034 (0.1132)	0.2732 (0.1278)	**0.4001 (0.0792)**	**0.2668 (0.1010)**
1st Count	15	15	0	0	0	0	0	0	4	4
2nd Count	1	1	0	0	0	0	3	3	15	15

Table 33: MVLin, Exp(1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.1356 (0.0224)**	**0.0387 (0.0159)**	0.3267 (0.0485)	0.2243 (0.1238)	0.3358 (0.0605)	0.2194 (0.0786)	2.0972 (0.3588)	6.8201 (1.8843)	0.2406 (0.0470)	0.1036 (0.0371)
0.10	**0.1216 (0.0185)**	**0.0299 (0.0106)**	0.3259 (0.0472)	0.2017 (0.0633)	0.3323 (0.0577)	0.2149 (0.0732)	0.5051 (0.0718)	0.4788 (0.1592)	0.2441 (0.0410)	0.1066 (0.0327)
0.15	**0.1139 (0.0154)**	**0.0260 (0.0075)**	0.3244 (0.0509)	0.1930 (0.0588)	0.3312 (0.0560)	0.2131 (0.0695)	0.3789 (0.0662)	0.2378 (0.0753)	0.2481 (0.0477)	0.1101 (0.0376)
0.20	**0.1125 (0.0139)**	**0.0257 (0.0062)**	0.3217 (0.0531)	0.1880 (0.0598)	0.3327 (0.0558)	0.2140 (0.0682)	0.3812 (0.0797)	0.2381 (0.0884)	0.2489 (0.0475)	0.1109 (0.0377)
0.25	**0.1165 (0.0138)**	**0.0283 (0.0065)**	0.3179 (0.0526)	0.1833 (0.0592)	0.3368 (0.0573)	0.2178 (0.0695)	0.3728 (0.0776)	0.2305 (0.0853)	0.2499 (0.0489)	0.1120 (0.0382)
0.30	**0.1234 (0.0149)**	**0.0329 (0.0077)**	0.3150 (0.0521)	0.1799 (0.0590)	0.3432 (0.0605)	0.2246 (0.0738)	0.3584 (0.0710)	0.2165 (0.0780)	0.2500 (0.0484)	0.1118 (0.0377)
0.35	**0.1317 (0.0163)**	**0.0388 (0.0094)**	0.3122 (0.0497)	0.1765 (0.0566)	0.3518 (0.0653)	0.2341 (0.0809)	0.3448 (0.0629)	0.2032 (0.0699)	0.2474 (0.0452)	0.1097 (0.0354)
0.40	**0.1405 (0.0180)**	**0.0457 (0.0112)**	0.3076 (0.0453)	0.1712 (0.0520)	0.3622 (0.0713)	0.2465 (0.0907)	0.3384 (0.0592)	0.1970 (0.0656)	0.2447 (0.0441)	0.1075 (0.0349)
0.45	**0.1493 (0.0198)**	**0.0529 (0.0133)**	0.3075 (0.0421)	0.1705 (0.0489)	0.3739 (0.0781)	0.2612 (0.1026)	0.3330 (0.0597)	0.1913 (0.0682)	0.2431 (0.0416)	0.1057 (0.0331)
0.50	**0.1579 (0.0218)**	**0.0605 (0.0158)**	0.3091 (0.0401)	0.1706 (0.0464)	0.3863 (0.0852)	0.2780 (0.1162)	0.3276 (0.0502)	0.1853 (0.0593)	0.2408 (0.0378)	0.1034 (0.0303)
0.55	**0.1670 (0.0242)**	**0.0683 (0.0188)**	0.3101 (0.0375)	0.1703 (0.0430)	0.3987 (0.0920)	0.2961 (0.1308)	0.3270 (0.0479)	0.1840 (0.0578)	0.2419 (0.0378)	0.1039 (0.0299)
0.60	**0.1783 (0.0274)**	**0.0770 (0.0227)**	0.3197 (0.0359)	0.1776 (0.0401)	0.4104 (0.0978)	0.3147 (0.1456)	0.3267 (0.0435)	0.1820 (0.0513)	0.2467 (0.0364)	0.1070 (0.0290)
0.65	**0.1945 (0.0328)**	**0.0879 (0.0283)**	0.3383 (0.0447)	0.1956 (0.0530)	0.4205 (0.1018)	0.3327 (0.1591)	0.3264 (0.0432)	0.1805 (0.0512)	0.2521 (0.0386)	0.1104 (0.0311)
0.70	**0.2202 (0.0413)**	**0.1043 (0.0365)**	0.3542 (0.0455)	0.2110 (0.0525)	0.4289 (0.1031)	0.3491 (0.1697)	0.3394 (0.0494)	0.1918 (0.0553)	0.2599 (0.0385)	0.1158 (0.0316)
0.75	**0.2601 (0.0523)**	0.1320 (0.0491)	0.3747 (0.0489)	0.2319 (0.0552)	0.4365 (0.1002)	0.3641 (0.1746)	0.3548 (0.0591)	0.2066 (0.0645)	0.2695 (0.0446)	**0.1236 (0.0375)**
0.80	0.3181 (0.0640)	0.1805 (0.0682)	0.4124 (0.0511)	0.2760 (0.0628)	0.4482 (0.0925)	0.3815 (0.1704)	0.3782 (0.0666)	0.2298 (0.0724)	**0.2835 (0.0487)**	**0.1357 (0.0419)**
0.85	0.3932 (0.0747)	0.2620 (0.0972)	0.4778 (0.0782)	0.3627 (0.1092)	0.4803 (0.0858)	0.4193 (0.1574)	0.3989 (0.0717)	0.2527 (0.0809)	**0.3069 (0.0490)**	**0.1567 (0.0459)**
0.90	0.4710 (0.0856)	0.3796 (0.1426)	0.5394 (0.0995)	0.4612 (0.1542)	0.5849 (0.1120)	0.5515 (0.1764)	0.4299 (0.0860)	0.2919 (0.1029)	**0.3426 (0.0574)**	**0.1920 (0.0582)**
0.95	0.5175 (0.1010)	0.4841 (0.2127)	0.6449 (0.1354)	0.6533 (0.2545)	0.9466 (0.2010)	1.1844 (0.4074)	0.4852 (0.1235)	0.3703 (0.1674)	**0.4154 (0.0869)**	**0.2782 (0.1022)**
1st Count	15	14	0	0	0	0	0	0	4	5
2nd Count	2	2	0	0	0	0	2	3	15	14

Table 34: MVLin, Exp(1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.0625 (0.0224)**	**0.0070 (0.0049)**	0.1836 (0.0479)	0.0582 (0.0268)	0.1584 (0.0516)	0.0459 (0.0327)	0.2690 (0.0648)	0.1629 (0.0906)	0.1423 (0.0561)	0.0354 (0.0285)
0.10	**0.0492 (0.0161)**	**0.0047 (0.0034)**	0.1769 (0.0386)	0.0545 (0.0202)	0.1564 (0.0514)	0.0447 (0.0317)	0.1626 (0.0302)	0.0448 (0.0150)	0.1325 (0.0369)	0.0303 (0.0152)
0.15	**0.0436 (0.0125)**	**0.0040 (0.0030)**	0.1763 (0.0387)	0.0542 (0.0202)	0.1558 (0.0514)	0.0443 (0.0309)	0.1602 (0.0286)	0.0436 (0.0142)	0.1289 (0.0316)	0.0289 (0.0123)
0.20	**0.0464 (0.0147)**	**0.0046 (0.0033)**	0.1757 (0.0388)	0.0539 (0.0203)	0.1564 (0.0515)	0.0445 (0.0303)	0.1609 (0.0277)	0.0439 (0.0137)	0.1288 (0.0303)	0.0289 (0.0118)
0.25	**0.0544 (0.0187)**	**0.0063 (0.0041)**	0.1747 (0.0385)	0.0533 (0.0201)	0.1578 (0.0517)	0.0452 (0.0300)	0.1645 (0.0314)	0.0458 (0.0155)	0.1293 (0.0324)	0.0292 (0.0130)
0.30	**0.0640 (0.0214)**	**0.0085 (0.0051)**	0.1734 (0.0366)	0.0525 (0.0191)	0.1600 (0.0522)	0.0465 (0.0300)	0.1653 (0.0300)	0.0462 (0.0152)	0.1296 (0.0295)	0.0291 (0.0117)
0.35	**0.0726 (0.0224)**	**0.0109 (0.0059)**	0.1727 (0.0344)	0.0519 (0.0179)	0.1630 (0.0528)	0.0482 (0.0305)	0.1650 (0.0323)	0.0463 (0.0162)	0.1310 (0.0291)	0.0296 (0.0113)
0.40	**0.0792 (0.0219)**	**0.0131 (0.0065)**	0.1725 (0.0320)	0.0516 (0.0166)	0.1667 (0.0538)	0.0505 (0.0317)	0.1652 (0.0285)	0.0462 (0.0146)	0.1310 (0.0282)	0.0295 (0.0108)
0.45	**0.0837 (0.0205)**	**0.0151 (0.0068)**	0.1736 (0.0297)	0.0520 (0.0155)	0.1714 (0.0553)	0.0535 (0.0335)	0.1655 (0.0288)	0.0464 (0.0148)	0.1318 (0.0276)	0.0298 (0.0106)
0.50	**0.0868 (0.0186)**	**0.0168 (0.0070)**	0.1756 (0.0277)	0.0529 (0.0145)	0.1776 (0.0575)	0.0574 (0.0364)	0.1673 (0.0295)	0.0474 (0.0155)	0.1340 (0.0310)	0.0308 (0.0123)
0.55	**0.0900 (0.0167)**	**0.0183 (0.0072)**	0.1788 (0.0258)	0.0546 (0.0138)	0.1856 (0.0609)	0.0625 (0.0404)	0.1697 (0.0297)	0.0485 (0.0154)	0.1351 (0.0312)	0.0313 (0.0125)
0.60	**0.0947 (0.0156)**	**0.0199 (0.0074)**	0.1834 (0.0244)	0.0573 (0.0135)	0.1959 (0.0660)	0.0692 (0.0462)	0.1698 (0.0270)	0.0488 (0.0148)	0.1388 (0.0327)	0.0329 (0.0136)
0.65	**0.1019 (0.0154)**	**0.0221 (0.0077)**	0.1901 (0.0239)	0.0615 (0.0138)	0.2084 (0.0728)	0.0778 (0.0542)	0.1753 (0.0289)	0.0519 (0.0158)	0.1459 (0.0367)	0.0360 (0.0168)
0.70	**0.1112 (0.0160)**	**0.0249 (0.0081)**	0.1992 (0.0232)	0.0673 (0.0142)	0.2229 (0.0813)	0.0885 (0.0646)	0.1794 (0.0300)	0.0543 (0.0170)	0.1508 (0.0407)	0.0385 (0.0192)
0.75	**0.1211 (0.0175)**	**0.0284 (0.0087)**	0.2130 (0.0229)	0.0769 (0.0155)	0.2378 (0.0904)	0.1007 (0.0770)	0.1845 (0.0295)	0.0573 (0.0173)	0.1567 (0.0418)	0.0414 (0.0194)
0.80	**0.1328 (0.0206)**	**0.0333 (0.0102)**	0.2335 (0.0240)	0.0928 (0.0188)	0.2508 (0.0985)	0.1129 (0.0898)	0.1920 (0.0320)	0.0622 (0.0197)	0.1646 (0.0362)	0.0452 (0.0167)
0.85	**0.1631 (0.0299)**	**0.0480 (0.0165)**	0.2650 (0.0282)	0.1201 (0.0258)	0.2612 (0.1017)	0.1232 (0.0989)	0.2025 (0.0352)	0.0691 (0.0229)	0.1772 (0.0447)	0.0527 (0.0247)
0.90	0.2561 (0.0655)	0.1101 (0.0506)	0.3128 (0.0361)	0.1672 (0.0390)	0.2888 (0.1004)	0.1427 (0.1008)	0.2178 (0.0425)	0.0797 (0.0296)	**0.1947 (0.0497)**	**0.0634 (0.0292)**
0.95	0.4320 (0.1296)	0.3120 (0.1738)	0.3885 (0.0541)	0.2571 (0.0713)	0.4912 (0.1744)	0.3388 (0.2061)	0.2399 (0.0529)	0.0959 (0.0407)	**0.2200 (0.0470)**	**0.0806 (0.0318)**
1st Count	17	17	0	0	0	0	0	0	2	2
2nd Count	0	0	0	0	0	0	2	2	17	17

Table 35: SIM, N(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.4911 (0.0416)	0.3708 (0.0877)	0.7290 (0.0624)	0.7954 (0.1309)	0.5847 (0.0427)	0.4648 (0.0693)	0.4659 (0.0289)	0.3414 (0.0599)	**0.4632 (0.0213)**	**0.3056 (0.0267)**
0.10	0.3875 (0.0245)	0.2361 (0.0358)	0.6211 (0.0566)	0.5928 (0.1051)	0.4303 (0.0326)	0.2558 (0.0400)	**0.3637 (0.0169)**	**0.1965 (0.0235)**	0.3836 (0.0249)	0.2070 (0.0287)
0.15	0.3273 (0.0259)	0.1692 (0.0339)	0.5457 (0.0518)	0.4689 (0.0879)	0.3427 (0.0271)	0.1675 (0.0280)	**0.3079 (0.0148)**	**0.1357 (0.0138)**	0.3361 (0.0240)	0.1619 (0.0259)
0.20	0.2831 (0.0325)	0.1265 (0.0351)	0.4893 (0.0476)	0.3853 (0.0750)	0.2841 (0.0244)	0.1206 (0.0224)	**0.2699 (0.0167)**	**0.1042 (0.0132)**	0.2978 (0.0236)	0.1306 (0.0233)
0.25	0.2445 (0.0371)	0.0942 (0.0323)	0.4468 (0.0444)	0.3269 (0.0656)	0.2427 (0.0232)	0.0929 (0.0196)	**0.2398 (0.0192)**	**0.0842 (0.0137)**	0.2632 (0.0228)	0.1050 (0.0199)
0.30	**0.2077 (0.0395)**	**0.0681 (0.0276)**	0.4138 (0.0413)	0.2845 (0.0580)	0.2140 (0.0230)	0.0760 (0.0180)	0.2138 (0.0212)	0.0689 (0.0139)	0.2312 (0.0224)	0.0838 (0.0177)
0.35	**0.1716 (0.0399)**	**0.0471 (0.0222)**	0.3894 (0.0385)	0.2546 (0.0520)	0.1952 (0.0237)	0.0658 (0.0172)	0.1899 (0.0223)	0.0561 (0.0134)	0.2021 (0.0233)	0.0664 (0.0163)
0.40	**0.1376 (0.0384)**	**0.0310 (0.0171)**	0.3726 (0.0360)	0.2345 (0.0474)	0.1848 (0.0250)	0.0603 (0.0170)	0.1706 (0.0226)	0.0467 (0.0128)	0.1788 (0.0236)	0.0538 (0.0148)
0.45	**0.1093 (0.0349)**	**0.0202 (0.0126)**	0.3627 (0.0340)	0.2226 (0.0441)	0.1813 (0.0267)	0.0586 (0.0175)	0.1593 (0.0240)	0.0419 (0.0135)	0.1641 (0.0249)	0.0465 (0.0146)
0.50	**0.0939 (0.0293)**	**0.0149 (0.0092)**	0.3596 (0.0328)	0.2184 (0.0420)	0.1840 (0.0283)	0.0604 (0.0186)	0.1564 (0.0232)	0.0411 (0.0136)	0.1612 (0.0258)	0.0449 (0.0146)
0.55	**0.0977 (0.0236)**	**0.0156 (0.0075)**	0.3628 (0.0322)	0.2210 (0.0411)	0.1925 (0.0295)	0.0657 (0.0204)	0.1642 (0.0242)	0.0452 (0.0151)	0.1680 (0.0248)	0.0477 (0.0143)
0.60	**0.1189 (0.0207)**	**0.0223 (0.0083)**	0.3733 (0.0322)	0.2318 (0.0417)	0.2070 (0.0304)	0.0748 (0.0230)	0.1813 (0.0246)	0.0544 (0.0167)	0.1859 (0.0254)	0.0568 (0.0162)
0.65	**0.1511 (0.0217)**	**0.0355 (0.0115)**	0.3904 (0.0333)	0.2508 (0.0440)	0.2275 (0.0313)	0.0884 (0.0263)	0.2053 (0.0244)	0.0682 (0.0183)	0.2112 (0.0240)	0.0712 (0.0169)
0.70	**0.1890 (0.0246)**	**0.0555 (0.0167)**	0.4149 (0.0348)	0.2794 (0.0478)	0.2539 (0.0322)	0.1075 (0.0305)	0.2358 (0.0242)	0.0878 (0.0202)	0.2413 (0.0240)	0.0905 (0.0188)
0.75	**0.2308 (0.0277)**	**0.0831 (0.0235)**	0.4474 (0.0371)	0.3199 (0.0536)	0.2864 (0.0330)	0.1335 (0.0355)	0.2704 (0.0248)	0.1128 (0.0232)	0.2763 (0.0242)	0.1160 (0.0212)
0.80	**0.2768 (0.0295)**	**0.1206 (0.0304)**	0.4897 (0.0407)	0.3768 (0.0626)	0.3256 (0.0336)	0.1687 (0.0409)	0.3099 (0.0232)	0.1450 (0.0239)	0.3159 (0.0241)	0.1490 (0.0241)
0.85	**0.3302 (0.0287)**	**0.1736 (0.0353)**	0.5445 (0.0450)	0.4574 (0.0745)	0.3737 (0.0339)	0.2170 (0.0464)	0.3559 (0.0237)	0.1879 (0.0272)	0.3610 (0.0250)	0.1913 (0.0280)
0.90	**0.3993 (0.0309)**	0.2563 (0.0468)	0.6169 (0.0506)	0.5762 (0.0923)	0.4373 (0.0346)	0.2885 (0.0516)	0.4120 (0.0236)	**0.2478 (0.0304)**	0.4179 (0.0252)	0.2534 (0.0318)
0.95	0.5048 (0.0536)	0.4055 (0.1132)	0.7222 (0.0566)	0.7730 (0.1183)	0.5419 (0.0424)	0.4250 (0.0661)	**0.4939 (0.0231)**	**0.3561 (0.0354)**	0.5103 (0.0264)	0.3776 (0.0399)
1st Count	13	12	0	0	0	0	5	6	1	1
2nd Count	3	0	0	0	1	2	14	13	1	4

Table 36: SIM, N(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.4675 (0.0275)	0.3612 (0.0534)	0.5297 (0.0229)	0.4181 (0.0356)	0.4649 (0.0227)	0.3233 (0.0508)	**0.4384 (0.0118)**	**0.2812 (0.0188)**	0.4471 (0.0142)	0.2946 (0.0236)
0.10	**0.3449 (0.0102)**	0.1734 (0.0152)	0.4448 (0.0230)	0.2996 (0.0306)	0.3630 (0.0141)	0.1927 (0.0239)	0.3480 (0.0101)	**0.1683 (0.0087)**	0.3585 (0.0138)	0.1814 (0.0140)
0.15	**0.2856 (0.0085)**	**0.1078 (0.0050)**	0.3869 (0.0230)	0.2309 (0.0273)	0.2990 (0.0110)	0.1257 (0.0097)	0.2926 (0.0121)	0.1161 (0.0079)	0.3029 (0.0161)	0.1285 (0.0135)
0.20	**0.2453 (0.0090)**	**0.0776 (0.0048)**	0.3416 (0.0227)	0.1839 (0.0246)	0.2536 (0.0135)	0.0888 (0.0090)	0.2503 (0.0153)	0.0856 (0.0099)	0.2591 (0.0186)	0.0950 (0.0138)
0.25	**0.2078 (0.0086)**	**0.0560 (0.0045)**	0.3046 (0.0225)	0.1498 (0.0223)	0.2150 (0.0157)	0.0643 (0.0097)	0.2152 (0.0187)	0.0647 (0.0113)	0.2221 (0.0203)	0.0713 (0.0139)
0.30	**0.1688 (0.0079)**	**0.0375 (0.0036)**	0.2744 (0.0221)	0.1246 (0.0202)	0.1770 (0.0167)	0.0447 (0.0089)	0.1834 (0.0214)	0.0488 (0.0114)	0.1888 (0.0218)	0.0532 (0.0129)
0.35	**0.1282 (0.0068)**	**0.0220 (0.0025)**	0.2505 (0.0217)	0.1063 (0.0186)	0.1392 (0.0166)	0.0291 (0.0072)	0.1538 (0.0236)	0.0359 (0.0107)	0.1568 (0.0230)	0.0382 (0.0118)
0.40	**0.0876 (0.0060)**	**0.0106 (0.0016)**	0.2330 (0.0215)	0.0939 (0.0174)	0.1064 (0.0151)	0.0185 (0.0055)	0.1285 (0.0238)	0.0264 (0.0092)	0.1289 (0.0219)	0.0271 (0.0096)
0.45	**0.0516 (0.0060)**	**0.0040 (0.0010)**	0.2223 (0.0216)	0.0867 (0.0168)	0.0863 (0.0144)	0.0133 (0.0044)	0.1109 (0.0220)	0.0206 (0.0075)	0.1082 (0.0201)	0.0200 (0.0078)
0.50	**0.0359 (0.0076)**	**0.0024 (0.0010)**	0.2191 (0.0222)	0.0846 (0.0169)	0.0845 (0.0173)	0.0128 (0.0047)	0.1062 (0.0214)	0.0192 (0.0071)	0.1014 (0.0167)	0.0177 (0.0061)
0.55	**0.0591 (0.0077)**	**0.0055 (0.0014)**	0.2233 (0.0227)	0.0874 (0.0177)	0.0977 (0.0198)	0.0163 (0.0059)	0.1153 (0.0231)	0.0224 (0.0082)	0.1120 (0.0172)	0.0210 (0.0062)
0.60	**0.0957 (0.0074)**	**0.0129 (0.0021)**	0.2349 (0.0233)	0.0952 (0.0189)	0.1195 (0.0203)	0.0230 (0.0074)	0.1357 (0.0251)	0.0299 (0.0104)	0.1356 (0.0193)	0.0296 (0.0077)
0.65	**0.1333 (0.0073)**	**0.0239 (0.0027)**	0.2533 (0.0237)	0.1083 (0.0205)	0.1464 (0.0197)	0.0329 (0.0089)	0.1630 (0.0261)	0.0415 (0.0127)	0.1657 (0.0201)	0.0425 (0.0096)
0.70	**0.1699 (0.0076)**	**0.0379 (0.0034)**	0.2780 (0.0238)	0.1272 (0.0223)	0.1771 (0.0186)	0.0467 (0.0100)	0.1943 (0.0258)	0.0570 (0.0147)	0.1986 (0.0202)	0.0594 (0.0114)
0.75	**0.2052 (0.0083)**	**0.0545 (0.0042)**	0.3086 (0.0236)	0.1529 (0.0243)	0.2121 (0.0174)	0.0659 (0.0110)	0.2287 (0.0246)	0.0768 (0.0159)	0.2338 (0.0195)	0.0806 (0.0127)
0.80	**0.2408 (0.0088)**	**0.0751 (0.0047)**	0.3456 (0.0234)	0.1874 (0.0264)	0.2528 (0.0164)	0.0932 (0.0130)	0.2670 (0.0224)	0.1027 (0.0162)	0.2718 (0.0191)	0.1071 (0.0141)
0.85	**0.2825 (0.0086)**	**0.1078 (0.0064)**	0.3907 (0.0231)	0.2346 (0.0286)	0.3024 (0.0167)	0.1333 (0.0175)	0.3114 (0.0194)	0.1384 (0.0154)	0.3171 (0.0173)	0.1445 (0.0147)
0.90	**0.3457 (0.0121)**	**0.1790 (0.0202)**	0.4480 (0.0226)	0.3030 (0.0309)	0.3667 (0.0182)	0.1949 (0.0245)	0.3688 (0.0164)	0.1957 (0.0169)	0.3751 (0.0159)	0.2036 (0.0183)
0.95	0.4647 (0.0278)	0.3567 (0.0572)	0.5313 (0.0213)	0.4199 (0.0341)	0.4661 (0.0217)	**0.3044 (0.0308)**	**0.4589 (0.0167)**	0.3140 (0.0335)	0.4664 (0.0176)	0.3246 (0.0327)
1st Count	17	16	0	0	0	1	2	2	0	0
2nd Count	1	1	0	0	14	14	3	3	1	1

Table 37: SIM, LN(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.4199 (0.0248)	0.3007 (0.0374)	0.6482 (0.0604)	0.7111 (0.1557)	**0.4107 (0.0158)**	**0.2369 (0.0167)**	0.5146 (0.0634)	0.4442 (0.0939)	0.4158 (0.0169)	0.2791 (0.0253)
0.10	**0.3831 (0.0231)**	0.2423 (0.0321)	0.6414 (0.0664)	0.7013 (0.1665)	0.3891 (0.0159)	**0.2157 (0.0166)**	0.4055 (0.0228)	0.2687 (0.0353)	0.3904 (0.0160)	0.2308 (0.0183)
0.15	**0.3543 (0.0222)**	**0.1979 (0.0309)**	0.6290 (0.0687)	0.6802 (0.1699)	0.3709 (0.0165)	0.1984 (0.0169)	0.3692 (0.0144)	0.2061 (0.0188)	0.3787 (0.0183)	0.2100 (0.0194)
0.20	**0.3312 (0.0208)**	**0.1649 (0.0290)**	0.6146 (0.0702)	0.6558 (0.1709)	0.3543 (0.0176)	0.1832 (0.0180)	0.3538 (0.0156)	0.1809 (0.0153)	0.3726 (0.0208)	0.2019 (0.0233)
0.25	**0.3114 (0.0191)**	**0.1402 (0.0257)**	0.5997 (0.0707)	0.6304 (0.1692)	0.3386 (0.0193)	0.1698 (0.0199)	0.3465 (0.0194)	0.1702 (0.0188)	0.3670 (0.0228)	0.1965 (0.0264)
0.30	**0.2931 (0.0177)**	**0.1209 (0.0217)**	0.5836 (0.0708)	0.6035 (0.1671)	0.3239 (0.0219)	0.1583 (0.0231)	0.3405 (0.0239)	0.1653 (0.0236)	0.3609 (0.0258)	0.1928 (0.0306)
0.35	**0.2747 (0.0171)**	**0.1045 (0.0179)**	0.5675 (0.0706)	0.5771 (0.1645)	0.3103 (0.0255)	0.1490 (0.0277)	0.3344 (0.0284)	0.1620 (0.0287)	0.3526 (0.0292)	0.1873 (0.0345)
0.40	**0.2549 (0.0172)**	**0.0896 (0.0150)**	0.5516 (0.0701)	0.5511 (0.1614)	0.2984 (0.0305)	0.1428 (0.0342)	0.3268 (0.0320)	0.1579 (0.0326)	0.3404 (0.0317)	0.1782 (0.0368)
0.45	**0.2326 (0.0178)**	**0.0750 (0.0132)**	0.5357 (0.0687)	0.5252 (0.1567)	0.2892 (0.0372)	0.1406 (0.0431)	0.3155 (0.0361)	0.1508 (0.0361)	0.3252 (0.0345)	0.1666 (0.0389)
0.50	**0.2077 (0.0192)**	**0.0609 (0.0126)**	0.5215 (0.0672)	0.5019 (0.1520)	0.2839 (0.0461)	0.1439 (0.0548)	0.3005 (0.0382)	0.1406 (0.0374)	0.3073 (0.0362)	0.1528 (0.0390)
0.55	**0.1812 (0.0228)**	**0.0480 (0.0141)**	0.5096 (0.0654)	0.4816 (0.1472)	0.2846 (0.0570)	0.1545 (0.0702)	0.2849 (0.0392)	0.1303 (0.0378)	0.2873 (0.0419)	0.1386 (0.0436)
0.60	**0.1578 (0.0313)**	**0.0384 (0.0194)**	0.5038 (0.0635)	0.4697 (0.1430)	0.2934 (0.0700)	0.1745 (0.0901)	0.2677 (0.0410)	0.1191 (0.0395)	0.2686 (0.0460)	0.1260 (0.0467)
0.65	**0.1474 (0.0454)**	**0.0365 (0.0299)**	0.5075 (0.0612)	0.4709 (0.1392)	0.3132 (0.0837)	0.2069 (0.1157)	0.2556 (0.0420)	0.1119 (0.0406)	0.2593 (0.0515)	0.1200 (0.0508)
0.70	**0.1659 (0.0623)**	**0.0514 (0.0476)**	0.5263 (0.0596)	0.4939 (0.1390)	0.3455 (0.0962)	0.2548 (0.1478)	0.2593 (0.0412)	0.1160 (0.0418)	0.2731 (0.0582)	0.1329 (0.0623)
0.75	**0.2307 (0.0767)**	**0.1064 (0.0785)**	0.5693 (0.0600)	0.5564 (0.1466)	0.3920 (0.1049)	0.3227 (0.1865)	0.2901 (0.0393)	0.1413 (0.0438)	0.3120 (0.0595)	0.1658 (0.0715)
0.80	**0.3603 (0.0895)**	0.2665 (0.1489)	0.6498 (0.0640)	0.6919 (0.1664)	0.4577 (0.1055)	0.4192 (0.2290)	0.3607 (0.0394)	**0.2077 (0.0499)**	0.3933 (0.0583)	0.2496 (0.0852)
0.85	0.5885 (0.1131)	0.7251 (0.3552)	0.7934 (0.0724)	0.9802 (0.2056)	0.5600 (0.0930)	0.5729 (0.2653)	**0.4808 (0.0450)**	**0.3501 (0.0666)**	0.5228 (0.0557)	0.4211 (0.1012)
0.90	0.9871 (0.1837)	1.9988 (0.9176)	1.0542 (0.0892)	1.6474 (0.2872)	0.7595 (0.0708)	0.9217 (0.2766)	**0.6840 (0.0554)**	**0.6722 (0.1077)**	0.7369 (0.0571)	0.8045 (0.1381)
0.95	1.6487 (0.2912)	4.9316 (1.9246)	1.6101 (0.1472)	3.6895 (0.6241)	1.3374 (0.1170)	2.5357 (0.4487)	**1.0868 (0.0852)**	**1.6074 (0.2450)**	1.2061 (0.0803)	2.0780 (0.3014)
1st Count	15	13	0	0	1	2	3	4	0	0
2nd Count	0	0	0	0	8	6	7	7	4	6

Table 38: SIM, LN(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.4326 (0.0197)	0.3329 (0.0284)	0.5006 (0.0284)	0.4129 (0.1155)	0.4273 (0.0297)	0.3108 (0.0484)	0.4206 (0.0190)	0.3044 (0.0282)	**0.4176 (0.0200)**	**0.2982 (0.0292)**
0.10	0.3806 (0.0145)	0.2432 (0.0190)	0.4730 (0.0240)	0.3603 (0.0776)	0.3847 (0.0200)	0.2405 (0.0320)	**0.3757 (0.0139)**	**0.2250 (0.0198)**	0.3777 (0.0155)	0.2274 (0.0210)
0.15	**0.3459 (0.0112)**	0.1835 (0.0124)	0.4577 (0.0255)	0.3336 (0.0561)	0.3546 (0.0138)	0.1917 (0.0203)	0.3498 (0.0112)	**0.1818 (0.0133)**	0.3541 (0.0124)	0.1871 (0.0149)
0.20	**0.3235 (0.0092)**	**0.1468 (0.0078)**	0.4451 (0.0276)	0.3147 (0.0482)	0.3336 (0.0112)	0.1598 (0.0127)	0.3324 (0.0113)	0.1556 (0.0096)	0.3388 (0.0115)	0.1636 (0.0114)
0.25	**0.3087 (0.0082)**	**0.1255 (0.0054)**	0.4329 (0.0296)	0.2985 (0.0462)	0.3182 (0.0119)	0.1395 (0.0101)	0.3194 (0.0140)	0.1394 (0.0104)	0.3271 (0.0126)	0.1490 (0.0110)
0.30	**0.2971 (0.0078)**	**0.1131 (0.0048)**	0.4201 (0.0310)	0.2827 (0.0460)	0.3053 (0.0142)	0.1261 (0.0115)	0.3086 (0.0176)	0.1290 (0.0141)	0.3167 (0.0152)	0.1390 (0.0132)
0.35	**0.2853 (0.0075)**	**0.1041 (0.0050)**	0.4061 (0.0320)	0.2666 (0.0458)	0.2919 (0.0167)	0.1153 (0.0140)	0.2981 (0.0220)	0.1213 (0.0183)	0.3052 (0.0182)	0.1301 (0.0158)
0.40	**0.2705 (0.0074)**	**0.0945 (0.0050)**	0.3903 (0.0326)	0.2493 (0.0449)	0.2755 (0.0193)	0.1040 (0.0160)	0.2860 (0.0265)	0.1139 (0.0223)	0.2920 (0.0216)	0.1208 (0.0184)
0.45	**0.2499 (0.0074)**	**0.0819 (0.0048)**	0.3733 (0.0328)	0.2314 (0.0432)	0.2544 (0.0219)	0.0907 (0.0172)	0.2706 (0.0312)	0.1046 (0.0254)	0.2745 (0.0246)	0.1089 (0.0199)
0.50	**0.2218 (0.0077)**	**0.0656 (0.0045)**	0.3547 (0.0328)	0.2127 (0.0411)	0.2279 (0.0244)	0.0753 (0.0175)	0.2506 (0.0362)	0.0926 (0.0277)	0.2518 (0.0266)	0.0938 (0.0198)
0.55	**0.1846 (0.0080)**	**0.0463 (0.0039)**	0.3357 (0.0324)	0.1944 (0.0387)	0.1970 (0.0266)	0.0592 (0.0171)	0.2256 (0.0395)	0.0782 (0.0277)	0.2247 (0.0282)	0.0773 (0.0186)
0.60	**0.1378 (0.0082)**	**0.0265 (0.0031)**	0.3185 (0.0319)	0.1786 (0.0369)	0.1648 (0.0281)	0.0453 (0.0161)	0.1977 (0.0421)	0.0633 (0.0270)	0.1936 (0.0291)	0.0604 (0.0171)
0.65	**0.0849 (0.0096)**	**0.0107 (0.0026)**	0.3078 (0.0316)	0.1692 (0.0365)	0.1407 (0.0284)	0.0370 (0.0154)	0.1734 (0.0416)	0.0519 (0.0252)	0.1687 (0.0288)	0.0488 (0.0157)
0.70	**0.0570 (0.0146)**	**0.0056 (0.0029)**	0.3124 (0.0311)	0.1737 (0.0386)	0.1448 (0.0292)	0.0406 (0.0165)	0.1675 (0.0381)	0.0500 (0.0234)	0.1726 (0.0262)	0.0519 (0.0156)
0.75	**0.1185 (0.0100)**	**0.0203 (0.0036)**	0.3458 (0.0314)	0.2077 (0.0448)	0.1933 (0.0311)	0.0664 (0.0223)	0.1987 (0.0348)	0.0668 (0.0241)	0.2215 (0.0250)	0.0817 (0.0192)
0.80	**0.2213 (0.0160)**	**0.0666 (0.0092)**	0.4249 (0.0331)	0.3018 (0.0608)	0.2863 (0.0356)	0.1351 (0.0380)	0.2749 (0.0347)	0.1172 (0.0305)	0.3146 (0.0290)	0.1570 (0.0304)
0.85	**0.3556 (0.0225)**	**0.1742 (0.0220)**	0.5711 (0.0389)	0.5266 (0.0950)	0.4315 (0.0467)	0.2953 (0.0786)	0.3991 (0.0362)	0.2304 (0.0434)	0.4536 (0.0340)	0.3147 (0.0500)
0.90	0.6111 (0.0667)	0.5795 (0.1702)	0.8160 (0.0479)	1.0372 (0.1576)	0.6638 (0.0655)	0.6781 (0.1760)	**0.5981 (0.0385)**	**0.4924 (0.0635)**	0.6694 (0.0389)	0.6634 (0.0824)
0.95	1.3183 (0.2442)	2.8061 (1.0124)	1.2810 (0.0705)	2.4289 (0.2941)	1.1416 (0.0979)	1.8775 (0.3767)	**0.9946 (0.0418)**	**1.3198 (0.0931)**	1.0973 (0.0515)	1.7257 (0.1588)
1st Count	15	14	0	0	0	0	3	4	1	1
2nd Count	1	2	0	0	11	10	5	5	2	2

Table 39: SIM, Exp(1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.2721 (0.0116)**	0.1252 (0.0105)	0.4208 (0.0415)	0.2930 (0.0653)	0.2748 (0.0128)	**0.1091 (0.0109)**	0.3469 (0.0539)	0.1975 (0.0546)	0.2745 (0.0118)	0.1197 (0.0110)
0.10	**0.2478 (0.0091)**	0.0991 (0.0075)	0.4201 (0.0451)	0.2933 (0.0695)	0.2624 (0.0130)	0.1003 (0.0109)	0.2638 (0.0133)	0.1099 (0.0143)	0.2580 (0.0110)	**0.0987 (0.0086)**
0.15	**0.2282 (0.0078)**	**0.0799 (0.0066)**	0.4125 (0.0466)	0.2846 (0.0703)	0.2499 (0.0135)	0.0919 (0.0110)	0.2451 (0.0104)	0.0885 (0.0079)	0.2513 (0.0133)	0.0910 (0.0101)
0.20	**0.2115 (0.0068)**	**0.0655 (0.0058)**	0.4009 (0.0470)	0.2714 (0.0690)	0.2373 (0.0143)	0.0839 (0.0113)	0.2366 (0.0130)	0.0801 (0.0090)	0.2465 (0.0153)	0.0875 (0.0116)
0.25	**0.1963 (0.0066)**	**0.0546 (0.0049)**	0.3881 (0.0470)	0.2569 (0.0675)	0.2248 (0.0154)	0.0765 (0.0119)	0.2299 (0.0165)	0.0754 (0.0114)	0.2411 (0.0172)	0.0848 (0.0127)
0.30	**0.1813 (0.0071)**	**0.0457 (0.0042)**	0.3740 (0.0467)	0.2413 (0.0651)	0.2125 (0.0168)	0.0698 (0.0127)	0.2219 (0.0190)	0.0711 (0.0129)	0.2323 (0.0202)	0.0802 (0.0150)
0.35	**0.1657 (0.0081)**	**0.0380 (0.0040)**	0.3598 (0.0465)	0.2259 (0.0630)	0.2008 (0.0188)	0.0641 (0.0138)	0.2126 (0.0202)	0.0663 (0.0132)	0.2215 (0.0219)	0.0744 (0.0158)
0.40	**0.1491 (0.0090)**	**0.0310 (0.0039)**	0.3458 (0.0459)	0.2107 (0.0606)	0.1904 (0.0214)	0.0596 (0.0153)	0.2006 (0.0222)	0.0604 (0.0136)	0.2077 (0.0222)	0.0670 (0.0150)
0.45	**0.1311 (0.0098)**	**0.0245 (0.0038)**	0.3326 (0.0450)	0.1966 (0.0577)	0.1820 (0.0246)	0.0567 (0.0172)	0.1874 (0.0236)	0.0540 (0.0137)	0.1916 (0.0226)	0.0586 (0.0148)
0.50	**0.1127 (0.0105)**	**0.0187 (0.0036)**	0.3213 (0.0441)	0.1844 (0.0551)	0.1767 (0.0283)	0.0558 (0.0196)	0.1728 (0.0244)	0.0473 (0.0135)	0.1757 (0.0246)	0.0510 (0.0153)
0.55	**0.0960 (0.0121)**	**0.0142 (0.0036)**	0.3137 (0.0426)	0.1754 (0.0525)	0.1759 (0.0323)	0.0573 (0.0227)	0.1589 (0.0254)	0.0414 (0.0134)	0.1614 (0.0265)	0.0445 (0.0153)
0.60	**0.0863 (0.0147)**	**0.0118 (0.0040)**	0.3116 (0.0411)	0.1711 (0.0502)	0.1808 (0.0364)	0.0621 (0.0263)	0.1527 (0.0252)	0.0390 (0.0132)	0.1539 (0.0284)	0.0412 (0.0159)
0.65	**0.0896 (0.0175)**	**0.0129 (0.0051)**	0.3180 (0.0395)	0.1744 (0.0481)	0.1930 (0.0396)	0.0709 (0.0307)	0.1566 (0.0249)	0.0411 (0.0135)	0.1580 (0.0284)	0.0428 (0.0162)
0.70	**0.1105 (0.0196)**	**0.0199 (0.0074)**	0.3357 (0.0385)	0.1889 (0.0475)	0.2135 (0.0413)	0.0852 (0.0356)	0.1754 (0.0243)	0.0503 (0.0143)	0.1783 (0.0265)	0.0523 (0.0166)
0.75	**0.1512 (0.0216)**	**0.0368 (0.0118)**	0.3685 (0.0383)	0.2202 (0.0482)	0.2439 (0.0408)	0.1070 (0.0406)	0.2106 (0.0244)	0.0702 (0.0165)	0.2160 (0.0250)	0.0736 (0.0180)
0.80	**0.2142 (0.0245)**	**0.0736 (0.0201)**	0.4222 (0.0401)	0.2798 (0.0531)	0.2877 (0.0380)	0.1412 (0.0448)	0.2632 (0.0258)	0.1061 (0.0208)	0.2701 (0.0246)	0.1108 (0.0208)
0.85	**0.3095 (0.0305)**	**0.1573 (0.0392)**	0.5051 (0.0451)	0.3886 (0.0662)	0.3534 (0.0338)	0.1997 (0.0470)	0.3353 (0.0287)	0.1669 (0.0282)	0.3442 (0.0273)	0.1750 (0.0287)
0.90	0.4654 (0.0448)	0.3717 (0.0968)	0.6341 (0.0571)	0.5953 (0.0991)	0.4662 (0.0362)	0.3233 (0.0537)	**0.4348 (0.0318)**	**0.2723 (0.0398)**	0.4500 (0.0304)	0.2920 (0.0412)
0.95	0.7438 (0.0806)	0.9715 (0.2856)	0.8561 (0.0851)	1.0582 (0.1911)	0.7172 (0.0641)	0.7212 (0.1262)	**0.6035 (0.0383)**	**0.5092 (0.0597)**	0.6382 (0.0349)	0.5786 (0.0679)
1st Count	17	15	0	0	0	1	2	2	0	1
2nd Count	0	1	0	0	5	3	10	12	4	3

Table 40: SIM, Exp(1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.2779 (0.0131)	0.1360 (0.0121)	0.3251 (0.0207)	0.1682 (0.0231)	0.2780 (0.0170)	0.1302 (0.0186)	0.2777 (0.0147)	0.1315 (0.0159)	**0.2726 (0.0133)**	**0.1253 (0.0146)**
0.10	**0.2444 (0.0092)**	0.0971 (0.0075)	0.3056 (0.0153)	0.1460 (0.0164)	0.2503 (0.0109)	0.0989 (0.0119)	0.2467 (0.0093)	0.0948 (0.0097)	0.2464 (0.0086)	**0.0937 (0.0080)**
0.15	**0.2230 (0.0068)**	**0.0731 (0.0045)**	0.2945 (0.0162)	0.1350 (0.0164)	0.2319 (0.0086)	0.0791 (0.0072)	0.2297 (0.0083)	0.0765 (0.0059)	0.2318 (0.0083)	0.0777 (0.0057)
0.20	**0.2085 (0.0057)**	**0.0588 (0.0027)**	0.2841 (0.0181)	0.1262 (0.0175)	0.2190 (0.0096)	0.0672 (0.0054)	0.2173 (0.0092)	0.0655 (0.0047)	0.2211 (0.0096)	0.0685 (0.0056)
0.25	**0.1972 (0.0053)**	**0.0504 (0.0022)**	0.2730 (0.0195)	0.1177 (0.0182)	0.2084 (0.0119)	0.0596 (0.0065)	0.2065 (0.0111)	0.0582 (0.0057)	0.2111 (0.0113)	0.0620 (0.0066)
0.30	**0.1866 (0.0053)**	**0.0445 (0.0022)**	0.2609 (0.0205)	0.1089 (0.0183)	0.1974 (0.0144)	0.0536 (0.0081)	0.1962 (0.0139)	0.0527 (0.0075)	0.2012 (0.0135)	0.0568 (0.0080)
0.35	**0.1742 (0.0054)**	**0.0392 (0.0023)**	0.2476 (0.0213)	0.0997 (0.0181)	0.1840 (0.0167)	0.0473 (0.0092)	0.1844 (0.0162)	0.0473 (0.0087)	0.1900 (0.0154)	0.0516 (0.0089)
0.40	**0.1586 (0.0053)**	**0.0330 (0.0022)**	0.2334 (0.0216)	0.0904 (0.0175)	0.1670 (0.0188)	0.0400 (0.0095)	0.1705 (0.0187)	0.0415 (0.0095)	0.1759 (0.0170)	0.0452 (0.0092)
0.45	**0.1383 (0.0052)**	**0.0256 (0.0019)**	0.2184 (0.0217)	0.0810 (0.0168)	0.1470 (0.0204)	0.0323 (0.0094)	0.1534 (0.0212)	0.0347 (0.0098)	0.1576 (0.0192)	0.0375 (0.0094)
0.50	**0.1128 (0.0050)**	**0.0174 (0.0016)**	0.2038 (0.0215)	0.0724 (0.0158)	0.1257 (0.0218)	0.0250 (0.0088)	0.1333 (0.0226)	0.0274 (0.0092)	0.1361 (0.0186)	0.0291 (0.0080)
0.55	**0.0825 (0.0050)**	**0.0095 (0.0012)**	0.1913 (0.0212)	0.0653 (0.0149)	0.1065 (0.0231)	0.0195 (0.0083)	0.1131 (0.0224)	0.0208 (0.0080)	0.1157 (0.0184)	0.0222 (0.0070)
0.60	**0.0510 (0.0057)**	**0.0038 (0.0009)**	0.1842 (0.0210)	0.0612 (0.0142)	0.0950 (0.0244)	0.0167 (0.0083)	0.0984 (0.0205)	0.0167 (0.0067)	0.1013 (0.0165)	0.0180 (0.0059)
0.65	**0.0376 (0.0065)**	**0.0024 (0.0008)**	0.1861 (0.0208)	0.0619 (0.0140)	0.0988 (0.0249)	0.0182 (0.0092)	0.0986 (0.0187)	0.0169 (0.0061)	0.1025 (0.0146)	0.0186 (0.0053)
0.70	**0.0736 (0.0052)**	**0.0079 (0.0012)**	0.2018 (0.0206)	0.0704 (0.0145)	0.1214 (0.0253)	0.0257 (0.0116)	0.1196 (0.0195)	0.0237 (0.0074)	0.1264 (0.0142)	0.0268 (0.0059)
0.75	**0.1294 (0.0062)**	**0.0226 (0.0022)**	0.2340 (0.0208)	0.0906 (0.0164)	0.1606 (0.0258)	0.0420 (0.0157)	0.1596 (0.0216)	0.0394 (0.0104)	0.1702 (0.0159)	0.0458 (0.0085)
0.80	**0.1920 (0.0086)**	**0.0485 (0.0043)**	0.2856 (0.0216)	0.1291 (0.0199)	0.2156 (0.0263)	0.0723 (0.0217)	0.2145 (0.0223)	0.0673 (0.0138)	0.2270 (0.0164)	0.0781 (0.0116)
0.85	**0.2624 (0.0106)**	**0.0906 (0.0063)**	0.3604 (0.0229)	0.1981 (0.0257)	0.2917 (0.0277)	0.1284 (0.0315)	0.2856 (0.0220)	0.1145 (0.0171)	0.3005 (0.0163)	0.1324 (0.0151)
0.90	**0.3641 (0.0128)**	**0.1884 (0.0198)**	0.4694 (0.0246)	0.3259 (0.0348)	0.4039 (0.0311)	0.2396 (0.0496)	0.3854 (0.0215)	0.2034 (0.0207)	0.4044 (0.0169)	0.2344 (0.0206)
0.95	0.6063 (0.0632)	0.5945 (0.1495)	0.6487 (0.0301)	0.6042 (0.0564)	0.6106 (0.0423)	0.5219 (0.0824)	**0.5547 (0.0207)**	**0.4222 (0.0301)**	0.5816 (0.0194)	0.4825 (0.0350)
1st Count	17	16	0	0	0	0	1	1	1	2
2nd Count	0	0	0	0	6	7	11	13	2	1

Table 41: Add, N(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.7421 (0.1029)**	**1.0513 (0.3626)**	1.6722 (0.1660)	4.1147 (1.1986)	0.8962 (0.1652)	1.2995 (0.4275)	0.8562 (0.1227)	1.1860 (0.3318)	1.2725 (0.2145)	2.4423 (0.6896)
0.10	**0.6997 (0.0981)**	**0.9025 (0.2833)**	1.3858 (0.1332)	2.8976 (0.5245)	0.7283 (0.1165)	0.9177 (0.2884)	0.9312 (0.1216)	1.3859 (0.3281)	1.1240 (0.1796)	1.9756 (0.5462)
0.15	0.6646 (0.0881)	0.7783 (0.2189)	1.2168 (0.1196)	2.3014 (0.4037)	**0.6625 (0.0930)**	**0.7729 (0.2256)**	0.9353 (0.1154)	1.4086 (0.3087)	1.0245 (0.1433)	1.6777 (0.4155)
0.20	0.6388 (0.0801)	**0.6905 (0.1753)**	1.0968 (0.1019)	1.9233 (0.3266)	**0.6296 (0.0804)**	0.6992 (0.1908)	0.9110 (0.1044)	1.3518 (0.2784)	0.9569 (0.1188)	1.4842 (0.3330)
0.25	0.6276 (0.0772)	**0.6451 (0.1543)**	1.0102 (0.0935)	1.6637 (0.2894)	**0.6110 (0.0731)**	0.6562 (0.1697)	0.8841 (0.0913)	1.2824 (0.2435)	0.9055 (0.1016)	1.3447 (0.2808)
0.30	0.6298 (0.0795)	0.6348 (0.1531)	0.9461 (0.0789)	1.4760 (0.2384)	**0.5997 (0.0685)**	**0.6296 (0.1563)**	0.8602 (0.0813)	1.2198 (0.2160)	0.8655 (0.0853)	1.2395 (0.2333)
0.35	0.6395 (0.0841)	0.6460 (0.1615)	0.9003 (0.0685)	1.3441 (0.2050)	**0.5928 (0.0657)**	**0.6132 (0.1478)**	0.8419 (0.0706)	1.1694 (0.1884)	0.8373 (0.0720)	1.1664 (0.1957)
0.40	0.6504 (0.0877)	0.6643 (0.1698)	0.8693 (0.0583)	1.2552 (0.1734)	**0.5889 (0.0641)**	**0.6041 (0.1428)**	0.8281 (0.0631)	1.1315 (0.1691)	0.8194 (0.0650)	1.1218 (0.1761)
0.45	0.6571 (0.0885)	0.6779 (0.1734)	0.8521 (0.0500)	1.2042 (0.1488)	**0.5875 (0.0634)**	**0.6007 (0.1405)**	0.8186 (0.0600)	1.1056 (0.1590)	0.8091 (0.0599)	1.0982 (0.1626)
0.50	0.6564 (0.0868)	0.6805 (0.1709)	0.8476 (0.0445)	1.1882 (0.1324)	**0.5882 (0.0636)**	**0.6025 (0.1404)**	0.8172 (0.0584)	1.1009 (0.1552)	0.8062 (0.0589)	1.0932 (0.1598)
0.55	0.6487 (0.0829)	0.6715 (0.1628)	0.8557 (0.0423)	1.2065 (0.1235)	**0.5911 (0.0645)**	**0.6091 (0.1424)**	0.8208 (0.0629)	1.1107 (0.1692)	0.8112 (0.0618)	1.1076 (0.1677)
0.60	0.6365 (0.0774)	0.6549 (0.1511)	0.8756 (0.0446)	1.2576 (0.1268)	**0.5963 (0.0657)**	**0.6206 (0.1460)**	0.8289 (0.0699)	1.1307 (0.1878)	0.8246 (0.0666)	1.1452 (0.1806)
0.65	0.6236 (0.0709)	**0.6374 (0.1393)**	0.9092 (0.0525)	1.3480 (0.1465)	**0.6038 (0.0671)**	0.6374 (0.1510)	0.8462 (0.0765)	1.1758 (0.2060)	0.8447 (0.0742)	1.2008 (0.2024)
0.70	0.6152 (0.0653)	**0.6280 (0.1323)**	0.9579 (0.0612)	1.4812 (0.1722)	**0.6138 (0.0686)**	0.6603 (0.1570)	0.8735 (0.0887)	1.2493 (0.2434)	0.8753 (0.0841)	1.2850 (0.2313)
0.75	**0.6159 (0.0647)**	**0.6357 (0.1357)**	1.0252 (0.0754)	1.6734 (0.2144)	0.6272 (0.0698)	0.6906 (0.1635)	0.9112 (0.1043)	1.3517 (0.2896)	0.9172 (0.0973)	1.4012 (0.2713)
0.80	**0.6301 (0.0703)**	**0.6684 (0.1532)**	1.1135 (0.0887)	1.9357 (0.2622)	0.6452 (0.0707)	0.7314 (0.1698)	0.9606 (0.1187)	1.4899 (0.3403)	0.9707 (0.1116)	1.5526 (0.3160)
0.85	**0.6575 (0.0802)**	**0.7301 (0.1864)**	1.2334 (0.1102)	2.3126 (0.3397)	0.6719 (0.0723)	0.7906 (0.1761)	1.0282 (0.1401)	1.6854 (0.4116)	1.0407 (0.1273)	1.7577 (0.3673)
0.90	**0.6905 (0.0924)**	**0.8157 (0.2372)**	1.4097 (0.1249)	2.9287 (0.4314)	0.7194 (0.0805)	0.8941 (0.1905)	1.1206 (0.1654)	1.9643 (0.5102)	1.1383 (0.1503)	2.0593 (0.4571)
0.95	**0.7201 (0.0981)**	**0.9080 (0.2929)**	1.7037 (0.1682)	4.2140 (1.2796)	0.8415 (0.1195)	1.1655 (0.2700)	1.2650 (0.2049)	2.4304 (0.6805)	1.2936 (0.1813)	2.5704 (0.5942)
1st Count	7	11	0	0	12	8	0	0	0	0
2nd Count	12	8	0	0	6	10	1	1	0	0

Table 42: Add, N(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.3761 (0.0852)**	**0.2530 (0.1177)**	1.1227 (0.1237)	2.5602 (1.1393)	0.4240 (0.1030)	0.3286 (0.1647)	0.5755 (0.0866)	0.5569 (0.1595)	0.6125 (0.0770)	0.6477 (0.1601)
0.10	**0.2933 (0.0561)**	**0.1506 (0.0630)**	0.9419 (0.0788)	1.4975 (0.2364)	0.3610 (0.0734)	0.2374 (0.1036)	0.5330 (0.0660)	0.5018 (0.1235)	0.5847 (0.0689)	0.5901 (0.1396)
0.15	**0.2445 (0.0376)**	**0.1017 (0.0352)**	0.8608 (0.0715)	1.2465 (0.1931)	0.3336 (0.0547)	0.1993 (0.0675)	0.5266 (0.0577)	0.4921 (0.1102)	0.5717 (0.0611)	0.5630 (0.1200)
0.20	**0.2214 (0.0278)**	**0.0808 (0.0219)**	0.8122 (0.0656)	1.1207 (0.1709)	0.3197 (0.0451)	0.1802 (0.0499)	0.5157 (0.0553)	0.4739 (0.1042)	0.5580 (0.0557)	0.5387 (0.1085)
0.25	**0.2113 (0.0233)**	**0.0723 (0.0162)**	0.7766 (0.0599)	1.0304 (0.1518)	0.3125 (0.0403)	0.1700 (0.0419)	0.5098 (0.0552)	0.4636 (0.1037)	0.5445 (0.0501)	0.5157 (0.0978)
0.30	**0.2093 (0.0230)**	**0.0707 (0.0154)**	0.7506 (0.0549)	0.9662 (0.1362)	0.3088 (0.0386)	0.1643 (0.0385)	0.5084 (0.0534)	0.4594 (0.1009)	0.5383 (0.0473)	0.5052 (0.0918)
0.35	**0.2140 (0.0247)**	**0.0739 (0.0173)**	0.7316 (0.0510)	0.9206 (0.1250)	0.3069 (0.0384)	0.1610 (0.0374)	0.5086 (0.0523)	0.4580 (0.0984)	0.5329 (0.0452)	0.4971 (0.0868)
0.40	**0.2224 (0.0265)**	**0.0798 (0.0197)**	0.7179 (0.0486)	0.8887 (0.1179)	0.3058 (0.0387)	0.1590 (0.0372)	0.5102 (0.0536)	0.4599 (0.1028)	0.5301 (0.0437)	0.4939 (0.0845)
0.45	**0.2310 (0.0282)**	**0.0860 (0.0217)**	0.7097 (0.0472)	0.8712 (0.1139)	0.3052 (0.0392)	0.1579 (0.0373)	0.5128 (0.0521)	0.4615 (0.0992)	0.5294 (0.0435)	0.4947 (0.0843)
0.50	**0.2361 (0.0295)**	**0.0897 (0.0229)**	0.7066 (0.0475)	0.8660 (0.1144)	0.3056 (0.0400)	0.1580 (0.0380)	0.5185 (0.0537)	0.4690 (0.1033)	0.5302 (0.0447)	0.4987 (0.0870)
0.55	**0.2360 (0.0295)**	**0.0895 (0.0227)**	0.7089 (0.0485)	0.8738 (0.1177)	0.3072 (0.0413)	0.1596 (0.0395)	0.5219 (0.0540)	0.4734 (0.1042)	0.5321 (0.0451)	0.5046 (0.0889)
0.60	**0.2318 (0.0279)**	**0.0861 (0.0209)**	0.7161 (0.0504)	0.8932 (0.1246)	0.3106 (0.0437)	0.1632 (0.0427)	0.5279 (0.0553)	0.4817 (0.1069)	0.5369 (0.0461)	0.5159 (0.0916)
0.65	**0.2261 (0.0247)**	**0.0818 (0.0179)**	0.7292 (0.0532)	0.9266 (0.1351)	0.3161 (0.0475)	0.1693 (0.0481)	0.5357 (0.0564)	0.4937 (0.1106)	0.5415 (0.0479)	0.5271 (0.0962)
0.70	**0.2227 (0.0218)**	**0.0791 (0.0152)**	0.7489 (0.0564)	0.9760 (0.1478)	0.3240 (0.0528)	0.1783 (0.0563)	0.5449 (0.0595)	0.5079 (0.1153)	0.5507 (0.0504)	0.5471 (0.1043)
0.75	**0.2265 (0.0236)**	**0.0814 (0.0161)**	0.7744 (0.0603)	1.0409 (0.1639)	0.3343 (0.0593)	0.1908 (0.0677)	0.5545 (0.0616)	0.5244 (0.1206)	0.5601 (0.0531)	0.5672 (0.1127)
0.80	**0.2433 (0.0341)**	**0.0937 (0.0248)**	0.8095 (0.0659)	1.1312 (0.1852)	0.3473 (0.0662)	0.2075 (0.0821)	0.5671 (0.0649)	0.5452 (0.1277)	0.5738 (0.0579)	0.5967 (0.1249)
0.85	**0.2774 (0.0512)**	**0.1218 (0.0423)**	0.8579 (0.0716)	1.2612 (0.2080)	0.3645 (0.0729)	0.2303 (0.0991)	0.5804 (0.0692)	0.5693 (0.1375)	0.5902 (0.0628)	0.6307 (0.1373)
0.90	**0.3223 (0.0709)**	**0.1660 (0.0686)**	0.9387 (0.0796)	1.5397 (0.2662)	0.3942 (0.0825)	0.2683 (0.1191)	0.5983 (0.0758)	0.6025 (0.1529)	0.6110 (0.0684)	0.6758 (0.1540)
0.95	**0.3456 (0.0806)**	**0.1989 (0.0918)**	1.1157 (0.1287)	2.7960 (1.6391)	0.4868 (0.1234)	0.3853 (0.1724)	0.6220 (0.0866)	0.6498 (0.1776)	0.6503 (0.0779)	0.7572 (0.1808)
1st Count	19	19	0	0	0	0	0	0	0	0
2nd Count	0	0	0	0	19	19	0	0	0	0

Table 43: Add, LN(0,1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.8967 (0.2322)	1.3205 (0.5782)	1.9527 (0.5760)	31.5138 (33.3949)	**0.6016 (0.1166)**	1.8734 (10.4859)	0.6709 (0.0637)	**0.7541 (0.1498)**	1.0087 (0.1870)	2.3731 (1.0020)
0.10	0.8039 (0.2049)	**1.0715 (0.4753)**	1.5986 (0.2736)	11.3456 (10.6298)	**0.5981 (0.1125)**	1.8218 (9.9116)	0.7854 (0.1132)	1.1858 (0.3756)	1.1110 (0.1941)	2.9638 (1.1574)
0.15	0.7375 (0.1834)	**0.9047 (0.4054)**	1.4043 (0.1975)	5.3943 (3.8834)	**0.5992 (0.1101)**	1.7925 (9.3787)	0.9191 (0.1529)	1.7933 (0.6406)	1.1663 (0.2001)	3.3549 (1.3068)
0.20	0.7017 (0.1622)	**0.8110 (0.3480)**	1.3770 (0.1765)	4.5822 (1.8864)	**0.6051 (0.1098)**	1.7865 (8.8941)	1.0034 (0.1790)	2.3350 (0.9224)	1.1988 (0.2035)	3.6415 (1.4521)
0.25	0.6925 (0.1391)	**0.7746 (0.2947)**	1.3619 (0.1722)	4.5185 (1.7151)	**0.6158 (0.1120)**	1.8055 (8.4648)	1.0537 (0.1957)	2.7648 (1.1852)	1.2177 (0.2019)	3.8338 (1.5135)
0.30	0.7025 (0.1154)	**0.7802 (0.2448)**	1.3416 (0.1703)	4.5504 (1.8880)	**0.6316 (0.1166)**	1.8511 (8.0885)	1.0841 (0.2038)	3.0818 (1.3708)	1.2303 (0.2025)	3.9818 (1.5777)
0.35	0.7240 (0.0940)	**0.8150 (0.2019)**	1.3185 (0.1732)	4.5349 (2.0668)	**0.6525 (0.1233)**	1.9265 (7.7644)	1.1049 (0.2085)	3.3404 (1.5567)	1.2389 (0.2045)	4.1220 (1.7101)
0.40	0.7510 (0.0786)	**0.8695 (0.1712)**	1.2870 (0.1662)	4.4392 (2.1642)	**0.6790 (0.1322)**	2.0364 (7.4940)	1.1095 (0.2106)	3.4981 (1.7316)	1.2420 (0.2030)	4.2109 (1.7782)
0.45	0.7797 (0.0702)	**0.9379 (0.1562)**	1.2576 (0.1606)	4.3176 (2.2199)	**0.7113 (0.1431)**	2.1876 (7.2803)	1.1223 (0.2133)	3.6515 (1.8662)	1.2454 (0.1987)	4.3054 (1.8926)
0.50	0.8096 (0.0684)	**1.0193 (0.1587)**	1.2314 (0.1547)	4.1733 (2.1973)	**0.7501 (0.1562)**	2.3892 (7.1288)	1.1327 (0.2067)	3.7490 (1.9472)	1.2496 (0.1942)	4.3723 (1.9970)
0.55	0.8433 (0.0738)	**1.1204 (0.1848)**	1.2115 (0.1474)	4.0356 (2.1783)	**0.7965 (0.1713)**	2.6538 (7.0481)	1.1393 (0.2059)	3.8170 (2.0507)	1.2593 (0.1907)	4.4420 (2.1318)
0.60	0.8852 (0.0900)	**1.2625 (0.2560)**	1.2052 (0.1387)	3.9134 (2.1392)	**0.8518 (0.1886)**	2.9985 (7.0543)	1.1544 (0.2022)	3.8789 (2.1195)	1.2778 (0.1834)	4.5163 (2.2669)
0.65	0.9447 (0.1211)	**1.4919 (0.4102)**	1.2171 (0.1297)	3.8206 (2.0987)	**0.9178 (0.2082)**	3.4464 (7.1703)	1.1829 (0.1967)	3.9540 (2.1847)	1.3144 (0.1755)	4.6367 (2.4553)
0.70	1.0373 (0.1692)	**1.9004 (0.6955)**	1.2619 (0.1204)	3.8157 (2.0524)	**0.9981 (0.2299)**	4.0287 (7.4284)	1.2356 (0.1866)	4.0686 (2.2613)	1.3764 (0.1676)	4.8083 (2.5786)
0.75	1.1860 (0.2300)	**2.6544 (1.1679)**	1.3594 (0.1116)	3.9534 (1.9896)	**1.0988 (0.2532)**	4.7891 (7.8727)	1.3272 (0.1761)	4.2843 (2.3233)	1.4786 (0.1585)	5.1185 (2.8030)
0.80	1.4166 (0.2984)	**4.0289 (1.8925)**	1.5416 (0.1080)	4.4067 (1.9171)	**1.2338 (0.2769)**	5.7940 (8.5544)	1.4870 (0.1736)	4.7206 (2.3936)	1.6543 (0.1538)	5.6864 (2.8911)
0.85	1.7499 (0.3742)	6.3586 (2.9483)	1.8737 (0.1149)	**5.5638 (1.8628)**	**1.4372 (0.2990)**	7.1796 (9.5261)	1.7609 (0.1791)	5.6673 (2.4821)	1.9538 (0.1577)	6.8662 (3.1398)
0.90	2.1680 (0.4669)	9.6271 (4.3362)	2.5148 (0.1409)	8.6327 (1.8319)	**1.8182 (0.3238)**	9.4150 (10.8127)	2.2827 (0.2278)	**8.0137 (2.6712)**	2.5208 (0.1949)	9.6967 (3.4630)
0.95	**2.6017 (0.5533)**	**12.3534 (5.2878)**	4.0362 (0.2201)	19.6013 (2.2543)	2.8959 (0.4160)	15.8010 (12.3299)	3.4943 (0.3540)	15.8739 (3.4771)	3.8594 (0.3089)	19.3363 (4.4700)
1st Count	1	16	0	1	18	0	2	2	0	0
2nd Count	16	1	0	4	1	12	2	2	0	0

Table 44: Add, LN(0,1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.2006 (0.0457)**	**0.0683 (0.0295)**	1.4296 (0.3670)	20.9273 (24.8255)	0.2879 (0.0645)	0.1558 (0.0754)	0.3141 (0.0487)	0.1704 (0.0549)	0.3355 (0.0358)	0.2080 (0.0581)
0.10	**0.1795 (0.0373)**	**0.0550 (0.0221)**	1.0475 (0.1440)	5.4924 (5.0492)	0.2815 (0.0605)	0.1457 (0.0637)	0.3238 (0.0496)	0.1999 (0.0682)	0.3648 (0.0445)	0.2727 (0.0993)
0.15	**0.1639 (0.0289)**	**0.0458 (0.0160)**	0.9196 (0.1045)	2.3285 (1.0360)	0.2784 (0.0582)	0.1406 (0.0573)	0.3530 (0.0518)	0.2790 (0.1151)	0.4040 (0.0543)	0.3641 (0.1474)
0.20	**0.1586 (0.0230)**	**0.0423 (0.0123)**	0.8969 (0.1062)	1.9676 (0.7039)	0.2790 (0.0580)	0.1402 (0.0561)	0.3923 (0.0608)	0.4000 (0.1750)	0.4479 (0.0651)	0.4819 (0.2094)
0.25	**0.1640 (0.0212)**	**0.0444 (0.0114)**	0.9095 (0.1055)	2.0364 (0.7135)	0.2830 (0.0594)	0.1437 (0.0586)	0.4318 (0.0694)	0.5292 (0.2421)	0.4925 (0.0734)	0.6167 (0.2784)
0.30	**0.1782 (0.0236)**	**0.0517 (0.0137)**	0.9153 (0.1071)	2.0931 (0.7556)	0.2894 (0.0614)	0.1503 (0.0630)	0.4710 (0.0812)	0.6966 (0.3692)	0.5396 (0.0811)	0.7726 (0.3454)
0.35	**0.1990 (0.0291)**	**0.0643 (0.0193)**	0.9140 (0.1068)	2.1101 (0.7721)	0.2972 (0.0635)	0.1590 (0.0687)	0.5072 (0.0852)	0.8600 (0.4401)	0.5837 (0.0857)	0.9338 (0.4100)
0.40	**0.2249 (0.0357)**	**0.0821 (0.0275)**	0.9113 (0.1037)	2.1167 (0.7876)	0.3056 (0.0654)	0.1694 (0.0755)	0.5453 (0.0924)	1.0286 (0.5074)	0.6294 (0.0907)	1.1066 (0.4740)
0.45	**0.2533 (0.0416)**	**0.1040 (0.0368)**	0.9053 (0.1017)	2.1079 (0.8101)	0.3141 (0.0674)	0.1816 (0.0838)	0.5821 (0.0972)	1.2012 (0.5966)	0.6755 (0.0954)	1.2923 (0.5384)
0.50	**0.2798 (0.0455)**	**0.1267 (0.0446)**	0.8971 (0.0972)	2.0814 (0.8189)	0.3235 (0.0699)	0.1967 (0.0952)	0.6220 (0.1053)	1.3775 (0.6855)	0.7269 (0.1005)	1.5030 (0.6052)
0.55	**0.2990 (0.0449)**	**0.1452 (0.0478)**	0.8946 (0.0930)	2.0727 (0.8349)	0.3357 (0.0739)	0.2175 (0.1120)	0.6692 (0.1061)	1.5905 (0.7853)	0.7795 (0.1026)	1.7219 (0.6718)
0.60	**0.3080 (0.0428)**	**0.1585 (0.0566)**	0.8934 (0.0875)	2.0447 (0.8406)	0.3534 (0.0809)	0.2488 (0.1388)	0.7172 (0.1112)	1.8032 (0.8724)	0.8367 (0.1078)	1.9641 (0.7535)
0.65	**0.3121 (0.0484)**	**0.1754 (0.1162)**	0.9034 (0.0823)	2.0369 (0.8255)	0.3803 (0.0924)	0.2991 (0.1842)	0.7820 (0.1160)	2.0535 (0.9736)	0.9049 (0.1131)	2.2512 (0.8445)
0.70	**0.3237 (0.0651)**	**0.2154 (0.2410)**	0.9290 (0.0768)	2.0583 (0.8164)	0.4215 (0.1111)	0.3837 (0.2666)	0.8551 (0.1186)	2.3452 (1.0897)	0.9861 (0.1183)	2.5938 (0.9487)
0.75	**0.3635 (0.0912)**	**0.3115 (0.4353)**	0.9831 (0.0755)	2.1518 (0.8099)	0.4854 (0.1414)	0.5335 (0.4238)	0.9529 (0.1300)	2.7216 (1.2395)	1.0852 (0.1212)	2.9879 (1.0375)
0.80	**0.4680 (0.1383)**	**0.5577 (0.7384)**	1.0862 (0.0799)	2.3857 (0.8329)	0.5864 (0.1894)	0.8131 (0.7317)	1.0852 (0.1335)	3.2429 (1.4179)	1.2200 (0.1263)	3.5401 (1.1624)
0.85	0.7557 (0.2552)	1.3777 (1.3243)	1.2713 (0.0959)	2.9119 (0.8647)	**0.7471 (0.2616)**	**1.3595 (1.3413)**	1.2716 (0.1447)	4.0039 (1.6264)	1.4227 (0.1337)	4.3673 (1.3108)
0.90	1.4289 (0.4536)	3.8019 (2.4676)	1.5911 (0.1165)	4.1526 (0.9482)	**1.0057 (0.3550)**	**2.4529 (2.5305)**	1.5695 (0.1672)	5.3195 (1.9058)	1.7649 (0.1464)	5.9051 (1.5078)
0.95	1.9271 (0.5216)	5.8475 (3.1392)	2.5543 (0.2113)	10.5286 (2.1681)	**1.5261 (0.4419)**	**4.8966 (4.5059)**	2.1828 (0.2445)	8.5389 (2.3379)	2.5300 (0.1930)	10.3441 (1.9391)
1st Count	16	16	0	0	3	3	0	0	0	0
2nd Count	3	3	0	0	16	16	0	0	0	0

Table 45: Add, Exp(1), N=512.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	0.5068 (0.1209)	0.4635 (0.2332)	1.2235 (0.3126)	7.6928 (8.6284)	**0.4683 (0.0649)**	**0.4126 (0.1491)**	0.5928 (0.0601)	0.5761 (0.1128)	0.6660 (0.1021)	0.8238 (0.2807)
0.10	0.4708 (0.0951)	**0.3935 (0.1641)**	1.0282 (0.1216)	2.5750 (1.1882)	**0.4701 (0.0675)**	0.4146 (0.1538)	0.6070 (0.0989)	0.6404 (0.2134)	0.7118 (0.1126)	0.9402 (0.3175)
0.15	**0.4457 (0.0751)**	**0.3459 (0.1178)**	0.9594 (0.1110)	1.7447 (0.4857)	0.4739 (0.0708)	0.4207 (0.1613)	0.6796 (0.1309)	0.8328 (0.3256)	0.7366 (0.1205)	1.0106 (0.3514)
0.20	**0.4361 (0.0633)**	**0.3244 (0.0933)**	0.9484 (0.1059)	1.6784 (0.3920)	0.4797 (0.0744)	0.4309 (0.1711)	0.7202 (0.1436)	0.9607 (0.3906)	0.7522 (0.1219)	1.0577 (0.3651)
0.25	**0.4464 (0.0610)**	**0.3327 (0.0886)**	0.9251 (0.1031)	1.6153 (0.3698)	0.4871 (0.0784)	0.4447 (0.1831)	0.7386 (0.1476)	1.0350 (0.4295)	0.7551 (0.1196)	1.0743 (0.3674)
0.30	**0.4774 (0.0669)**	**0.3719 (0.1003)**	0.9031 (0.1017)	1.5628 (0.3662)	0.4960 (0.0825)	0.4624 (0.1971)	0.7456 (0.1486)	1.0787 (0.4540)	0.7547 (0.1220)	1.0847 (0.3859)
0.35	0.5211 (0.0761)	**0.4360 (0.1213)**	0.8751 (0.0967)	1.4904 (0.3523)	**0.5064 (0.0869)**	0.4840 (0.2133)	0.7473 (0.1461)	1.1019 (0.4648)	0.7500 (0.1183)	1.0817 (0.3835)
0.40	0.5664 (0.0843)	0.5121 (0.1442)	0.8469 (0.0907)	1.4105 (0.3365)	**0.5182 (0.0912)**	**0.5100 (0.2319)**	0.7441 (0.1399)	1.1074 (0.4622)	0.7458 (0.1147)	1.0785 (0.3851)
0.45	0.6045 (0.0890)	0.5850 (0.1635)	0.8212 (0.0853)	1.3349 (0.3197)	**0.5316 (0.0957)**	**0.5410 (0.2530)**	0.7439 (0.1353)	1.1132 (0.4650)	0.7421 (0.1122)	1.0733 (0.3890)
0.50	0.6314 (0.0896)	0.6432 (0.1755)	0.7980 (0.0793)	1.2601 (0.3052)	**0.5466 (0.1003)**	**0.5776 (0.2770)**	0.7413 (0.1251)	1.1046 (0.4435)	0.7388 (0.1036)	1.0629 (0.3735)
0.55	0.6469 (0.0873)	0.6826 (0.1804)	0.7816 (0.0711)	1.1957 (0.2844)	**0.5637 (0.1047)**	**0.6209 (0.3040)**	0.7412 (0.1176)	1.1005 (0.4363)	0.7415 (0.0983)	1.0651 (0.3686)
0.60	0.6549 (0.0848)	0.7090 (0.1831)	0.7760 (0.0622)	1.1502 (0.2579)	**0.5832 (0.1087)**	**0.6724 (0.3342)**	0.7471 (0.1047)	1.0972 (0.4063)	0.7512 (0.0920)	1.0776 (0.3619)
0.65	0.6630 (0.0845)	0.7365 (0.1912)	0.7855 (0.0548)	1.1342 (0.2348)	**0.6057 (0.1119)**	**0.7338 (0.3676)**	0.7635 (0.0941)	1.1139 (0.3774)	0.7700 (0.0837)	1.1025 (0.3412)
0.70	0.6797 (0.0881)	**0.7857 (0.2133)**	0.8170 (0.0515)	1.1651 (0.2110)	**0.6328 (0.1136)**	0.8083 (0.4034)	0.7919 (0.0821)	1.1487 (0.3354)	0.8052 (0.0789)	1.1606 (0.3269)
0.75	0.7135 (0.0953)	**0.8821 (0.2568)**	0.8799 (0.0555)	1.2699 (0.1957)	**0.6674 (0.1134)**	0.9020 (0.4398)	0.8434 (0.0782)	1.2363 (0.3050)	0.8645 (0.0796)	1.2770 (0.3132)
0.80	0.7702 (0.1068)	1.0578 (0.3320)	0.9903 (0.0704)	1.5053 (0.2104)	**0.7169 (0.1113)**	**1.0281 (0.4731)**	0.9284 (0.0905)	1.4047 (0.2881)	0.9581 (0.0919)	1.4841 (0.3161)
0.85	0.8527 (0.1261)	1.3541 (0.4678)	1.1734 (0.0937)	1.9762 (0.2723)	**0.7980 (0.1125)**	**1.2237 (0.4981)**	1.0607 (0.1211)	1.7202 (0.3312)	1.1065 (0.1182)	1.8723 (0.3699)
0.90	0.9637 (0.1560)	1.8182 (0.7147)	1.4884 (0.1256)	2.9695 (0.4111)	**0.9585 (0.1402)**	**1.6170 (0.5278)**	1.2803 (0.1712)	2.3511 (0.4988)	1.3556 (0.1610)	2.6377 (0.5204)
0.95	**1.1447 (0.1852)**	**2.5337 (1.0403)**	2.1214 (0.1710)	5.5754 (0.7367)	1.3903 (0.2415)	2.8787 (0.7811)	1.7041 (0.2564)	3.8945 (0.9433)	1.8285 (0.2356)	4.4905 (0.9479)
1st Count	5	9	0	0	14	10	0	0	0	0
2nd Count	14	10	0	0	5	9	0	0	0	0

Table 46: Add, Exp(1), N=5120.

τ	CMQNet (Ours)		NQ-Net (2025)		DQRP (2024)		DQR* (2022)		DQR (2021)	
	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE
0.05	**0.1544 (0.0307)**	**0.0405 (0.0156)**	0.7487 (0.1606)	3.0757 (3.5399)	0.2318 (0.0547)	0.0988 (0.0472)	0.2652 (0.0338)	0.1207 (0.0311)	0.2654 (0.0319)	0.1206 (0.0274)
0.10	**0.1325 (0.0202)**	**0.0298 (0.0091)**	0.6335 (0.0606)	0.9656 (0.3500)	0.2267 (0.0509)	0.0931 (0.0413)	0.2728 (0.0378)	0.1362 (0.0385)	0.2809 (0.0300)	0.1409 (0.0330)
0.15	**0.1237 (0.0141)**	**0.0258 (0.0061)**	0.6141 (0.0619)	0.7605 (0.1650)	0.2248 (0.0491)	0.0906 (0.0386)	0.2892 (0.0419)	0.1609 (0.0476)	0.3023 (0.0372)	0.1666 (0.0444)
0.20	**0.1232 (0.0128)**	**0.0253 (0.0055)**	0.6169 (0.0642)	0.7569 (0.1594)	0.2254 (0.0487)	0.0905 (0.0381)	0.3088 (0.0457)	0.1905 (0.0585)	0.3232 (0.0423)	0.1934 (0.0542)
0.25	**0.1283 (0.0136)**	**0.0272 (0.0059)**	0.6184 (0.0643)	0.7595 (0.1577)	0.2279 (0.0494)	0.0923 (0.0391)	0.3280 (0.0501)	0.2221 (0.0721)	0.3428 (0.0473)	0.2205 (0.0651)
0.30	**0.1386 (0.0147)**	**0.0313 (0.0068)**	0.6143 (0.0620)	0.7509 (0.1531)	0.2317 (0.0504)	0.0953 (0.0407)	0.3463 (0.0517)	0.2535 (0.0822)	0.3618 (0.0497)	0.2485 (0.0746)
0.35	**0.1541 (0.0160)**	**0.0383 (0.0082)**	0.6083 (0.0597)	0.7372 (0.1493)	0.2363 (0.0513)	0.0992 (0.0426)	0.3670 (0.0543)	0.2919 (0.0970)	0.3823 (0.0549)	0.2804 (0.0881)
0.40	**0.1750 (0.0172)**	**0.0489 (0.0103)**	0.5988 (0.0562)	0.7150 (0.1422)	0.2413 (0.0520)	0.1039 (0.0444)	0.3842 (0.0530)	0.3233 (0.1035)	0.4038 (0.0575)	0.3141 (0.0987)
0.45	**0.1988 (0.0186)**	**0.0626 (0.0127)**	0.5894 (0.0526)	0.6924 (0.1359)	0.2467 (0.0522)	0.1092 (0.0461)	0.4061 (0.0569)	0.3607 (0.1167)	0.4242 (0.0576)	0.3481 (0.1065)
0.50	**0.2188 (0.0200)**	**0.0753 (0.0145)**	0.5818 (0.0484)	0.6710 (0.1277)	0.2529 (0.0520)	0.1153 (0.0478)	0.4263 (0.0547)	0.3967 (0.1208)	0.4466 (0.0590)	0.3847 (0.1167)
0.55	**0.2272 (0.0200)**	**0.0811 (0.0149)**	0.5758 (0.0445)	0.6511 (0.1197)	0.2606 (0.0518)	0.1231 (0.0497)	0.4520 (0.0570)	0.4379 (0.1330)	0.4712 (0.0588)	0.4250 (0.1237)
0.60	**0.2256 (0.0199)**	**0.0809 (0.0152)**	0.5754 (0.0406)	0.6392 (0.1114)	0.2707 (0.0521)	0.1336 (0.0526)	0.4765 (0.0561)	0.4761 (0.1352)	0.4975 (0.0596)	0.4691 (0.1318)
0.65	**0.2235 (0.0226)**	**0.0813 (0.0183)**	0.5825 (0.0390)	0.6392 (0.1048)	0.2849 (0.0539)	0.1486 (0.0580)	0.5079 (0.0561)	0.5255 (0.1442)	0.5283 (0.0609)	0.5217 (0.1414)
0.70	**0.2269 (0.0268)**	**0.0862 (0.0245)**	0.6008 (0.0396)	0.6583 (0.1002)	0.3047 (0.0586)	0.1707 (0.0681)	0.5429 (0.0573)	0.5790 (0.1449)	0.5634 (0.0602)	0.5820 (0.1451)
0.75	**0.2396 (0.0323)**	**0.0987 (0.0341)**	0.6347 (0.0442)	0.7071 (0.1028)	0.3327 (0.0673)	0.2047 (0.0868)	0.5850 (0.0587)	0.6453 (0.1498)	0.6107 (0.0610)	0.6670 (0.1557)
0.80	**0.2688 (0.0418)**	**0.1268 (0.0501)**	0.6913 (0.0532)	0.8049 (0.1155)	0.3732 (0.0819)	0.2589 (0.1207)	0.6383 (0.0658)	0.7354 (0.1572)	0.6701 (0.0642)	0.7831 (0.1727)
0.85	**0.3358 (0.0629)**	**0.2006 (0.0870)**	0.7821 (0.0656)	0.9898 (0.1465)	0.4319 (0.1027)	0.3481 (0.1788)	0.7051 (0.0766)	0.8599 (0.1729)	0.7521 (0.0651)	0.9560 (0.1798)
0.90	**0.4844 (0.1119)**	**0.4153 (0.1984)**	0.9380 (0.0782)	1.3819 (0.2065)	0.5198 (0.1279)	0.5013 (0.2702)	0.8007 (0.0990)	1.0644 (0.2222)	0.8761 (0.0772)	1.2536 (0.2059)
0.95	0.7218 (0.1937)	0.9239 (0.4947)	1.3144 (0.1292)	2.8268 (0.8907)	**0.7067 (0.1829)**	**0.8604 (0.4227)**	0.9618 (0.1433)	1.4789 (0.3657)	1.0947 (0.1022)	1.8904 (0.2914)
1st Count	18	18	0	0	1	1	0	0	0	0
2nd Count	1	1	0	0	18	18	0	0	0	0