UNIVERSAL 2.030-ROUNDING FOR MULTILAYER CORRELATION CLUSTERING WITH PROBABILITY WEIGHTS

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ABSTRACT

Multilayer correlation clustering seeks one partition for conflicting preferences across layers. For probability weights, where joining and separating each pair have costs summing to one, we establish universal 2.030-rounding. Specifically, any explicit feasible rational cluster-LP cover can be rounded deterministically to one partition costing at most 2.030 times the cover's fractional cost for all probability-weight vectors simultaneously, including those not supplied to the algorithm. We achieve this through a single auxiliary objective and potential, conditional variance bounds, and computer-assisted verification using an exact rational certificate. Consequently, combining this rounding with scalar fractional optimization and averaging yields a randomized $(2.030 + \varepsilon)$-approximation for each fixed ℓ_p objective, $p \in [1, \infty]$, with a factor independent of the number of layers. Together with a two-vertex lower bound, the rounding theorem places the worst-layer cluster-relaxation gap in $[2, 2.030]$.

1 INTRODUCTION

Pairwise similarity predictions need not be mutually consistent: two objects may each resemble a third while differing from one another. Correlation clustering resolves such conflicts by finding a partition that minimizes disagreement with the pairwise evidence, without requiring the number of clusters in advance (Bansal et al., 2004). This formulation has been used for coreference resolution (Elsner & Schudy, 2009) and community detection (Veldt et al., 2018). In many settings, however, the same objects admit several sources of pairwise evidence, such as different relations, measurements, or similarity models. Combining these sources into a single score can obscure how well the resulting partition represents each source.

Multilayer correlation clustering retains this distinction. Each layer defines a disagreement cost, and one partition minimizes the ℓ_p norm of the vector of layer costs (Miyauchi et al., 2026). The case $p = 1$ minimizes total disagreement, while $p = \infty$ minimizes the largest layer cost. We study the probability-weighted model: for every pair in every layer, the cost w^+ of separating its endpoints and the cost w^- of joining them are nonnegative and sum to one. This includes signed inputs and permits confidence scores between the two extremes.

For this model, Miyauchi et al. (2026) give a 4-approximation and a general transfer from a single-layer α-approximation to a multilayer $(\alpha+2)$-approximation. With more recent single-layer results, the latter yields $3.3865 + \varepsilon$; Table 1 makes the composition explicit. These results leave open how much more can be gained by exploiting a stronger relaxation of the common partition.

The obstacle: controlling one realized partition. A good expected cost in every layer does not by itself control the worst-layer cost of an output. Consider two vertices with opposite signs in two layers. Joining or separating them with equal probability gives expected cost $1/2$ in each layer, yet every realized partition has maximum cost one. Thus a rounding analysis for one prescribed objective cannot simply be applied separately to each layer. We seek a guarantee that holds for all layers on the *same* output.

Our starting point is the cluster LP of Cao et al. (2025a), which assigns weights to possibly overlapping vertex sets. In addition to pairwise distances, a feasible cover specifies a distribution on sets

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containing each vertex. These conditional distributions retain correlations that a metric description alone does not provide. We ask whether they can support a stronger form of simultaneous rounding: a partition determined solely by the cover, with a guarantee for *every* probability-weight vector, including vectors never supplied to the algorithm.

Our results. We answer this question affirmatively with factor 2.030. Given any explicit feasible rational cover z with induced distances $\boldsymbol{x}$, our deterministic algorithm returns a partition $C(z)$ satisfying

$$D_a(C(z)) \leq 2.030\, D_a(\boldsymbol{x}) \qquad \text{for every } a \in [0,1]^{\mathcal{E}}.$$

Here D_a is the disagreement cost defined in Section 2. The algorithm is polynomial in the encoding length of the supplied cover (Theorem 1, Algorithm 1).

To turn this rounding theorem into an approximation algorithm, we construct a common cover for the chosen norm. We repeatedly solve scalar combinations of the layers using the fractional solver of Abbasi et al. (2026), and average the returned covers. Rounding that average gives a randomized $(2.030+\varepsilon)$-approximation for each fixed $p \in [1,\infty]$, with success probability at least $1-\delta$ (Theorem 2, Algorithm 2). The two-vertex example also gives a lower bound of two on the worst-layer relaxation gap, placing it in $[2, 2.030]$ (Proposition 3).

Technical approach. We first express the maximum excess over all probability weights as a single auxiliary clustering objective minus a fixed budget. This removes the layer weights from the rounding problem. A deterministic procedure then chooses clusters from two families: intersections with support sets, and pivot candidates using a shared threshold and a conditional support set. The minimum of two residual budgets is a potential that pays for each chosen cluster. A bound for every triangle separately would fail; two conditional variances instead connect the triangles of a common cover and control their total excess. The proof is computer-assisted: an exact rational certificate establishes Lemma 5 at $\rho = 203/100$. Appendix G.1 identifies the proof obligation, and Appendix G.3 maps it to the verification files. Finally, a shifted-norm analysis of the scalar responses constructs the common cover without an approximation factor depending on the number of layers.

1.1 Related work

Single-layer and multilayer approximation. For complete signed graphs, stronger relaxations have improved on metric-LP rounding (Chawla et al., 2015; Cohen-Addad et al., 2022; 2023). The corrected cluster-LP analysis of Cao et al. (2025a) gives $1.485+\varepsilon$; we use this version because the earlier 1.437 proof has a gap. García-Soriano & Schohn (2026) improve the factor to $1.3865+\varepsilon$ and approximately optimize the fractional cluster LP. Theorem 20 of Chawla et al. (2015) transfers an unweighted approximation to probability weights. Combining this transfer with García-Soriano & Schohn (2026) and Miyauchi et al. (2026, Theorem 2) yields the $3.3865+\varepsilon$ multilayer bound in Table 1. For $p=1$, averaging layer weights reduces the problem directly to one layer, so $1.3865+\varepsilon$ already applies. Our improved general guarantee is relevant to objectives that preserve individual layer costs, especially $p=\infty$.

Table 1: Approximation guarantees for complete probability-weighted inputs. "Composition" is a consequence of the cited results. All factors use the integer optimum for the chosen p.

Objective or input	Factor	Source
One layer; also multilayer $p=1$	$1.3865+\varepsilon$	Composition: García-Soriano & Schohn (2026); Chawla et al. (2015)
Multilayer, fixed p	4	Miyauchi et al. (2026)
Multilayer, fixed p	$3.3865+\varepsilon$	Composition with Miyauchi et al. (2026)
Multilayer, fixed p	$2.030+\varepsilon$	Theorem 2

Simultaneous guarantees. The metric rounding of Miyauchi et al. (2026, Algorithm 3 and Appendix C.5) depends only on the metric and gives a factor-four bound in every layer. Its analysis therefore already implies universal factor-four rounding; we improve the factor using the additional

information in a cluster cover. A different line of work controls the disagreements incident to each vertex (Puleo & Milenkovic, 2016; Charikar et al., 2017; Kalhan et al., 2019). Davies et al. (2024) approximate all ℓ_p norms of that vertex-cost vector, and Cao et al. (2025b) extend this to monotone symmetric norms. Our coordinates are whole layers with potentially conflicting signs. The probability constraint is also essential to our model; unrestricted weighted correlation clustering admits logarithmic bounds (Demaine et al., 2006).

Rounding and optimization ingredients. Our analysis builds on the support-set rounding, residual budgets, and conditional covariance constraints of Cao et al. (2025a), and on the weighted-variance and exact polynomial techniques of García-Soriano & Schohn (2026). Deterministic pivot selection by averaging has earlier precedents (van Zuylen & Williamson, 2009). For fractional optimization, we use the one-color specialization of Abbasi et al. (2026, Section 6 and Appendix C.3) and the shifted-norm method of Molinaro (2017); Kesselheim & Singla (2020). Our contribution is their adaptation to the universal auxiliary objective, with explicit candidates and a certificate giving factor 2.030, together with a common-cover construction for multilayer costs.

2 Preliminaries and main results

2.1 Model and notation

Probability-weighted multilayer costs. Let V be a set of n vertices, $\mathcal{E} = \binom{V}{2}$, and $m = |\mathcal{E}|$. Each of $L \geq 1$ layers specifies rational weights $w^+_{\ell,e}, w^-_{\ell,e} \geq 0$ with $w^+_{\ell,e} + w^-_{\ell,e} = 1$ for every $e \in \mathcal{E}$. Write $a_{\ell,e} = w^-_{\ell,e}$. For a fractional separation value $t \in [0,1]$, define

$$d_a(t) = (1-a)t + a(1-t) = a + (1-2a)t, \qquad D_\ell(\boldsymbol{x}) = \sum_{e \in \mathcal{E}} d_{a_{\ell,e}}(x_e), \qquad \Phi_p(\boldsymbol{x}) = \|\boldsymbol{D}(\boldsymbol{x})\|_p, \tag{1}$$

where $\boldsymbol{x} = (x_e)_{e \in \mathcal{E}}$ and $\boldsymbol{D}(\boldsymbol{x}) = (D_1(\boldsymbol{x}), \ldots, D_L(\boldsymbol{x}))$. For a partition C into nonempty, disjoint parts, set $c_{uv} = 1$ if u, v lie in different parts and $c_{uv} = 0$ otherwise, and evaluate these costs at $\boldsymbol{c}$. Thus joining a pair costs $a_{\ell,e}$ and separating it costs $1 - a_{\ell,e}$. We use $\|d\|_p = (\sum_\ell d_\ell^p)^{1/p}$ for finite p and $\|d\|_\infty = \max_\ell d_\ell$, and write $\mathrm{OPT}_p = \min_C \Phi_p(C)$. For an arbitrary single probability vector $a \in [0,1]^{\mathcal{E}}$, let $D_a(\boldsymbol{x}) = \sum_e d_{a_e}(x_e)$. This notation includes objectives that are not among the input layers.

The cluster relaxation and explicit covers. Following Cao et al. (2025a), associate a nonnegative weight z_S with each nonempty $S \subseteq V$, subject to

$$z_S \geq 0, \qquad \sum_{S \ni u} z_S = 1 \quad (u \in V), \qquad x_{uv} = 1 - \sum_{S \supseteq \{u,v\}} z_S. \tag{2}$$

We call such a z a *fractional cluster cover*. Every partition induces a cover by assigning weight one to each part, so minimizing $\Phi_p(\boldsymbol{x})$ over (2) is a relaxation of the integer problem. It is an LP for $p = 1$, and for $p = \infty$ after introducing an upper bound on all layer costs; other finite norms give a convex objective over the same feasible region. An *explicit cover* lists its positive-weight sets and their rational weights. The rounding algorithm is polynomial in this list's encoding length. Producing an appropriate list from the layer input is a separate optimization step (Section 5).

Conditional marginals. A cover may contain overlapping sets and need not be a distribution over partitions. For each vertex u, however, $\{z_S : S \ni u\}$ is a probability distribution. If S is drawn from this distribution, then

$$\Pr[v \in S] = 1 - x_{uv}, \qquad \Pr[v, w \in S] = q_{uvw} := \sum_{S \supseteq \{u,v,w\}} z_S.$$

The union bound gives $q_{uvw} \geq 1 - x_{uv} - x_{uw}$, while $q_{uvw} \leq 1 - x_{vw}$, proving $x_{vw} \leq x_{uv} + x_{uw}$. Together with $x_{uu} = 0$, the induced distances therefore form a pseudometric. The conditional marginals carry additional information used by our rounding. On a residual vertex set U, intersecting support sets with U, merging duplicates, and discarding empty sets preserves coverage and all pair and triple marginals on U (Appendix A).

2.2 MAIN RESULTS

The first theorem controls every probability-weighted objective for one fixed cover. The second supplies a common cover suitable for a chosen multilayer objective. Together they separate the rounding guarantee from the cost of fractional optimization.

Theorem 1 (Universal simultaneous rounding)**.** *Given any explicit rational cover satisfying* (2)*, a deterministic algorithm, polynomial in its encoding length, returns a partition $C(z)$ such that*

$$D_a(C(z)) \le \frac{203}{100} D_a(\boldsymbol{x}) \quad \textit{for every } a \in [0,1]^{\mathcal{E}}. \tag{3}$$

The algorithm depends on z.

Theorem 2 (Multilayer approximation)**.** *For every fixed $\varepsilon > 0$ and fixed $p \in [1, \infty]$, and every $\delta \in (0,1)$, probability-weighted multilayer correlation clustering admits a randomized polynomial time algorithm that returns, with probability at least $1-\delta$, a partition C satisfying*

$$\Phi_p(C) \le (2.030 + \varepsilon)\,\mathrm{OPT}_p\,. \tag{4}$$

The running time is polynomial in the rational input length and $\log(1/\delta)$ for fixed precision parameters and fixed p.

Sections 3–4 prove Theorem 1; Section 5 proves Theorem 2. The rounding in Theorem 1 is deterministic. Randomness in Theorem 2 is used to construct a common fractional cover. The fractional optimization depends on the chosen p.

Proposition 3 (Gap of the worst-layer relaxation)**.** *The integrality gap of minimizing $\Phi_\infty(\boldsymbol{x})$ over* (2) *lies in* $[2, 2.030]$.

Proof. For the lower bound, use two vertices and two layers with opposite unit signs. Every partition has maximum layer cost one. The cover with weight $1/2$ on the pair and $1/2$ on each singleton gives $x = 1/2$ and maximum fractional cost $1/2$, which is optimal since the two costs sum to one. The upper bound follows from Theorem 1: for rational input, the finite worst-layer LP has an optimal rational cover. □

3 ONE OBJECTIVE FOR EVERY PROBABILITY LAYER

To prove Theorem 1, we first replace all probability-weighted objectives by one exact test. For $\rho \ge 2$, define

$$b_\rho(t) = \mathrm{clip}_{[0,1]}\left(\rho t - \frac{\rho-1}{2}\right), \qquad k_\rho(t) = \min\left\{\rho t, \rho(1-t), \frac{\rho-1}{2}\right\}. \tag{5}$$

The auxiliary instance assigns negative weight $b_\rho(x_e)$ to pair e. Hereafter $\rho = 203/100$, and subscripts on b and k are omitted.

Lemma 4 (Universal excess identity)**.** *For every $t \in [0,1]$ and $c \in \{0,1\}$,*

$$\max_{a \in [0,1]} \{d_a(c) - \rho d_a(t)\} = d_{b_\rho(t)}(c) - k_\rho(t). \tag{6}$$

Consequently, a partition with $D_b(C) \le K := \sum_e k(x_e)$ satisfies (3)*.*

Proof. The expression is affine in a, so its maximum is attained at 0 or 1:

$$\max_a \{d_a(c) - \rho d_a(t)\} = \max\{c - \rho t,\ 1 - c - \rho(1-t)\}.$$

For $c = 0, 1$, these values are respectively $b_\rho(t) - k_\rho(t)$ and $1 - b_\rho(t) - k_\rho(t)$, by (5). Since the weights on different pairs can be chosen independently,

$$\max_{a \in [0,1]^{\mathcal{E}}} \{D_a(C) - \rho D_a(\boldsymbol{x})\} = D_b(C) - K. \tag{7}$$

Thus $D_b(C) \le K$ is also necessary for the universal guarantee. □

The maximizer can always be chosen in $\{0,1\}^{\mathcal{E}}$. Consequently, controlling every complete signed graph on V is equivalent to controlling every probability-weighted layer. Equation (7) avoids enumerating those sign assignments: it tests the entire requirement by evaluating one objective.

3.1 One potential from two residual bounds

It remains to find a partition of auxiliary cost at most K. Neither of the following pair budgets needs to be smaller than k on every pair. What matters is that their smaller total is at most K.

For a pair at distance t, support sets containing both endpoints have total weight $1 - t$; those containing exactly one have total weight $2t$. Dividing their cost by the total weight $1 + t$ gives the budget

$$e(t) = \frac{(1-t)b(t) + 2t(1-b(t))}{1+t}, \qquad \Delta(t) = e(t) - k(t). \tag{8}$$

This is the cluster-based rounding formula of Cao et al. (2025a, Lemma 6). We use two pair budgets, $e(t)$ and

$$\lambda = \frac{39}{4}, \qquad B(t) = k(t) - \lambda\Delta(t). \tag{9}$$

For the unassigned vertices U, take their smaller total as the potential:

$$\mathcal{P}(U) = \min\left\{ \sum_{e\in\binom{U}{2}} e(x_e), \sum_{e\in\binom{U}{2}} B(x_e) \right\} \le \sum_{e\in\binom{U}{2}} k(x_e). \tag{10}$$

Indeed, the two sums differ from $\sum k(x_e)$ by $\sum \Delta(x_e)$ and $-\lambda \sum \Delta(x_e)$; at least one is nonpositive.

Outputting a part $A \subseteq U$ incurs the auxiliary cost

$$\mathrm{cost}_U(A) = \sum_{e\in\binom{A}{2}} b(x_e) + \sum_{e\in\delta_U(A)} (1 - b(x_e)),$$

where $\delta_U(A)$ consists of pairs with exactly one endpoint in A. For pair budgets β_e, write

$$G_U^\beta(A) = \mathrm{cost}_U(A) + \sum_{e\in\binom{U\setminus A}{2}} \beta_e - \sum_{e\in\binom{U}{2}} \beta_e. \tag{11}$$

Suppose each of the two budgets admits a nonempty candidate with $G_U^\beta(A) \le 0$. Use a candidate for the budget attaining the minimum in (10). Since the next potential is at most that same budget on $U \setminus A$,

$$\mathrm{cost}_U(A) + \mathcal{P}(U \setminus A) \le \mathcal{P}(U). \tag{12}$$

Telescoping gives $D_b(C) \le \mathcal{P}(V) \le K$, since $\mathcal{P}(\varnothing) = 0$. Thus the two budgets can be used in one clustering process, even if the smaller one changes between steps.

For support intersections, the definition of e gives

$$\sum_S z_S G_U^e(S \cap U) = \sum_{e\in\binom{U}{2}} [(1 - x_e)b(x_e) + 2x_e(1 - b(x_e)) - (1 + x_e)e(x_e)] = 0. \tag{13}$$

Empty intersections contribute zero; some nonempty intersection has nonpositive score. The support sets thus justify the first budget. For the second, use $\beta_e = B(x_e)$ and write $G_U = G_U^B$. We next construct candidates whose average score is nonpositive, using the correlations supplied by the cover.

4 Correlated rounding

The second budget requires more than support intersections. We add candidates that combine a metric ball with a set drawn conditionally on the pivot. Write $\mathcal{B}_u(r) = \{v \in U : x_{uv} \le r\}$, with $x_{uu} = 0$. For each pivot $u \in U$, choose a mode from Table 2. A ball mode outputs $\mathcal{B}_u(r)$. A conditional mode draws $S \ni u$ with probability z_S, independently draws its threshold R, and outputs

$$A = \mathcal{B}_u(R) \cup \big(S \cap \mathcal{B}_u(4/5)\big). \tag{14}$$

Both S and R are shared by all neighbors of the pivot, so their correlations must be retained in the analysis. All candidates contain u. Continuous thresholds induce only finitely many distinct candidates on an explicit input; Algorithm 1 enumerates them together with the support intersections.

Table 2: The candidate distribution. The final column gives $1000w_i$, so every displayed weight is exact. Conditional modes use (14).

Mode	Radius or threshold interval	$1000w_i$
Ball	$21/40$	22
Conditional, uniform	$[17/40, 23/50]$	227
Conditional, uniform	$[47/100, 19/40]$	106
Conditional, uniform	$[12/25, 13/25]$	544
Conditional, uniform	$[12/25, 1/2]$	80
Conditional, fixed	$47/100$	21

Algorithm 1 Universal simultaneous rounding

Require: Explicit feasible cover z.
1: Compute $\boldsymbol{x}$ and the pair values b, k, e, B.
2: $U \leftarrow V$, $C \leftarrow \varnothing$.
3: **while** $U \neq \varnothing$ **do**
4: Enumerate all nonempty support intersections and the candidates of Table 2.
5: Choose A minimizing $\mathrm{cost}_U(A) + \mathcal{P}(U \setminus A)$.
6: Add A to C; set $U \leftarrow U \setminus A$.
7: **end while**
8: **return** C.

4.1 Inclusion probabilities

Let W be the total conditional-mode weight and define

$$g(t) = \sum_{i \text{ conditional}} w_i \Pr[R_i < t], \qquad \pi_b(t) = \sum_{i \text{ ball}} w_i \mathbf{1}[t \le r_i].$$

The marginal inclusion probability at distance t is

$$\pi(t) = \pi_b(t) + \mathbf{1}[t \le 4/5]\big(W - tg(t)\big). \tag{15}$$

Indeed, a conditional mode excludes the vertex exactly when $R < t$ and the support set misses it. For pivot distances $s \le t$ and $q = \sum_{S \supseteq \{u,v,w\}} z_S$, the joint inclusion probability is

$$J(s,t,q) = \mathbf{1}[t \le 4/5]\big[\pi(t) - g(s)(1 - t - q)\big]. \tag{16}$$

For $t \le 4/5$, subtract from $\pi(t)$ the probability of including only the farther vertex. This event requires $R < s$ and a support set containing the farther vertex only; its probability is $g(s)(1-t-q)$. For $t > 4/5$ the farther vertex is always excluded.

For an edge at distance r opposite the pivot, with other distances s, t, the contribution is

$$T_r = (b(r) - B(r))J + (1 - b(r) - B(r))(\pi(s) + \pi(t) - 2J), \tag{17}$$

where $J = J(\min\{s,t\}, \max\{s,t\}, q)$. The two terms correspond to including both endpoints and exactly one endpoint, respectively. Let $\mathcal{T}(x,y,z,q)$ be the sum of the three opposite-edge contributions. We allow repeated vertices in an ordered triple $(u,v,w) \in U^3$, setting $x_{uu} = 0$ and $q_{uvw} = \sum_{S \supseteq \{u,v,w\}} z_S$ as before. In particular, $q_{uuv} = 1 - x_{uv}$ and $q_{uuu} = 1$.

4.2 Why conditional variance is enough

A separate bound $\mathcal{T} \le 0$ for every feasible triple is too strong for these candidates. For example, the feasible cover $z_{\{1,2\}} = z_{\{1,3\}} = z_{\{2\}} = z_{\{3\}} = 1/2$ has $(x,y,z,q) = (1/2, 1/2, 1, 0)$ and $\mathcal{T} = 34137/400000 > 0$. We instead add local covariance terms whose sum over the whole cover is nonnegative. Proving that each compensated term is nonpositive then bounds the original total.

Write $\boldsymbol{f}(t) = (f_1(t), f_2(t))$, with the values in Table 3, and define

$$\Psi(t_1,t_2,t_3,q) = \sum_{1 \le a < b \le 3} \langle \boldsymbol{f}(t_a), \boldsymbol{f}(t_b) \rangle \big(q - (1 - t_a)(1 - t_b)\big). \tag{18}$$

For each u, draw $S \ni u$ with probability z_S and set

$$Z_u = \sum_{v \in U} \boldsymbol{f}(x_{uv})\big(\mathbf{1}[v \in S] - (1 - x_{uv})\big).$$

The summand for $v = u$ is zero. Since $\mathbb{E}\mathbf{1}[v \in S] = 1 - x_{uv}$ and $\mathbb{E}\mathbf{1}[v, w \in S] = q_{uvw}$, expanding the square and summing the three pivot positions gives

$$\sum_{u \in U} \mathbb{E}[G_U(A) \mid u] + \frac{1}{2} \sum_{u \in U} \mathbb{E}\|Z_u\|_2^2 = \frac{1}{6} \sum_{(u,v,w) \in U^3} (\mathcal{T} + \Psi)(x_{uv}, x_{uw}, x_{vw}, q_{uvw}). \tag{19}$$

The first expectation uses the full candidate law; the second uses the conditional support set. Appendix A verifies the counting, including repeated vertices. The square is nonnegative, so a nonpositive right side implies that some candidate has $G_U(A) \le 0$. For the example above, the compensated term is $\mathcal{T} + \Psi = -1087/500000 < 0$. Thus the added terms retain a valid global bound while allowing positive individual excesses. This is the conditional covariance method of Cao et al. (2025a, Section 6) and the variance compensation of García-Soriano & Schohn (2026, Lemma 7.2), written as one vector square. We use it for the universal auxiliary objective.

Lemma 5 (Computer-assisted triangle inequality)**.** *For the rational parameters in Table 2 and Appendix E,*

$$\mathcal{T}(x, y, z, q) + \Psi(x, y, z, q) \le 0 \tag{20}$$

for every

$$0 \le x \le y \le z \le 1, \qquad z \le x + y, \qquad \max\{0, 1 - x - y\} \le q \le 1 - z. \tag{21}$$

The proof uses computer-assisted exact rational polynomial calculations (Section 4.3 and Appendix F).

Domain (21) is exactly the set of three-vertex cover marginals (Appendix A). The square identity links the different triples of a larger cover.

Proof of Theorem 1, given Lemma 5. Restricting the cover to any remaining set U preserves feasibility and the marginals on its vertices. Every ordered triple, including repeated vertices, satisfies (21) after sorting its distances. Lemma 5 and (19) give $\sum_u \mathbb{E}[G_U(A) \mid u] \le 0$, so some candidate has $G_U(A) \le 0$. Together with (13), each budget admits such a candidate. Algorithm 1 minimizes the left side of (12) over both families, so each step decreases the potential by at least its cost. Summing gives $D_b(C) \le \mathcal{P}(V) \le K$. Lemma 4 proves the simultaneous guarantee. For support size s, there are $O(n^2 s)$ candidates per step and at most n steps. Their scores are rational and can be evaluated in polynomial time. □

Algorithm 1 uses random candidates only to prove existence. It enumerates their possible outputs. For a uniform threshold, one rational midpoint between each pair of consecutive pivot distances suffices; for a fixed threshold, equality is included. Appendix B gives the finite enumeration and bit bounds.

4.3 Exact verification of the local inequality

Lemma 5 is the sole computer-assisted proof obligation in our analysis. Its verification uses exact rational arithmetic. The expression is piecewise rational and affine in q, so checking the two endpoints in (21) suffices. The breakpoints divide each distance interval into 18 pieces. On each feasible sorted cell and each endpoint, multiplying by $(1+x)(1+y)(1+z) > 0$ gives a rational polynomial P. For each tetrahedron in a cell triangulation, we check

$$(\textstyle\sum_i u_i)^d P\Big(\frac{\sum_i u_i v_i}{\sum_i u_i}\Big) = \sum_{|\alpha|=d} c_\alpha u^\alpha, \qquad c_\alpha \le 0, \tag{22}$$

where $d = \deg P$. Nonnegative barycentric coordinates prove $P \le 0$ throughout the tetrahedron. Rational bisection refines inconclusive cases, following García-Soriano & Schohn (2026, Section 7.4).

The rechecker reconstructs the subdivisions and verifies coverage of 2,261 cells and 24,153 terminal tetrahedra, with 1,871,766 integer coefficients. Appendix F gives the full argument, including

threshold equality, one-sided limits, and lower-dimensional boundaries. The analytic bounds in Appendix H leave the region in Appendix G.2 unresolved; the proof therefore uses the full certificate. Theorem 1, Theorem 2, and the upper bound in Proposition 3 inherit this dependence. Appendix G.3 identifies the verifier and its inputs; Appendix G.5 separates optional tests from the proof.

5 Constructing a common fractional cover

Theorem 1 applies to any supplied cover. To prove Theorem 2, we must construct one with small cost for the chosen norm, despite the exponentially many sets in (2). We first specialize the weighted chromatic cluster-LP solver of Abbasi et al. (2026, Section 6 and Appendix C.3).

Lemma 6 (Scalar fractional solution; specialization of Abbasi et al. (2026))**.** *For every fixed $\eta > 0$ and rational probability-weighted instance, a randomized polynomial time procedure returns an explicit feasible cover satisfying*

$$D_a(\boldsymbol{x}) \le (1+\eta) \min_C D_a(C) \tag{23}$$

with any prescribed success probability $1-\delta$.

Proof sketch. Use one color, with $w^+(e) = 1 - a_e$ and $w^-(e) = a_e$. The cited LP becomes (2) with objective $D_a(\boldsymbol{x})$. Its solver returns a polynomial-support cover within $1 + O(\theta)$ of the integer optimum; choosing θ sufficiently small and amplifying success gives the claim. Appendix C gives the full specialization, exact feasibility check, and finite sampling argument. □

5.1 Averaging scalar covers

We assign larger weights to layers with larger cumulative costs, then average the returned covers. Hölder's inequality controls each step. The scalar comparison is with the integer optimum, which is all this argument needs.

Parameters and easy instances. First choose each pair's cheaper relation, breaking ties toward joining. If all layers agree and these relations form a partition, return it: it is optimal in every layer. Otherwise every partition pays at least $1/2$ in some layer, so $\mathrm{OPT}_r \ge 1/2$ for every $r \ge 1$; also $m > 0$. Choose rational $0 < \eta \le \min\{1/4, \varepsilon/(4\rho)\}$. For finite p, set $r = p$; for $p = \infty$, let r be the least positive integer with $(1+\eta)^r \ge L$. Choose rational $L^{1/r} \le M \le 2L^{1/r}$, using $M = 1 + \eta$ when $p = \infty$. If $r = 1$, one scalar call with equal weights suffices. For $r > 1$, set

$$\nu = \lceil r - 1 \rceil, \qquad \tau = \frac{4\nu m}{\eta}, \qquad T = \max\left\{1, \left\lceil \frac{2\tau(M-1)}{\eta} \right\rceil \right\}. \tag{24}$$

Scalar responses and the average cover. Start with $s_0 = \tau \mathbf{1}$. At step t, write $s = s_{t-1}$ and use rational weights satisfying

$$w_\ell = s_\ell^{r-1}, \qquad (1 - \eta/4) w_\ell \le \widehat{w}_\ell \le w_\ell. \tag{25}$$

All weights are positive. Normalize $\widehat{w}$ to form a scalar probability instance and apply Lemma 6 with accuracy $\eta/4$. Its response cost vector $v \in [0, m]^L$ obeys

$$y \cdot v \le \kappa\, \mathrm{OPT}_r, \qquad y = \frac{w}{\|s\|_r^{r-1}}, \qquad \kappa = \frac{1 + \eta/4}{1 - \eta/4}. \tag{26}$$

Indeed, $\|y\|_{r/(r-1)} = 1$, so $y \cdot \boldsymbol{D}(C_r^*) \le \mathrm{OPT}_r$ for an optimal partition C_r^*. Nonnegative costs and (25) transfer the scalar guarantee to y.

Accumulate each response z_t and return their average:

$$s_t = s_{t-1} + \boldsymbol{D}(\boldsymbol{x}_t), \qquad \bar{z} = \frac{1}{T} \sum_{t=1}^{T} z_t, \qquad \boldsymbol{D}(\bar{\boldsymbol{x}}) = \frac{s_T - \tau \mathbf{1}}{T}. \tag{27}$$

The cover $\bar{z}$ is feasible because every vertex has total weight one in every response. Algorithm 2 gives the complete procedure.

Algorithm 2 Multilayer approximation by averaging covers

Require: Rational probability layers, fixed p, accuracy $\varepsilon > 0$, failure bound $\delta \in (0, 1)$.

If the cheaper pair relations in all layers agree and form a partition, return it.
Choose η, r, M as above.
if $r = 1$ **then**
  Obtain an equal-weight scalar cover with accuracy $\eta/4$ and failure bound δ.
  return Algorithm 1 applied to this cover.
end if
Set τ, T by (24) and $s = \tau\mathbf{1}$.
for $t = 1, \dots, T$ **do**
  Compute rational $\widehat{w}$ satisfying (25).
  Set $a_e = \sum_\ell \widehat{w}_\ell a_{\ell,e} / \sum_\ell \widehat{w}_\ell$.
  Apply Lemma 6 with accuracy $\eta/4$ and failure bound δ/T, obtaining z_t.
  Set $s = s + \boldsymbol{D}(\boldsymbol{x}_t)$.
end for
return Algorithm 1 applied to $\bar{z} = T^{-1} \sum_{t=1}^{T} z_t$.

Proof sketch of Theorem 2. Condition on all scalar calls succeeding. Since $s_\ell \geq \tau$ and $v_\ell \leq m$, the shifted norm obeys

$$\|s + v\|_r - \|s\|_r \leq (1 + m/\tau)^{r-1} y \cdot v \leq (1 + \eta)\,\mathrm{OPT}_r\,.$$

Summing over the T responses and removing the initial shift gives $\|\boldsymbol{D}(\bar{\boldsymbol{x}})\|_r \leq (1+2\eta)\,\mathrm{OPT}_r$. For $p = \infty$, the chosen r satisfies $\|d\|_\infty \leq \|d\|_r \leq (1 + \eta)\|d\|_\infty$. Thus $\|\boldsymbol{D}(\bar{\boldsymbol{x}})\|_p \leq (1 + 4\eta)\,\mathrm{OPT}_p$, and universal rounding yields $\Phi_p(C) \leq \rho(1 + 4\eta)\,\mathrm{OPT}_p \leq (\rho + \varepsilon)\,\mathrm{OPT}_p$. A union bound over the scalar calls gives success probability $1 - \delta$. Appendix D.1 supplies the full proof, including the shift bound, number of calls, and rational computation. □

6 Conclusion

We have shown that one fractional cluster cover can support simultaneous rounding for all probability-weighted layers. By separating universal 2.030-rounding from norm-specific cover construction, we obtain a $(2.030 + \varepsilon)$-approximation for each fixed ℓ_p objective with no loss depending on the number of layers. Thus, the same rounding procedure serves different multilayer objectives through the choice of cover.

Improving the factor to 2, however, requires a change in the local analysis. With the current variance functions, $\rho = 2$ and $x = y = z = q = 1/2$ give $b(1/2) = B(1/2) = 1/2$, so $\mathcal{T} = 0$ while $\Psi > 0$. Hence, merely lowering ρ cannot certify factor two; different variance compensation or a stronger analysis is needed. This leaves universal factor-two rounding open.

Even at 2.030, a fully analytic proof remains incomplete because the candidate inclusion law and variance vectors change pieces at different thresholds. In particular, the bounds in Appendix H do not cover the mixed threshold region identified in Appendix G.2, where the proof still uses an exact rational certificate. Removing that dependence requires uniform bounds on the combined triangle and covariance terms across these changes.

Beyond rounding, faster common-cover construction must address both the cost of scalar optimization over exponentially many possible clusters and the precision-dependent number of calls required by averaging (Appendix D.1). Improvements within this framework must therefore preserve the explicit feasible covers and weighted cost guarantees used in the approximation proof.

REPRODUCIBILITY STATEMENT

The proofs appear in the main text and appendices. The anonymous supplement includes the rational parameters, full certificate, exact rechecker, separate symbolic spot checker, and deterministic rounding checks. Appendix G gives commands and dependencies.

AI USE STATEMENT

OpenAI generative AI assistants were used to develop and refine mathematical claims and proof strategies, and search and analyze related work. Verification included the full exact rechecker described in the paper. The authors are responsible for the final content of this work, including text, claims or artifacts produced with the aid of generative AI.

ETHICS STATEMENT

This work studies an abstract clustering objective and uses no personal or sensitive data.

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A COVER IDENTITIES AND LOCAL FEASIBILITY

Put $y_{uv} = \sum_{S \supseteq \{u,v\}} z_S = 1 - x_{uv}$ and $q_{uvw} = \sum_{S \supseteq \{u,v,w\}} z_S$. For each pivot u, the weights z_S on sets containing u sum to one. Under this probability distribution, $\mathbb{E} I_v = y_{uv}$ and $\mathbb{E} I_v I_w = q_{uvw}$, where $I_v = \mathbf{1}[v \in S]$.

Lemma 7. *After labeling the three distances $x = x_{12}$, $y = x_{13}$, $z = x_{23}$ so that $x \le y \le z$, their feasible domain with triple marginal q is exactly* (21).

Proof. The pair marginals lie in $[0,1]$, and $q \le 1 - z$. At vertex 1 the union bound gives $(1-x) + (1-y) - q \le 1$, so $q \ge 1 - x - y$. Nonnegativity gives $q \ge 0$. These inequalities also imply $z \le x + y$. Conversely, a point satisfying (21) is realized by

$$
\begin{aligned}
z_{\{1,2,3\}} &= q, \\
z_{\{1,2\}} &= 1 - x - q, & z_{\{1,3\}} &= 1 - y - q, & z_{\{2,3\}} &= 1 - z - q, \\
z_{\{1\}} &= x + y + q - 1, & z_{\{2\}} &= x + z + q - 1, & z_{\{3\}} &= y + z + q - 1.
\end{aligned}
$$

All seven weights are nonnegative, every vertex has coverage one, and the specified pair and triple marginals are recovered. □

For a residual set U, intersect all support sets with U, merge duplicates, and discard the empty set. This preserves coverage one for vertices in U and preserves their pair and triple marginals.

Derivation of (19)**.** For pivot u, let $h_u(v,w)$ be the expected excess charged to pair vw. Set $h_u(v,v) = 0$. Formula (17) remains valid when $u = v$ or $u = w$, since the pivot is always included. It also gives zero when $v = w$: then $b(0) = B(0) = 0$ and $J(t,t,1-t) = \pi(t)$. Ordering pair endpoints and expanding the square give, respectively,

$$
\begin{aligned}
\sum_{u \in U} \mathbb{E}[G_U(A) \mid u] &= \frac{1}{2} \sum_{u,v,w \in U} h_u(v,w), \\
\frac{1}{2} \sum_{u \in U} \mathbb{E}\|Z_u\|_2^2 &= \frac{1}{2} \sum_{u,v,w \in U} \langle \boldsymbol{f}(x_{uv}), \boldsymbol{f}(x_{uw}) \rangle [q_{uvw} - (1 - x_{uv})(1 - x_{uw})].
\end{aligned}
$$

Average each sum over the three pivot positions. The summands become $\mathcal{T}/6$ and $\Psi/6$, proving (19). Repeated vertices include the diagonal variance terms: $\Psi(0,t,t,1-t) = \|\boldsymbol{f}(t)\|_2^2 t(1-t)$. Terms involving $\boldsymbol{f}(0)$ have zero covariance, and terms with three equal vertices vanish.

B EXPLICIT DETERMINISTIC ROUNDING

Algorithm 1 includes every nonempty intersection $S \cap U$ of a positive-weight support set with the remaining vertices. Its other candidates are enumerated as follows. For a uniform threshold on $[a,b]$, sort $\{a,b\} \cup \{x_{uv} : v \in U,\ a < x_{uv} < b\}$. Between two successive values, the candidate set is constant; use the rational midpoint and probability equal to the interval length divided by $b - a$. Isolated threshold values have probability zero under a uniform law. For an atom, use its exact radius and include equality. Combine each radius with every positive-weight support set containing the pivot. Duplicates may be retained.

With support size s, there are $O(n^2 s)$ candidates per step: at most n pivots, a constant number of modes, $O(n)$ threshold intervals, and at most s sets. Scoring each by a direct sum over pairs and taking at most n steps uses $O(n^5 s)$ rational operations. Evaluating $\mathcal{P}$ uses two rational sums and their minimum. Distances, midpoints, and scores have polynomial bit length because all fixed parameters are rational.

C THE EXISTING PROBABILITY-WEIGHTED FRACTIONAL SOLVER

Lemma 6 is a one-color specialization of Abbasi et al. (2026, Sections 4–6 and Appendix C.3), using their July 14, 2026 version. Their model has a set of colors $\mathcal{L}$, weights satisfying $\sum_{j \in \mathcal{L}} w^j(e) +$

$w^-(e) = 1$, and a variable z_S^j for every set and color. The specialization is

$$\mathcal{L} = \{+\}, \qquad w^+(e) = 1 - a_e, \qquad w^-(e) = a_e, \qquad z_S = z_S^+.$$

Their vertex constraints become $\sum_{S \ni u} z_S = 1$, and their objective becomes $D_a(\boldsymbol{x})$. Thus both the feasible covers and the integer optimum are the same as ours.

The cited analysis bounds the returned fractional cost by $(1 + O(\theta))\,\mathrm{OPT}$, where OPT is the *integer* optimum. We use precisely this comparison. For fixed θ, their bounded LP and sampling construction take polynomial time and produce polynomially many sets. The construction gives rational weights explicitly: after selecting sampled sets $\widetilde{C}_1, \ldots, \widetilde{C}_J$ so that every vertex appears exactly k times, it sets

$$z_S = \frac{|\{j : \widetilde{C}_j = S\}|}{k}, \qquad \sum_{S \ni u} z_S = \frac{k}{k} = 1.$$

Empty sets can be discarded and duplicates merged. This gives the explicit rational input required by Theorem 1.

Success probability and finite sampling. Choose the cited solver's constants for success probability at least $2/3$. Run it independently $r = \lceil \log_3(2/\delta) \rceil$ times and return the least-cost output with nonnegative weights and exact vertex coverage; if none is feasible, return any integral cover. Failure of all runs has probability at most $3^{-r} \le \delta/2$. For a rational distribution with common denominator d, draw uniformly from $\{0, \ldots, 2^{\lceil \log_2 d \rceil} - 1\}$ and reject integers at least d. Each attempt succeeds with probability at least $1/2$. If J_0 bounds the total number of draws, cap each at $\lceil \log_2(2J_0/\delta) \rceil$ attempts; abort the run if all attempts fail. A union bound contributes at most another $\delta/2$ failure probability. The output length and running time are polynomial in the rational input length and $\log(1/\delta)$ for fixed η.

D Norm bounds and rational weights

D.1 Full proof of the multilayer approximation

Proof of Theorem 2. Condition on all scalar calls meeting their stated guarantees. Since $s_\ell \ge \tau$ and $0 \le v_\ell \le m$, Hölder's inequality gives

$$\|s + v\|_r - \|s\|_r \le \frac{\sum_\ell (s_\ell + v_\ell)^{r-1} v_\ell}{\|s + v\|_r^{r-1}} \le (1 + m/\tau)^{r-1} y \cdot v \le (1 + \eta)\,\mathrm{OPT}_r\,. \tag{28}$$

The last step uses $(1 + m/\tau)^{r-1}\kappa \le 1 + \eta$. Also, $\|u + \tau \mathbf{1}\|_r \ge \|u\|_r + \tau$ for $u \ge 0$. Summing (28) therefore gives

$$\|\boldsymbol{D}(\bar{\boldsymbol{x}})\|_r \le \frac{\|s_T\|_r - \tau}{T} \le (1 + \eta)\,\mathrm{OPT}_r + \frac{\tau(M-1)}{T} \le (1 + 2\eta)\,\mathrm{OPT}_r\,. \tag{29}$$

Appendix D.2 verifies the two elementary inequalities and the rational computation.

For $p = \infty$ and $L > 1$, Bernoulli's inequality gives $(1 + \eta)^{\lceil 1/\eta \rceil} \ge 2$, so $r \le \lceil 1/\eta \rceil \lceil \log_2 L \rceil$. The defining inequality for r gives $\|d\|_\infty \le \|d\|_r \le (1 + \eta)\|d\|_\infty$. Thus in all cases $\|\boldsymbol{D}(\bar{\boldsymbol{x}})\|_p \le (1 + 4\eta)\,\mathrm{OPT}_p$. Theorem 1 now gives

$$\Phi_p(C) \le \rho(1 + 4\eta)\,\mathrm{OPT}_p \le (\rho + \varepsilon)\,\mathrm{OPT}_p\,.$$

There are polynomially many scalar calls, each with conditional failure probability at most δ/T. A union bound proves Theorem 2. The number of scalar calls is $O(1 + \lceil p - 1 \rceil m L^{1/p}/\eta^2)$ for finite p and $O(1 + m \log(2L)/\eta^2)$ for $p = \infty$; here $T = 1$ when $r = 1$. □

D.2 Auxiliary inequalities and finite precision

Two elementary bounds. Set $a = \eta/4$ and $\nu = \lceil r - 1 \rceil$. The binomial theorem gives $(1 + a/\nu)^\nu \le \sum_{j \ge 0} a^j = (1 - a)^{-1}$. Hence

$$(1 + m/\tau)^{r-1}\kappa \le \frac{1 + \eta/4}{(1 - \eta/4)^2} \le 1 + \eta,$$

where the last inequality is equivalent to $\eta(4 - 7\eta + \eta^2) \geq 0$. For $u \geq 0$, $u \neq 0$, the coordinatewise inequality $\mathbf{1} \geq u/\|u\|_r$ gives $\|u + \tau\mathbf{1}\|_r \geq \|u\|_r + \tau$; the case $u = 0$ is immediate. Finally, $\tau(M-1)/T \leq \eta/2$ and $\mathrm{OPT}_r \geq 1/2$ give (29). For $p = \infty$, $(1+\eta)(1+2\eta) \leq 1 + 4\eta$.

Rational computation. For nonnegative costs, the bounds on $\widehat{w}$ imply

$$w \cdot v \leq \frac{\widehat{w} \cdot v}{1 - \eta/4} \leq \frac{1 + \eta/4}{1 - \eta/4} \min_C \widehat{w} \cdot \boldsymbol{D}(C) \leq \kappa \min_C w \cdot \boldsymbol{D}(C).$$

Dividing by $\|s\|_r^{r-1}$ and applying Hölder proves (26). Choose a fixed dyadic mesh $h \leq \eta/8$ with $\log(1/h) = O(\log(1/\eta))$. Compute each power to absolute error at most h from below, then round down to this mesh. Since $\tau \geq 1$, we have $w_\ell \geq 1$, so the total error is at most $2h \leq \eta w_\ell/4$. For fixed rational r, integer powering and root bisection suffice, also for choosing M; repeated rational multiplication finds r when $p = \infty$. The bound $w_\ell \leq (\tau + Tm)^{r-1}$ gives polynomial bit length independently of earlier responses. The same holds for the scalar inputs and returned covers; (27) therefore has polynomial support and encoding length.

E RATIONAL PARAMETERS OF THE CERTIFICATE

Table 2 specifies the candidate distribution for the B-budget bound. Its weights sum exactly to one. The cutoff is $4/5$, $\rho = 203/100$, and $\lambda = 39/4$. Table 3 gives both variance functions on their 15 intervals, including every endpoint. Both variances have coefficient one. All candidate probabilities and function values are integer multiples of $1/1000$.

Table 3: The two variance functions. Divide the final two columns by 1000; both variances have coefficient one.

Interval	$1000 f_1$	$1000 f_2$
$[0, 3/8]$	0	898
$(3/8, 2/5]$	2	-58
$(2/5, 17/40]$	201	104
$(17/40, 9/20]$	179	85
$(9/20, 23/50]$	66	182
$(23/50, 47/100]$	115	77
$(47/100, 19/40]$	109	-103
$(19/40, 12/25]$	185	-269
$(12/25, 49/100]$	269	-452
$(49/100, 1/2]$	155	-571
$(1/2, 51/100]$	189	-615
$(51/100, 13/25]$	189	-619
$(13/25, 21/40]$	116	-512
$(21/40, 11/20]$	-123	-243
$(11/20, 1]$	0	0

F THE EXACT CONTINUOUS CERTIFICATE

F.1 PIECEWISE POLYNOMIALS

The sorted union of all relevant breakpoints is

$$\mathcal{A} = \{0, \tfrac{103}{406}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{17}{40}, \tfrac{9}{20}, \tfrac{23}{50}, \tfrac{47}{100}, \tfrac{19}{40}, \tfrac{12}{25}, \tfrac{49}{100}, \tfrac{1}{2}, \tfrac{51}{100}, \tfrac{13}{25}, \tfrac{21}{40}, \tfrac{11}{20}, \tfrac{303}{406}, \tfrac{4}{5}, 1\}. \tag{30}$$

The two clipping breakpoints arise from b and k. For each nondecreasing triple of interval indices, intersect the closed box with $x \le y \le z$ and $z \le x + y$. For the upper endpoint take $q = 1 - z$. For the lower endpoint form separate cells with $q = 0$, $x + y \ge 1$, and with $q = 1 - x - y$, $x + y \le 1$. Empty cells are discarded only after exact feasibility analysis. On a cell interior, uniform-threshold survival functions are affine or constant, and variance functions are constant. Therefore

$$P(x, y, z) = (1 + x)(1 + y)(1 + z)\big(\mathcal{T}(x, y, z, q) + \Psi(x, y, z, q)\big) \tag{31}$$

is a rational polynomial after endpoint substitution. The denominator-clearing factor is strictly positive on the domain.

F.2 GEOMETRY AND HOMOGENEOUS COEFFICIENTS

Enumerate intersections of every independent triple of bounding planes and retain the points satisfying every halfspace. This produces all vertices of a bounded three-dimensional cell. Exact determinants determine dimension. For a full-dimensional cell, its vertex mean is interior. Order each facet's vertices by a rational planar convex-hull computation, triangulate by a fan, and cone these triangles to the interior point. The resulting tetrahedra cover the cell with disjoint interiors. Duplicate facets are removed.

For a tetrahedron with vertices $v_0, \dots, v_3$, let $d = \deg P$, $S = u_0 + \cdots + u_3$, and homogenize:

$$\widehat{P}(\boldsymbol{u}) = S^d P\left(\frac{\sum_{i=0}^{3} u_i v_i}{S}\right) = \sum_{|\alpha|=d} c_\alpha \boldsymbol{u}^\alpha. \tag{32}$$

The expansion defines a polynomial, including at $S = 0$. If every $c_\alpha \le 0$, then $P \le 0$ on the tetrahedron by taking $u_i \ge 0$ and $S = 1$. Bernstein coefficients differ only by positive multinomial factors. Clear rational denominators by a positive integer scale and divide by a positive common divisor, preserving all signs.

If this sufficient sign test fails, bisect the longest edge and test the two child tetrahedra. Rational midpoints preserve exact coverage. The certificate stores all terminal tetrahedra and normalized coefficients. The full rechecker regenerates cells and initial triangulations, re-expands polynomials, recomputes all coefficients, and reconstructs the longest-edge subdivision trees. Every saved leaf must be consumed exactly once, with no uncovered branch.

For the redundant repeated-vertex check, define

$$Q(t) = \frac{1 + t}{2}(\mathcal{T} + \Psi)(0, t, t, 1 - t).$$

For a one-dimensional interval $[a, b]$, substitute $t = a + (b - a)u$ in Q. If the resulting power coefficients are q_i and its degree is d, its Bernstein coefficients are

$$\beta_k = \sum_{i=0}^{k} q_i \frac{\binom{k}{i}}{\binom{d}{i}}.$$

The verifier recomputes these rational values, checks their signs, and checks interval coverage. Identically zero polynomials need no subdivision.

F.3 BOUNDARY VALUES

Fixed thresholds include equality. A step function takes the value of the interval ending at the positive breakpoint; at zero it takes the first interval value. The actual formulas at positive breakpoints are therefore their limits from below.

Table 4: Exact certificate totals for $\rho = 203/100$. Counts refer to proof objects and verification operations.

Object or check	Count
Full-dimensional cells	2,261
Nonzero cell polynomials	2,259
Nonempty lower-dimensional descriptions	64
Certified terminal tetrahedra	24,153
Maximum bisection depth	6
Integer homogeneous coefficients	1,871,766
One-dimensional Bernstein coefficients	77
Direct three-point probability checks	2,261
Direct one-dimensional evaluations	77
Separate symbolic leaf expansions	19
Nonzero coefficients in those expansions	1,484

A short perturbation verifies all actual boundary values. For any feasible triple $t = (x, y, z)$, set

$$t^{(\epsilon)} = (1 - \epsilon)t + \epsilon^2(1, 5/4, 3/2), \qquad \epsilon \downarrow 0.$$

For small $\epsilon > 0$, the new coordinates are strictly ordered and form a strict triangle. Every originally positive coordinate decreases, so it stays in its assigned piece even at a breakpoint; zero coordinates enter the first piece. A strict sign of $x + y - 1$ is preserved; at equality the perturbation approaches from $x + y < 1$. The endpoint functions $1 - z$ and $\max\{0, 1 - x - y\}$ are continuous. Hence the certified cell closures give the actual one-sided values of every term in (20). This argument also covers repeated distances and zero distances.

The rechecker also enumerates all 64 nonempty lower-dimensional cell descriptions and checks containment in the appropriate adjacent certified closures. It uses the above endpoint convention rather than assuming continuity across jumps. Affineness in q extends the result from both endpoints to the whole feasible interval. Together with the passing coefficient checks, this proves Lemma 5.

G Computer-assisted proof: scope, files, and reproduction

G.1 Which claims depend on computation?

The sole computer-assisted proof obligation in this paper is Lemma 5: the inequality $\mathcal{T} + \Psi \leq 0$ on the entire continuous domain (21), for the fixed rational parameters in Appendix E. The computer checks polynomial coefficients and exact geometric coverage. The reduction to these finite checks, including the treatment of discontinuous thresholds and all boundary values, is proved in Appendix F.

The dependence of the final results is as follows. Theorem 1 combines this lemma with the analytically proved square identity (19), support average (13), and potential inequality (12). Theorem 2 then uses that rounding theorem and the analytic common-cover argument in Appendix D.1, together with the cited solver specialized in Appendix C. The upper bound in Proposition 3 also uses Theorem 1; its lower bound is the elementary two-vertex construction. Thus both approximation guarantees and the gap upper bound are computer-assisted, through the same local lemma.

G.2 What remains after the analytic bounds?

Appendix H supplies optional direct proofs on the following regions, always subject to (21):

Region proved analytically	Location
$z \leq 17/40$	Appendix H.3
$x \geq 12/25$	Appendix H.11
$y > 11/20$	The argument for two long distances in Appendix H.6
$3/8 < x \leq y \leq 23/50$	The combined conclusion in Appendix H.12

Consequently, the region not covered by these direct arguments is

$$x < 12/25, \qquad y \le 11/20, \qquad z > 17/40, \\ x \le 3/8 \quad \text{or} \quad y > 23/50, \tag{33}$$

intersected with (21), with the full feasible interval of q. The argument in Appendix H.4 further reduces the case $z > 4/5$ to boundary cases; that reduction does not establish the inequality on all of (33). The certificate checks the *whole* domain, including the portions already proved analytically. Hence the proof of Lemma 5 does not require stitching the optional analytic arguments to a smaller computer certificate.

G.3 Files that carry the computer-assisted proof

All filenames in the following table are relative to `verification/`. The verification entry point is `mixture_rounding_verification.py`; its function `verify_saved` performs the full continuous certificate check.

Table 5: Proof data, verifier, and supporting routines for Lemma 5.

File	Role in the proof
`covariance_2030_config.json`	Exact parameters of the candidate law, budgets, and two variance functions; input for certificate and table generation.
`covariance_2030_certificate.json`	Saved proof object: embedded configuration, cells, polynomials, leaf tetrahedra, and integer coefficients.
`mixture_rounding_verification.py`	`verify_saved` enumerates all regimes; `verify_one_cell` recomputes each polynomial and coefficient, and reconstructs the subdivision tree.
`threshold_mixture_certificate.py`	`parse_config`, `expressions`, and `inequalities` produce the breakpoints, endpoint polynomials, and defining halfspaces.
`exact_polytope_certificate.py`	`tetrahedra` constructs exact cell geometry; `prepared` and `homogeneous_coefficients` recompute the integer coefficient lists.
`smooth_pivot_certificate.py`	`auxiliary_polys` supplies the rational pieces of the auxiliary pair cost and budgets.
`mixed_pivot_certificate.py`	`rat` converts exact fractions; `univariate` supplies interval certification during generation.

The verifier reads the configuration embedded in the certificate, then regenerates all regimes instead of trusting the saved list of successful cells. It checks all 2,261 full-dimensional cells, 24,153 terminal tetrahedra, and 1,871,766 integer coefficients, reconstructs subdivision coverage, and checks the containment of 64 lower-dimensional descriptions in certified closures. It also checks 77 one-dimensional coefficients; these are redundant boundary checks, as explained in Appendix F. The mathematical step from nonpositive coefficients to a continuous inequality is (32).

`parallel_threshold_certificate.py` generates the saved proof object from `covariance_2030_config.json`. Regeneration is optional: the full verifier recomputes the saved coefficients and geometry.

G.4 Reproducing the full check

The supplement includes the local verification modules and their inputs. The checks were rerun with Python 3.12, NumPy 1.26.4, SciPy 1.13.1, and SymPy 1.12; these package versions are pinned in `verification/requirements.txt`. NumPy and SciPy construct small LPs. Exact checks use Python integers, rational fractions, and SymPy. From the `verification` directory run:

```
python -m pip install -r requirements.txt
python check_manifest.py
python mixture_rounding_verification.py \
  covariance_2030_certificate.json \
  ../revision_audit/full_certificate_recheck.json --workers 4
```

The last command runs the full continuous check before the driver's additional finite tests.

Optionally regenerate a certificate from the fixed configuration:

```
python parallel_threshold_certificate.py \
  covariance_2030_config.json \
  regenerated_certificate.json --workers 4
```

Elapsed-time metadata may differ. Run `python generate_tables.py` from the source root to rebuild the parameter tables. `check_manifest.py` compares 19 supplied files with the SHA-256 digests in `source_manifest.json`, covering the verification sources, configuration, certificate, and original recorded results. A matching digest establishes file integrity; it does not replace a mathematical check.

G.5 Additional diagnostics and their limits

The full rechecker verifies continuous coefficients and coverage. Direct probability checks instantiate the seven-set cover of Lemma 7 at a rational point of every cell and enumerate the candidate law, testing agreement of formulas with the implementation. The separate `independent_bernstein_check.py` checker uses SymPy expansion; it checks 19 selected leaves and 1,484 nonzero coefficients.

The full rechecker also exercises its original two-procedure rounding on 225 rational covers on 2–10 vertices, 48 multilayer LP instances on 3–6 vertices, and one 11-vertex cover. Its 37 variance checks compare conditional second moments with the edge–triangle expansion, including residual subsets. These tests complement the continuous certificate.

The driver imports `near_two_verification.py`, `hybrid_rounding_verification.py`, `toward_two_search.py`, and `adaptive_two_counterexample.py` for these finite tests; the last also imports `mixed_rounding_verification.py`. These modules supply test covers, small LPs, and rounding utilities.

The separate `revision_checks/check_potential_rounding.py` harness, using `verification/potential_rounding.py`, tests the single-potential Algorithm 1 on those same classes of instances and five degenerate cases, including an empty graph: 279 cases and 718 steps in total. Each step checks (12) and both candidate-averaging bounds with exact fractions. Each output is checked by

$$\sum_e \max_{a_e \in \{0,1\}} \{d_{a_e}(c_e) - \rho d_{a_e}(x_e)\} = D_b(C) - K \le 0.$$

This tests every probability layer for the particular cover, rather than only its generating layers. The smaller residual budget changes in 19 cases, while the potential inequality continues to hold. Finite tests check implementation consistency; the continuous certificate establishes the universal local inequalities. The common-cover averaging guarantee is proved in Appendix D.1.

H DIRECT BOUNDS FOR PARTS OF THE LOCAL INEQUALITY

The certificate in Appendix F proves the whole local inequality. The following optional direct bounds explain several regions using elementary inequalities. Appendix G.2 lists the covered regions and states the region still requiring the certificate; Appendix G.3 identifies the proof files.

Write $F = \mathcal{T} + \Psi$, $\gamma = B - b$, $w = 1 - 2b$, $H = W - g$, $\alpha = 103/406$, and $\beta = 303/406$. We give direct proofs on several continuous regions and a boundary reduction for $z > 4/5$. Throughout, $x \le y \le z$ and q satisfies (21). Every terminating decimal below denotes an exact rational number.

H.1 QUADRATIC BOUNDS USED BELOW

The following elementary lemma combines the Taylor expansions and completion of squares used in the direct proofs.

Lemma 8. *Let P be twice continuously differentiable on a convex set and suppose $\nabla^2 P \preceq -\operatorname{diag}(m_1, \dots, m_d)$, with $m_i > 0$. For a point c in the set, write $p_i = \partial_i P(c)$. If $p_i(t_i - c_i) \le 0$ for every $i \notin I$, then*

$$P(t) \le P(c) + \sum_{i \in I} \frac{p_i^2}{2m_i}. \tag{34}$$

Also, if $U'' \ge -M$ on $[l, u]$, with $M \ge 0$, then

$$U(z) \le \max\{U(l), U(u)\} + \frac{M(u-l)^2}{8}. \tag{35}$$

Proof. Taylor's inequality bounds $P(t)$ by $P(c) + \sum_i [p_i(t_i - c_i) - m_i(t_i - c_i)^2/2]$. Omit nonpositive summands and complete the remaining squares. For the second claim, $U(z) + Mz^2/2$ is convex, so its chord gives an error at most $M(z-l)(u-z)/2 \le M(u-l)^2/8$. □

All applications keep the formula of the interval being proved at its endpoints. This is essential when a variance vector or threshold law jumps.

H.2 TWO ELEMENTARY BOUNDS

The Bernoulli variables defining a conditional covariance have variances $s(1-s)$ and $t(1-t)$. Cauchy–Schwarz and $2ab \le a^2 + b^2$ give

$$\Psi \le \sum_{t=x,y,z} \|\boldsymbol{f}(t)\|_2^2 t(1-t). \tag{36}$$

This bound permits different vectors at the three distances and either sign of their inner products. On the middle interval $[\alpha, \beta]$, direct substitution gives

$$\gamma(t) = \frac{22939t^2 - 20134t + 4429}{400(1+t)} = \frac{22939(t - 10067/22939)^2 + 11 + 13/22939}{400(1+t)}, \tag{37}$$

$$\gamma'(t) = \frac{22939 - 47502/(1+t)^2}{400}, \qquad \gamma''(t) = \frac{95004}{400(1+t)^3} > 0. \tag{38}$$

For $t \ge \beta$,

$$B(t) = (1-t)\left[(1+\lambda)\rho - \frac{\lambda}{1+t}\right] \ge 12(1-t). \tag{39}$$

H.3 ALL DISTANCES AT MOST 17/40

All candidates contain all three vertices, so $\pi = J = 1$ and $\mathcal{T} = -\sum_t \gamma(t)$. We show $\gamma(t) \ge \|\boldsymbol{f}(t)\|_2^2 t(1-t)$ on this interval; then (36) proves $F \le 0$. For $t \le \alpha$, use $b(t) = 0$, $B(t) \ge (929/400)t$, and $\|\boldsymbol{f}(t)\|_2^2 \le 81/100$. For $\alpha \le t \le 3/8$, the same norm bound applies, and

$$(1+t)\left[\gamma(t) - \frac{81}{100}t(1-t)\right] = \frac{324t^3 + 22939t^2 - 20458t + 4429}{400} =: p(t).$$

Since $p'' > 0$ and $p'(3/8) < 0$, we have $p(t) \geq p(3/8) = 17/51200 > 0$. For $3/8 < t \leq 17/40$, the table gives $\|\boldsymbol{f}(t)\|_2^2 < 13/250$; (37) gives $\gamma(t) > 11/600 > 13/1000$, whereas $\|\boldsymbol{f}(t)\|_2^2 t(1-t) < 13/1000$.

H.4 A LARGEST DISTANCE ABOVE $4/5$

If $y > 4/5$, only $\pi(x)$ can be nonzero, so $F = -(B(y) + B(z))\pi(x) \leq 0$ by (39). Otherwise, fix $x \leq y \leq 4/5$ and vary z in $(4/5, \min\{1, x+y\}]$. Then $\pi(z) = \boldsymbol{f}(z) = 0$. Set $P = \pi(x) + \pi(y)$ and $J = J(x, y, q) = a + cq$, where $c \geq 0$. Apart from terms affine in q, the only varying contribution to F is $B(z)(J - P) + J$. Since

$$B'(z) = -(1+\lambda)\rho + \frac{2\lambda}{(1+z)^2} < 0, \qquad B''(z) = -\frac{4\lambda}{(1+z)^3} < 0,$$

the lower endpoint $q = \max\{0, 1 - x - y\}$ gives $F'' = B''(J - P) \geq 0$. At the upper endpoint $q = 1 - z$, $F'' = B''(J - P) - 2cB' \geq 0$. Both restrictions are convex. Their maxima therefore occur at the right limit $z \downarrow 4/5$ or at $z = \min\{1, x + y\}$. Equality at $z = 4/5$ uses the actual inclusion rule, not its right limit.

H.5 ALL DISTANCES AT LEAST $1/2$

First suppose $z \leq 4/5$. Let $a(t) = \gamma(\min\{t, 11/20\})$ on $[1/2, 4/5]$. Equations (38)–(39) show that a is nonnegative and nondecreasing, and $a \leq \gamma$. Above β, use $\gamma = B - 1 \geq 7/5 > \gamma(11/20)$. The marginal π is nonincreasing. For the edge of length r, write $d_r = \pi(s) + \pi(t) - J(s, t, q)$ and $s_r = \pi(s) + \pi(t) - 2J(s, t, q)$. Since $b(r) \geq 1/2$,

$$T_r = -\gamma(r) d_r + (1 - 2b(r)) s_r \leq -\frac{a(r)}{2}\big(\pi(s) + \pi(t)\big).$$

Here $s_r \geq 0$ and $J \leq \min\{\pi(s), \pi(t)\}$. Writing $t_1 = x$, $t_2 = y$, $t_3 = z$, opposite monotonicity gives

$$\frac{1}{2}\sum_i a(t_i) \sum_{j \neq i} \pi(t_j) - \sum_i a(t_i)\pi(t_i) = \frac{1}{2}\sum_{i<j} (a(t_j) - a(t_i))(\pi(t_i) - \pi(t_j)) \geq 0.$$

Together with (36), this proves $F \leq 0$ once

$$a(t)\pi(t) \geq \|\boldsymbol{f}(t)\|_2^2 t(1-t) \qquad (1/2 \leq t \leq 4/5). \tag{40}$$

Above $11/20$ the right side vanishes. For the remaining intervals, the following rational bounds suffice; every entry in the last column is greater than one quarter of the preceding entry.

Interval	Upper bound on $\|\boldsymbol{f}(t)\|_2^2$	Lower bound on $\gamma(t)\pi(t)$
$[1/2, 51/100]$	$414/1000$	$104/1000$
$(51/100, 13/25]$	$419/1000$	$120/1000$
$(13/25, 21/40]$	$276/1000$	$129/1000$
$(21/40, 11/20]$	$75/1000$	$130/1000$

For completeness, these bounds require only the following arithmetic. On $[1/2, 13/25]$,

$$\pi(t) = 1 + \frac{3047}{500}t - \frac{68}{5}t^2, \quad \gamma' \geq 4.56, \quad \pi \geq 0.49, \quad \gamma \leq 0.267, \quad -\pi' \leq 8.05.$$

Thus $(\gamma\pi)' \geq 4.56(0.49) - 0.267(8.05) > 0$. At $1/2$ and $51/100$, respectively, $\gamma\pi$ equals $83463/800000 > 0.104$ and $36258618731/302000000000 > 0.120$. On the last two intervals, monotonicity of γ and π gives $\gamma\pi \geq 0.266(0.486) > 0.129$ and $\gamma\pi \geq 0.297(0.440) > 0.130$.

Now suppose $z > 4/5$. We may assume $y \leq 4/5$. Since $x + y \geq 1$, the preceding convexity argument reduces both q endpoints to $z \downarrow 4/5$ or $z = 1$. For $t \geq 1/2$, the table gives $f_2(t) \leq 0$ and $|f_1(t)| \leq |f_2(t)|$, so $\langle \boldsymbol{f}(x), \boldsymbol{f}(y) \rangle \geq 0$. If both vectors are nonzero, $x, y \leq 11/20$, and $(1-x)(1-y) \geq 81/400 > 1/5 \geq q$. Hence $\Psi \leq 0$ throughout this case. At $z \downarrow 4/5$, the two terms T_x, T_y are nonpositive, and $T_z = J - B(4/5)(P - J) \leq 0$ because $B(4/5) > 1$ and $J \leq P - J$.

At $z = 1$, necessarily $q = 0$. Set $A(t) = B(t) + b(t) - 1$. It remains to prove

$$J(x, y, 0) \le A(x)\pi(y) + A(y)\pi(x) + \langle \boldsymbol{f}(x), \boldsymbol{f}(y)\rangle (1 - x)(1 - y). \tag{41}$$

Here $A \ge 0$. If $y \ge 13/25$, then $H(y) = 0$ and $A(y) > 1/3$. Also $H(x) \le 0.272 < (7/25)W$ and $\pi(x) \ge W(1 - x)$. The ball contribution to J is either zero, or 0.022 with $x \le y \le 21/40$ and $\pi(x) > 0.440$. Therefore

$$J(x, y, 0) = \pi_b(y) + H(x)(1 - y) \le \left(\frac{1}{20} + \frac{7}{25}\right)\pi(x) = \frac{33}{100}\pi(x) < A(y)\pi(x),$$

which proves (41).

It remains that $1/2 \le x \le y \le 13/25$. The table gives $\langle \boldsymbol{f}(x), \boldsymbol{f}(y)\rangle \ge 7/20$. Define

$$R(x, y) = A(x)\pi(y) + A(y)\pi(x) + \frac{7}{20}(1 - x)(1 - y).$$

On this square, $A' \ge 8.6$, $\pi \ge 0.49$, $A \le 0.35$, and $-\pi' \le 8.1$. Thus both partial derivatives of R are at least $8.6(0.49) - 0.35(8.1) - 0.175 > 0$. In contrast,

$$J(x, y, 0) = \frac{3547}{500} - \frac{68}{5}(x - xy + y^2)$$

decreases in each coordinate on the sorted square. Consequently,

$$R(x, y) - J(x, y, 0) \ge R(1/2, 1/2) - J(1/2, 1/2, 0) = \frac{863}{400000} > 0.$$

This proves (41) and completes the region $x \ge 1/2$.

H.6 Two distances above $11/20$

Suppose $y > 11/20$. The cases $x \ge 1/2$ and $y > 4/5$ are already proved, so assume $x < 1/2$ and $y \le 4/5$. Since $\boldsymbol{f}(y) = \boldsymbol{f}(z) = 0$, we have $\Psi = 0$. For $11/20 < t \le 4/5$, the preceding formulas give

$$\gamma(t) > 0.47, \qquad A(t) = B(t) + b(t) - 1 > 0.67, \qquad \pi(t) < 0.441. \tag{42}$$

Indeed, γ and A increase on $[11/20, \beta]$; above β, use $B \ge 12(1 - t) \ge 2.4$. Also $A(t) > 1.35$ for $3/5 < t \le 4/5$.

If $z \le 4/5$, then $b(y), b(z) \ge 1/2$ and $T_y + T_z < -0.94\pi(x)$. Since $s_x \le d_x$ and $\gamma(x) \ge 0$,

$$T_x \le [1 - 2b(x) - \gamma(x)]_+\big(\pi(y) + \pi(z)\big).$$

For $x \le 17/40$, this is less than 0.882, whereas $\pi(x) = 1$. For $17/40 < x < 1/2$, use $1 - 2b(x) < 0.305$ and $\pi(y) + \pi(z) \le 2\pi(x)$. In either case, $F < 0$.

If $z > 4/5$, then $J(x, z, q) = J(y, z, q) = \pi(z) = 0$. Using $J(x, y, q) \le \pi(y)$ and dropping a nonpositive term gives

$$F \le (2 - b(x) - B(x))\pi(y) - A(y)\pi(x). \tag{43}$$

When $x < 1/5$, feasibility forces $y > 3/5$, so the right side is less than $2(0.978)(0.4) - 1.35 < 0$. For $1/5 \le x < 1/2$, we claim

$$2 - b(x) - B(x) \le \tfrac{3}{2}\pi(x). \tag{44}$$

On $[1/5, \alpha]$, B increases and $B(1/5) > 1/2$. On $[\alpha, 17/40]$, $\pi = 1$ and

$$(1 + t)(B(t) + b(t) - 1/2) = \frac{24563t^2 - 19122t + 3817}{400} > 0,$$

since the quadratic has positive leading coefficient and discriminant -9377000. On the last three intervals the following bounds suffice:

t	$(17/40, 47/100]$	$(47/100, 49/100]$	$(49/100, 1/2]$
$2 - b(t) - B(t)$, upper bound	1.3045	1.0658	0.9216
$\pi(t)$, lower bound	0.893	0.740	0.647

The upper bounds follow from $2 - b - B = 1 + (1 - 2b) - \gamma$, using $\gamma \ge 0$, $\gamma(0.47) > 0.056$, and $\gamma(0.49) > 0.119$, respectively. The lower bounds follow from monotonicity of π. Each upper bound is less than $3/2$ times the lower bound. Finally, (43) is at most $(1.5(0.441) - 0.67)\pi(x) < 0$.

H.7 TWO DISTANCES IN $(2/5, 17/40]$

Suppose $2/5 < x \le y \le 17/40$. Set

$$r = 17/40, \quad s = 21/50, \quad \boldsymbol{v} = (201, 104)/1000,$$
$$\kappa = \|\boldsymbol{v}\|^2 = 0.051217, \quad j(z) = \langle \boldsymbol{v}, \boldsymbol{f}(z) \rangle, \quad h(z) = 1 - \pi(z).$$

Both short edges have $\pi = 1$ and vector $\boldsymbol{v}$. Their incident pairs satisfy $J(x, z, q) = J(y, z, q) = \pi(z)$ and $J(x, y, q) = 1$. Thus

$$\mathcal{T} = 2\rho(1 - x - y)h(z) - \gamma(x) - \gamma(y) - \gamma(z). \tag{45}$$

The case $z \le r$ is already proved. Since $x + y < 1$, the feasible q endpoints are $1 - x - y$ and $1 - z$, and its coefficient in F is $\kappa + 2j(z)$.

First let $r < z \le 0.49$. Here $j(z) > 0$, so take $q = 1 - z$. With $L = -2\rho h + \kappa + j(1 - z)$, substitution yields

$$F = 2\rho h - \gamma(z) - \kappa z - \gamma(x) - \gamma(y) + L(x + y) - \kappa xy.$$

Each partial derivative in x, y is positive: for example,

$$\partial_x F \ge -4.06(0.2597) - \gamma'(r) + \kappa(1 - r) > 0, \qquad \gamma'(r) = -163821/144400.$$

We used $h \le 0.2597$ and increasing γ'. It therefore suffices to set $x = y = r$, giving

$$F_0(z) = 0.609h(z) - 2\gamma(r) - \gamma(z) + \kappa(2r - r^2 - z) + 2j(z)r(1 - z). \tag{46}$$

On each row below, $F_0'' = 0.609h'' - \gamma'' < 0$, since $h'' \le 42.4$ and $\gamma'' > 70$. The derivative at the indicated endpoint has the displayed sign, so the maximum is attained there. Direct substitution in (46) gives $F_0(e) < -0.001$ in every row. At a left endpoint, use the formula in that row, namely its right limit.

z interval	$10^6 j$	$h(z)$	e	Bound on $F_0'(e)$
$(0.425, 0.45]$	44819	$\frac{227}{35} z(z - r)$	0.45	> 0.92
$(0.45, 0.46]$	32194	$\frac{227}{35} z(z - r)$	0.46	> 0.24
$(0.46, 0.47]$	31123	$0.227z$	0.46	< -1.57
$(0.47, 0.475]$	11197	$z(21.2z - 9.716)$	0.475	> 3.52
$(0.475, 0.48]$	9209	$0.354z$	0.475	< -2.60
$(0.48, 0.49]$	7061	$17.6z^2 - 8.094z$	0.49	> 1.66

Next let $0.49 < z \le 0.55$. Here $\kappa + 2j < 0$, so take $q = 1 - x - y$. With $L = -2\rho h - j(1 + z)$,

$$F = 2\rho h - \gamma(z) + 2jz - \gamma(x) - \gamma(y) + L(x + y) - \kappa xy. \tag{47}$$

Let $m = (x + y)/2$ and $d = (y - x)/2$. Since $\gamma'' \ge 82 > \kappa$ on $[2/5, r]$, replacing x, y by m, m increases the right side by at least $(82 - \kappa)d^2$. Write the common value as t. Taylor's inequality gives $\gamma(t) \ge \gamma(s) + \gamma'(s)(t - s) + 41(t - s)^2$. Completing a square now gives

$$F \le U(z) - (82 + \kappa)\left(t - s - \frac{D(z)}{82 + \kappa}\right)^2 \le U(z), \tag{48}$$
$$D = L - \gamma'(s) - \kappa s,$$
$$U = 2\rho h - \gamma(z) + 2jz - 2\gamma(s) + 2Ls - \kappa s^2 + \frac{D^2}{82 + \kappa}.$$

The coefficient of j in (47) is $2z - (1 + z)(x + y) < 0$. We may therefore replace j by a lower bound. Use $j = -0.028229$ on $(0.49, 0.52]$, and retain its exact values -0.029932 and -0.049995 on $(0.52, 0.525]$ and $(0.525, 0.55]$.

For these choices,

$$U'' = 0.6496h'' - \gamma'' + \frac{2(D'^2 + DD'')}{82 + \kappa}, \quad D' = -4.06h' - j, \quad D'' = -4.06h''. \tag{49}$$

On $(0.49, 0.5]$, use $h = 17.6z^2 - 8.094z$. Here $D > 0$, $D'' < 0$, $|D'| < 39$, and $\gamma'' > 70$, so $U'' < 0.6496(35.2) - 70 + 2(39^2)/82 < 0$. Substitution gives $U'(0.5) > 1.4$ and $U(0.5) < -0.0008$; hence $U < 0$ on this interval.

On $(0.5, 0.52]$, use $h = 13.6z^2 - 6.094z$. The bounds $|D| < 1/2$, $|D'| < 33$, $|D''| < 111$, and $\gamma'' > 67$ give

$$U'' < 0.6496(27.2) - 67 + \frac{2(33^2 + 111/2)}{82} < -20.$$

At $z_0 = 101/200$, substitution gives $U(z_0) < -0.00037$ and $|U'(z_0)| < 0.03$. Lemma 8, with $m_1 = 20$, gives $U(z) < -0.00037 + 0.03^2/40 < 0$. On the last two intervals, respectively, $h = 0.978z$ and $h = 0.022 + 0.978z$. Here $|D| < 1$, $|D'| < 4.1$, $j < 0$, and $\gamma' > 5.9$, so $U' < 0.65 - 5.9 + 2(4.1)/82 < 0$. The right limits satisfy $U(0.52+) < -0.002$ and $U(0.525+) < -0.01$, which finishes this range.

Finally let $z > 0.55$. Now $\boldsymbol{f}(z) = 0$, so Cauchy–Schwarz gives $\Psi \le \kappa/4 < 0.013$. Also $2\rho(1 - x - y) \le 0.812$ and $\gamma(x) + \gamma(y) \ge 2\gamma(r) > 0.054$. For $z \le \beta$, (45) gives

$$F < 0.812(0.022 + 0.978z) - 0.054 - \gamma(z) + 0.013 < 0.$$

The last expression is negative at 0.55 and decreases, since $\gamma'(z) > 7$. For $\beta \le z \le x + y \le 0.85$, use $\gamma(z) \ge 12(1 - z) - 1 \ge 0.8$ and $h \le 1$ to obtain $F < 0.812 - 0.054 - 0.8 + 0.013 < 0$.

H.8 Short distances on opposite sides of $17/40$

Suppose $2/5 < x \le 17/40 < y \le 9/20$. Put

$$r = 0.425, \quad a = 227/35, \quad \boldsymbol{v}_1 = (201, 104)/1000, \quad \boldsymbol{v}_2 = (179, 85)/1000,$$
$$\kappa = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0.044819, \quad j_i(z) = \langle \boldsymbol{v}_i, \boldsymbol{f}(z) \rangle, \quad g_y = a(y - r), \quad h_y = y g_y.$$

Write $w(t) = 1 - 2b(t)$ and $h(z) = 1 - \pi(z)$. Thus $w(t) = \rho(1 - 2t)$ on the middle interval, $\pi(x) = 1$, and $\pi(y) = 1 - h_y$. For $z \le 4/5$, the two q endpoints give the following expressions, where $\Phi = (w(x) + w(y))h - \gamma(x) - \gamma(y) - \gamma(z)$:

$$\begin{aligned} P_+ &:= F(x, y, z, 1 - z) \\ &= \Phi + (\gamma(x) - w(x) + w(z))h_y + \kappa(x + y - xy - z) + j_1 x(1 - z) + j_2 y(1 - z), \qquad (50) \\ P_- &:= F(x, y, z, 1 - x - y) \\ &= \Phi + g_y\big[(w(x) + w(z))y + (2w(x) - \gamma(x))(x - z)\big] - \kappa xy \\ &\quad + j_1(z - y - xz) + j_2(z - x - yz). \qquad (51) \end{aligned}$$

These follow from $J(x, y, q) = \pi(y)$, $J(x, z, q) = \pi(z)$, and $J(y, z, q) = \pi(z) - g_y(1 - z - q)$. Since F is affine in q, both endpoints suffice. Boundary evaluations below use the displayed formulas with vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ retained; at $y = r$ this is the right limit.

The range $z \le 0.44$. Here $\boldsymbol{f}(z) = \boldsymbol{v}_2$. The coefficient of q is $2\kappa + \|\boldsymbol{v}_2\|^2 - (2w(x) - \gamma(x))g_y > 0$: the subtracted term is at most $0.812a(0.44 - r) < 0.08$, whereas $2\kappa + \|\boldsymbol{v}_2\|^2 > 0.12$. Thus use P_+. The function $\gamma - w$ increases on $[0.4, r]$, giving $\gamma(x) - w(x) + w(z) \le \gamma(r) < 0.028$. Also $h \le 0.043$, $w(x) + w(y) < 0.711$, $\gamma(x) > 0.027$, and $\gamma(y), \gamma(z) > 0.019$ by (37). The covariance factors are at most $0.245, 0.245, 0.247$. Using $\kappa < 0.045$ and $\|\boldsymbol{v}_2\|^2 < 0.04$ gives

$$P_+ < (0.711 + 0.028)(0.043) + 2(0.045)(0.245) + 0.04(0.247) - 0.065 < 0.$$

Curvature on $0.44 < z \le 0.55$. We bound the Hessians in x, y, keeping z fixed. For $0.4 \le x \le r \le y \le \min\{0.45, z\}$, direct differentiation gives

$$\begin{aligned} (P_+)_{xx} &= -(1 - h_y)\gamma''(x), \\ (P_+)_{yy} &= -\gamma''(y) + 2a(\gamma(x) - w(x) + w(z)), \\ (P_+)_{xy} &= (\gamma'(x) + 2\rho)h'_y - \kappa, \\ (P_-)_{xx} &= -[1 - g_y(z - x)]\gamma''(x) - 2g_y(\gamma'(x) + 4\rho), \\ (P_-)_{yy} &= -\gamma''(y) + 2a(w(x) + w(z)), \\ (P_-)_{xy} &= a\big[(\gamma'(x) + 4\rho)(z - x) - \gamma(x) + 2\rho(1 + r) - 4\rho(x + y)\big] - \kappa. \end{aligned}$$

The following bounds are uniform in this region:

	P_{xx} at most	P_{yy} at most	$\lvert P_{xy}\rvert$ at most	m_x	m_y
P_+	-76	-77	9	67	68
P_-	-80	-70	8	72	62

Here is how to check the bounds without a multivariable search. For the first row, use

$$(1-h_{0.45})\gamma''(r)>76,\quad \gamma''(0.45)-2a\gamma(r)>77,\quad (\gamma'(r)+2\rho)a(0.9-r)-\kappa<9,$$

together with $\gamma'(0.4)+2\rho>0$. For the second row, $[1-a(0.45-r)(0.15)]\gamma''(r)>80$ and $\gamma'(0.4)+4\rho>0$ control P_{xx}. The function $\gamma''(y)-2a(w(0.4)+w(y))$ decreases in y and exceeds 70 at 0.45, which controls P_{yy}. For the mixed derivative, use $\gamma'(x)+4\rho<7$ to obtain $P_{xy}<a(7(0.15)-0.9135)-\kappa<1$. Its lower bound follows by setting $z=y$, then $y=0.45$. The resulting expression is $aE(x)-\kappa$, where

$$E(x)=\gamma'(x)(0.45-x)-\gamma(x)+5.7855-16.24x.$$

Since $E''=\gamma'''(x)(0.45-x)-3\gamma''(x)<0$ and $E'(0.4)<0$, its minimum is at r, where $aE(r)-\kappa>-8$. All these evaluations use (38).

The inequality $2|uv|\le u^2+v^2$ now gives $\nabla^2 P_\pm \preceq -\operatorname{diag}(m_x,m_y)$. Use these centers and retained coordinates:

	$z\le 0.49$	$z>0.49$
P_+	$c=(r,0.43),\ I=\{y\}$	$c=(0.42,r),\ I=\{x\}$
P_-	$c=(r,0.43),\ I=\{y\}$	$c=(0.42,0.43),\ I=\{x,y\}$

For the first column, $\partial_x P_\pm(c,z)>0.02$ and $x-r\le 0$. For the second column of P_+, $\partial_y P_+(c,z)<-0.58$ and $y-r\ge 0$. Their derivatives in z are negative as shown below, so substitution at the relevant endpoints proves these bounds. Lemma 8 therefore gives

$$P_\pm(x,y,z)\le U_\pm(z):=P_\pm(c,z)+\sum_{i\in I}\frac{(\partial_i P_\pm(c,z))^2}{2m_i}. \tag{52}$$

One-variable bounds. On each interval in the next table, j_1,j_2 are constant and h is quadratic or affine. Set $p_i(z)=\partial_i P_\pm(c,z)$. Differentiating (50)–(51) gives $p_i''=-2\rho h''\le 0$ and $p_i'<0$. To check the latter signs, use $h'\ge 0.227$, $|j_1|<0.05$, $|j_2|<0.043$, $g_y(c)<0.033$, and $\gamma'(c_x)+4\rho<7$. Consequently,

$$U_\pm'''=-\gamma'''(z)+3\sum_{i\in I}\frac{p_i'p_i''}{m_i}>0.$$

For an interval $(l,u]$, the table lists rational numbers $e_\pm, M_\pm$ such that substitution in (52) gives

$$\max\{U_\pm(l),U_\pm(u)\}<e_\pm,\qquad U_\pm''(l)>-M_\pm. \tag{53}$$

As before, endpoint values use the formula of their own row. Since $U_\pm''$ increases, (35) gives

$$U_\pm(z)<e_\pm+\frac{M_\pm(u-l)^2}{8}<0. \tag{54}$$

$(l,u]$	e_+	M_+	e_-	M_-
$(0.44,0.45]$	-0.0011	65	-0.023	62
$(0.45,0.46]$	-0.0026	63	-0.0186	60
$(0.46,0.47]$	-0.0029	75	-0.0198	73
$(0.47,0.475]$	-0.0072	9	-0.0192	2
$(0.475,0.48]$	-0.0077	72	-0.0188	70
$(0.48,0.49]$	-0.00025	19	-0.0216	13
$(0.49,0.5]$	-0.0026	31	-0.00063	0
$(0.5,0.51]$	-0.0016	40	-0.0012	8
$(0.51,0.52]$	-0.0030	37	-0.0021	2
$(0.52,0.525]$	-0.0094	68	-0.0025	62
$(0.525,0.55]$	-0.030	67	-0.0090	61

For example, the closest bound in (54) is $-0.00025+19(0.01)^2/8=-1/80000<0$.

The range $z > 0.55$. Here $\boldsymbol{f}(z) = 0$, so $\Psi \le \kappa/4 < 0.012$. For $z \le 4/5$, both extra terms after Φ in the triangle contributions of (50)–(51) are nonpositive. For P_+, use $\gamma(x) < 0.09$, $w(x) \ge 0.3045$, and $w(z) \le -0.203$. For P_-, set $z = 0.55$, $y = 0.45$ to bound its bracket by

$$w(x)(2x - 0.65) - 0.09135 + \gamma(x)(0.55 - x) \le \rho(7/40)^2 - 0.09135 + 0.09(0.15) < 0.$$

Since $\gamma(x) + \gamma(y) > 0.037$, it follows that $F < 0.711h(z) - 0.037 - \gamma(z) + 0.012$. For $z \le \beta$, this is negative at 0.55 and decreases because $h' = 0.978$ and $\gamma' > 7$. For $\beta \le z \le 4/5$, use $h \le 1$ and $\gamma(z) \ge 1.4$.

Finally, suppose $z > 4/5$. Then $\Psi \le 0$, since $q \le 0.2 < (1 - x)(1 - y)$ and $\kappa > 0$. The joint probabilities are $\pi(y), 0, 0$. Using $z \le x + y \le 0.875$, $B(z) \ge 12(1 - z)$, and $\pi(y) \le 1$ gives

$$F \le (1 + w(x) - \gamma(x))\pi(y) + w(y) - \gamma(y) - B(z) \le 1 - (12 - 2\rho)(1 - x - y) - 0.037 < 0.$$

H.9 Both shorter distances in the first threshold interval

We next prove $F \le 0$ throughout

$$r < x \le y \le s, \qquad r = 17/40, \quad s = 23/50. \tag{55}$$

The proof uses the same curvature bound for all three ways in which x, y can lie on either side of $9/20$. Set

$$a = 227/35, \quad g_t = a(t - r), \quad h_t = tg_t \quad (t = x, y), \qquad h = 1 - \pi(z).$$

Write $D = (r, 0.45]$, $E = (0.45, s]$, with respective variance vectors $v_D = (0.179, 0.085)$ and $v_E = (0.066, 0.182)$. For the applicable pair of intervals, put $\kappa = v_x \cdot v_y$ and $j_i = v_i \cdot \boldsymbol{f}(z)$. In particular, $0 < \kappa < 0.04$ and $|j_i| < 0.11$ for $z \le 0.55$. All decimals below are exact rationals.

The two endpoint formulas. For $z \le 4/5$, the joint probabilities are

$$J_{xy} = \pi(y) - g_x(1 - y - q), \quad J_{xz} = \pi(z) - g_x(1 - z - q), \quad J_{yz} = \pi(z) - g_y(1 - z - q).$$

Since $x + y < 1$, it suffices to use $q = 1 - z$ and $q = 1 - x - y$. Let $\Phi = (w(x) + w(y))h - \gamma(x) - \gamma(y) - \gamma(z)$. Substitution gives

$$\begin{aligned} P_+ &= \Phi + h_y[\gamma(x) - w(x) + w(z)] + h_x[\gamma(y) + \gamma(z) - w(y) - w(z)] \\ &\quad + (2w(z) - \gamma(z))g_x(z - y) + \kappa(x + y - xy - z) + (j_1 x + j_2 y)(1 - z), \qquad (56) \\ P_- &= \Phi + g_y[(w(x) + w(z))y + (2w(x) - \gamma(x))(x - z)] \\ &\quad + g_x[(w(y) + w(z))x + (2w(y) - \gamma(y))(y - z)] \\ &\quad - \kappa xy + j_1(z - y - xz) + j_2(z - x - yz). \qquad (57) \end{aligned}$$

These are F at its two q endpoints. On $[r, s]$,

$$0.019 < \gamma(t) < 0.037, \qquad -2 < \gamma'(t) < 2. \tag{58}$$

For the lower bound, the numerator of $2000(1 + t)(\gamma(t) - 0.019)$ is $114695t^2 - 100708t + 22107$, which has negative discriminant. The upper bound follows from convexity and the values at r, s.

The range $z \le 0.44$. Here all three variance vectors equal v_D and $h < 0.043$. For P_+, the term multiplying h_x is negative. The remaining two extra triangle terms are at most $0.028h$ and $2w(r)a(z - r)^2/4 < 0.000223$. The covariance sum is at most $3\kappa z(1 - z)$: for fixed z it increases with x, y, so set $x = y = z$. Hence

$$P_+ < 0.609(0.043) + 0.028(0.043) + 0.000223 + 3(0.0393)(0.44)(0.56) - 0.057 < 0.$$

For P_-, discard the negative factors $x - z, y - z$ in its two brackets. Since $(w(r) + w(z))z \le 2w(r)r$, their sum is at most $4w(r)ra(0.015)$. Its covariance sum is $\kappa[2z - (1 + z)(x + y) - xy]$, at most its value at $(r, r, 0.44)$. As $\kappa > 0.039$, this gives

$$P_- < 0.609(0.043) + 4w(r)ra(0.015) + 0.039[0.88 - 2.88r - r^2] - 0.057 < 0.$$

Uniform curvature bounds. For $0.44 \le z \le 0.55$, extend each fixed-interval formula to the square $[r,s]^2$. This permits Taylor expansion at a boundary point while retaining the variance vectors of the interval being proved. Let $h'_t = a(2t-r)$. Direct differentiation gives

$$\begin{aligned}
P_{+,xx} &= -(1-h_y)\gamma''(x) + 2a[\gamma(y)+\gamma(z)-w(y)-w(z)],\\
P_{+,yy} &= -(1-h_x)\gamma''(y) + 2a[\gamma(x)-w(x)+w(z)],\\
P_{+,xy} &= h'_y(\gamma'(x)+2\rho) + h'_x(\gamma'(y)+2\rho) - a[2w(z)-\gamma(z)] - \kappa,\\
P_{-,xx} &= -\gamma''(x) + g_y[(z-x)\gamma''(x) - 2(\gamma'(x)+4\rho)] + 2a[w(y)+w(z)],\\
P_{-,xy} &= a[E_z(x)+E_z(y)+4\rho(1+r)] - \kappa, \qquad E_z(t) = (\gamma'(t)+4\rho)(z-t) - \gamma(t) - 8\rho t.
\end{aligned}$$

The formula for $P_{-,yy}$ exchanges x, y. These derivatives satisfy

$$\begin{aligned}
P_{+,xx} < -61, \quad P_{+,yy} < -66, \quad 0 < P_{+,xy} < 43,\\
P_{-,xx}, P_{-,yy} < -69, \qquad -25 < P_{-,xy} < 0.
\end{aligned} \tag{59}$$

Here are explicit bounds for checking these constants. Since γ'' decreases and $\gamma - w$ increases,

$$\begin{aligned}
P_{+,xx} &\le -(1-h_s)\gamma''(s) + 2a[\gamma(s)-w(s)+\gamma(0.55)-w(0.55)] < -61,\\
P_{+,yy} &\le -(1-h_s)\gamma''(s) + 2a[\gamma(s)-w(s)+w(0.44)] < -66,\\
P_{+,xy} &< 2h'_s(\gamma'(s)+2\rho) - a[2w(0.55)-\gamma(0.55)] < 43.
\end{aligned}$$

Its lower bound follows from $2h'_r(\gamma'(r)+2\rho) - 2aw(0.44) - 0.04 > 0$. For P_-, use

$$0.125\gamma''(r) - 2(\gamma'(r)+4\rho) < 0, \qquad -\gamma''(s) + 2a[w(r)+w(0.44)] < -69.$$

Also E_z decreases in t, because $E'_z(t) \le 0.125\gamma''(r) - 2\gamma'(r) - 12\rho < 0$, and increases in z. Thus

$$-25 < a[2E_{0.44}(s) + 4\rho(1+r)] - 0.04 < P_{-,xy} < a[2E_{0.55}(r) + 4\rho(1+r)] < 0.$$

Using $2|uv| \le u^2 + v^2$ in (59) now gives

$$\nabla^2 P_+ \preceq -\operatorname{diag}(18,23), \qquad \nabla^2 P_- \preceq -44I. \tag{60}$$

Removing the two shorter distances. Apply Lemma 8 with $(m_x, m_y) = (18,23)$ or $(44,44)$, writing $p_i(z) = \partial_i P_\pm(c,z)$. Choose the centers and retained coordinates as follows. At a distance breakpoint the vectors remain those of the stated case; for example, an E coordinate equal to 0.45 uses v_E.

Case	Range of z	c	Retain
$DD,+$	$z \le 0.475$	$(r, 0.43)$	y
	$z > 0.475$	(r,r)	none
$DD,-$	$z \le 0.46$	$(0.448, 0.448)$	x, y
	$0.46 < z \le 0.48$	$(0.44, 0.44)$	x, y
	$0.48 < z \le 0.51$	$(0.43, 0.43)$	x, y
	$z > 0.51$	(r,r)	none
$DE,+$	all	$(r, 0.45)$	x, using $p_x < 1/2$
$DE,-$	$z \le 0.47$	$(0.445, 0.45)$	x
	$0.47 < z \le 0.48$	$(0.44, 0.45)$	x
	$0.48 < z \le 0.51$	$(0.43, 0.45)$	x
	$z > 0.51$	$(r, 0.45)$	none
$EE,\pm$	all	$(0.45, 0.45)$	none

All omitted linear terms are nonpositive. The following derivative bounds give a short way to check their signs on whole intervals. On each row of the next table, $0 \le h'' \le 42.4$ and $h' \ge 0.227$. Differentiating the endpoint formulas, using (58), $g_t \le 0.227$ and $|j_i| < 0.11$, gives $p'_y < -8$ for P_+ and $p'_x, p'_y < -10$ for P_-. Their omitted derivatives are negative at the left endpoints by substitution. For p_x of P_+,

$$p''_x = -2\rho h'' + a[(2c_x - r + c_y - z)\gamma''(z) - 2\gamma'(z) - 8\rho] > -300.$$

Indeed, the coefficient of γ'' is at least 0.3, $\gamma''(z) > 63$, and $\gamma'(z) < 8$. At both endpoints of every applicable row, $p_x < -0.07, 0.3, -1.6$ for DD, DE, EE, respectively. The chord bound adds at most $300(0.025)^2/8 < 0.024$. Thus $p_x < 0$ in DD, EE, and $p_x < 1/2$ in DE. The corresponding displacements are nonnegative.

The lemma gives

$$P_\pm(x, y, z) \le U(z) := P_\pm(c, z) + \sum_{i \text{ retained}} \frac{p_i(z)^2}{2m_i}. \tag{61}$$

For $DE, +$, replace its single square term by $(1/2)^2/(2 \cdot 18) = 1/144$.

One-variable verification. Every function U just defined has $U''' > 0$ on each row. For retained derivatives, $p_i' < 0$ and $p_i'' = -2\rho h'' \le 0$, so

$$U''' = -\gamma'''(z) + 3\sum_i p_i' p_i''/m_i > 0.$$

This formula also applies when the sum is empty or the correction is constant, except for $EE, +$. In that case, direct differentiation gives $U''' = -[1 - g_{0.45}(1 + 2 \cdot 0.45)]\gamma'''(z) > 0$.

The next table gives e with $U(l), U(u) < e$ for each interval $(l, u]$. Substitution also gives $U''(l) > -M$, with $M = 80$ except that $M = 0$ for $DD, -$ on $(0.48, 0.51]$, and $M = 60$ for $DE, -$ on $(0.45, 0.46]$. Endpoint values always use the formula of their row.

$(l, u]$	$DD, +$	$DD, -$	$DE, +$	$DE, -$	$EE, +$	$EE, -$
$(0.44, 0.45]$	$-.0026$	$-.0060$	—	—	—	—
$(0.45, 0.46]$	$-.0049$	$-.0013$	$-.014$	$-.00096$	$-.037$	$-.0055$
$(0.46, 0.47]$	$-.0050$	$-.0034$	$-.018$	$-.0034$	$-.045$	$-.0057$
$(0.47, 0.475]$	$-.0058$	$-.0107$	$-.035$	$-.0079$	$-.066$	$-.0107$
$(0.475, 0.48]$	$-.0088$	$-.0106$	$-.041$	$-.0033$	$-.078$	$-.00128$
$(0.48, 0.49]$	$-.0037$	$-.0171$	$-.055$	$-.0130$	$-.100$	$-.0087$
$(0.49, 0.5]$	$-.0025$	$-.00055$	$-.066$	$-.0037$	$-.114$	$-.0046$
$(0.5, 0.51]$	$-.0015$	$-.00046$	$-.067$	$-.0046$	$-.118$	$-.0150$
$(0.51, 0.52]$	$-.0045$	$-.0051$	$-.080$	$-.0199$	$-.134$	$-.0410$
$(0.52, 0.525]$	$-.014$	$-.0121$	$-.097$	$-.0418$	$-.149$	$-.0780$
$(0.525, 0.55]$	$-.036$	$-.0216$	$-.112$	$-.0626$	$-.153$	$-.112$

Equation (35) now gives

$$U(z) < e + M(u - l)^2/8 < 0.$$

The closest displayed bound is $-0.00096 + 60(0.01)^2/8 = -21/100000$. Thus these one-variable bounds cover the whole rectangle of shorter distances in each variance case, and hence (55).

The range $z > 0.55$. Here $\boldsymbol{f}(z) = 0$, so $\Psi \le \kappa/4 < 0.01$. First suppose $z \le 4/5$. In P_-, each of the two extra triangle brackets in (57) is nonpositive. To see this for the first bracket, increase it by setting $z = 0.55$. If $x \ge 0.45$, both terms are nonpositive. Otherwise set $y = s$ and use

$$w(x)(2x - 0.64) - 0.09338 + \gamma(x)(0.55 - x) \le w(r)(0.21) - 0.09338 + 0.037(0.125) < 0.$$

The same argument applies with x, y exchanged. For P_+, regroup its last two triangle terms. The coefficient of $-\gamma(z)$ becomes $1 - g_x(x + y - z) > 0.916$. The terms $h_y[\gamma(x) - w(x) + w(z)]$ and $h_x[\gamma(y) - w(y)]$ are negative. The only remaining positive term is at most

$$g_s\rho(1.38 - 2z)_+(2z - 1) \le g_s\rho(0.38)^2/4 < 0.017.$$

Consequently both endpoints satisfy

$$F < 0.609h - 0.011 - 0.916\gamma(z).$$

For $0.55 < z \le \beta$, this is negative at 0.55 and decreases, as $h' = 0.978$ and γ' increases. For $\beta \le z \le 4/5$, use $h \le 1$ and $\gamma(z) \ge 1.4$.

Finally let $z > 4/5$. Now $\Psi \le 0$ because $q \le 0.2 < (1 - x)(1 - y)$ and $\kappa > 0$. Writing $J = J_{xy} \le \pi(y) \le \pi(x)$ gives

$$\mathcal{T} = (w(x) - \gamma(x))\pi(y) + (w(y) - \gamma(y))\pi(x) + J - B(z)[\pi(x) + \pi(y) - J].$$

The explicit formula for B yields $B(z) > 16.4(1 - z)$ on this range. Since $z \le x + y \le 0.92$,

$$F \le [1 + w(x) + w(y) - \gamma(x) - \gamma(y) - B(z)]\pi(x) < [1 - (16.4 - 2\rho)(0.08) - 0.038]\pi(x) < 0.$$

H.10 Closing the interval between the preceding regions

It remains to connect these bounds with the preceding short-distance proofs by treating

$$0.4 < x \le r, \qquad 0.45 < y \le s.$$

Use (50)–(51), now with $v_x = (0.201, 0.104)$ and $v_y = v_E$, so $\kappa = 0.032194$. For $0.45 \le z \le 0.55$, their displayed second-derivative formulas give

	P_{xx}	P_{yy}	P_{xy}	upper bound on $\nabla^2 P$
P_+	< -73	< -77	$0 < P_{xy} < 10$	$-\operatorname{diag}(63, 67)$
P_-	< -79	< -68	$-9 < P_{xy} < 0$	$-\operatorname{diag}(70, 59)$.

For the first row, use $(1 - h_s)\gamma''(r) > 73$, $\gamma''(s) - 2a[\gamma(r) - w(r) + w(0.45)] > 77$, and $(\gamma'(r) + 2\rho)h'_s - \kappa < 10$; the mixed derivative is positive. For the second, use

$$[1 - g_s(0.15)]\gamma''(r) > 79, \qquad \gamma''(s) - 2a[w(0.4) + w(0.45)] > 68.$$

The upper bound on the mixed derivative follows from $a[7(0.15) + 2\rho(1 + r) - 4\rho(0.85)] - \kappa < 0$. For its lower bound, set $z = 0.45$, $y = s$ in the bracket of (51)'s mixed derivative. The resulting function of x decreases, since its derivative is at most $0.05\gamma''(0.4) - 2\gamma'(0.4) - 8\rho < 0$. Its value at r, after multiplication by a and subtraction of κ, is greater than -9.

Expand at $c = (0.42, 0.45)$. Retain x in both cases, and also retain y for P_- when $z \le 0.46$. Apply (34) with the diagonal bounds just given. Every omitted p_y decreases and is negative at the left endpoint of its row. For retained derivatives, $p_i'' = -2\rho h''$. When $h'' > 0$, we have $h' \ge 2$, which gives $p_i' < 0$ directly; when $h'' = 0$, their contribution to U''' vanishes. Thus $U''' > 0$ again.

Use the same z intervals as in the preceding table, starting at 0.45. The endpoint substitutions can be bounded uniformly:

Rows	$\max\{U(l), U(u)\}$	$U''(l)$
$(0.45, 0.46], \ldots, (0.52, 0.525]$	< -0.004	> -80
$(0.525, 0.55]$	< -0.04	> -80.

Both bounds apply to P_+ and P_-. The earlier rows have length at most 0.01, so $U < -0.004 + 80(0.01)^2/8 = -0.003$. The last row gives $U < -0.04 + 80(0.025)^2/8 < 0$.

For $0.55 < z \le 4/5$, the extra triangle terms in both endpoint formulas are nonpositive. The P_+ coefficient is at most $0.09 - w(r) + w(0.55) < 0$. For P_-, setting $z = 0.55$, $y = s$ bounds its bracket by

$$w(x)(2x - 0.64) - 0.09338 + \gamma(x)(0.55 - x) \le \rho(0.18)^2 - 0.09338 + 0.09(0.15) < 0.$$

Now $\gamma(x) + \gamma(y) > 0.046$ and $\Psi < 0.009$, giving $F < 0.711h - 0.037 - \gamma(z)$. This is negative at 0.55 and decreases up to β; for $\beta \le z \le 4/5$, use $\gamma(z) \ge 1.4$. For $z > 4/5$, $\Psi \le 0$, $J_{xy} = \pi(y)$, and $x + y \le 0.885$. The same $B(z) > 16.4(1 - z)$ bound gives

$$F < 1 - (16.4 - 2\rho)(0.115) - 0.046 < 0.$$

Together, the preceding four subsections prove $F \le 0$ for every feasible z, q whenever $2/5 < x \le y \le 23/50$.

H.11 Extending the large-distance proof to $12/25$

We now prove $F \le 0$ whenever $x \ge 12/25$. The preceding results leave only

$$r := 0.48 \le x < 0.5, \qquad x \le y \le 0.55.$$

Write $w = 1 - 2b$, $A = \gamma - w$, $g = W - H$, and $\kappa_{st} = \langle \boldsymbol{f}(s), \boldsymbol{f}(t) \rangle$. For $r \le t \le 0.52$, the marginal is continuous and equals

$$\pi_0(t) = 1 + 8.094t - 17.6t^2 \quad (t \le 0.5), \qquad \pi_1(t) = 1 + 6.094t - 13.6t^2 \quad (t \ge 0.5). \tag{62}$$

On $[r, 0.5]$, $g(t) = 17.6t - 8.094$. Also A increases on $[r, 0.55]$, with $0 < A(r) < 0.01$.

A small surplus at each distance. Set $\xi(t) = \|\boldsymbol{f}(t)\|^2 t(1-t)$ and $\eta = 0.0006$. The following bounds give $\gamma(t)\pi(t) - \xi(t) > \eta$ throughout $[r, 0.55]$, since $t(1-t) \le 1/4$.

Interval	Upper bound on $\|\boldsymbol{f}(t)\|^2$	Lower bound on $\gamma(t)\pi(t)$
$[0.48, 0.49]$	0.276665	0.06985
$(0.49, 0.5]$	0.350066	0.08817
$(0.5, 0.51]$	0.413946	0.1043
$(0.51, 0.52]$	0.418882	0.120
$(0.52, 0.525]$	0.275600	0.129
$(0.525, 0.55]$	0.074178	0.130

For the first two rows, $(\gamma\pi)'$ is positive: on $[l, u]$, it is at least $\gamma'(l)\pi(u) + \gamma(u)\pi'(u) > 0$. Substitution at l gives the displayed bounds. The last four rows use the estimates in Appendix H.5; at 0.5, $\gamma\pi = 83463/800000 > 0.1043$. On the first row we will also use

$$\gamma(t)\pi(t) - 0.276665t(1-t) > \eta + 1.2(t-r). \tag{63}$$

Indeed, its derivative exceeds $\gamma'(r)\pi_0(0.49) + \gamma(0.49)\pi_0'(0.49) - 0.276665(0.04) > 1.2$.

The overlap is largest at the difficult endpoint. Suppose first that $z \le 0.8$. The coefficient of q in F is

$$C = \kappa_{xy} + \kappa_{xz} + \kappa_{yz} - (2w_x - \gamma_x)g_y - (2w_y - \gamma_y + 2w_z - \gamma_z)g_x, \tag{64}$$

where subscripts denote evaluation. All inner products on $[r, 0.55]$ exceed 0.042; below 0.49 they exceed 0.106. To check the first bound without a pairwise table, the only vector with negative first coordinate is $(-0.123, -0.243)$. Its product with $(0.185, -0.269)$ is 0.042612, its squared norm is 0.074178, and every other product with it is at least $0.243(0.452) - 0.123(0.269) > 0.042$. All remaining products are at least 0.269^2. Below 0.49, both coordinate magnitudes are at least those of $(0.185, -0.269)$, whose squared norm exceeds 0.106.

Further, $(2w_t - \gamma_t)_+ < 0.079$ and it vanishes for $t \ge 0.49$. Use $g_r = 0.354$, $g_{0.49} = 0.53$, and $g \le 0.978$. When $z > 0.55$, also use $\boldsymbol{f}(z) = 0$ and $\gamma_z - 2w_z > 0.87$. Equation (64) gives

$$C > \begin{cases} 3(0.106) - 3(0.079)(0.53) > 0, & z < 0.49, \\ 3(0.042) - 0.079(0.978 + 0.53) > 0, & 0.49 \le z \le 0.55, \\ 0.042 - 0.079(0.978 + 0.53) + 0.354(0.87) > 0, & 0.55 < z \le 0.8. \end{cases}$$

Thus it suffices to take $q = 1 - z$.

A common identity at $q = 1 - z$. Here $J_{xz} = J_{yz} = \pi_z$ and $J_{xy} = \pi_y - g_x(z - y)$. Writing $\Delta_1 = \pi_x - \pi_y \ge 0$ and $\Delta_2 = \pi_y - \pi_z \ge 0$, direct expansion gives

$$\begin{aligned} \mathcal{T} + \sum_{t=x,y,z} \gamma_t \pi_t = (\gamma_x - \gamma_y - \gamma_z + w_y + w_z)\Delta_1 + (w_x + w_y - \gamma_z)\Delta_2 \\ + g_x(z - y)(2w_z - \gamma_z). \end{aligned} \tag{65}$$

For $0.5 \le z \le 0.55$, the first two coefficients are at most $2w_r - \gamma_{0.5} = 0.00115 < 3\eta$, and the last term is nonpositive. Since $\Delta_1 + \Delta_2 \le 1$, the surplus table and (36) give $F < 0$. For $0.55 < z \le 0.8$, all three terms are nonpositive, because $\gamma_z > 0.47 > 2w_r$. Here $\xi(z) = 0$ and $\gamma_z \pi_z \ge 0$, so the two shorter distances already suffice.

It remains that $z < 0.5$. At any distance equal to r, replace its vector by $v_a = (0.269, -0.452)$ to obtain an upper bound on Ψ: all covariance factors are positive, and all products increase. If $z \le 0.49$, the vectors are now identical. The right side of (65) is at most

$$0.079[9.2 + 0.53](z - x) \le 1.2(z - x).$$

This uses $-\pi_0' < 9.2$ and $g_x \le 0.53$ on $[r, 0.49]$. The three bounds (63) absorb this expression. If $0.49 < x \le z < 0.5$, all coefficients in (65) are nonpositive, since $2w_{0.49} < \gamma_{0.49}$.

In the remaining case, $x \le 0.49 < z < 0.5$, both v_a and $v_b = (0.155, -0.571)$ occur. Index the distances by $t_1 = x, t_2 = y, t_3 = z$, with vectors v_i, and put $C_{ij} = 1 - z - (1 - t_i)(1 - t_j)$. Then

$0 < C_{ij} \le \min\{t_i(1-t_i), t_j(1-t_j)\}$. Using $2\langle v_i, v_j\rangle = \|v_i\|^2 + \|v_j\|^2 - \|v_i - v_j\|^2$ therefore improves (36) to

$$\Psi \le \sum_t \xi(t) - \frac{1}{2}\sum_{i<j} \|v_i - v_j\|^2 C_{ij} \le \sum_t \xi(t) - 0.234(0.027157).$$

Here ξ uses the replacement vector at r. There are two mixed pairs, each with $C_{ij} > 0.234$, and $\|v_a - v_b\|^2 = 0.027157$. The first two coefficients in (65) are less than 0.0433, the last term is nonpositive, and $\pi_x - \pi_z < 0.1831$. Hence

$$F < 0.0433(0.1831) - 3\eta - 0.234(0.027157) = -\frac{56627}{250000000} < 0.$$

This completes $z \le 0.8$.

The face $z = x + y \le 1$. For $z > 0.8$, Appendix H.4 reduces both overlap endpoints to $z \downarrow 0.8$ or $z = \min\{1, x+y\}$. At the right limit, $T_x, T_y \le 0$ since $A_x, A_y > 0$; $T_z = J - B(0.8)(P - J) \le 0$ since $B(0.8) > 1$ and $J \le P - J$. Also $\Psi \le 0$, because $q \le 0.2 < (1-x)(1-y)$.

On $z = x + y \le 1$, necessarily $y \le 0.52$ and $q = 1 - x - y$. Then $J_{xy} = \pi_x + \pi_y - 1$ and, with $\kappa = \kappa_{xy}$,

$$F = H_\kappa(x, y) := (1 - A_x)\pi_y + (1 - A_y)\pi_x - 1 - B(x + y) - \kappa xy. \tag{66}$$

On each rectangle with fixed vectors, $0 \le \kappa < 0.383$ and both partial derivatives are positive. Indeed, either partial derivative, in coordinate t, is bounded below by

$$D(t) = (1 - A_r)\pi'(t) - A'(t)\pi_r - B'(r + t) - 0.383(0.52).$$

On each marginal piece, $D' < -0.99(27.2) + 5.2 < 0$, using $-B''(r+t) = 39/(1+r+t)^3 < 5.2$. The endpoint values satisfy

$$D(0.5-) = \frac{6060001943}{604395000000} > 0.01, \qquad D(0.52-) = \frac{27738458209}{66785000000} > 0.41.$$

The maximum on each rectangle intersected with $x + y \le 1$ is therefore at its upper corner or on $x + y = 1$. The upper corners are $(0.49, 0.49)$, $(0.49, 0.5)$, $(0.49, 0.51)$, and $(0.5, 0.5)$; substitution into (66) gives $H_\kappa < -0.0019$ at each. The separate boundary $x = r$, whose vector is $(0.185, -0.269)$, has upper endpoints $y = r, 0.49, 0.5, 0.51, 0.52$; all give $H_\kappa < -0.018$.

On $x + y = 1$, the vector products are at least 0.328 for $r < x \le 0.49$, and at least 0.38 for $0.49 < x < 0.5$. For either lower bound k, define

$$G_k(x) = (1 - A_x)\pi_1(1 - x) + (1 - A_{1-x})\pi_0(x) - 1 - kx(1 - x),$$
$$P_k(u) = -(1 + x)(2 - x)G_k(x), \qquad u = 0.5 - x.$$

Direct differentiation gives

$$P_k(0) = 9k/16 - 307233/1600000, \qquad P_k'(0) = -6039/1600 > -4,$$

$$P_k''(u) = 983.29278 - 5k + 494.22u + 12(k - 3540.93739)u^2 + 4912.6u^3.$$

For $0 \le u \le 0.02$ and $k \in \{0.328, 0.38\}$, $P_k'' > 983.29 - 5(0.38) - 12(3541)(0.02)^2 > 960$. Since $P_{0.328}(0) > -0.008$ and $P_{0.38}(0) > 0.02$,

$$P_{0.328}(u) > 480u^2 - 4u - 0.008 \ge 0 \quad (0.01 \le u \le 0.02),$$
$$P_{0.38}(u) > 480\left(u - \frac{1}{240}\right)^2 + \frac{7}{600} > 0.$$

The first quadratic is zero at 0.01 and increases thereafter. Thus $G_k < 0$, completing this face, including the one-sided limits of its vector pieces.

The face $z = 1$. Here $x + y \geq 1$, $y > 0.5$, and $q = 0$. For a lower bound $k \leq \kappa_{xy}$,

$$F \leq L_k(x, y) := (1 - A_x)\pi_y - A_y\pi_x - g_x(1 - y) - k(1 - x)(1 - y). \tag{67}$$

For $0 \leq k \leq 0.42$, this expression decreases with y, since

$$\partial_y L_k < -8.6(0.647) + 0.706 + 0.42(0.52) < 0.$$

We used $A'_y > 8.6$, $\pi_x > 0.647$, $g_x \leq 0.706$, and dropped the nonpositive term $(1 - A_x)\pi'_y$; the jump at 0.525 also decreases L_k. For $y \leq 0.52$ and $x > r$, take $k = 0.328$ when $x \leq 0.49$ and $k = 0.38$ otherwise. The preceding face gives $L_k(x, y) \leq L_k(x, 1 - x) = G_k(x) < 0$.

For $0.52 \leq y \leq 0.525$, including $x = r$, use $k = 0.159$. First decrease y to 0.52. Then $L_k(x, 0.52)$ decreases with x: its derivative is at most $0.35(9.51) - 17.6(0.48) + 0.159(0.48) < 0$. Consequently,

$$F \leq L_{0.159}(r, 0.52) = -\frac{723521581}{87875000000} < -0.008.$$

Finally, for $y \geq 0.525$, discard the nonpositive covariance and compare J_{xy}/π_x with A_y. The numerator $J_{xy} = \pi_y - g_x(1 - y)$ decreases with y, since $\pi'_y + g_x \leq -0.978 + 0.706 < 0$. The ratio also decreases with x: the numerator of its x derivative is

$$g_x J_{xy} + 17.6[x\pi_y - (1 - y)] \leq 0.706(0.487) + 17.6[0.5(0.487) - 0.45] < 0.$$

Thus $J_{xy}/\pi_x \leq J(r, 0.525, 0)/\pi_r < 0.385$, whereas $A_y \geq A_{0.525} > 0.397$. Since $A_x > 0$, this proves $F < 0$ and finishes the region $x \geq 12/25$.

H.12 Extending the short-distance proof to $3/8$

The following argument adds the region

$$0.375 < x \leq 0.4, \qquad x \leq y \leq 0.46.$$

Together with the preceding results, this proves $F \leq 0$ whenever $3/8 < x \leq y \leq 23/50$. Write

$$s = 0.4, \quad r = 0.425, \quad c = 0.415, \quad a = 227/35, \quad v = (0.002, -0.058), \quad h = 1 - \pi(z).$$

Here $\pi(x) = 1$ and $\boldsymbol{f}(x) = v$. On the range of y, put $g_y = a(y - r)_+$, so $\pi(y) = 1 - yg_y$. As before, let $w = 1 - 2b$, $\kappa = \langle v, \boldsymbol{f}(y)\rangle$, $j_x = \langle v, \boldsymbol{f}(z)\rangle$, and $j_y = \langle \boldsymbol{f}(y), \boldsymbol{f}(z)\rangle$. For $z \leq 0.8$, the two endpoint expressions are (50)–(51), with these vectors. All their dependence on the variance functions is through κ, j_x, j_y.

First eliminate the shortest distance. Only $z > r$ remains. Suppose initially that $z \leq 0.55$, and write $P_+ = F(x, y, z, 1 - z)$ and $P_- = F(x, y, z, 1 - x - y)$. Their derivatives in x satisfy

$$(P_+)_x = -\gamma'(x)\pi(y) - 2\rho[\pi(y) - \pi(z)] + \kappa(1 - y) + j_x(1 - z),$$
$$(P_-)_x = (P_+)_x - \kappa - j_x - j_y + g_y[2w(x) - \gamma(x) - (x + y - z)(4\rho + \gamma'(x))].$$

On this range, $\gamma'(x) < -3.24$, $\pi(z) > 0.44$, $|\kappa| < 0.011$, $|j_x| < 0.037$, and $|j_y| < 0.101$. Also $g_y \leq 0.227$, $x + y - z \leq 0.4$, $2w(x) - \gamma(x) \geq 0.7305$, and $4\rho + \gamma'(x) < 4.879$. Consequently,

$$(P_+)_x > -0.82 + 4.06(0.44) - 0.011 - 0.037 > 0.918,$$
$$(P_-)_x > 0.9184 - 0.149 + 0.227[0.7305 - 0.4(4.879)] > 0.49.$$

It therefore suffices to set $x = s$. If $y \leq s$, symmetry increases both shorter distances to s; both vectors then remain equal to v. Notice that $\gamma(s) = 0.0815$ and $w(s) = 0.406$.

Variance bounds at $x = s$. For $y > s$, the two endpoint expressions for the covariance sum are

$$\Psi_+ = \kappa(s + y - sy - z) + (1 - z)\langle \boldsymbol{f}(z), sv + y\boldsymbol{f}(y)\rangle,$$
$$\Psi_- = -sy\kappa - \langle \boldsymbol{f}(z), (M, N)\rangle,$$
$$(M, N) = (y - 0.6z)v + [0.4 - (1 - y)z]\boldsymbol{f}(y).$$

The first covariance factor is positive, and $\kappa < 0$. We use the following small bounds, uniformly over both endpoints:

$$\Psi < \begin{cases} 0.009, & s < y \leq r,\ r < z < 0.48, \\ 0.008, & s < y \leq r,\ 0.48 \leq z \leq 0.55, \\ 0.012, & r < y \leq 0.46,\ y \leq z \leq 0.55. \end{cases} \tag{68}$$

Here are direct bounds for checking them from Table 3. When $s < y \le r$, $\kappa = -0.00563$ and

$$sv + y\boldsymbol{f}(y) \in [0.0812, 0.086225] \times [0.0184, 0.021].$$

Its product with $\boldsymbol{f}(z)$ is below 0.0173 for $z \le 0.47$ and below 0.015 for $z > 0.47$. Thus

$$\Psi_+ < 0.575(0.0173) - 0.00563(0.17) < 0.009 \quad (z \le 0.47),$$
$$\Psi_+ < 0.53(0.015) - 0.00563(0.09) < 0.008 \quad (z > 0.47).$$

The coefficients M, N are positive and have bounds

	M	N
$z > r$	< 0.031621	≤ 0.00667
$z \ge 0.48$	≤ 0.025198	$\le 0.005152.$

Since $f_1(z) \ge -0.123$ and $f_2(z) \ge -0.619$, the corresponding bounds on Ψ_- are $0.4(0.00563)r + 0.123M + 0.619N < 0.009$ and < 0.008.

When $r < y \le 0.46$, the vector $sv + y\boldsymbol{f}(y)$ has positive coordinates and its product with every relevant $\boldsymbol{f}(z)$ is below 0.017. Hence $\Psi_+ < 0.575(0.017) < 0.012$. For the other endpoint, $M, N > 0$ and $-sy\kappa \le 0.4(0.010424)(0.46)$. If $z \le 0.47$, both coordinates of $\boldsymbol{f}(z)$ are nonnegative. If $0.47 < z \le 0.525$, use $f_1(z) \ge 0$ and $N \le 0.0162844$ to get

$$\Psi_- \le 0.4(0.010424)(0.46) + 0.619(0.0162844) < 0.012.$$

If $z > 0.525$, use $\boldsymbol{f}(z) = (-0.123, -0.243)$, $M < 0.032$, and $N < 0.0163$; this gives an even smaller bound. All the displayed bounds on M, N follow by evaluating their bilinear formulas at the endpoints of the stated intervals.

The range $y \le r$. First take $x = y = s$. Put $k_0 = \|v\|^2 = 0.003368$ and $j = \langle v, \boldsymbol{f}(z)\rangle$. Table 3 gives $-0.010424 \le j \le 0.03628$, and

$$\Psi_+ = k_0(0.64 - z) + 0.8j(1 - z), \qquad \Psi_- = -0.16k_0 + 2j(0.6z - 0.4) < 0.0025.$$

Both endpoints are therefore bounded by

$$U_0(z) = 0.812h - 0.163 - \gamma(z) + 0.003368(0.64 - z) + 0.8(0.03628)(1 - z).$$

For $s < y \le r$, use $\gamma''(y) > 82$ and expand in y at c. Completing one square and applying (68) gives

$$F \le U_1(z) := 0.7511h - 0.0815 - \gamma(c) - \gamma(z) + \frac{[-4.06h - \gamma'(c)]^2}{164} + e, \tag{69}$$

where $e = 0.009$ for $z < 0.48$ and $e = 0.008$ for $z \ge 0.48$.

For $r < z \le 0.48$, use $h \le 0.17$ and $\gamma(z) > 0.019$ to prove $U_0 < 0$ directly. For U_1, use $\gamma(c) > 0.0425$ and $|4.06h + \gamma'(c)| < 1.965$. If $z \le 0.46$, take $h < 0.105$, $\gamma(z) > 0.019$; otherwise take $h < 0.17$, $\gamma(z) > 0.03639$. Substitution in both cases gives $U_1 < 0$.

On $[0.48, 0.5]$ and $[0.5, 0.52]$, respectively,

$$h = 17.6z^2 - 8.094z, \qquad h = 13.6z^2 - 6.094z.$$

Both U_i have $U_i'' < -25$ on each piece. For U_1, writing $d = -4.06h - \gamma'(c)$ and $K = 0.7511 - 4.06d/82$ gives

$$U_1' = Kh' - \gamma', \qquad U_1'' = Kh'' + \frac{4.06^2}{82}(h')^2 - \gamma''.$$

On the first piece, this gives $U_1'' < 0.757(35.2) + 0.202(9.51)^2 - 70.3 < -25$. On the second, replace $(35.2, 9.51, 70.3)$ by $(27.2, 8.05, 67.6)$. Here $K < 0.757$ and $4.06^2/82 < 0.202$. The bounds for U_0 follow immediately from its formula. The following substitutions prove that both functions increase up to 0.52 and remain negative:

	$U_i'(0.5-)$	$U_i'(0.52-)$	$U_i(0.52)$	$U_i(0.525+)$
U_0	> 3.1	> 0.55	< -0.0022	< -0.01
U_1	> 2.2	> 0.13	< -0.00045	$< -0.01.$

On $(0.52, 0.525]$ and $(0.525, 0.55]$, $h' = 0.978$ and $U_i' < -5$. The right-limit column handles the jump at 0.525.

The range $r < y \le 0.46$. Let $T_-(y, z)$ denote the triangle part of the lower endpoint at $x = s$. For $z \le 0.51$, the upper-endpoint triangle part is no larger: its extra term is nonpositive, whereas the lower-endpoint bracket is at least $0.3654r - 0.7305(0.11) > 0$. Furthermore,

$$(T_-)_{yy} = -\gamma''(y) + 2a[w(s) + w(z)] < -67.$$

Expand at $y = r$, where $g_y = 0$, and complete a square. With

$$V(z) = 0.7105h - 0.0815 - \gamma(r) - \gamma(z) + 0.012,$$
$$D(z) = -4.06h - \gamma'(r) + a(1.3275 - 2.456z), \qquad U(z) = V(z) + D(z)^2/134,$$

we obtain $F \le U(z)$ for $r < z \le 0.51$. On every marginal piece, $h'' \ge 0$, $D < 3$, $D'' = -4.06h''$, and

$$U'' = 0.7105h'' - \gamma'' + \frac{(D')^2 + DD''}{67} > -82.1 - \frac{3(4.06)(42.4)}{67} > -90.$$

For completeness, the earlier marginal pieces are

$$h = \begin{cases} az(z - r), & r \le z \le 0.46, \\ 0.227z, & 0.46 < z \le 0.47, \\ z(21.2z - 9.716), & 0.47 < z \le 0.475, \\ 0.354z, & 0.475 < z \le 0.48. \end{cases}$$

Substitution at the endpoints of each piece gives the three bounds below. At a jump, each endpoint uses its own piece's formula.

Pieces	$\max\{U(l), U(u)\}$	$u - l$ at most
$(r, 0.46]$, $(0.46, 0.47]$, $(0.47, 0.475]$, $(0.475, 0.48]$	< -0.029	0.035
$(0.48, 0.5]$	< -0.006	0.02
$(0.5, 0.51]$	< -0.0017	0.01.

Equation (35) gives $U(z) < \max\{U(l), U(u)\} + 90(u - l)^2/8 < 0$ in every row.

For $z \ge 0.51$, $D(z) \le D(0.51) < -0.12$. The triangle part decreases in y at both endpoints: for the lower endpoint use its negative second derivative; for the upper endpoint use $-4.06h - \gamma'(r) < 0$, and note that its extra term also decreases with y. Thus $F \le V(z)$. On $[0.51, 0.52]$, $V'' < -48$. At $z_0 = 0.515$, $V(z_0) < -0.00115$ and $|V'(z_0)| < 0.016$, so (34) gives $V(z) < -0.00115 + 0.016^2/96 = -1721/1500000 < 0$. Above 0.52, use $V' < -5$ on each piece and $V(0.525+) < -0.01$.

Finally let $z > 0.55$. We return to the original $0.375 < x \le 0.4$, $y \le 0.46$. Here $\boldsymbol{f}(z) = 0$ and $\Psi \le |\kappa|/4 < 0.0027$. Both extra triangle terms are nonpositive. For the upper endpoint, convexity gives $\gamma(x) - w(x) < -0.31$. For the lower endpoint, maximize its bracket by setting $z = 0.55$ and $y = 0.46$. The result is

$$Q(x) = w(x)(2x - 0.64) - 0.09338 + \gamma(x)(0.55 - x) < 0.$$

Indeed, $Q'' = -16.24 + (0.55 - x)\gamma'' - 2\gamma' > 0$, and both endpoint values on $[0.375, 0.4]$ are negative. Thus, for $z \le 0.8$,

$$F < 1.015h - 0.0815 - 0.019 - \gamma(z) + 0.0027.$$

This is negative at 0.55, where $\gamma(0.55) > 0.474$, and decreases up to β, since $h' = 0.978$ and $\gamma' > 7$. For $z \ge \beta$, use $w(x) + w(y) = 2\rho(1 - x - y) \le 2\rho(1 - z)$, $\gamma(z) \ge 12(1 - z) - 1$, and $z \le x + y \le 0.86$ to get

$$F < 1 - (12 - 2\rho)(0.14) - 0.0815 - 0.019 + 0.0027 < 0.$$

For $z > 0.8$, the joint marginals involving z vanish. The triangle part is then exactly the upper-endpoint expression with $h = 1$, so the same bound applies to every q. This completes the extension.

The remaining region is stated explicitly in (33) of Appendix G.2. The exact certificate in Appendix F is still needed there. It also verifies the regions proved here, so the main proof does not depend on these optional analytic arguments.