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1 Introduction

Theorem 1.1 (Estimation Error Bound for Regularized Parameter Estimator)**.** *Let $\lambda > 0$ be the regularization parameter, $d \in \mathbb{N}$ the dimension of the parameter vectors, and $\delta \in (0,1)$ the confidence level. Assume that the drift term $f(x, u(x))$ satisfies the linear assumption* **??**. *For each iteration $n = 1, \ldots, n$ and time step $j = 1, \ldots, \tau_n$, define*

$$F_{n,j} = (\hat{t}_{n,j} - \hat{t}_{n,j-1}) F(x_n(\hat{t}_{n,j}), u_n(x_n(\hat{t}_{n,j})))$$

where $\hat{t}_{n,j}$ is the time j-th sampling at the n-th episode, and set the weighting factor where the matrix V_N is given by

$$V_N = \sum_{n=1}^{N} \sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top F_{n,j} + \lambda I,$$

Utilizing these definitions, construct the estimator θ_n for the true parameter θ^* as

$$\theta_n = V_N^{-1} \sum_{n=1}^{N} \sum_{j=1}^{\tau_n} F_{n,j}^\top \left(x_n(\hat{t}_{n,j}) - x_n(\hat{t}_{n,j-1}) \right),$$

with I being the identity matrix. Then under the assumption that the sampling interval $\forall n \in [N] and j \in [\tau_n]$, the sampling interval $[t_{n,j-1}, t_{n,j}]$ is bounded by $\frac{1}{\max(n, \sqrt{n\tau_n})}$, then with probability at least $1 - \left(\log_2 \left(\sum_{n=1}^{N} \tau_n \right) + 2 \right) \delta$, the

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estimation error measured in the norm induced by $V_N^{1/2}$ satisfies

$$\|V_N^{\frac{1}{2}}(\theta_N - \theta^*)\|_2 \leq \frac{2\sqrt{2\varphi_{\hat{F}}}\log\left(\frac{d+1}{\delta}\right)}{3} + \sqrt{2\ln\left(\frac{d+1}{\delta}\right)\left\|\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}}\int g^*(x,u_n(x))g^*(x,u_n(x))^\top\pi^*(x,t|u_n)dx\right\|_2} \tag{1}$$

$$+ 8L_F T\sqrt{e^{(1+2(L_f+L_g)(1+L_u))}(\varphi_F^2+\sqrt{d}\varphi_{\hat{F}})}\left(2\left(1+\log\left(\sum_{n=1}^{N}\tau_n\right)\right)+\log\left(\frac{4}{\delta}\right)\right) \tag{2}$$

Proof. We begin by considering the estimation of the parameter θ^* in the drift term $f(x,u(x))$. Before proving this theorem, we first introduce two notations. In the following proof, we let

$$\bar{F}_{n,j} := \int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} F(x_n(t),u_n(x_n(t)))dt$$

Recall that the estimator θ_N is defined as

$$\theta_N = V_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top\left(x_n(\hat{t}_{n,j}) - x_n(\hat{t}_{n,j-1})\right), \tag{3}$$

To analyze the estimation error, we decompose $\theta_n - \theta^*$ as follows:

$$\theta_N - \theta^* = \left(V_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2\left[F_{n,j}^\top\left(x_n(\hat{t}_{n,j}) - x_n(\hat{t}_{n,j-1})\right)\right]\right) - \theta^*$$

$$= \left(V_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2\left[F_{n,j}^\top\left(x_n(\hat{t}_{n,j}) - x_n(\hat{t}_{n,j-1})\right)\right]\right) - \left(V_N^{-1}\left(\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top\bar{F}_{n,j} + \lambda I\right)\right)\theta^* \tag{4}$$

$$- \left(V_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2\left(F_{n,j}^\top(F_{n,j} - \bar{F}_{n,j})\right)\right)\theta^*. \tag{5}$$

Multiplying both sides by $V_N^{\frac{1}{2}}$ and taking the ℓ_2-norm, by triangle inequality

we obtain

$$\left\|V_N^{\frac{1}{2}}(4)\right\|_2 \leq \left(\left\|V_N^{-\frac{1}{2}}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top \left[\left(\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right)\right]\right\|_2 \tag{6}$$

$$+\left\|V_N^{-\frac{1}{2}}\lambda\theta^*\right\|_2\Big) \tag{7}$$

where $g^*(x_n(t),u_n(x_n(t)))$ is the diffusion term, and W_t is a standard Brownian motion.

Then we start by bounding

$$\left\|V_N^{-\frac{1}{2}}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top \left[\left(\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right)\right]\right\|_2 \tag{8}$$

Then we shall try to bound $\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\left\|\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right\|_2^2$
Note that

$$\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\left\|\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right\|_2^2 \leq \tag{9}$$

$$\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\mathrm{tr}\left(\left(\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right)\left(\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right)^\top\right) \tag{10}$$

Then by Lemma 1.5, with possibility $1-\delta$, there holds

$$\left\|\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right\|_2 \leq \sqrt{2\sqrt{d}\varphi_{\hat{F}}\log\left(\frac{2}{\delta}\right)} \tag{11}$$

Then

$$\left\|V_N^{-\frac{1}{2}} w_{n,j}^2 F_{n,j}^\top \left[\left(\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right)\right]\right\|_2 \leq \sqrt{2\sqrt{d}\varphi_{\hat{F}}\log\left(\frac{2}{\delta}\right)} \tag{12}$$

Then

$$\mathbb{E}\left[\left(V_N^{-\frac{1}{2}} w_{n,j}^2 F_{n,j}^\top \left[\left(\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right)\right]\right)\left(V_N^{-\frac{1}{2}} w_{n,j}^2 F_{n,j}^\top \left[\left(\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t),u_n(x_n(t)))\,dW_t\right)\right.\right. \tag{13}$$

$$\preceq \int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} \mathbb{E}[g^*(x_n(t),u_n(x_n(t)))g^*(x_n(t),u_n(x_n(t)))^\top]dt \tag{14}$$

which is due to $\|V_N^{-\frac{1}{2}}\ w_{n,j}^2 F_{n,j}^\top\| \leq 1$ Then by matrix freedman inequality, with possibility at least R,

$$\left\| V_N^{-\frac{1}{2}} \sum_{n=1}^{N} \sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top \left[\left(\int_{t_{n,j-1}}^{t_{n,j}} g^*(x_n(t), u_n(x_n(t)))\, dW_t \right) \right] \right\|_2 \tag{15}$$

$$\leq \sqrt{2 \left\| \sum_{n=1}^{N} \int_0^T g^*(x_n(t), u_n(x_n(t))) g^*(x_n(t), u_n(x_n(t)))^\top \pi^*(x, u_n, t) dt \right\|_2 \log\left(\frac{d+1}{\delta}\right)} \tag{16}$$

$$+ \frac{2\sqrt{2\sqrt{d}\varphi_{\hat{F}} \log\left(\frac{2}{\delta}\right)}}{3} \log\left(\frac{d+1}{\delta}\right) \tag{17}$$

Using Ito lemma, we have

$$d\mathbb{E}\left[\|x_n(t) - x_n(t_0)\|_2^2\right] \leq 2\mathbb{E}\left[(x_n(t) - x_n(t_0))^\top f^*(x_n(t), u(x_n(t)))\right] \tag{18}$$

$$+ 2\mathbb{E}\left[\mathrm{tr}\left(g^*(x_n(t), u(x_n(t))) g^*(x_n(t), u(x_n(t)))^\top\right)\right] \tag{19}$$

$$\leq \mathbb{E}\| \left[x_n(t) - x_n(t_0)\|_2^2 + \varphi_F^2\right] + 2\sqrt{d}\varphi_{\hat{F}} \tag{20}$$

Then given we have

$$\mathbb{E}\left[\|x_n(t) - x_n(t_0)\|_2^2\right] \leq e^{(t-t_0)}(\varphi_F^2 + 2\sqrt{d}\varphi_{\hat{F}})(t - t_0) \tag{21}$$

Note that $\forall n, j, t_{n,j} - t_{n,j-1} \leq 1$, then for $t_{n,j-1} \leq t \leq t_{n,j}$,

$$\mathbb{E}\left[\|x_n(t) - x_n(\hat{t}_{n,j-1})\|_2^2\right] \leq e^2(\varphi_F^2 + 2\sqrt{d}\varphi_{\hat{F}})(t - \hat{t}_{n,j-1}) \tag{22}$$

Then utilizing the lipschitz condition of $F(x, u(x))$, we have

$$\|F_{n,j} - \bar{F}_{n,j}\|_2^2 \leq L_F^2(1 + L_u)^2 (\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} \|x_n(\hat{t}_{n,j}) - x_n(t)\|_2 dt)^2 \tag{23}$$

$$\leq L_F^2(1 + L_u)^2 \left(\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} \|x_n(\hat{t}_{n,j}) - x_n(t))\|_2^2 dt \right) (\hat{t}_{n,j} - \hat{t}_{n,j-1}) \tag{24}$$

Then using Lemma 1.6, with possibility at least $1 - \log_2(\sum_{n=1}^{N} \tau_n + 1)$, there holds

$$\sum_{n=1}^{N} \sum_{j=1}^{\tau_n} \|F_{n,j} - \bar{F}_{n,j}\|_2^2 \leq 32e^2(\varphi_F^2 + 2\sqrt{d}\varphi_{\hat{F}}) \sum_{n=1}^{N} \sum_{j=1}^{\tau_n} ((t_{n,j} - t_{n,j-1})^3 + (t_{n,j-1} - t_{n,j-2})^3) + 64\varphi_F^2 \log\left(\frac{4}{\delta}\right) \tag{25}$$

Then with probability at least $1-\left(\log_2\left(\sum_{n=1}^N \tau_n\right)+1+\sum_{n=1}^N \tau_n\right)\delta$, under assumption that $t_{n,j}-t_{n,j-1}\le \frac{1}{\max(n,\tau_n)}$, we have

$$\|V_N^{\frac{1}{2}}(\theta_N-\theta^*)\|_2 \le \sqrt{2\left\|\sum_{n=1}^N\int_0^T g^*(x_n(t),u_n(x_n(t)))g^*(x_n(t),u_n(x_n(t)))^\top \pi^*(x,u_n,t)dt\right\|_2 \log\left(\frac{d+1}{\delta}\right)} \tag{26}$$

$$+\frac{2\sqrt{2\sqrt{d}\varphi_{\hat{F}}\log\left(\frac{2}{\delta}\right)}}{3}\log\left(\frac{d+1}{\delta}\right) \tag{27}$$

$$+L_F^2(1+L_u)^2\left(32e^2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})F\sum_{j=1}^{\tau_n}((t_{n,j}-t_{n,j-1})^3+(t_{n,j-1}-t_{n,j-2})^3)+64\varphi_F^2\log\left(\frac{4}{\delta}\right.\right. \tag{28}$$

$$=\tilde{O}\left(\sqrt{\left\|\sum_{n=1}^N\int_0^T g^*(x_n(t),u_n(x_n(t)))g^*(x_n(t),u_n(x_n(t)))^\top \pi^*(x,u_n,t)dt\right\|_2}+d^{\frac{1}{4}}\right. \tag{29}$$

$$\left.+\ L_F^2(1+L_u)^2\left(\sqrt{d(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})}\sqrt{\sum_{n=1}^N\sum_{j=1}^{\tau_n}((t_{n,j}-t_{n,j-1})^3+(t_{n,j-1}-t_{n,j-2})^3)}+\varphi_F\right)\right) \tag{30}$$

□

Lemma 1.2 (Trace-exponential supermartingale and maximal inequality)**.** *Let $X_n\in\mathbb{S}^d$ be $\mathcal{F}_n$-measurable and satisfy*

$$0\preceq X_n\preceq I_d \qquad \textit{almost surely,} \qquad n\ge 1.$$

Define

$$\mu_n:=\mathbb{E}[X_n\mid\mathcal{F}_{n-1}], \qquad S_n:=\sum_{i=1}^n X_i, \qquad P_n:=\sum_{i=1}^n \mu_i,$$

with $S_0=P_0=0$.
For every fixed $\theta\in\mathbb{R}$, let

$$\alpha_\theta:=e^\theta-1, \qquad Z_n(\theta):=\operatorname{tr}\exp(\theta S_n-\alpha_\theta P_n).$$

Then $(Z_n(\theta))_{n\ge0}$ is a nonnegative scalar supermartingale with $Z_0(\theta)=d$. Moreover, for every $t>0$,

$$\boxed{\mathbb{P}(\exists n\ge1:\lambda_{\max}(\theta S_n-\alpha_\theta P_n)\ge t)\le de^{-t}.} \tag{A}$$

Proof. First, conditional expectation preserves the semidefinite order. Indeed, for every deterministic $v \in \mathbb{R}^d$,

$$v^\top \mu_n v = \mathbb{E}[v^\top X_n v \mid \mathcal{F}_{n-1}] \in [0, \|v\|_2^2] \quad \text{almost surely.}$$

Applying this to a countable dense collection of vectors and using continuity gives

$$0 \preceq \mu_n \preceq I_d \quad \text{almost surely.} \tag{1}$$

In particular, P_n is $\mathcal{F}_{n-1}$-measurable and

$$0 \preceq S_n, P_n \preceq nI_d. \tag{2}$$

We next establish the conditional matrix moment-generating-function bound. Convexity of $x \mapsto e^{\theta x}$ gives, for every real θ,

$$e^{\theta x} \leq (1-x)e^0 + xe^\theta = 1 + \alpha_\theta x, \qquad 0 \leq x \leq 1.$$

Diagonalizing X_n and applying this inequality to each eigenvalue yields

$$e^{\theta X_n} \preceq I_d + \alpha_\theta X_n.$$

Taking conditional expectations,

$$M_n(\theta) := \mathbb{E}[e^{\theta X_n} \mid \mathcal{F}_{n-1}] \preceq I_d + \alpha_\theta \mu_n. \tag{3}$$

Both matrices in (3) are positive definite. For example,

$$e^{\theta X_n} \succeq \min\{1, e^\theta\} I_d,$$

and hence the same lower bound holds for $M_n(\theta)$.

The matrix logarithm is operator monotone on the positive-definite cone. One way to see this is through

$$\log B - \log A = \int_0^\infty \left((A + sI_d)^{-1} - (B + sI_d)^{-1} \right) ds \succeq 0$$

whenever $0 \prec A \preceq B$, because inversion reverses the semidefinite order. Consequently, (3) implies

$$\log M_n(\theta) \preceq \log(I_d + \alpha_\theta \mu_n).$$

For every eigenvalue $u \in [0,1]$ of μ_n, we have $1 + \alpha_\theta u > 0$ and

$$\log(1 + \alpha_\theta u) \leq \alpha_\theta u.$$

Another application of spectral calculus therefore gives

$$\log \mathbb{E}[e^{\theta X_n} \mid \mathcal{F}_{n-1}] \preceq \alpha_\theta \mu_n. \tag{4}$$

This calculation includes negative θ.

The matrix-analysis input is **Lieb's concavity theorem**: for every fixed symmetric H, the function

$$A \longmapsto \operatorname{tr} \exp(H + \log A)$$

is concave on the positive-definite cone. Its conditional Jensen consequence is

$$\mathbb{E}[\operatorname{tr} e^{H+Y} \mid \mathcal{G}] \leq \operatorname{tr} \exp\bigl(H + \log \mathbb{E}[e^{Y} \mid \mathcal{G}]\bigr)\,, \tag{5}$$

when H is $\mathcal{G}$-measurable and the relevant quantities are integrable. See [**?**, Theorem 1.4 and Corollary 1.5].

We also need monotonicity of the **trace exponential**:

$$A \preceq B \quad \Longrightarrow \quad \operatorname{tr} e^{A} \leq \operatorname{tr} e^{B}. \tag{6}$$

For completeness, set $D = B - A \succeq 0$ and

$$h(s) := \operatorname{tr} e^{A+sD}, \qquad 0 \leq s \leq 1.$$

Differentiation of the matrix exponential, followed by cyclicity of the trace, gives

$$h'(s) = \operatorname{tr}\bigl(De^{A+sD}\bigr) = \operatorname{tr}\Bigl(D^{1/2} e^{A+sD} D^{1/2}\Bigr) \geq 0.$$

Thus $h(1) \geq h(0)$, proving (6).

We can now verify the supermartingale property. Fix θ, and abbreviate $\alpha = \alpha_\theta$. The matrix

$$H_n := \theta S_{n-1} - \alpha P_n$$

is $\mathcal{F}_{n-1}$-measurable. Applying (5), then (4) and (6), gives

$$\begin{aligned} \mathbb{E}[Z_n(\theta) \mid \mathcal{F}_{n-1}] &= \mathbb{E}[\operatorname{tr} \exp(H_n + \theta X_n) \mid \mathcal{F}_{n-1}] \\ &\leq \operatorname{tr} \exp\bigl(H_n + \log \mathbb{E}[e^{\theta X_n} \mid \mathcal{F}_{n-1}]\bigr) \\ &\leq \operatorname{tr} \exp(H_n + \alpha \mu_n) \\ &= \operatorname{tr} \exp(\theta S_{n-1} - \alpha P_{n-1}) \\ &= Z_{n-1}(\theta). \end{aligned} \tag{7}$$

The process is adapted and positive. Its integrability follows from (2):

$$0 < Z_n(\theta) \leq d \exp(n(|\theta| + |\alpha_\theta|))\,.$$

Also,

$$Z_0(\theta) = \operatorname{tr} I_d = d.$$

Finally, we prove the maximal inequality using bounded stopping times. Set

$$W_n := \theta S_n - \alpha_\theta P_n, \qquad \tau := \inf\{n \geq 1 : \lambda_{\max}(W_n) \geq t\},$$

with $\inf \varnothing = \infty$. Since W_n is adapted, τ is a stopping time.
For every integer $N \geq 1$,

$$Z_{\tau \wedge N} = Z_0 + \sum_{k=1}^{N} \mathbf{1}_{\{\tau \geq k\}} (Z_k - Z_{k-1}).$$

Because $\{\tau \geq k\} \in \mathcal{F}_{k-1}$, (7) implies

$$\begin{aligned} \mathbb{E}[Z_{\tau \wedge N}] &= d + \sum_{k=1}^{N} \mathbb{E}\big[\mathbf{1}_{\{\tau \geq k\}} \mathbb{E}[Z_k - Z_{k-1} \mid \mathcal{F}_{k-1}]\big] \\ &\leq d. \end{aligned} \tag{8}$$

On $\{\tau \leq N\}$, spectral decomposition gives

$$Z_{\tau \wedge N} = \operatorname{tr} e^{W_\tau} \geq e^{\lambda_{\max}(W_\tau)} \geq e^t.$$

Since $Z_{\tau \wedge N} \geq 0$ everywhere, (8) yields

$$e^t \mathbb{P}(\tau \leq N) \leq \mathbb{E}[Z_{\tau \wedge N}] \leq d.$$

Letting $N \to \infty$ and using continuity of probability for increasing events,

$$\mathbb{P}(\tau < \infty) \leq d e^{-t}.$$

This is exactly (A). □

Put

$$b := \log 2.$$

The choices $\theta = -b$ and $\theta = b$, for which

$$e^{-b} - 1 = -\tfrac{1}{2}, \qquad e^b - 1 = 1,$$

give the two consequences

$$\boxed{\mathbb{P}\big(\exists n \geq 1 : \lambda_{\max}\big(\tfrac{1}{2} P_n - b S_n\big) \geq t\big) \leq d e^{-t}} \tag{B}$$

and

$$\boxed{\mathbb{P}(\exists n \geq 1 : \lambda_{\max}(b S_n - P_n) \geq t) \leq d e^{-t}.} \tag{C}$$

Lemma 1.3 (Predictable second moments bounded by observed second moments)**.** *For every $\delta \in (0, 1)$,*

$$\boxed{\mathbb{P}\left(\forall n \geq 1 :\ V_n \preceq 2(\log 2) Q_n + 2R^2 \log \frac{d}{\delta}\, I_d\right) \geq 1 - \delta.} \tag{9-M}$$

Proof. Define

$$X_i := \frac{Y_i Y_i^\top}{R^2}.$$

For every $v \in \mathbb{R}^d$,

$$v^\top X_i v = \frac{\|Y_i^\top v\|_2^2}{R^2} \le \frac{\|Y_i\|_{\mathrm{op}}^2 \|v\|_2^2}{R^2} \le \|v\|_2^2.$$

Thus $0 \preceq X_i \preceq I_d$, so the auxiliary lemma applies. Its partial sums are

$$S_n = \frac{Q_n}{R^2}, \qquad P_n = \frac{V_n}{R^2}.$$

Substitution into (B) gives, for every $t > 0$,

$$\mathbb{P}\left(\exists n \ge 1 : \lambda_{\max}\left(\frac{V_n}{2R^2} - \frac{bQ_n}{R^2} \right) \ge t \right) \le d e^{-t}.$$

Since multiplication by a positive scalar scales all eigenvalues,

$$\mathbb{P}\big(\exists n \ge 1 : \lambda_{\max}(V_n - 2bQ_n) \ge 2R^2 t \big) \le d e^{-t}. \tag{10}$$

Take

$$t = \log \frac{d}{\delta} > 0.$$

Then $d e^{-t} = \delta$. Outside the exceptional event in (10), every n satisfies

$$\lambda_{\max}(V_n - 2bQ_n) < 2R^2 \log \frac{d}{\delta}.$$

For a symmetric matrix H, the condition $\lambda_{\max}(H) < a$ implies $H \preceq a I_d$. Hence

$$V_n - 2bQ_n \preceq 2R^2 \log \frac{d}{\delta}\, I_d$$

simultaneously for all n. Rearranging proves (9-M). □

Lemma 1.4 (Observed second moments bounded by predictable second moments)**.** *Under the same assumptions,*

$$\boxed{\mathbb{P}\left(\forall n \ge 1 :\ Q_n \preceq \frac{1}{\log 2} V_n + \frac{R^2}{\log 2} \log \frac{d}{\delta}\, I_d \right) \ge 1 - \delta.} \tag{10-M}$$

Proof. Again apply the auxiliary lemma to

$$X_i = \frac{Y_i Y_i^\top}{R^2}.$$

This time (C) yields

$$\mathbb{P}\Big(\exists n \geq 1 : \lambda_{\max}\Big(\frac{bQ_n - V_n}{R^2}\Big) \geq t\Big) \leq de^{-t}.$$

Equivalently,

$$\mathbb{P}\big(\exists n \geq 1 : \lambda_{\max}(bQ_n - V_n) \geq R^2 t\big) \leq de^{-t}. \tag{11}$$

Taking $t = \log(d/\delta)$, we obtain, with probability at least $1 - \delta$,

$$bQ_n - V_n \preceq R^2 \log\frac{d}{\delta}\, I_d \qquad \text{for every } n \geq 1.$$

Dividing by $b = \log 2 > 0$ proves (10-M). □

Lemma 1.5 (Bounds for the diffusion integral)**.** *Let $T > 0$, and let $(x_t)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued solution of*

$$dx_t = f(x_t, u_t)\, dt + g(x_t, u_t)\, dW_t,$$

on a filtered probability space satisfying the usual conditions, where W_t is an m-dimensional Brownian motion. Suppose that

$$G_t := g(x_t, u_t) \in \mathbb{R}^{d\times m}$$

is progressively measurable and that, almost surely, for almost every $t \in [0, T]$,

$$\big\|\mathrm{vec}(G_t G_t^\top)\big\|_2 \leq a$$

for a deterministic constant $a > 0$. Define

$$M_t := \int_0^t G_s\, dW_s.$$

Then

$$\mathbb{E}\|M_T\|_2^2 \leq \sqrt{d}\, aT. \tag{31}$$

Moreover, for every $r > 0$,

$$\mathbb{P}\left(\sup_{0\leq t\leq T} \|M_t\|_2 \geq r\right) \leq 2\exp\left(-\frac{r^2}{2\sqrt{d}\, aT}\right). \tag{32}$$

Consequently, for every $\delta \in (0, 1)$, with probability at least $1 - \delta$,

$$\sup_{0\leq t\leq T} \left\|\int_0^t g(x_s, u_s)\, dW_s\right\|_2 \leq \sqrt{2\sqrt{d}\, aT \log\left(\frac{2}{\delta}\right)}. \tag{33}$$

Proof. By the identity between the Euclidean norm of a vectorized matrix and its Frobenius norm,

$$\|G_t G_t^\top\|_{\mathrm{F}} = \left\|\mathrm{vec}(G_t G_t^\top)\right\|_2 \le a.$$

Hence, the Cauchy–Schwarz inequality for the Frobenius inner product gives

$$\begin{aligned}\mathrm{tr}(G_t G_t^\top) &= \langle I_d, G_t G_t^\top\rangle_{\mathrm{F}} \\ &\le \|I_d\|_{\mathrm{F}}\, \|G_t G_t^\top\|_{\mathrm{F}} \\ &\le \sqrt{d}\, a. \end{aligned} \tag{34}$$

In particular,

$$\mathbb{E}\int_0^T \|G_s\|_{\mathrm{F}}^2\, ds = \mathbb{E}\int_0^T \mathrm{tr}(G_s G_s^\top)\, ds \le \sqrt{d}\, aT < \infty,$$

so M_t is a continuous square-integrable martingale. Its matrix quadratic variation is

$$Q_t := \langle M\rangle_t = \int_0^t G_s G_s^\top\, ds.$$

Define

$$A_t := \mathrm{tr}(Q_t) = \int_0^t \mathrm{tr}(G_s G_s^\top)\, ds.$$

By (34), almost surely,

$$A_t \le \sqrt{d}\, at, \qquad 0 \le t \le T. \tag{35}$$

Itô's isometry therefore yields

$$\mathbb{E}\|M_T\|_2^2 = \mathbb{E}\int_0^T \mathrm{tr}(G_s G_s^\top)\, ds \le \sqrt{d}\, aT,$$

which proves (31).

To establish the exponential tail bound, fix $\theta > 0$ and define

$$\Phi_\theta(z) := \cosh(\theta\|z\|_2), \qquad z \in \mathbb{R}^d.$$

This function is smooth on $\mathbb{R}^d$, including at the origin, as follows from the expansion

$$\Phi_\theta(z) = \sum_{k=0}^{\infty} \frac{\theta^{2k}(z^\top z)^k}{(2k)!}.$$

For $z \neq 0$, write $\rho = \|z\|_2$ and $e = z/\rho$. Direct differentiation gives

$$\nabla^2\Phi_\theta(z) = \theta^2 \cosh(\theta\rho)\, ee^\top + \frac{\theta \sinh(\theta\rho)}{\rho}(I_d - ee^\top).$$

For every $y \geq 0$,

$$\sinh y \leq y \cosh y,$$

because the function $y \cosh y - \sinh y$ vanishes at zero and has derivative $y \sinh y \geq 0$. Consequently,

$$\frac{\theta \sinh(\theta\rho)}{\rho} \leq \theta^2 \cosh(\theta\rho),$$

and therefore

$$\nabla^2 \Phi_\theta(z) \preceq \theta^2 \Phi_\theta(z) I_d, \qquad z \in \mathbb{R}^d. \tag{36}$$

At $z = 0$, this follows from $\nabla^2 \Phi_\theta(0) = \theta^2 I_d$.

Now define

$$Z_t := \exp\left(-\frac{\theta^2}{2} A_t\right) \Phi_\theta(M_t).$$

Since A_t is a continuous finite-variation process, Itô's formula gives

$$\begin{aligned} dZ_t = {} & e^{-\theta^2 A_t/2} \nabla \Phi_\theta(M_t)^\top G_t \, dW_t \\ & + \frac{1}{2} e^{-\theta^2 A_t/2} \Big[\operatorname{tr}\left(\nabla^2 \Phi_\theta(M_t) G_t G_t^\top\right) - \theta^2 \Phi_\theta(M_t) \operatorname{tr}(G_t G_t^\top) \Big] \, dt. \end{aligned}$$

Because $G_t G_t^\top$ is positive semidefinite, (36) implies

$$\operatorname{tr}\left(\nabla^2 \Phi_\theta(M_t) G_t G_t^\top\right) \leq \theta^2 \Phi_\theta(M_t) \operatorname{tr}(G_t G_t^\top).$$

Thus the drift of Z_t is nonpositive. It follows that Z_t is a nonnegative local supermartingale and hence a supermartingale, with $Z_0 = 1$.

For $r > 0$, let

$$\tau_r := \inf\{t \in [0, T] : \|M_t\|_2 \geq r\}, \qquad \inf \varnothing := \infty,$$

and set

$$v := \sqrt{d}\, aT.$$

On the event $\{\tau_r \leq T\}$, continuity of M_t gives $\|M_{\tau_r}\|_2 = r$, while (35) gives $A_{\tau_r} \leq v$. Optional sampling at the bounded stopping time $\tau_r \wedge T$ therefore yields

$$\begin{aligned} 1 &\geq \mathbb{E}[Z_{\tau_r \wedge T}] \\ &\geq \mathbb{E}\left[Z_{\tau_r} \mathbf{1}_{\{\tau_r \leq T\}}\right] \\ &\geq e^{-\theta^2 v/2} \cosh(\theta r)\, \mathbb{P}(\tau_r \leq T). \end{aligned}$$

Using $\cosh(\theta r) \geq \frac{1}{2} e^{\theta r}$, we obtain

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} \|M_t\|_2 \geq r\right) &= \mathbb{P}(\tau_r \leq T) \\ &\leq \frac{e^{\theta^2 v/2}}{\cosh(\theta r)} \\ &\leq 2 \exp\left(-\theta r + \frac{\theta^2 v}{2}\right). \end{aligned}$$

Choosing $\theta = r/v$ proves

$$\mathbb{P}\left(\sup_{0\le t\le T}\|M_t\|_2 \ge r\right) \le 2\exp\left(-\frac{r^2}{2\sqrt{d}\,aT}\right),$$

which is (32). Finally, taking

$$r = \sqrt{2\sqrt{d}\,aT\log\left(\frac{2}{\delta}\right)}$$

proves (33). □

Lemma 1.6. *Let $\{\mathcal{F}_i\}_{i\ge 0}$ be a filtration, and let $\{X_i\}_{i=1}^n$ be a sequence of random variables such that X_i is $\mathcal{F}_i$-measurable and $|X_i| \le 1$ almost surely. Then, for every $\delta \in (0,1)$,*

$$\mathbb{P}\left(\sum_{i=1}^n X_i^2 \ge 8\sum_{i=1}^n \mathbb{E}\left[X_i^2 \mid \mathcal{F}_{i-1}\right] + 4\ln\frac{4}{\delta}\right) \le (\lceil\log_2 n\rceil + 1)\,\delta.$$

Proof. Before proceeding to prove our result, we first introduce some notations to simplify the expression.

$$\hat{F}_{n,j} = (\hat{t}_{n,j} - \hat{t}_{n,j-1})\hat{F}(x_n(\hat{t}_{n,j}), u_n(x_n(\hat{t}_{n,j})))$$

$$\hat{\hat{F}}_{n,j} := \int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} \hat{F}(x_n(t), u_n(x_n(t)))dt$$

$$\bar{F}_{n,j} := \int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} F(x_n(t), u_n(x_n(t)))dt$$

We construct an estimator for Θ^* as

$$\Theta_N = \hat{V}_N^{-1}\sum_{n=1}^N\sum_{j=1}^{\tau_n}\bar{w}_{n,j}^2\hat{V}_{N-1}^{\frac{1}{2}}\hat{V}_{n-1}^{-\frac{1}{2}}\hat{F}_{n,j}^\top \operatorname{vec}\left(\left(x_n(\hat{t}_{n,j}) - x_n(\hat{t}_{n,j-1}) - F_{n,j}\theta_N\right)\left(x_n(\hat{t}_{n,j}) - x_n(\hat{t}_{n,j-1}) - F_{n,j}\theta_N\right)^\top\right. \tag{37}$$

where $V_N = \sum_{n=1}^N\sum_{j=1}^{\tau_n} w_{n,j}^2\bar{V}_{N-1}^{\frac{1}{2}}V_{n-1}^{-\frac{1}{2}}F_{n,j}^\top F_{n,j} + \lambda I$, $\bar{V}_N = V_{N-1}^{-\frac{1}{2}}V_{N-2}^{\frac{1}{2}}(V_{N-1} - \lambda I) + \lambda I$, $\bar{\hat{V}}_N = \hat{V}_{N-1}^{-\frac{1}{2}}\hat{V}_{N-2}^{\frac{1}{2}}(V_{N-1} - \lambda I) + \lambda I$, $\hat{V}_N = \sum_{n=1}^N\sum_{j=1}^{\tau_n}\bar{w}_{n,j}^2\bar{\hat{V}}_N^{\frac{1}{2}}\hat{V}_{n-1}^{-\frac{1}{2}}\hat{F}_{n,j}^\top\hat{F}_{n,j} + \hat{\lambda}I$, $\theta_N = V_N^{-1}\sum_{n=1}^N\sum_{j=1}^{\tau_n} w_{n,j}^2\bar{V}_N^{\frac{1}{2}}V_{n-1}^{-\frac{1}{2}}F_{n,j}^\top\left(x_n(\hat{t}_{n,j}) - x_n(\hat{t}_{n,j-1})\right)$.

For readability, we denote

$$\Delta x_{n,j} = x(\hat{t}_{n,j}) - x(\hat{t}_{n,j-1}) \tag{38}$$

$$\xi_{n,j} = \Delta x_{n,j}\bar{F}\theta^*, e_{n,j} = F_{n,j}\theta_n - \bar{F}_{n,j}\theta^* \tag{39}$$

Then Θ_N could be simplified as follows.

$$\Theta_N = \hat{V}_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\bar{w}_{n,j}^2\bar{\hat{V}}_N^{\frac{1}{2}}\hat{V}_{n-1}^{-\frac{1}{2}}\hat{F}_{n,j}\Big[\mathrm{vec}(\xi_{n,j}\xi_{n,j}^\top) - \mathrm{vec}(e_{n,j}\xi_{n,j}^\top) - \mathrm{vec}(\xi_{n,j}e_{n,j}^\top) + \mathrm{vec}(e_{n,j}e_{n,j}^\top)\Big]. \tag{40}$$

For ease of expression, we define the first term of (40) as S_N, shown as follows.

$$S_N = \hat{V}_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\bar{w}_{n,j}^2\hat{F}_{n,j}\,\mathrm{vec}(\xi_{n,j}\xi_{n,j}^\top) \tag{41}$$

By multiplying $S_N - \theta_N$ with $\hat{V}_N^{\frac{1}{2}}$, we have

$$\hat{V}_{N-1}^{-\frac{1}{2}}\hat{V}_N(T_N - \Theta^*) = \hat{V}_{N-1}^{-\frac{1}{2}}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\bar{w}_{n,j}^2\hat{F}_{n,j}^\top\,\mathrm{vec}(\xi_{n,j}\xi_{n,j}^\top) - \hat{V}_{N-1}^{-\frac{1}{2}}\hat{V}_N\Theta^*. \tag{42}$$

Under the parameterization assumption of the drifting term, there holds

$$\xi_{n,j} = \int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} g^*(x_n(t), u_n((x_n(t)))dw_t \tag{43}$$

For ease of expression, we define

$$\mathcal{D}_{n,j} = \xi_{n,j}\xi_{n,j}^\top - \int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} g^*(x_n(t), u_n((x_n(t)))g^*(x_n(t), u_n((x_n(t)))^\top dt \tag{44}$$

Then by the parameterization assumption of the diffusion term and the definition of $\hat{\hat{F}}_{n,j}$, we have

$$\mathrm{vec}\left(\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} g^*(x_n(t), u_n((x_n(t)))g^*(x_n(t), u_n((x_n(t)))^\top dt\right) = \hat{\hat{F}}_{n,j}\Theta^* \tag{45}$$

Then substituting (45) and (44), we have

$$\hat{V}_N^{1/2}(T_N - \Theta^*) = \hat{V}_N^{-1/2}\sum_{n,j}\bar{w}_{n,j}^2\bar{\hat{V}}_N^{\frac{1}{2}}\hat{V}_{n-1}^{-\frac{1}{2}}\hat{F}_{n,j}^\top\mathcal{D}_{n,j} + \hat{V}_N^{-1/2}\sum_{n,j}\bar{w}_{n,j}^2\hat{F}_{n,j}^\top\hat{\hat{F}}_{n,j}\Theta^* - \hat{V}_N^{1/2}\Theta^*. \tag{46}$$

Then using

$$\hat{V}_N^{1/2}\Theta^* = \hat{V}_N^{-1/2}\hat{V}_N\Theta^* = \hat{V}_N^{-1/2}\sum_{n,j}\bar{w}_{n,j}^2\bar{\hat{V}}_N^{\frac{1}{2}}\hat{V}_{n-1}^{-\frac{1}{2}}\hat{F}_{n,j}^\top\hat{F}_{n,j}\Theta^* + \lambda\hat{V}_N^{-1/2}\Theta^*. \tag{47}$$

we have

$$\hat{V}_N^{1/2}(T_N-\Theta^*)=\hat{V}_N^{-1/2}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\bar{w}_{n,j}^2\bar{\hat{V}}_N^{\frac{1}{2}}\hat{V}_{n-1}^{-\frac{1}{2}}\hat{F}_{n,j}^\top(\hat{\hat{F}}_{n,j}-\hat{F}_{n,j})\Theta^*+\hat{V}_N^{-1/2}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\bar{\hat{V}}_N^{\frac{1}{2}}\hat{V}_{n-1}^{-\frac{1}{2}}\bar{w}_{n,j}^2\hat{F}_{n,j}^\top\mathcal{D}_{n,j}-\hat{\lambda}\hat{V}_N^- \tag{48}$$

Here, we define event

$$\mathcal{E}_{n,j}:=\left\{\|\xi_{n,j}\|_2\le\sqrt{2\sqrt{d}\,\varphi_{\hat{F}}\log\left(\frac{2}{\delta}\right)(\hat{t}_{n,j}-\hat{t}_{n,j-1})}\right\},\qquad\mathcal{E}:=\bigcap_{n=1}^{N}\bigcap_{j=1}^{\tau_n}\mathcal{E}_{n,j}. \tag{49}$$

$\mathcal{E}_{n,j}$ as $\|\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}}g(x_n(t),u_n(x_n(t))dw_t\|_2\le\sqrt{2\sqrt{d}\log\left(\frac{2}{\delta}\right)\varphi_{\hat{F}}}$. By Lemma 1.5, the event $\mathcal{E}_{n,j}$ holds with possibility at least $1-\sum_{n=1}^{N}\tau_n\delta$. Then under the event $\mathcal{E}$ and the sampling condition $t_{n,j}-t_{n,j-1}\le 1$, we could construct the following results.

$$\left\|\hat{V}_{n-1}^{-\frac{1}{2}}\bar{w}_{n,j}^2\hat{F}_{n,j}\mathcal{D}_{n,j}\right\|_2\le\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}} \tag{50}$$

Note that $\left\|\hat{V}_{n-1}^{-\frac{1}{2}}\bar{w}_{n,j}^2\hat{F}_{n,j}\right\|_2\le 1$ holds, then we have

$$E[(\hat{V}_{n-1}^{-\frac{1}{2}}\bar{w}_{n,j}^2\hat{F}_{n,j}\mathcal{D}_{n,j})(\hat{V}_{n-1}^{-\frac{1}{2}}\bar{w}_{n,j}^2\hat{F}_{n,j}\mathcal{D}_{n,j})^\top]\preceq\mathbb{E}(\|\mathcal{D}\|_2^2)I \tag{51}$$

$$\preceq\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)^2\varphi_{\hat{F}}^2(\hat{t}_{n,j}-\hat{t}_{n,j-1})^2I \tag{52}$$

By definition,

$$\|\hat{V}_N^{-1}\bar{\hat{V}}_N\|_2\le 1 \tag{53}$$

Then by freedman inequality, with possibility at least $1-\delta$, there holds

$$\left\|\hat{V}_N^{-1/2}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\bar{\hat{V}}_N^{\frac{1}{2}}\hat{V}_{n-1}^{-\frac{1}{2}}\bar{w}_{n,j}^2\hat{F}_{n,j}^\top\mathcal{D}_{n,j}\right\|_2\le\left\|\hat{V}_N^{-\frac{1}{2}}\bar{\hat{V}}_{N-1}^{\frac{1}{2}}\right\|_2\left\|\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\hat{V}_{n-1}^{-\frac{1}{2}}\bar{w}_{n,j}^2\hat{F}_{n,j}\mathcal{D}_{n,j}\right\|_2 \tag{54}$$

$$\le\left(\sqrt{2\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}(\hat{t}_{n,j}-\hat{t}_{n,j-1})^2\log\left(\frac{d^2+1}{\delta}\right)}\right. \tag{55}$$

$$\left.+\frac{2}{3}\log\left(\frac{d^2+1}{\delta}\right)\right)\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}} \tag{56}$$

Before proceeding to prove, we start by identifying the upper bound of $F_{n,j}\theta_N - F_{n,j}\theta_N^*$. By the Cauthy-Schwarz inequality, we have the following.

$$\|F_{n,j}\theta_N - F_{n,j}\theta_N^*\|_2 \leq \left\| F_{n,j}V_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top \left(\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} g^*(x_n(t),u_n(x_n(t)))dw_t\right) + F_{n,j}V_N^{-1}\lambda\theta^* \right\|_2 \tag{57}$$

$$\leq \left\| F_{n,j}V_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top \left(\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} g^*(x_n(t),u_n(x_n(t)))dw_t\right) \right\|_2 + \left\|F_{n,j}V_N^{-1}\lambda\theta^*\right\|_2 \tag{58}$$

Note that, with possibility at least $1-\delta$,

$$\left\| F_{n,j}V_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top \left(\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} g^*(x_n(t),u_n(x_n(t)))dw_t\right) \right\|_2 \tag{59}$$

$$\leq \left\|V_N^{-\frac{1}{2}}F_{n,j}\right\|_2 \sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}} \tag{60}$$

$$\leq w_{n,j}^2 \left\|V_N^{-\frac{1}{2}}F_{n,j}\right\|_2^2 \sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}} \tag{61}$$

Then using Freedman inequality, with possibility at least $1-\delta$

$$\left\|F_{n,j}\theta_N - \bar{F}_{n,j}\theta_N^*\right\|_2 \leq \left\| F_{n,j}V_N^{-1}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top \left(\int_{\hat{t}_{n,j-1}}^{\hat{t}_{n,j}} g^*(x_n(t),u_n(x_n(t)))dw_t\right) \right\|_2 + \left\|F_{n,j}V_N^{-1}\lambda\theta^*\right\|_2 \tag{62}$$

$$+ \left\|F_{n,j}\theta^* - \bar{F}_{n,j}\theta^*\right\|_2 \tag{63}$$

$$\leq w_{n,j}^2 \left\|V_N^{-\frac{1}{2}}F_{n,j}\right\|_2^2 \sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}} + \sqrt{\lambda}w_{n,j}^2\left\|V_N^{-\frac{1}{2}}F_{n,j}\right\|_2^2 + \left\|F_{n,j}\theta^* - \bar{F}_{n,j}\theta^*\right\| \tag{64}$$

Note that

$$V_0 := \lambda I_d, \tag{65}$$

$$V_k := \lambda I_d + \sum_{\ell=1}^{k}\sum_{j=1}^{\tau_\ell} w_{\ell,j}^2 F_{\ell,j}^\top F_{\ell,j}, \qquad k = 1,\ldots,N. \tag{66}$$

For every $n \in \{1,\ldots,N\}$, define

$$S_n := \sum_{j=1}^{\tau_n} w_{n,j}^2 F_{n,j}^\top F_{n,j} = V_n - V_0. \tag{67}$$

Then

$$\sum_{n=1}^{N}\sum_{j=1}^{\tau_n} w_{n,j}^2 \left\|V_N^{-1/2}F_{n,j}^\top\right\|_2^2 \le \operatorname{tr}\big(V_N^{-1}S_n\big) \tag{68}$$

$$\le d - \lambda \operatorname{tr}\big(V_N^{-1}\big) \le d. \tag{69}$$

Then by inequalities (21) and (24), we have

$$\|F_{n,j}\theta_N - F_{n,j}\theta_N^*\|_2 \le \left(\left(\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}} + \sqrt{\lambda}\right) w_{n,j}^2 \left\|V_N^{-\frac{1}{2}}F_{n,j}\right\|_2^2 \right. \tag{70}$$

Similarly, we define $\hat{\mathcal{D}}_{n,j} = \hat{V}_N^{-\frac{1}{2}}\bar{w}_{n,j}^2\hat{F}_{n,j}\operatorname{vec}\left(\left(F_{n,j}\theta^* - \bar{F}_{n,j}\theta^*\right)\left(x_n(\hat{t}_{n,j}) - x_n(\hat{t}_{n,j-1}) - \bar{F}_{n,j}\theta^*\right)^\top\right)$

$$\operatorname{Var}(\hat{\mathcal{D}}_{n,j}) \preceq \sqrt{d}\varphi_{\hat{F}}(\hat{t}_{n,j} - \hat{t}_{n,j-1})\left\|F_{n,j} - \bar{F}_{n,j}\right\|_2^2 I \tag{71}$$

Note that under the event,

$$\|\hat{\mathcal{D}}_{n,j}\|_2 \le \left\|F_{n,j}\theta^* - \bar{F}_{n,j}\theta^*\right\|_2 \sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}} \le 2\varphi_F\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}} \tag{72}$$

Then we have

$$\left\|\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\hat{\mathcal{D}}_{n,j}\right\|_2 \le L_F(1+L_u)e\sqrt{2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})}\sqrt{2\sqrt{d}\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\varphi_{\hat{F}}(\hat{t}_{n,j}-\hat{t}_{n,j-1})\left\|F_{n,j}\theta^*-\bar{F}_{n,j}\theta^*\right\|_2^2\log\left(\frac{d+1}{\delta}\right.} \tag{73}$$

$$+\frac{4\varphi_F\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}\log\left(\frac{d+1}{\delta}\right)}{3} \tag{74}$$

$$\le L_F(1+L_u)e\sqrt{64e^2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}((t_{n,j}-t_{n,j-1})^3+(t_{n,j-1}-t_{n,j-2})^3)+128\varphi_F^2\log(} \tag{75}$$

$$+\frac{4\varphi_F\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}\log\left(\frac{d+1}{\delta}\right)}{3} \tag{76}$$

Then we have

$$\|\hat{V}_N^{\frac{1}{2}}(\mathbf{??})\|_2 \leq \sum_{n=1}^{N}\sum_{j=1}^{\tau_n} \|F_{n,j}\theta_N - F_{n,j}\theta^*\|_2 \sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}} \tag{77}$$

$$+ L_F(1+L_u)e\sqrt{2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})}\sqrt{2\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\varphi_{\hat{F}}(\hat{t}_{n,j}-\hat{t}_{n,j-1})^3\log\left(\frac{d+1}{\delta}\right)} \tag{78}$$

$$+ \frac{2L_F(1+L_u)e\sqrt{2\sqrt{d}(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}\log\left(\frac{d+1}{\delta}\right)}{3} \tag{79}$$

$$\leq L_F(1+L_u)e\sqrt{2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})}\sqrt{2\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\varphi_{\hat{F}}(\hat{t}_{n,j}-\hat{t}_{n,j-1})^3\log\left(\frac{d+1}{\delta}\right)} \tag{80}$$

$$+ \frac{2L_F(1+L_u)e\sqrt{2\sqrt{d}(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}\log\left(\frac{d+1}{\delta}\right)}{3} \tag{81}$$

$$+ d\left(\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}+\sqrt{\lambda}\right) \tag{82}$$

By the same taken,

$$\|\hat{V}_N^{\frac{1}{2}}(\mathbf{??})\|_2 \leq L_F(1+L_u)e\sqrt{2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})}\sqrt{2\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\varphi_{\hat{F}}(\hat{t}_{n,j}-\hat{t}_{n,j-1})^3\log\left(\frac{d+1}{\delta}\right)} \tag{83}$$

$$+ \frac{2L_F(1+L_u)e\sqrt{2\sqrt{d}(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}\log\left(\frac{d+1}{\delta}\right)}{3} \tag{84}$$

$$+ d\left(\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}+\sqrt{\lambda}\right) \tag{85}$$

$$\|\hat{V}_N^{\frac{1}{2}}(\mathbf{??})\|_2 \leq \sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\left\|F_{n,j}\theta_N - \bar{F}_{n,j}\theta^*\right\|_2^2 \leq \sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\left(2\left\|F_{n,j}\theta_N - F_{n,j}\theta^*\right\|_2^2 + 2\left\|F_{n,j}\theta^* - \bar{F}_{n,j}\theta^*\right\|_2^2\right) \tag{86}$$

$$\tag{87}$$

By Cauthy-Schwarz inequality, we have

$$\|\hat{V}_N^{\frac{1}{2}}(\mathbf{??})\|_2 \leq 2\left((\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}+\sqrt{\lambda}\right)^2 d + 64e^2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}((t_{n,j}-t_{n,j-1})^3+(t_{n,j-1} \tag{88}$$

Then we have

$$\left\|\hat{V}_N^{\frac{1}{2}}(\Theta_n-\Theta^*)\right\|_2 \leq \sqrt{8\varphi_{\hat{F}}(\hat{t}_{n,j}-\hat{t}_{n,j-1})^2\log\left(\frac{d^2+1}{\delta}\right)}+\frac{\left(4\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}{3}\log\left(\frac{d^2+1}{\delta}\right) \tag{89}$$

$$+2L_F(1+L_u)e\sqrt{2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})}\sqrt{2\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\varphi_{\hat{F}}(\hat{t}_{n,j}-\hat{t}_{n,j-1})^3\log\left(\frac{d+1}{\delta}\right)} \tag{90}$$

$$+\frac{4L_F(1+L_u)e\sqrt{2\sqrt{d}(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}\log\left(\frac{d+1}{\delta}\right)}{3} \tag{91}$$

$$+2d\left(\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}+\sqrt{\lambda}\right) \tag{92}$$

$$+2\left((\sqrt{\left(2\sqrt{d}\log\left(\frac{2}{\delta}\right)+1\right)\varphi_{\hat{F}}}+\sqrt{\lambda}\right)^2 d + 64e^2(\varphi_F^2+2\sqrt{d}\varphi_{\hat{F}})\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}((t_{n,j}-t_{n,j-1})^3+ \tag{93}$$

$$+\sum_{n=1}^{N}\sum_{j=1}^{\tau_n}\|\hat{F}_{i,j}-\hat{\hat{F}}_{i,j}\|_2 \tag{94}$$

□