Fully Distributed Tâtonnement for Chores Markets

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Abstract

We study price-adjustment dynamics for computing competitive equilibria (CE) in Fisher markets with chores. Unlike in classical goods markets, prices in chores markets are payments for taking on undesirable tasks, and natural excess-demand dynamics can fail; even the naïve analogue of Walrasian tâtonnement may diverge. Recent work of [12] overcomes this obstacle via *relative tâtonnement*, which subtracts the average excess-demand signal from the excess demand vector. This recovers convergence, but at the cost of coupling the price updates across all chores. This leaves open whether such global coupling is inherent, or whether convergent tâtonnement can be recovered through a genuinely local update in which each chore reacts only to its own excess demand.

We answer this question affirmatively through *multiplicative tâtonnement*, a fully distributed dynamics in which each chore price is updated using only its current price and its own excess-demand signal. Although the update contains no explicit normalization term, Walras' law and the multiplicative form of the update implicitly preserve the relevant aggregate price geometry. We prove that multiplicative tâtonnement converges to a CE in any chores Fisher market with continuous, convex, and 1-homogeneous (CCH) disutilities. For convex CES disutilities, we further prove an approximate-CE convergence rate with the same $O(1/\varepsilon^2)$ dependence as relative tâtonnement, but with improved dependence on problem constants. Experiments on real-world and simulated instances show that multiplicative tâtonnement is substantially faster in practice, often by an order of magnitude.

1 Introduction

Competitive equilibrium (CE) is one of the central solution concepts in market design and microeconomic theory. The *Fisher market* is one of the most fundamental market settings. In a classical Fisher market with *goods*, there is a set of m goods and n agents. Each agent is endowed with a fixed budget of money, and once prices are assigned to the goods, agents spend their budgets to obtain utility-maximizing bundles. Prices are at equilibrium when these individual demands exactly clear the market. Beyond the static existence and welfare properties of CE, a classical question asks whether natural market dynamics can find such prices. Walrasian tâtonnement is the canonical example: prices are adjusted in response to excess demand, increasing when demand exceeds supply and decreasing when supply exceeds demand [46]. The stability and convergence of such dynamics have been studied since the foundational work of [43, 3, 2], and remain active topics in algorithmic game theory [19, 20, 15, 31, 40].

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This paper studies tâtonnement dynamics in Fisher markets with *chores*. In a chores market, agents do not pay to receive goods; rather, they are paid to perform undesirable tasks. Thus prices represent payments per unit of work, and the natural response to imbalance is reversed: when a chore is under-demanded, its price should increase to attract agents, while an over-demanded chore should become less lucrative. This reversal leads to a substantially different dynamical landscape. Chores markets can have disconnected sets of equilibria, and excess demand no longer has the monotonic structure that supports many goods-market analyses. Correspondingly, the computational literature on chores and mixed manna deals with non-convex optimization issues and is more nuanced than in the goods case [6, 29, 7, 10, 9, 11].

Relative Tâtonnement. The recent work of [12] initiated the study of tâtonnement dynamics for chores markets. They show that the naïve additive analogue of tâtonnement can diverge, even in simple chores markets, and propose *relative tâtonnement*, which restores convergence by subtracting the average excess-demand signal from each coordinate. Relative tâtonnement shows that it is indeed possible to recover convergence for a tâtonnement-like procedure, but at the cost of a global normalization to excess demand. Crucially, this normalization requires each price to be updated with knowledge of the excess-demand signals of all other chores, making relative tâtonnement not fully distributed: it is a centrally coordinated adjustment process rather than a truly decentralized one.

The Appeal of Decoupled Price-Adjustment Dynamics. Economically, the tatonnement process simulates how prices adjust in *real markets*: each good has a manager, auctioneer, or a market-maker who observes only the imbalance between supply and demand for that good, and adjusts its price accordingly. This decentralized vision is precisely what makes tatonnement a compelling model of market dynamics — it does not require a central planner who aggregates information across all goods, nor does it require any individual agent to know the global state of the market. Each price responds only to its own local signal, and equilibrium emerges as a *collective consequence* of many independent local adjustments. This is, in Hayek's terms [33], the informational miracle of the price system: global co-ordination is achieved through purely local price updates.

Our paper asks whether there exist truly decoupled tâtonnement dynamics for the chores setting. We answer this question in the affirmative by giving a multiplicative update in which each chore reacts only to its own excess demand signal. Our dynamics, which we call *multiplicative tâtonnement*, updates each chore price according to

$$p_j^{k+1} = p_j^k(1 + \eta^k y_j^k),$$

where y_j^k is the excess supply of chore j at the current prices p_j^ks. The update is fully local: the manager of chore j only needs to know the current payment p_j^k and the imbalance signal y_j^k for that chore. No average excess-demand term, price-simplex projection, or information about other chores is used. The key observation is that the correction imposed explicitly by relative tâtonnement emerges implicitly from the multiplicative update. When prices are initialized in the budget simplex, Walras' law makes the multiplicative dynamics preserve the relevant aggregate price geometry while still allowing each coordinate to update independently.

1.1 Our Contributions

We make four main contributions.

1. First, we introduce multiplicative tâtonnement as a fully distributed market dynamics for chores Fisher markets.
2. Second, we prove that this dynamics converges to a CE for general continuous, convex, and 1-homogeneous (CCH) disutilities (Theorem 1). The proof relates the update to entropic mirror descent, while controlling the perturbation created by using the simple multiplicative rule rather than the exact exponential update that is induced by entropic mirror descent.
3. Third, for convex CES disutilities, we prove convergence rates to approximate CE using a relative-smoothness argument on the price simplex (Theorem 2). This yields the same $O(1/\varepsilon^2)$ dependence on the approximation parameter ϵ as relative tâtonnement, with improved dependence on problem constants, while avoiding any coupling.
4. Finally, we compare the dynamics empirically on real-world and simulated instances, and find that multiplicative tâtonnement substantially outperforms relative tâtonnement in practice (Section 5).

1.2 Related work

Economic dynamics in goods markets. The study of economic dynamics in goods markets has been prevalent in the theoretical computer science community since the 2000s, often with a focus on algorithmic complexity. The first polynomial-time algorithm for linear Fisher markets, introduced by [23], uses a primal-dual approach that can be interpreted as an economic dynamics. In the more general setting of exchange economies satisfying weak gross substitutability (WGS), [19] showed that discrete-time additive tâtonnement dynamics converges to an approximate equilibrium. [20, 16] showed that multiplicative tâtonnement dynamics with artificial upper bounds on excess demand converges in certain nonlinear *constant elasticity of substitution* (CES) markets. Along this line of research, [15, 21, 31, 40] further improved convergence rates for a wide range of markets.

Economic dynamics and optimization. Tâtonnement dynamics is often connected to first-order methods through the well-known Eisenberg–Gale convex program [27, 30, 14]. In [39], the authors show that even classic proximal gradient descent can have an economic-dynamics interpretation. There is also recent work on price-adjustment dynamics related to other optimization methods [49, 13]. Proportional response (PR) dynamics is another popular class of economic dynamics, known to converge to a CE for CES utilities in goods markets [47, 50]. [5] shows that PR dynamics is equivalent to mirror descent on the convex Shmyrev program [44]. Better convergence rates and generalized versions of PR dynamics have also been established [17, 18]. For a detailed survey of proportional response dynamics, we refer the reader to [8]. Recent work [45] also studies market dynamics, including tâtonnement and proportional response dynamics, for public goods.

CE in chores markets. In contrast to goods markets, computing a CE in chores markets is known to be harder [10]. This difficulty stems from the "poles" in the classic Eisenberg-Gale (EG) program for chores markets, and economic dynamics inherits this difficulty [6, 7, 12]. [11] opened up a direction for developing useful optimization algorithms by introducing a dual EG program without the "pole" issue. Later, a general version of this dual EG program was discovered for continuous, convex, 1-homogeneous chores markets [12, 45]. [12] prove a rate of $\tilde{\mathcal{O}}(1/\varepsilon^2)$ for computing an approximate CE in a wide range of CES chores markets.

2 Preliminaries

Notation. For a positive integer m, write $[m] = \{1, \ldots, m\}$. The vectors $\mathbf{1}_m$ and $\mathbf{0}_m$ denote the all-one and all-zero vectors in $\mathbb{R}^m$, respectively, and $\mathbf{e}_j$ denotes the jth standard basis vector for $j \in [m]$. For vectors $u, v \in \mathbb{R}^m$, $u \odot v$ denotes their coordinate-wise product, thus $(u \odot v)_j = u_j v_j$. For a convex set C, $\operatorname{relint}(C)$ denotes its relative interior, namely its interior within its affine hull.

Chores Fisher Markets. A chores Fisher market consists of a set of n agents and a set of m divisible chores. Agents are deciding on what chores to take on from in order to earn money. Each agent i incurs a disutility $d_i(\mathbf{x}_i)$ for a bundle $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im})$ of chores that is assigned to her, where x_{ij} represents the amount of chore j allocated to agent i. Each agent i has an earning requirement of $B_i > 0$. We consider each $d_i : \mathbb{R}^m_+ \to \mathbb{R}_+$ to be a general *continuous, convex,* 1-*homogeneous* (CCH) disutility function, and $d_i(\mathbf{x}_i) > 0$ for any nonzero $\mathbf{x}_i$. We call such markets as *CCH chores Fisher markets*.

Each chore j has a *fixed* supply $s_j > 0$. An allocation is denoted by a matrix $\mathbf{x} \in \mathbb{R}^{n \times m}_+$ whose ith row is $\mathbf{x}_i$. Given prices $\mathbf{p} = (p_1, \ldots, p_m)$ of chores, where p_j represents the payment-per-unit of chore j done, agent i will demand a bundle of chores that minimizes her disutility subject to satisfying her earning requirements, i.e., the demand of agent i is denoted by

$$X_i(\mathbf{p}) := \operatorname{argmin}_{\mathbf{x}'_i \in \mathbb{R}^m_+} \{d_i(\mathbf{x}'_i) \mid \langle \mathbf{p}, \mathbf{x}'_i \rangle \geq B_i\}.$$

A competitive equilibrium of a chores Fisher market is defined as follows.

Definition 1 (Competitive equilibrium)**.** A pair of price $\mathbf{p}^* \in \mathbb{R}^m_+$ and allocation $\mathbf{x}^* \in \mathbb{R}^{n \times m}_+$ is a competitive equilibrium (CE) if

- (optimal bundles) $\mathbf{x}^*_i \in X_i(\mathbf{p}^*)$ for each agent i, and
- (market clearing) $\sum_{i=1}^n x^*_{ij} \geq s_j$ and $p_j(\sum_{i=1}^n x^*_{ij} - s_j) = 0$ for each chore j.

Without loss of generality, we make the following assumptions.

Assumption 1. (1) $s_j = 1$ for any chore j. (2) $\sum_{i=1}^n B_i = 1$. (3) $d_i(\mathbf{1}_m) = 1$ for any agent i.

We show in Section B that any CCH chores Fisher market can be scaled to satisfy the above assumption, and that this scaling induces a bijection between the sets of CE of the two markets. We further make the following natural assumption on disutility functions.

Assumption 2. For each agent i, her disutility funciton $d_i(\cdot)$ is strictly increasing in its domain in each coordinate.

Under Assumption 2, no agent would do a zero-priced task, and therefore, it must be that $p_j^* > 0$ for all $j \in [m]$ at equilibrium to ensure that every chore is done. As a result, the market clearing condition reduces to $\sum_{i=1}^n x_{ij} = 1$ for all $j \in [m]$. In this paper, we consider the following approximate CE [12].

Definition 2 (ε-approximate CE)**.** A price vector $\mathbf{p} \in \mathbb{R}_+^m$ and an allocation $\mathbf{x} \in \mathbb{R}_+^{n\times m}$ satisfy ε-competitive equilibrium (CE) if and only if:

1. $\langle \mathbf{p}, \mathbf{x}_i \rangle = B_i$ for all $i \in [n]$;
2. $d_i(\mathbf{x}_i) \le d_i(\mathbf{y}_i)$ for all $\mathbf{y}_i$ such that $\langle \mathbf{p}, \mathbf{y}_i \rangle \ge \langle \mathbf{p}, \mathbf{x}_i \rangle,\ \forall i \in [n]$;
3. $1 - \varepsilon \le \sum_{i=1}^n x_{ij} \le \frac{1}{1-\varepsilon}$ for all $j \in [m]$.

That is, only the market clearing condition is approximately satisfied, the rest of the equilibrium conditions are satisfied exactly.[1] A common class of CCH disutility functions is the class of convex CES disutilities:

$$d_i(\mathbf{x}_i) = \Big(\sum\nolimits_{j=1}^m (d_{ij} x_{ij})^\rho \Big)^{\frac{1}{\rho}}, \quad d_{ij} > 0 \ \forall\, j \in [m],\ \rho \in (1, \infty). \qquad \text{(Convex CES Disutilities)}$$

Intuitively, ρ controls an agent's aversion to concentrating work on a few chores: $\rho = 1$ gives linear disutility, while larger values of ρ encourage spreading work across chores. Convex CES disutilities are CCH, strictly positive for any nonzero $\mathbf{x}_i$, and satisfy Assumption 2.

Excess Supply and Tâtonnement Dynamics. For a given price vector $\mathbf{p} \in \mathbb{R}_+^m$, the excess supply mapping is defined as *(excess-demand is negative of excess-supply)*

$$Y(\mathbf{p}) = \Big\{ \mathbf{1}_m - \sum\nolimits_{i=1}^n \mathbf{x}_i \ \Big|\ \mathbf{x}_i \in X_i(\mathbf{p}) \Big\}.$$

We call any $\mathbf{y} \in Y(\mathbf{p})$ an excess supply (vector). If there exists a $\mathbf{y} \in Y(\mathbf{p})$ such that $\mathbf{y} = \mathbf{0}_m$, then $\mathbf{p}$ corresponds to a CE. If there exists a $\mathbf{y} \in Y(\mathbf{p})$ such that $\|\mathbf{y}\|_2 \le \varepsilon$, then $\mathbf{p}$ corresponds to an ε-approximate CE defined in Definition 2; we add a proof these facts in Section B for completeness.

Tâtonnement dynamics is a price-adjustment process driven by excess supply. Intuitively, in a chores Fisher market, if the supply of a chore exceeds the demand for it, then the price of the chore should be increased to attract more agents to take on the chore. Conversely, if the supply is below the demand, then the price should be decreased so that agents become less willing to perform the chore. [12] considered two types of tâtonnement dynamics:

$$\begin{aligned} \mathbf{y}^k &\in Y(\mathbf{p}^k) \\ \mathbf{p}^{k+1} &= \mathbf{p}^k + \eta^k \mathbf{y}^k, \end{aligned} \qquad \text{(Naïve Tâtonnement)}$$

and

$$\begin{aligned} \mathbf{y}^k &\in Y(\mathbf{p}^k) \\ \tilde{\mathbf{y}}^k &= \mathbf{y}^k - \big(\frac{1}{m} \mathbf{1}_m^\top \mathbf{y}^k\big) \cdot \mathbf{1}_m \\ \mathbf{p}^{k+1} &= \mathbf{p}^k + \eta^k \tilde{\mathbf{y}}^k. \end{aligned} \qquad \text{(Relative Tâtonnement)}$$

They show that, (Naïve Tâtonnement) dynamics can diverge in chores Fisher markets with convex CES disutilities, whereas (Relative Tâtonnement) dynamics converges in any CCH chores Fisher market satisfying Assumption 2.

[1] Alternatively, one may relax the market-clearing condition based on complementarity, e.g., requiring $|p_j(\sum_{i=1}^n x_{ij}^* - 1)| \le \varepsilon$ for all $j \in [m]$ instead. We do not use this definition because the current definition of an approximate CE is more common in the chores Fisher market literature, making it more direct to compare our results with existing results.

Mirror descent. Let $h : \mathbb{R}^m \to \mathbb{R}$ be a function that is: a) strictly convex, b) continuously differentiable, c) defined on a closed convex set. Then, the Bregman divergence is defined as

$$D_h(\mathbf{p}, \mathbf{q}) = h(\mathbf{p}) - h(\mathbf{q}) - \langle \nabla h(\mathbf{q}), \mathbf{p} - \mathbf{q} \rangle.$$

If we takes $h_{\text{KL}}(\mathbf{p}) = \sum_{j=1}^m p_j \log p_j$, $D_{\text{KL}}(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^m p_j \log \frac{p_j}{q_j} - p_j + q_j$, which is classic relative entropy, or KL divergence. Mirror descent is a first-order optimization method that uses a Bregman divergence as its proximal geometry. Given a differentiable objective $f : \mathcal{P} \to \mathbb{R}$ and $\mathcal{P} \subset \mathbb{R}^m$, mirror descent updates

$$\mathbf{p}^{k+1} = \text{argmin}_{\mathbf{p} \in \mathcal{P}} \left\{ \eta^k \langle \nabla f(\mathbf{p}^k), \mathbf{p} \rangle + D_h(\mathbf{p}, \mathbf{p}^k) \right\}.$$

Relative smoothness generalizes the usual Lipschitz-gradient smoothness condition. We say that f is L-smooth relative to h if

$$f(\mathbf{q}) \leq f(\mathbf{p}) + \langle \nabla f(\mathbf{p}), \mathbf{q} - \mathbf{p} \rangle + L D_h(\mathbf{q}, \mathbf{p}), \quad \text{for all } \mathbf{p}, \mathbf{q} \in \mathcal{P}.$$

3 Multiplicative tâtonnement and its convergence

3.1 Multiplicative tâtonnement dynamics

We begin by introducing a new dynamics for chores Fisher market. For each chore j, suppose there is a manager j responsible for allocating that chore among the agents. In each round k, manager j submits a payment-per-unit p_j^k to the market and receive an excess-supply signal y_j^k. She then updates the payment-per-unit multiplicatively in response to this signal: if there is positive excess supply (more of the chore is supplied than agents want to perform), she increases the payment, and vice versa. The update in round k can be written compactly as follows, where $\odot$ denotes coordinate-wise product, and η^k is the step size:

$$\begin{aligned} \mathbf{y}^k &\in Y(\mathbf{p}^k) \\ \mathbf{p}^{k+1} &= \mathbf{p}^k \odot (\mathbf{1}_m + \eta^k \mathbf{y}^k). \end{aligned} \qquad \text{(Multiplicative Tâtonnement)}$$

This multiplicative dynamics has two key advantages:

- It is *fully distributed*: each manager observes only the feedback associated with her own chore, and not the feedback associated with other chores.
- It scales the adjustment by both the excess-supply signal and the current payment-per-unit.

Remark 1. The dynamics proposed in this paper is fully distributed in the following sense: each price updates in response to its current price and its own excess-supply signal, without access to the prices or excess-supply signals of other chores. The dynamics is not currently fully decentralized in the sense of hyperparameter selection. That said, our definition is in line with prior work on tâtonnement, which also defines "distributed" in terms of the update structure, rather than the step size. We remark that this is true in tâtonnement for both goods (see, e.g., Theorem 5.1 in [14], Theorem 4.4 in [31]) and chores Fisher markets (see, Theorem 4.3 in [12]].

3.2 Asymptotic convergence of multiplicative tâtonnement

Next, we prove that, (Multiplicative Tâtonnement) dynamics converges to a CE in any CCH chores Fisher market. This fact is in contrast to its additive analogue as Naïve Tâtonnement is known to provably diverge across a family of CCH chores Fisher markets [12].

One major obstacle to guaranteeing convergence of dynamics to a CE in chores markets is the divergent behavior of prices once they leave the "price simplex": $\Delta_B := \{\mathbf{p} \in \mathbb{R}_+^m \mid \sum_{j=1}^m p_j = \sum_{i=1}^n B_i\}$, as pointed out by [12]. In contrast to approaches that impose a correction on the excess-supply signal, multiplicative tâtonnement prevents price divergence by exploiting Walras' law, a fundamental identity in general equilibrium theory. All proofs in this section are deferred to Section C.

Proposition 1 (Walras's law)**.** *If $\mathbf{p} \in \Delta_B$, then we have $\langle \mathbf{p}, \mathbf{y} \rangle = 0$ for all $\mathbf{y} \in Y(\mathbf{p})$.*

By leveraging this intrinsic property of excess supply, the price vectors generated by multiplicative tâtonnement automatically stay within the affine hull of Δ_B, if the initial price vector lies in Δ_B. Beyond this fact, to ensure convergence of the dynamics, we additionally require boundedness of the excess supply and strict positivity of the prices.

Lemma 1 (Boundedness of excess supply)**.** *In a CCH chores Fisher market that satisfies Assumption 2, denote* $d_{\min}^{(i)} := \min_{j\in[m]} d_i(\mathbf{e}_j) > 0$ *where* $\mathbf{e}_j$ *is the basis vector with a* 1 *in coordinate* j *and* 0 *elsewhere. Then, for any* $\mathbf{p} \in \Delta_{\mathbf{B}}$, $1 - (d_{\min})^{-1} \le y_j \le 1$ *for any* $j \in [m]$ *and* $\mathbf{y} \in Y(\mathbf{p})$, *where* $d_{\min} := \min_i d_{\min}^{(i)}$. *If* $d_i(\cdot)$ *is a convex CES disutility function for each agent* i, *then* $1 - (\min_{i,j} d_{ij})^{-1} \le y_j \le 1$ *for any* $j \in [m]$ *and* $\mathbf{y} \in Y(\mathbf{p})$.

Define $\ell_0 = \frac{1}{3m} \min_i \left\{ \min\{d'_{ij}/d'_{ij'} \mid j, j' \in [m], \tilde{x}_{ij} \ge \frac{1}{2nm}, \tilde{x}_{ij'} \le 2mB_i, \mathbf{d}'_i \in \partial d_i(\tilde{\mathbf{x}}_i)\}\right\} > 0.$

Lemma 2 (Strict positiveness of prices)**.** *Let* $\beta := 1 + (d_{\min})^{-1} > 0$. *Let* $\{\mathbf{p}^k\}_{k\ge 0}$ *be a sequence of iterates generated by* (Multiplicative Tâtonnement) *with the initial point* $\mathbf{p}^0 \in \text{relint}\,(\Delta_B)$ *and* $\eta^k \le \frac{1}{2\beta}$ *for all* k. *Then, we have* (i) $\mathbf{p}^k \in \Delta_B$ *for all* $k \ge 0$, (ii) $p_j^{k+1} > p_j^k(1 + \frac{1}{2}\eta^k)$ *if* $p_j^k \le \ell_0$ *for any* $k \ge 0$ *and* $j \in [m]$, (iii) *if additionally* $\sum_{k=0}^{\infty} \eta^k = \infty$, *then there exists a finite index* $k_0 \ge 0$ *such that* $p_j^k \ge \frac{\ell_0}{2}\ \forall\, j \in [m]$ *for all* $k \ge k_0$. *In particular, one can choose* $k_0 = \lceil \frac{\log(\ell_0/(2\min_j p_j^0))}{\log(1+\eta/2)} \rceil$.

We then prove the convergence of multiplicative tâtonnement by linking it to the following updates:

$$\bar{\mathbf{p}}^{k+1} := \mathbf{p}^k \odot \exp(\eta \mathbf{y}^k), \tag{1}$$

which is mirror descent w.r.t. the relative entropy for the following objective function [12, Lemma 6]

$$f(\mathbf{p}) := -\sum_{j=1}^{m} p_j + \sum_{i=1}^{n} B_i \log\left(\max_{\mathbf{x}_i \ge 0 : d_i(\mathbf{x}_i) \le 1} \langle \mathbf{p}, \mathbf{x}_i \rangle\right). \tag{2}$$

Note that (1) itself does not converge since the function f is not globally lower bounded. In particular, the objective value goes to $-\infty$ as $\mathbf{p} \to \mathbf{0}_m$, or as $p_j \to \infty$ for any $j \in [m]$. However, it serves as a useful reference process for proving the per-step improvement: multiplicative tâtonnement tracks (1) within the price simplex Δ_B, and thus inherits the iterate descent property up to a small perturbation. Since f is lower bounded within Δ_B, the iterate converges to a stationary point asymptotically.

Formally, (Multiplicative Tâtonnement) dynamics fits into the following Bregman subgradient update scheme [24]:

$$\mathbf{p}^{k+1} \approx \text{argmin}_{\mathbf{q}} \{\eta^k \langle -\mathbf{y}^k, \mathbf{q} \rangle + D_{\text{KL}}(\mathbf{q} \| \mathbf{p}^k)\}, \tag{3}$$

and the perturbation error is controlled by

$$\left\| \frac{\nabla h_{\text{KL}}(\mathbf{p}^{k+1}) - \nabla h_{\text{KL}}(\mathbf{p}^k)}{\eta^k} - \mathbf{y}^k \right\|_2 \le \nu^k. \tag{4}$$

[24] prove the convergence of the above process under a set of regularity conditions. We adapt their result to our setting and state it as the following proposition.

Proposition 2. *Let* $\{\mathbf{p}^k\}_{k\ge 0}$ *be a sequence of iterates generated by Bregman subgradient update described in Eqs.* (3) *and* (4)*, and suppose that the process satisfies that* (1) *the sequences* $\{\mathbf{p}^k\}_{k\ge 0}$, $\{\log(\mathbf{p}^k)\}_{k\ge 0}$, *and* $\{\mathbf{y}^k\}_{k\ge 0}$ *are uniformly bounded almost surely,* (2) *the stepsizes satisfies* $\sum_{k=0}^{\infty} \eta^k = \infty$, *and either* $\eta^k = o(\frac{1}{\log k})$ *or* $\sum_{k=0}^{\infty} (\eta^k)^2 < \infty$, (3) *the perturbation error satisfies* $\lim_{k\to\infty} \nu^k \to 0$, (4) *there is a potential function* f *such that* $-\mathbf{y}^k \in \partial f(\mathbf{p}^k)$[2], *and* f *is lower bounded,* (5) *the critical value set* $\{f(\mathbf{p}) \mid \mathbf{0}_m \in \partial f(\mathbf{p})\}$ *has empty interior in* $\mathbb{R}$. *Then, almost surely, any cluster point of* $\{\mathbf{p}^k\}_{k\ge 0}$ *is a critical point and the function values* $\{f(\mathbf{p}^k)\}_{k\ge 0}$ *converge.*

Therefore, by combining Lemmas 1 and 2, and choosing the proper stepsizes to control the perturbation error, we can prove the convergence of (Multiplicative Tâtonnement).

Theorem 1 (Asymptotic convergence)**.** *Let* $\{\eta^k\}_{k\ge 0}$ *be a sequence of stepsizes satisfying (i)* $0 < \eta^k \le \frac{1}{2\beta}$, *(ii)* $\sum_{k=0}^{\infty} \eta^k = \infty$, *and (iii)* $\eta^k = o(\frac{1}{\log k})$ *or* $\sum_{k=0}^{\infty} (\eta^k)^2 < \infty$. *Let* $\{\mathbf{p}^k\}_{k\ge 0}$ *be a sequence of iterates generated by* (Multiplicative Tâtonnement) *with* $\{\eta^k\}_{k\ge 0}$ *and any initial point* $\mathbf{p}^0 \in \text{relint}(\Delta_B)$. *Then, every limit point of* $\{\mathbf{p}^k\}_{k\ge 0}$ *is a CE of the chores Fisher market.*

[2] $\partial f(\mathbf{p}^k)$ denotes Clarke differential of function f at $\mathbf{p}^k$.

Remark 2. Notice that the modulus ℓ_0 can be very small in the worst case. For example, convex CES disutilities with a large ρ cause $\ell_0 = O((D/m)^{\rho-1})$ where $D = \max_i(\max_j d_{ij}/\min_j d_{ij})$. This can be problematic for relative tâtonnement, since its stepsize must scale with ℓ_0 to keep prices strictly positive[3]. In contrast, multiplicative tâtonnement dynamics admits stepsizes independent of ℓ_0, since its updates scale multiplicatively with the prices.

4 Convergence rates to an approximate CE under convex CES disutilities

We further study convergence rates of multiplicative tâtonnement to find an ε-approximate CE. We focus on the class of convex CES disutility function, i.e., $d_i(\mathbf{x}_i) = (\sum_{j=1}^m (d_{ij}x_{ij})^\rho)^{1/\rho}$ with $\rho \in (1, \infty)$. Since the inner-log maximization problem in Eq. (2) ensures its value bounded away from zero, and has a unique maximizer, function f is differentiable:

$$f(\mathbf{p}) = -\sum_{j=1}^m p_j + \sum_{i=1}^n \frac{B_i}{\sigma} \log\Big(\sum_{j=1}^m d_{ij}^{-\sigma} p_j^\sigma\Big), \quad \sigma = \frac{\rho}{\rho-1} \in (1,\infty).$$

To prove the convergence rate of (Relative Tâtonnement), one utilizes the smoothness property of the potential function with respect to the squared Euclidean norm, i.e., the standard smoothness setup. Then, the rate depends on the modulus of the standard smoothness: a better smoothness modulus yields a better convergence rate. In contrast, (Multiplicative Tâtonnement) allows us to measure the smoothness modulus of our potential function with the KL divergence as the "distance" used in the smoothness definition. Because the KL divergence is better aligned with the curvature of our potential function on Δ_B, we get a better convergence rate.

By leveraging the fact that $\mathbf{p} \in \Delta_B$, we obtain an improved relative smoothness than the standard one. In particular, it establishes the smoothness over entire Δ_B for the full spectrum of $\rho \in (1, \infty)$, and the modulus does *not depend* on a lower bound on the prices. This relative smoothness then lays the foundation for a tighter convergence rate. All proofs in this section is in Section D.

Lemma 3 (Relative smoothness of f w.r.t. h_{KL})**.** *For any two price vectors $\mathbf{p}, \mathbf{q} \in \Delta_B$, we have*

$$f(\mathbf{p}) \le f(\mathbf{q}) + \langle \nabla f(\mathbf{q}), \mathbf{p} - \mathbf{q}\rangle + (\sigma-1)m^{\sigma-1}(\min_{i,j} d_{ij})^{-\sigma} D_{\mathrm{KL}}(\mathbf{p}, \mathbf{q}).$$

Remark 3. Note that the naïve global KL-relative smoothness of f still has an $O(1/\ell_0)$ smoothness modulus, which means that the constraint $\sum_{j=1}^m p_j = \|\mathbf{B}\|_1$ is necessary to eliminate the singularity caused by the logarithm. However, even under this constraint, f still has an $O(1/\ell_0)$ smoothness modulus with respect to $\frac{1}{2}\|\cdot\|^2$, due to the mismatch between the Euclidean geometry and the geometry induced by h_{KL}.

With this tool in hand, we next prove a convergence rate to compute an ε-approximate CE. We have

$$\mathbf{p}^{k+1} = \mathrm{argmin}_{\mathbf{q}\ge 0}\{\eta^k\langle -\mathbf{y}^k + \nu^k, \mathbf{q}\rangle + D_{\mathrm{KL}}(\mathbf{q}\|\mathbf{p}^k)\}$$

where the perturbation ν^k is defined as

$$\nu^k := -\frac{\log(1+\eta^k\mathbf{y}^k)}{\eta^k} + \mathbf{y}^k.$$

By standard mirror descent analysis and Lemma 3, we have the following descent lemma:

Lemma 4 (Descent property)**.** *Let $\{\mathbf{p}^k\}_{k\ge 0}$ be a sequence of iterates generated by* (Multiplicative Tâtonnement) *dynamics with any positive stepsizes $\{\eta^k\}_{k\ge 0}$ and any initial point $\mathbf{p}^0 \in \mathrm{relint}(\Delta_B)$. We have that*

$$f(\mathbf{p}^{k+1}) \le f(\mathbf{p}^k) - \frac{1}{\eta^k}\left(D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k)\right) - \langle \nu^k, \mathbf{p}^{k+1} - \mathbf{p}^k\rangle + L D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k). \tag{5}$$

[3]We do not consider price projection, since it would require each chore manager to have access to the prices of other chores.

Define $\|\mathbf{y}\|_{\mathbf{p}} := \sqrt{\sum_{j=1}^m p_j (y_j)^2}$, we can then link the last two terms in Eq. (14) to $\|\mathbf{y}^k\|^2_{\mathbf{p}^k}$. To do so, we need the stepsizes to be uniformly small to control $\|\eta^k \mathbf{y}^k\|_\infty$.

Lemma 5. *If $\eta^k \leq \frac{1}{2\beta}$ for all $k \geq 0$, then we have $-\langle \nu^k, \mathbf{p}^{k+1} - \mathbf{p}^k \rangle \leq \beta(\eta^k)^2 \|\mathbf{y}^k\|^2_{\mathbf{p}^k}$ and $D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) \geq \frac{1}{3}(\eta^k)^2 \|\mathbf{y}^k\|^2_{\mathbf{p}^k}$.*

Combining Lemmas 4 and 5 yields the convergence rate.

Theorem 2 (Convergence rate guarantee)**.** *Let $\{\mathbf{p}^k\}_{k\geq 0}$ be a sequence of iterates generated by* (Multiplicative Tâtonnement) *with any initial point $\mathbf{p}^0 \in \mathrm{relint}(\Delta_B)$ and any positive stepsizes $\{\eta^k\}_{k\geq 0}$ satisfying*

$$\eta^k \equiv \eta \leq \frac{1}{12(L+\beta)},$$

then we can find an iterate $(\mathbf{p}, \mathbf{y})$ satisfies $\|\mathbf{y}\|_{\mathbf{p}} \leq \varepsilon$ in $\lceil 12(f(\mathbf{p}^0) - \underline{f})/\eta\varepsilon^2 \rceil$ iterations. Here, $L = (\sigma - 1)\, m^{\sigma-1} (\min_{i,j} d_{ij})^{-\sigma}$ and $\beta = 1 + (\min_{i,j} d_{ij})^{-1}$.

By Lemma 2, after a warm-up time that is at most $\max\left\{ \left\lceil \frac{(2+\eta)\log(\ell_0/(2\min_j p_j^0))}{\eta} \right\rceil, 0 \right\}$, we have $p_j^k \geq \frac{\ell_0}{2}\ \forall\, j \in [m]$ for all k afterward. Therefore, by taking $\varepsilon = \sqrt{\ell_0/2}\varepsilon$ and $\eta = 1/(12(L+B))$, we directly have the following corollary.

Corollary 1. (Multiplicative Tâtonnement) *dynamics with $\eta = 1/(12(L+\beta))$ can find an ε-approximate CE in $\lceil 288(L+\beta)(f(\mathbf{p}^0) - \underline{f})/\ell_0\varepsilon^2 \rceil + \max\{\lceil (2+\eta)\log(\ell_0/(2\min_j p_j^0))/\eta \rceil, 0\}$ iterations.*

Discussions. *First*, Theorem 2 shows that we can find a price vector for which $\sum_{j=1}^m p_j (y_j)^2 \leq \varepsilon$ in $\mathcal{O}(1/\varepsilon^2)$ iterations, without any dependence on $1/\ell_0$. In other words, for any price coordinate that is not close to zero, an ε-small excess supply is guaranteed. We emphasize that as $\rho \to \infty$, $1/\ell_0$ grows exponentially fast and can therefore be problematic. Since $L + \beta = (\sigma-1)m^{\sigma-1}(\min_{i,j} d_{ij})^{-\sigma} + 1 + (\min_{i,j} d_{ij})^{-1}$, and $\sigma \to 1$ as $\rho \to \infty$, this convergence rate ensures that multiplicative tâtonnement finds a meaningful price vector even as $\rho \to \infty$.

Second, as we detail in Section D.1, relative tâtonnement has a dependence of $\mathcal{O}\big(\frac{(\sigma-1)(\min_{i,j} d_{ij})^{-(\sigma+1)}}{\sqrt{m}\ell_0\varepsilon^2}\big)$, even under an analysis sharper than the original one in [12]. In contrast, multiplicative tâtonnement improves this rate by a factor of $(\min_{i,j} d_{ij})^{-1}$. As we will see in Section 5, this factor dominates in most real-world instances and can be very large.

Third, Corollary 1 indicates that we can implement multiplicative tâtonnement using a constant stepsize independent of ℓ_0. In contrast, as shown in Section D.1, relative tâtonnement requires a constant stepsize of order $O(\ell_0)$, where ℓ_0 can be very small when ρ is large. In Section 5, we show that this theoretical comparison aligns with the practical performance: the best admissible stepsize for multiplicative tâtonnement is much larger than that for relative tâtonnement in practice, which accounts for the superior speedup of multiplicative tâtonnement over relative tâtonnement.

5 Numerical experiments

We conduct numerical experiments on three types of datasets: Spliddit instances, AAMAS bidding instances, and synthetic instances. For each type, we test three convex CES disutilities with $\rho \in \{1.2,\ 2,\ 5\}$. We compare the number of iterations required to compute an ε-approximate equilibrium, where $\varepsilon \in \{0.1,\ 0.05,\ 0.01,\ 0.005,\ 0.001\}$. The experiments were implemented on a personal desktop with Apple M1 chip.

Spliddit instances. Spliddit is a research-driven online platform that helps users divide resources fairly and transparently [32]. We use a dataset containing 670 task-distribution instances from Spliddit. Each Spliddit instance includes the disutility values for all pairs of agent and chore, which is collected by Spliddit when users enter their preferences. The instance sizes range from 2 to 6 agents and from 2 to 37 tasks. We test all instances in the dataset and report the average number of iterations required to compute an approximate CE, across different approximation tolerances ε in Fig. 1.

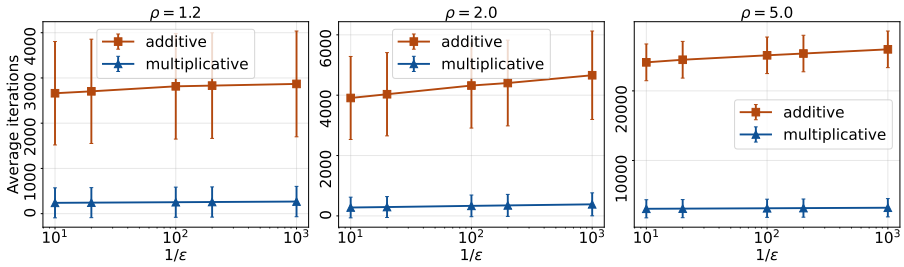

Figure 1: Multiplicative Tâtonnement v.s. Relative Tâtonnement on Spliddit instances. The marker denotes the mean value of the number of iterations required to compute an ε-approximate CE over all instances. The error bar corresponds to 0.1 standard deviation.

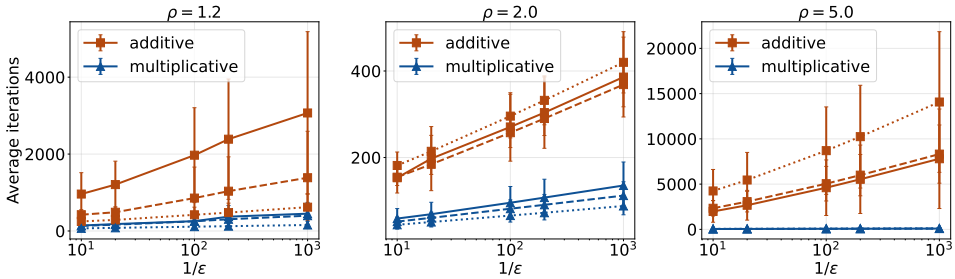

Figure 2: Multiplicative Tâtonnement v.s. Relative Tâtonnement on synthetic instances with truncated normal distributions. Solid lines correspond to 100×500 instances, dashed lines correspond to 100×1000 instances, dotted lines correspond to 100×2000 instances. The error bar corresponds to 0.1 standard deviation.

We use the following heuristic to select practical stepsizes for both algorithms. For each instance, we start with a large stepsize and iteratively search for an upper bound on the admissible stepsizes. If a trajectory leaves the price simplex, we set the current stepsize as an upper bound on the admissible range. We then perform a grid search (excluding the stepsize upper bound) over the admissible range using 9 grid points, from the largest to the smallest. If all tested stepsizes are admissible, we choose the one that achieves the best performance, measured by the smallest approximation tolerance. We also use this method to select practical stepsizes for other types of instances.

AAMAS bidding instances. We construct a set of AAMAS bidding instances from a dataset of bids submitted by PC members at the AAMAS conference for potential papers to review. This dataset is obtained from PrefLib [38] and is also used by [11]. We convert reviewer bids into a disutility matrix using the mapping `yes:1`, `maybe:3`, `missing:5`, `no:7`, and `conflict:1401`. For each instance, we choose a random seed paper, take its nearest `100` papers under the squared distance between bidding profiles, and then select the `200` reviewers with the most positive responses on this paper set. Finally, we optionally add Gaussian noise to the resulting $\texttt{100}\times\texttt{200}$ disutility matrix. For each ρ, we sample `10` such instances and set all reviewers' budgets to one, as is standard in fair-division settings. The results are shown in Fig. 4.

Synthetic instances. We simulate a set of large instances to compare the empirical convergence rates of the two dynamics. We generate instances with 100 agents and $500, 1000, 2000$ chores, reflecting the fact that, in many real applications, the number of chores is much larger than the number of agents. For each problem size, we test the dynamics on 20 instances generated from 20 random seeds. We consider two valuation distributions: (1) a lognormal distribution with location parameter 0 and scale parameter 1; and (2) a truncated normal distribution with mean 0.5 and standard deviation 0.2, truncated below at 0.01. We set each agent's budget to one. The results are shown in Figs. 2 and 5.

As can be seen from Figs. 2 and 5, different disutility generating distributions can yield different ranges of required iterations: the truncated normal instances requires more iterations to compute an approximate CE than the lognormal instances. This can be partially explained by the difference in

the distribution of $\max_{i,j} d_{ij} / \min_{i,j} d_{ij}$ values between the two classes of instances. Also, different distributions lead to different structures, which causes varying difficulty to find an approximate CE.

In Section E, we present preliminary experimental results on some non-CES disutilites. In particular, we consider $d_i(\mathbf{x}_i) = \|A_i \mathbf{x}_i\|_2$, where each agent has a dense matrix $A_i \in \mathbb{R}_{++}^{100 \times m}$.

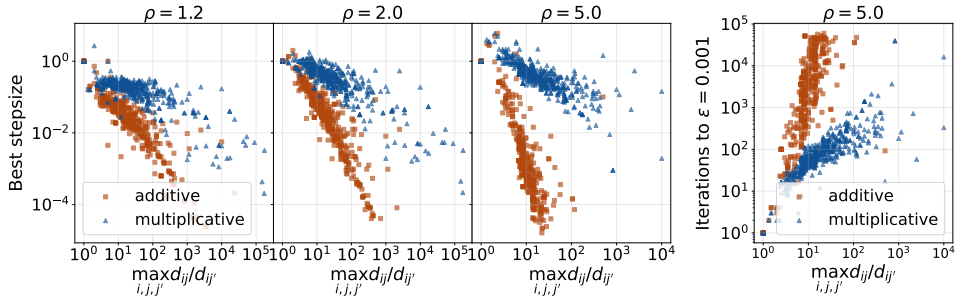

Figure 3: The best stepsizes in practice *or* the number of iterations to compute an approximate CE v.s. maximum ratio of the disutility coefficients with $\rho \in \{1.2,\ 2,\ 5\}$. Each point in the plots corresponding to a Spliddit instance that finds an approximate CE within the iteration budget.

Discussions. Overall, multiplicative tâtonnement significantly outperforms relative tâtonnement. On Spliddit instances, multiplicative tâtonnement requires fewer than 100 iterations to find a 0.001-approximate CE, even when $\rho = 5$. In contrast, relative tâtonnement may require a very large number of iterations to find an approximate CE, and this number increases as ρ grows. In particular, relative tâtonnement can take more than $20{,}000$ iterations to find a 0.1-approximate CE. On AMMAS bidding instances, although both dynamics compute approximate CE within 100 iterations, multiplicative tâtonnement still reduces the iteration cost by roughly half. These instances are relatively easy to solve because their value sets are sparse, containing only 5 distinct values. On synthetic instances, multiplicative tâtonnement typically achieves a $10\times$ speedup, even for small values of ρ.

An interesting observation is the relationship between the best practical stepsizes and the maximum value ratio $\max_{i,j,j'} d_{ij}/d_{ij'}$. We illustrate this relationship based on the Spliddit instances with scatter plots in Fig. 3. The plots reveal a clear pattern: as $\max_{i,j,j'} d_{ij}/d_{ij'}$ increases, the best stepsize decreases at a rate consistent with our analysis. In particular, the largest admissible stepsize appears to be inversely proportional to $(\max_{i,j,j'} d_{ij}/d_{ij'})^{\rho}$.

In Fig. 3, we plots the best practical stepsize versus $\max_{i,j,j'} d_{ij}/d_{ij'}$ for both dynamics on a log-log plot. Assuming that $\eta_{\text{best}} = 1/(\max_{i,j,j'} d_{ij}/d_{ij'})^{c}$, the slope in the left three subplots of Fig. 3 corresponds to c. By Fig. 3, we see the changes of slopes as ρ changes show an empirical observation: the best practical stepsize of multiplicative tâtonnement depends on $\max_{i,j,j'} d_{ij}/d_{ij'}$ but not coupled with ρ, while that of relative tâtonnement does empirically depend on $(\max_{i,j,j'} d_{ij}/d_{ij'})^{\rho}$. This observation aligns with our theoretical bounds.

Additionally, from the right plot in Fig. 3, as $\max_{i,j,j'} d_{ij}/d_{ij'}$ increases, the increase of the number of iterations appearantly has a separation between two dynamics: the iteration cost of multiplicative tâtonnement dynamics scale up at a $\frac{1}{2}$ rate of that of relative tâtonnement, which also matches our theoretical results as $\rho \to \infty$ and thus $\sigma \to 1$: multiplicative tâtonnement has a dependency $(d_{\min})^{-1} + (d_{\min})^{-\sigma}$ and relative tâtonnement has a dependency $(d_{\min})^{-1-\sigma}$.

6 Conclusion

In this paper, we propose multiplicative tâtonnement, the first distributed price-adjustment dynamics for computing a CE in Fisher markets with chores. We show that multiplicative tâtonnement converges to a CE for general CCH disutilities and achieves an $O(1/\varepsilon^2)$ convergence rate to an ε-approximate CE, with a better constant than the prior relative tâtonnement dynamics. We evaluate both tâtonnement dynamics on a wide range of datasets, including real-world data from Spliddit. Interesting future directions include (1) developing dynamics whose step sizes can be selected in a distributed manner and (2) proposing an adaptive step-size rule that performs better in practice than conservative theoretical step sizes while preserving similar theoretical guarantees.

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Appendices

A Additional related work

Computation of CE in Fisher markets. A series of foundational works established convex programming and linear complementarity problem (LCP) characterizations of the CE set across various economic models [27, 41, 26]. Building on these formulations, [42] obtained the first strongly polynomial-time algorithm in this setting. Subsequent research has produced a diverse body of algorithms for computing CE in broader market models—such as exchange and Arrow-Debreu markets—going well beyond the linear Fisher case [28, 36, 1, 51]. These include interior point methods [34, 48], as well as combinatorial approaches [25]. For *CES markets* specifically, polynomial-time computation has been established under various restrictions, including weak gross substitutes regimes ($\sigma > 1$) [19, 15], fixed numbers of agents or goods [22], and more general homothetic utilities via convex programming [35]. In the chores market, a separate line of work has yielded polynomial-time algorithms achieving arbitrarily good approximations of CE [7, 10, 11]. More recently, [9] gave a polynomial-time algorithm for exactly computing a CE in linear chores Fisher markets whenever the number of agents or chores is constant.

B Proofs of Section 2

Proposition 3. *Given a chores Fisher market instance $\mathcal{M} = (n, m, (d_i)_{i=1}^n, (s_j)_{j=1}^m, (B_i)_{i=1}^n)$, there is another instance $\tilde{\mathcal{M}}$ that satisfies Assumption 1, and there is a bijection between the sets of CE of $\mathcal{M}$ and $\tilde{\mathcal{M}}$.*

Proof of Proposition 3. Given a chores Fisher market instance $\mathcal{M} = (n, m, (d_i)_{i=1}^n, (s_j)_{j=1}^m, (B_i)_{i=1}^n)$, there is another instance $\tilde{\mathcal{M}} = (n, m, (\tilde{d}_i)_{i=1}^n, (\tilde{s}_j)_{j=1}^m, (\tilde{B}_i)_{i=1}^n)$ where

$$\tilde{s}_j := 1, \qquad \tilde{B}_i := \frac{B_i}{\sum_{i'=1}^n B_{i'}}, \qquad \tilde{d}_i(\mathbf{x}_i) := \frac{d_i(\mathbf{s} \odot \mathbf{x}_i)}{d_i(\mathbf{s})}.$$

One can verify that each $\tilde{d}_i$ is CCH, and $\tilde{d}_i(\mathbf{x}_i) = 0$ if and only if $\mathbf{x}_i = \mathbf{0}_m$. Moreover, $\tilde{\mathcal{M}}$ satisfies Assumption 1. Next, we claim that $(\mathbf{p}, \mathbf{x})$ is a CE of $\mathcal{M}$ if and only if $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ is a CE of $\tilde{\mathcal{M}}$ where

$$(\tilde{\mathbf{p}}, \tilde{\mathbf{x}}) := \left(\frac{\mathbf{s} \odot \mathbf{p}}{\sum_{i=1}^n B_i}, \mathbf{s}^{-1} \odot \mathbf{x} \right), \quad \mathbf{s}^{-1} = (s_1^{-1}, \ldots, s_m^{-1}),$$

which is true because of the following correspondences.

$$\begin{aligned}
\tilde{\mathbf{x}}_i \in \operatorname*{argmin}_{\mathbf{x}_i' \in \mathbb{R}_+^m} \left\{ \tilde{d}_i(\mathbf{x}_i') \,\middle|\, \langle \tilde{\mathbf{p}}, \mathbf{x}_i' \rangle \geq \tilde{B}_i \right\} &\Leftrightarrow \mathbf{s}^{-1} \odot \mathbf{x} \in \operatorname*{argmin}_{\mathbf{x}_i' \in \mathbb{R}_+^m} \{ d_i(\mathbf{s} \odot \mathbf{x}_i') \mid \langle \mathbf{s} \odot \mathbf{p}, \mathbf{x}_i' \rangle \geq B_i \} \\
&\Leftrightarrow \mathbf{s}^{-1} \odot \mathbf{x} \in \operatorname*{argmin}_{\mathbf{s}^{-1} \odot \mathbf{x}_i'' \in \mathbb{R}_+^m} \{ d_i(\mathbf{x}_i'') \mid \langle \mathbf{p}, \mathbf{x}_i'' \rangle \geq B_i \} \\
&\Leftrightarrow \mathbf{x} \in \operatorname*{argmin}_{\mathbf{x}_i'' \in \mathbb{R}_+^m} \{ d_i(\mathbf{x}_i'') \mid \langle \mathbf{p}, \mathbf{x}_i'' \rangle \geq B_i \}.
\end{aligned}$$

For any $j \in [m]$,

$$\sum_{i=1}^n \tilde{x}_{ij} \geq 1 \Leftrightarrow \sum_{i=1}^n x_{ij} \geq s_j \quad \tilde{p}_j \left(\sum_{i=1}^n \tilde{x}_{ij} - 1 \right) \Leftrightarrow \frac{p_j}{\sum_{i=1}^n B_i} \left(\sum_{i=1}^n x_{ij} - s_j \right). \qquad \square$$

C Proofs in Section 3

Proposition 1 (Walras's law)**.** *If $\mathbf{p} \in \Delta_B$, then we have $\langle \mathbf{p}, \mathbf{y} \rangle = 0$ for all $\mathbf{y} \in Y(\mathbf{p})$.*

Proof of Proposition 1. Let $\mathbf{y} \in Y(\mathbf{p})$ be any excess supply vector, and let $\mathbf{b} \in \mathbb{R}_+^{n \times m}$ be the earnings for each agent i and chore j corresponding to $\mathbf{y}$. That is, by definition $y_j = 1 - \frac{\sum_{i=1}^n b_{ij}}{p_j}$

for all $j \in [m]$. By agent's optimality, we have $\sum_{j=1}^m b_{ij} = B_i$ for all $i \in [n]$. Then, since $\mathbf{p} \in \Delta_B$, we have

$$\langle \mathbf{p}, \mathbf{y} \rangle = \sum_{j=1}^m p_j \left(1 - \frac{\sum_{i=1}^n b_{ij}}{p_j}\right) = \sum_{j=1}^m p_j - \sum_{i=1}^n \sum_{j=1}^m b_{ij} = \sum_{j=1}^m p_j - \sum_{i=1}^n B_i = 0. \qquad \square$$

We repeat the definition of ℓ_0 (for all disutility functions satisfying Assumption 2) here:

$$\ell_0 = \frac{1}{3m} \min_i \left\{ \min_{\substack{\tilde{\mathbf{x}}_i \geq 0, \mathbf{d}_i' > 0, \\ j,j' \in [m]}} \left\{ \frac{d_{ij}'}{d_{ij'}'} \;\middle|\; j, j' \in [m], \tilde{x}_{ij} \geq \frac{1}{2nm}, \tilde{x}_{ij'} \leq 2mB_i, \mathbf{d}_i' \in \partial d_i(\tilde{\mathbf{x}}_i) \right\} \right\} > 0.$$

For CES disutility function with $\rho \in (1, \infty)$, i.e., $d_i(\mathbf{x}_i) = (\sum_{j=1}^m (d_{ij} x_{ij})^\rho)^{1/\rho}$, then one can show that

$$\ell_0 = \frac{1}{3m} \min_{i \in [n]} \left\{ \min_{j,j' \in [m]} \left(\frac{d_{ij}}{d_{ij'}}\right)^\rho \left(\frac{\|\mathbf{B}\|_1}{4nm^2 B_i}\right)^{\rho - 1} \right\} > 0. \tag{6}$$

The quantity ℓ_0 is useful in the proof as the excess supply is positive whenever a price is lower than ℓ_0, which is shown by [12, Lemma 3].

Lemma 6. *In a chores Fisher market with CCH disutilities satisfying Assumption 2, given any $\mathbf{p} \in \mathbb{R}_+^m$ such that $\max_{j \in [m]} p_j \geq \frac{\|\mathbf{B}\|_1}{2m}$, for any chore j, if $p_j \leq \ell_0$ then $y_j > 1 - \frac{1}{2m} \geq \frac{1}{2}$ for all $\mathbf{y} \in Y(\mathbf{p})$.*

Lemma 1 (Boundedness of excess supply)**.** *In a CCH chores Fisher market that satisfies Assumption 2, denote $d_{\min}^{(i)} := \min_{j \in [m]} d_i(\mathbf{e}_j) > 0$ where $\mathbf{e}_j$ is the basis vector with a 1 in coordinate j and 0 elsewhere. Then, for any $\mathbf{p} \in \Delta_{\mathbf{B}}$, $1 - (d_{\min})^{-1} \leq y_j \leq 1$ for any $j \in [m]$ and $\mathbf{y} \in Y(\mathbf{p})$, where $d_{\min} := \min_i d_{\min}^{(i)}$. If $d_i(\cdot)$ is a convex CES disutility function for each agent i, then $1 - (\min_{i,j} d_{ij})^{-1} \leq y_j \leq 1$ for any $j \in [m]$ and $\mathbf{y} \in Y(\mathbf{p})$.*

Proof of Lemma 1. For each agent $i \in [n]$ and any

$$\mathbf{x}_i^* \in \operatorname{argmin}_{\mathbf{x} \geq 0} \{ d_i(\mathbf{x}_i) \mid \sum\nolimits_{j=1}^m p_j x_{ij} \geq B_i \}, \tag{7}$$

it follows that

$$d_i(\mathbf{x}_i^*) \leq d_i \left(\frac{B_i}{\|\mathbf{p}\|_1} \cdot \mathbf{1}_m \right) = d_i \left(B_i \cdot \mathbf{1}_m \right) = B_i, \tag{8}$$

where the first inequality holds because $(B_i / \|\mathbf{p}\|_1) \cdot \mathbf{1}_m$ is a feasible solution to the minimization problem in Eq. (7), and the inequalities follows by $\mathbf{p} \in \Delta_B$ and $\|\mathbf{B}\|_1 = 1, d_i(\mathbf{1}_m) = 1$ by Assumption 1.

Since d_i is increasing in each coordinate, by Eq. (8), we have

$$x_{ij}^* d_{\min}^{(i)} \leq x_{ij}^* d_i(\mathbf{e}_j) = d_i(x_{ij}^* \mathbf{e}_j) \leq d_i(\mathbf{x}_i^*) \leq B_i \; \forall \, j \in [m],$$

where the first equality is because d_i is 1-homogeneous, and the second inequality is because of Assumption 2. Thus, we have $x_{ij}^* \leq B_i / d_{\min}^{(i)}$.

Therefore, for any $\mathbf{y} \in Y(\mathbf{p})$, we have $1 \geq y_j = 1 - \sum_{i=1}^n x_{ij}^* \geq 1 - \sum_{i=1}^n B_i / d_{\min}^{(i)}$. Applying a uniform bound for $d_{\min}^{(i)}$ over all $i \in [n]$, we have

$$1 - \sum_{i=1}^n \frac{B_i}{d_{\min}^{(i)}} \geq 1 - \frac{1}{\min_i d_{\min}^{(i)}} \sum_{i=1}^n B_i = 1 - \frac{1}{d_{\min}}.$$

For CES disutility function with $\rho \in (1, \infty)$, i.e., $d_i(\mathbf{x}_i) = (\sum_{j=1}^m (d_{ij} x_{ij})^\rho)^{1/\rho}$,

$$d_{\min}^{(i)} = \min_j d_{ij} \text{ and } d_{\min} = \min_{i,j} d_{ij}. \tag{9}$$

This completes the proof. $\square$

Leveraging this sharper boundedness of the excess demand, we can show that, the desired behaviors of the discrete-time multiplicative tâtonnement holds with a larger stepsize than a naïve bound. Moreover, this admissible stepsize will not blow up when $\rho \to \infty$.

Lemma 2 (Strict positiveness of prices)**.** *Let $\beta := 1 + (d_{\min})^{-1} > 0$. Let $\{\mathbf{p}^k\}_{k\geq 0}$ be a sequence of iterates generated by* (Multiplicative Tâtonnement) *with the initial point $\mathbf{p}^0 \in$ relint (Δ_B) and $\eta^k \leq \frac{1}{2\beta}$ for all k. Then, we have (i) $\mathbf{p}^k \in \Delta_B$ for all $k \geq 0$, (ii) $p_j^{k+1} > p_j^k(1 + \frac{1}{2}\eta^k)$ if $p_j^k \leq \ell_0$ for any $k \geq 0$ and $j \in [m]$, (iii) if additionally $\sum_{k=0}^{\infty} \eta^k = \infty$, then there exists a finite index $k_0 \geq 0$ such that $p_j^k \geq \frac{\ell_0}{2}$ $\forall\, j \in [m]$ for all $k \geq k_0$. In particular, one can choose $k_0 = \lceil \frac{\log(\ell_0/(2\min_j p_j^0))}{\log(1+\eta/2)} \rceil$.*

Proof of Lemma 2. The above lemma follows from the following three facts. Informally, these points create a discrete-time threshold that prevents any price from approaching zero, provided the step size is chosen small enough.

1. For any $\mathbf{p}^k \in \Delta_B$, $\mathbf{p}^{k+1} \in \Delta_B$;
2. For any $\mathbf{p}^k \in \Delta_B$ and each $j \in [m]$, if $p_j^k \leq \ell_0$ then $p_j^{k+1} > p_j^k(1 + \frac{1}{2}\eta^k)$;
3. For any $\mathbf{p}^k \in \mathbb{R}_+^m$ and each $j \in [m]$, if $p_j^k > \ell_0$, then $p_j^{k+1} > \frac{\ell_0}{2}$.

The first fact is true because of Proposition 1. The second fact follows from Lemma 6. The third fact is true because of the following: for any $\mathbf{y}^k \in Y(\mathbf{p}^k)$, we have $|y_j^k| \leq \beta$. Thus, $p_j^{k+1} = p_j^k(1 + \eta^k y_j^k) > p_j^k(1 - \eta^k\beta) \geq \frac{1}{2}p_j^k > \frac{\ell_0}{2}$.

Then, we prove Lemma 2. The first statement in Lemma 2 follows from the first fact by induction, and second statement was proved as the second fact above. For the third statement, we only need to consider the case where there exists a nonempty set of chores J such that $p_j^0 \in [0, \ell_0)$ for any $j \in J$. Since $\sum_{k=0}^{\infty} \eta^k = \infty$ and $\eta^k \leq \frac{1}{2\beta}$ for all k, we have

$$\prod_{k=0}^{\infty}\left(1 + \frac{1}{2}\eta^k\right) = \exp\left(\sum_{k=0}^{\infty}\log\left(1 + \frac{1}{2}\eta^k\right)\right) \geq \exp\left(\sum_{k=0}^{\infty}\frac{\eta^k}{2+\eta^k}\right) \geq \exp\left(\sum_{k=0}^{\infty}\frac{2\beta\eta^k}{4\beta+1}\right) = \infty,$$

where we use $\log(1+x) \geq \frac{x}{1+x}$ for $x > -1$. Hence, since $p_j^{k+1} > p_j^k(1 + \frac{1}{2}\eta^k)$ if $p_j^k < \ell_0$, $p^k \in \Delta_B$, we have that there is a time point κ_j for each $j \in J$ such that it exceeds the barrier ℓ_0 for the first time, i.e., $\kappa_j := \min\{k \mid p_j^k > \ell_0\}$. The above facts then imply that $p_j^k \geq \frac{\ell_0}{2}$ for all $k \geq \kappa_j$. Hence, letting $k_0 := \max_{j\in J}\kappa_j$, we show the third statement in Lemma 2 is correct.

In particular, we can choose $\kappa_j = \max\left\{\left\lceil \frac{(2+\eta)\log(\ell_0/(2p_j^0))}{\eta}\right\rceil, 0\right\}$ and thus $k_0 = \max\left\{\left\lceil \frac{(2+\eta)\log(\ell_0/(2\min_j p_j^0))}{\eta}\right\rceil, 0\right\}$. □

We restate Proposition 2 here for reference.

Proposition 2. *Let $\{\mathbf{p}^k\}_{k\geq 0}$ be a sequence of iterates generated by Bregman subgradient update described in Eqs.* (3) *and* (4)*, and suppose that the process satisfies that* (1) *the sequences $\{\mathbf{p}^k\}_{k\geq 0}$, $\{\log(\mathbf{p}^k)\}_{k\geq 0}$, and $\{\mathbf{y}^k\}_{k\geq 0}$ are uniformly bounded almost surely,* (2) *the stepsizes satisfies $\sum_{k=0}^{\infty}\eta^k = \infty$, and either $\eta^k = o(\frac{1}{\log k})$ or $\sum_{k=0}^{\infty}(\eta^k)^2 < \infty$,* (3) *the perturbation error satisfies $\lim_{k\to\infty}\nu^k \to 0$,* (4) *there is a potential function f such that $-\mathbf{y}^k \in \partial f(\mathbf{p}^k)$[4], and f is lower bounded,* (5) *the critical value set $\{f(\mathbf{p}) \mid \mathbf{0}_m \in \partial f(\mathbf{p})\}$ has empty interior in $\mathbb{R}$. Then, almost surely, any cluster point of $\{\mathbf{p}^k\}_{k\geq 0}$ is a critical point and the function values $\{f(\mathbf{p}^k)\}_{k\geq 0}$ converge.*

Theorem 1 (Asymptotic convergence)**.** *Let $\{\eta^k\}_{k\geq 0}$ be a sequence of stepsizes satisfying (i) $0 < \eta^k \leq 1/(2\beta)$, (ii) $\sum_{k=0}^{\infty}\eta^k = \infty$, and (iii) $\eta^k = o(\frac{1}{\log k})$ or $\sum_{k=0}^{\infty}(\eta^k)^2 < \infty$. Let $\{\mathbf{p}^k\}_{k\geq 0}$ be a sequence of iterates generated by* (Multiplicative Tâtonnement) *with $\{\eta^k\}_{k\geq 0}$ and any initial point $\mathbf{p}^0 \in$ relint(Δ_B). Then, every limit point of $\{\mathbf{p}^k\}_{k\geq 0}$ is a CE of the chores Fisher market.*

[4] $\partial f(\mathbf{p}^k)$ denotes Clarke differential of function f at $\mathbf{p}^k$.

Proof. By Lemma 6, we show that the sequence of iterates $\left\{\mathbf{p}^k\right\}_{k\geq 0}$ lie in relint(Δ_B) and have a positive lower bound, if the stepsizes are properly upper bounded.

Combining Lemma 2 with $\mathbf{p}^0 \in \mathbb{R}^m_{++}$, we show a price lower bound for (Multiplicative Tâtonnement) dynamics under stated small stepsizes. Since $\mathbf{p}^k \in$ relint(Δ_B) holds, there is a trivial upper bound $\|\mathbf{B}\|_1 = 1$. It follows that the prices and the excess demands are bounded, and therefore Proposition 2.(1) is satisfied. Note that, we need the concrete lower bound to make sure the iterates $\{\nabla h_{\mathrm{KL}}(\mathbf{p}^k)\}_{k\geq 1}$ is uniformly bounded almost surely. Proposition 2.(2) are true by the stepsize selection rule. Note that

$$\begin{aligned}\left\|\frac{\nabla h_{\mathrm{KL}}(\mathbf{p}^{k+1}) - \nabla h_{\mathrm{KL}}(\mathbf{p}^k)}{\eta^k} - \mathbf{y}^k\right\|_2^2 =& \left\|\frac{\log\left(1+\eta^k\mathbf{y}^k\right)}{\eta^k} - \mathbf{y}^k\right\|_2^2 \\ =& \sum_{j=1}^m \frac{1}{(\eta^k)^2}\left(\log\left(1+\eta^k y_j^k\right) - \eta^k y_j^k\right)^2 \\ \leq& \sum_{j=1}^m \frac{1}{(\eta^k)^2}\left(\eta^k y_j^k\right)^4 \\ =& \sum_{j=1}^m (\eta^k)^2\left(y_j^k\right)^4 \leq \beta^4 m(\eta^k)^2 \to 0 \ \text{ as } k\to\infty,\end{aligned}$$

where the first inequality holds because $|\log(1+x) - x| \leq x^2$ for all $|x| \leq \frac{1}{2}$ and $|\eta^k y_j^k| \leq \frac{1}{2}$, and the second inequality holds by $|y_j^k| \leq \beta$. Thus, Proposition 2.(3) is satisfied.

To see Proposition 2.(4), by [12, Lemma 6], the chores potential function f is a valid potential function satisfies $\partial f(\mathbf{p}) = -Y(\mathbf{p})$, and has a lower bound over $\mathbf{p} \in \Delta_B$. The second one is true because stationary points of f are disconnected and the number of critical points is finite, and hence the critical value set cannot include an open interval.

Therefore, the theorem holds by Proposition 2. □

D Proofs of Section 4

Lemma 3 (Relative smoothness of f w.r.t. h_{KL})**.** *For any two price vectors $\mathbf{p}, \mathbf{q} \in \Delta_B$, we have*

$$f(\mathbf{p}) \leq f(\mathbf{q}) + \langle \nabla f(\mathbf{q}), \mathbf{p} - \mathbf{q}\rangle + (\sigma-1)m^{\sigma-1}(\min_{i,j} d_{ij})^{-\sigma} D_{\mathrm{KL}}(\mathbf{p},\mathbf{q}). \tag{10}$$

Proof. For any $\mathbf{p} \in \Delta_B$, for any $i \in [n]$, denote $w_{ij}(\mathbf{p}) := \frac{d_{ij}^{-\sigma} p_j^\sigma}{\sum_{\ell=1}^m d_{i\ell}^{-\sigma} p_\ell^\sigma}$. By calculation,

$$\begin{aligned}\nabla^2 f(\mathbf{p}) =& \sum_{i=1}^n B_i \mathrm{diag}(\mathbf{p})^{-1}\left((\sigma-1)\mathrm{diag}(\mathbf{w}_i(\mathbf{p})) - \sigma \mathbf{w}_i(\mathbf{p})\mathbf{w}_i(\mathbf{p})^\top\right)\mathrm{diag}(\mathbf{p})^{-1} \\ \preceq& \sum_{i=1}^n B_i \mathrm{diag}(\mathbf{p})^{-1}\left((\sigma-1)\mathrm{diag}(\mathbf{w}_i(\mathbf{p}))\right)\mathrm{diag}(\mathbf{p})^{-1},\end{aligned}$$

since the second term is rank-one and thus positive semidefinite. To show

$$\nabla^2 f(\mathbf{p}) \preceq L\nabla^2 h_{\mathrm{KL}}(p) = L\mathrm{diag}(\mathbf{p})^{-1}, \tag{11}$$

it suffices to prove that

$$(\sigma-1)p_\ell \sum_{i=1}^n B_i \frac{w_{i\ell}}{p_\ell^2} \leq L \quad \forall\, \ell \in [m].$$

By Jensen's inequality and the convexity of x^σ, we have

$$\left(\frac{1}{m}\sum_{j=1}^m p_j\right)^\sigma \leq \frac{1}{m}\sum_{j=1}^m p_j^\sigma,$$

hence

$$\sum_{j=1}^{m} p_j^{\sigma} \geq \frac{1}{m^{\sigma-1}} \sum_{j=1}^{m} p_j = \frac{1}{m^{\sigma-1}} \sum_{i=1}^{n} B_i = \frac{1}{m^{\sigma-1}}.$$

Then, we have

$$\sum_{\ell=1}^{m} d_{i\ell}^{-\sigma} p_{\ell}^{\sigma} \geq (\max_j d_{ij})^{-\sigma} \sum_{j=1}^{m} p_j^{\sigma} \geq \frac{1}{m^{\sigma-1}(\max_j d_{ij})^{\sigma}}. \tag{12}$$

Since

$$p_j \sum_{i=1}^{n} B_i \frac{w_{ij}}{p_j^2} = \sum_{i=1}^{n} B_i \frac{d_{ij}^{-\sigma} p_j^{\sigma-1}}{\sum_{\ell=1}^{m} d_{i\ell}^{-\sigma} p_{\ell}^{\sigma}} \leq \sum_{i=1}^{n} B_i m^{\sigma-1} \left(\frac{\max_j d_{ij}}{\min_j d_{ij}} \right)^{\sigma},$$

where the last inequality holds because Eq. (12), $d_{i\ell} \leq \min_j d_{ij}\ \forall \ell$, and $p_j^{\sigma-1} \leq \|\mathbf{p}\|_1^{\sigma-1} = \|\mathbf{B}\|_1^{\sigma-1} = 1$. Thus, we have that Eq. (11) holds with $L = (\sigma-1)\sum_{i=1}^{n} B_i m^{\sigma-1} (\frac{\max_j d_{ij}}{\min_j d_{ij}})^{\sigma} \leq (\sigma-1)m^{\sigma-1} \max_i (\frac{\max_j d_{ij}}{\min_j d_{ij}})^{\sigma} \sum_{i=1}^{n} B_i = (\sigma-1)m^{\sigma-1} \max_i (\frac{\max_j d_{ij}}{\min_j d_{ij}})^{\sigma}$.

Therefore, we have

$$f(\mathbf{p}) \leq f(\mathbf{q}) + \langle \nabla f(\mathbf{q}), \mathbf{p} - \mathbf{q} \rangle + (\sigma-1)m^{\sigma-1} \max_i \left(\frac{\max_j d_{ij}}{\min_j d_{ij}} \right)^{\sigma} D_{\mathrm{KL}}(\mathbf{p}, \mathbf{q}). \tag{13}$$

because $\nabla^2 f(\mathbf{p}) \preceq L \nabla^2 h_{\mathrm{KL}}(p)$ is equivalent to relative L-smoothness of f with respect to h_{KL} [37, Proposition 1.1.][4, Proposition 1]. Furthermore, since $(\max_j d_{ij})^{\rho} \leq \sum_{j=1}^{m} d_{ij}^{\rho} = (d_i(\mathbf{1}_m))^{\rho} = 1$, we have $(\max_j d_{ij})^{\sigma} \leq 1$. Combining this uniform bound with Eq. (13), we completes the proof. □

Lemma 4 (Descent property)**.** *Let* $\{\mathbf{p}^k\}_{k\geq 0}$ *be a sequence of iterates generated by* (Multiplicative Tâtonnement) *with any positive stepsizes* $\{\eta^k\}_{k\geq 0}$ *and any initial point* $\mathbf{p}^0 \in \mathrm{relint}(\Delta_B)$*. We have that*

$$f(\mathbf{p}^{k+1}) \leq f(\mathbf{p}^k) - \frac{1}{\eta^k} \left(D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) \right) - \langle \nu^k, \mathbf{p}^{k+1} - \mathbf{p}^k \rangle + L D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k). \tag{14}$$

Proof. The first order optimality then provides

$$\mathbf{0}_m = \eta^k (\mathbf{y}^k - \nu^k) - \left(\nabla h_{\mathrm{KL}}(\mathbf{p}^{k+1}) - \nabla h_{\mathrm{KL}}(\mathbf{p}^k) \right).$$

and then by the definition of Bregman divergence

$$\begin{aligned} \eta^k \langle \mathbf{y}^k - \nu^k, \mathbf{p}^{k+1} - \mathbf{p}^k \rangle =& \langle \nabla h_{\mathrm{KL}}(\mathbf{p}^{k+1}) - \nabla h_{\mathrm{KL}}(\mathbf{p}^k), \mathbf{p}^{k+1} - \mathbf{p}^k \rangle \\ =& D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) + D_{\mathrm{KL}}(\mathbf{p}^k, \mathbf{p}^{k+1}). \end{aligned}$$

Then, we have

$$\begin{aligned} & f(\mathbf{p}^{k+1}) \\ \leq & f(\mathbf{p}^k) + \langle \nabla f(\mathbf{p}^k), \mathbf{p}^{k+1} - \mathbf{p}^k \rangle + L D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) \\ = & f(\mathbf{p}^k) - \langle \mathbf{y}^k, \mathbf{p}^{k+1} - \mathbf{p}^k \rangle + L D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) \\ = & f(\mathbf{p}^k) - \frac{1}{\eta^k} \left(D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) + D_{\mathrm{KL}}(\mathbf{p}^k, \mathbf{p}^{k+1}) \right) - \langle \nu^k, \mathbf{p}^{k+1} - \mathbf{p}^k \rangle + L D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) \\ \leq & f(\mathbf{p}^k) - \frac{1}{\eta^k} D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) - \langle \nu^k, \mathbf{p}^{k+1} - \mathbf{p}^k \rangle + L D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k), \end{aligned}$$

where we use the relative smoothness in the first inequality. □

Lemma 5. *If* $\eta^k \leq \frac{1}{2\beta}$ *for all* $k \geq 0$*, then we have* $-\langle \nu^k, \mathbf{p}^{k+1} - \mathbf{p}^k \rangle \leq \beta (\eta^k)^2 \|\mathbf{y}^k\|_{\mathbf{p}^k}^2$ *and* $D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) \geq \frac{(\eta^k)^2}{3} \|\mathbf{y}^k\|_{\mathbf{p}^k}^2$.

Proof. Since $\eta^k \le \frac{1}{2\beta}$ thus $|\eta^k y_j^k| \le \frac{1}{2}$,

$$\begin{aligned}
-\langle \nu^k, \mathbf{p}^{k+1} - \mathbf{p}^k \rangle &= \sum_{j=1}^m \left(\frac{\log\left(1+\eta^k y_j^k\right)}{\eta^k} - y_j^k \right) \eta^k p_j^k y_j^k \\
&\le \sum_{j=1}^m \left| \frac{\log\left(1+\eta^k y_j^k\right)}{\eta^k} - y_j^k \right| \eta^k p_j^k |y_j^k| \\
&= \sum_{j=1}^m \left| \log\left(1+\eta^k y_j^k\right) - \eta^k y_j^k \right| p_j^k |y_j^k| \\
&\le \sum_{j=1}^m (\eta^k y_j^k)^2 p_j^k \beta = \beta (\eta^k)^2 \sum_{j=1}^m p_j^k (y_j^k)^2,
\end{aligned}$$

where the last inequality holds because $|\log(1+x) - x| \le x^2$ for all $|x| \le \frac{1}{2}$, together with $|\eta^k y_j^k| \le \frac{1}{2}$ and $|y_j^k| \le \beta$.

Also, since $(1+x)\log(1+x) - x \ge \frac{x^2}{3}$ for all $|x| \le \frac{1}{2}$, we have

$$\begin{aligned}
D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) &= \sum_{j=1}^m p_j^k (1+\eta^k y_j^k) \log(1+\eta^k y_j^k) \\
&= \eta^k \sum_{j=1}^m p_j^k y_j^k + \sum_{j=1}^m p_j^k \left((1+\eta^k y_j^k) \log(1+\eta^k y_j^k) - \eta^k y_j^k \right) \\
&\ge \frac{1}{3} \sum_{j=1}^m p_j^k \eta^k (y_j^k)^2 = \frac{(\eta^k)^2}{3} \sum_{j=1}^m p_j^k (y_j^k)^2.
\end{aligned}$$

□

Theorem 2 (Convergence rate guarantee)**.** *Let* $\{\mathbf{p}^k\}_{k\ge 0}$ *be a sequence of iterates generated by* (Multiplicative Tâtonnement) *with any initial point* $\mathbf{p}^0 \in \mathrm{relint}(\Delta_B)$ *and any positive stepsizes* $\{\eta^k\}_{k\ge 0}$ *satisfying*

$$\eta^k \equiv \eta \le \frac{1}{12(L+\beta)},$$

then we can find an iterate $(\mathbf{p}, \mathbf{y})$ *satisfies* $\|\mathbf{y}\|_{\mathbf{p}} \le \varepsilon$ *in* $\lceil 12(f(\mathbf{p}^0) - \underline{f})/\eta\varepsilon^2 \rceil$ *iterations. Here,* $L = (\sigma-1)\, m^{\sigma-1} (\min_{i,j} d_{ij})^{-\sigma}$ *and* $\beta = 1 + (\min_{i,j} d_{ij})^{-1}$.

Proof of Theorem 2. Since $\eta^k = \eta \le \frac{1}{12(L+\beta)} \le \min\{\frac{1}{2L}, \frac{1}{12\beta}\}$, we have

$$\begin{aligned}
f(\mathbf{p}^{k+1}) \le& f(\mathbf{p}^k) - \left(\frac{1}{\eta^k} - L \right) D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) + \beta (\eta^k)^2 \sum_{j=1}^m p_j^k (y_j^k)^2 \\
\le& f(\mathbf{p}^k) - \frac{1}{2\eta^k} D_{\mathrm{KL}}(\mathbf{p}^{k+1}, \mathbf{p}^k) + \frac{\eta^k}{12} \sum_{j=1}^m p_j^k (y_j^k)^2 \\
\le& f(\mathbf{p}^k) - \frac{\eta^k}{6} \sum_{j=1}^m p_j^k (y_j^k)^2 + \frac{\eta^k}{12} \sum_{j=1}^m p_j^k (y_j^k)^2 \\
\le& f(\mathbf{p}^k) - \frac{\eta^k}{12} \sum_{j=1}^m p_j^k (y_j^k)^2.
\end{aligned}$$

Equivalently,

$$T \cdot \left(\min_{0\le k\le T-1} \sum_{j=1}^m p_j^k (y_j^k)^2 \right) \le \sum_{k=0}^{T-1} \sum_{j=1}^m p_j^k (y_j^k)^2 \le \frac{12(f(\mathbf{p}^0) - f(\mathbf{p}^T))}{\eta} \le \frac{12(f(\mathbf{p}^0) - \underline{f})}{\eta},$$

where $\mathbf{p}^0$ is the initial price vector and $\underline{f}$ is the minimum value of f over Δ_B. To obtain that $\min_{0\le k\le T-1} \sum_{j=1}^m p_j^k (y_j^k)^2 \le \varepsilon^2$, we need $T \ge \lceil \frac{12(f(\mathbf{p}^0) - \underline{f})}{\eta\varepsilon^2} \rceil$ iterations. □

D.1 Details on the rate of relative tâtonnement

Here, we focus on the case where each convex CES disutility functions are with $\rho > 2$. Note that we use a different parameterization ($d_i(\mathbf{x}_i) = (\sum_{j=1}^m (d_{ij}x_{ij})^\rho)^{1/\rho}$) for the convex CES disutility functions from that in [12] ($d_i(\mathbf{x}_i) = (\sum_{j=1}^m d_{ij}x_{ij}^\rho)^{1/\rho}$).

We consider the terms with $1/\ell_0$ and lower bound all other nonnegative terms in the expression of the smoothness modulus. In their Lemma 15, r_i can equal 1 in our setting (by Assumption 1), then we have

$$\begin{aligned} L \geq \sum_{i=1}^n B_i R_i L_i &\geq \sum_{i=1}^n B_i R_i ((\sigma-1)\frac{(\max_j d_{ij}^\rho)^{\frac{(\sigma-1)^2}{\sigma}}}{(\min_j d_{ij}^\rho)^{\sigma-1}})\frac{1}{2\ell_0} \\ &= \sum_{i=1}^n B_i (\min_j d_{ij}^\rho)^{-1/\rho}((\sigma-1)\frac{(\max_j d_{ij}^\rho)^{\frac{(\sigma-1)^2}{\sigma}}}{(\min_j d_{ij}^\rho)^{\sigma-1}})\frac{1}{2\ell_0}. \end{aligned}$$

Since we have $d_i(\mathbf{1}_m) = 1$ by Assumption 1, we have

$$\max_j d_{ij}^\rho \geq \frac{1}{m}\sum_{j=1}^m d_{ij}^\rho = \frac{1}{m}.$$

Therefore,

$$L \geq \sum_{i=1}^n B_i (\min_j d_{ij})^{-1}((\sigma-1)\frac{1}{m^{\frac{(\sigma-1)^2}{\sigma}}(\min_j d_{ij}^\rho)^{\sigma-1}})\frac{1}{2\ell_0}.$$

Since $\rho(\sigma-1) = \rho(\frac{\rho}{\rho-1}-1) = \rho\frac{1}{\rho-1} = \sigma$, and $\frac{(\sigma-1)^2}{\sigma} = \frac{1}{\rho(\rho-1)} \leq \frac{1}{2}$, we have

$$L \geq (\sigma-1)m^{-\frac{1}{2}}(\min_j d_{ij})^{-(\sigma+1)}\frac{1}{2\ell_0}.$$

Since we can take $\eta = \frac{1}{L}$, and their convergence guarantee leads to $\lceil 2L(f(\mathbf{p}^0) - \underline{f})/\varepsilon^2 \rceil$ iterations required to compute an ε-approximate CE.

The following lemma shows that $f(\mathbf{p}^0) - \underline{f}$ is always bounded by a number that is polynomial in problem parameters.

Proposition 4 (Boundedness of f)**.** *For convex CES disutility function with $\rho > 1$, i.e., $d_i(\mathbf{x}_i) = (\sum_{j=1}^m (d_{ij}x_{ij})^\rho)^{1/\rho}$, $\log\frac{1}{e\max_i\|\mathbf{d}_i\|_\rho} \leq f(\mathbf{p}) \leq \log\frac{1}{e\min_{i,j} d_{ij}}$ for any $\mathbf{p} \in \Delta_B$.*

Proof. By solving $\min_{\mathbf{p}\geq 0, \|\mathbf{p}\|_1 = C}\sum_{j=1}^m d_{ij}^{-\sigma}p_j^\sigma$, we have

$$\sum_{j=1}^m d_{ij}^{-\sigma}p_j^\sigma \geq \left(\frac{\|\mathbf{p}\|_1}{\|\mathbf{d}_i\|_\rho}\right)^\sigma, \tag{15}$$

therefore $f(\mathbf{p})$ is differentiable when $\mathbf{p}$ is away from $\mathbf{0}_m$. By Eq. (15), we have

$$f(\mathbf{p}) \geq -\sum_{j=1}^m p_j + \sum_{i=1}^n \frac{B_i}{\sigma}\log\left(\left(\frac{\|\mathbf{p}\|_1}{\|\mathbf{d}_i\|_\rho}\right)^\sigma\right) \geq \|\mathbf{B}\|_1 \log\frac{\|\mathbf{B}\|_1}{e\max_i\|\mathbf{d}_i\|_\rho}.$$

Similarly, by solving $\max_{\mathbf{p}\geq 0, \|\mathbf{p}\|_1 = C}\sum_{j=1}^m d_{ij}^{-\sigma}p_j^\sigma$, we have

$$\sum_{j=1}^m d_{ij}^{-\sigma}p_j^\sigma \leq \left(\frac{\|\mathbf{p}\|_1}{\min_j d_{ij}}\right)^\sigma, \tag{16}$$

thus

$$f(\mathbf{p}) \leq -\sum_{j=1}^m p_j + \sum_{i=1}^n \frac{B_i}{\sigma}\log\left(\left(\frac{\|\mathbf{p}\|_1}{\min_j d_{ij}}\right)^\sigma\right) \leq \|\mathbf{B}\|_1 \log\frac{\|\mathbf{B}\|_1}{e\min_{i,j} d_{ij}}. \qquad \square$$

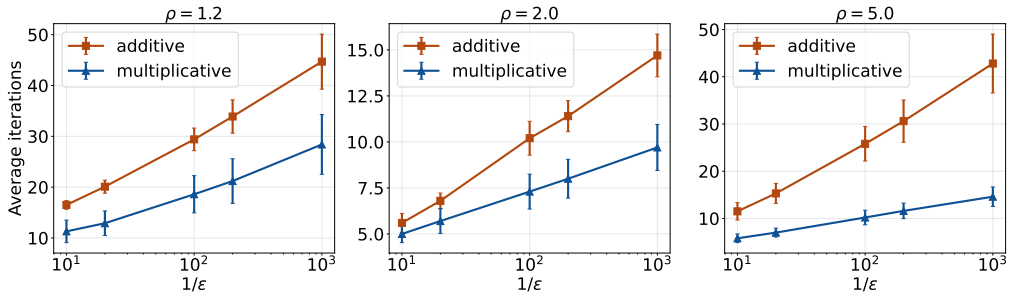

Figure 4: Multiplicative Tâtonnement v.s. Relative Tâtonnement on AAMAS bidding instances. The error bar corresponds to 1 standard deviation.

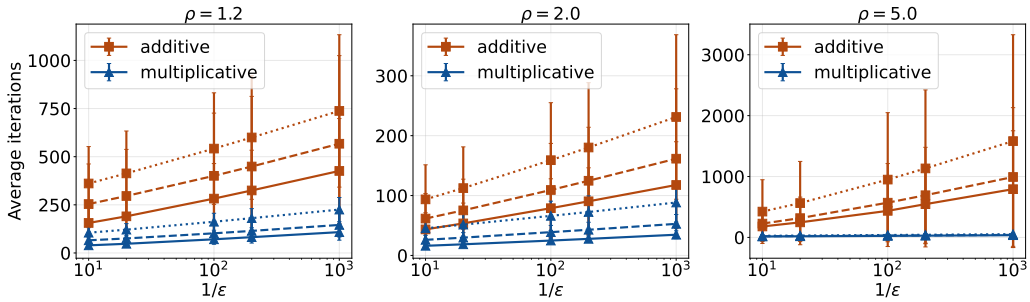

Figure 5: Multiplicative Tâtonnement v.s. Relative Tâtonnement on synthetic instances with lognormal value distributions. Solid lines correspond to 100×500 instances, dashed lines correspond to 100×1000 instances, dotted lines correspond to 100×2000 instances. The error bar corresponds to 0.1 standard deviation.

E Additional numerical experiments

Non-CES disutility instances. We additionally consider non-CES disutilities of the form $d_i(\mathbf{x}_i) = \|A_i\mathbf{x}_i\|_2$, where each agent has a dense matrix $A_i \in \mathbb{R}_{++}^{100\times m}$. We generate instances with 100 agents and $m \in \{500, 1000, 2000\}$ chores, using 20 random seeds for each size and distribution. Matrix entries are drawn independently either from a lognormal distribution with underlying normal distribution $\mathcal{N}(0,1)$, or from $\mathcal{N}(0.5, 0.2^2)$ with values below 0.01 replaced by 0.01. All chores have unit supply, all agents have unit earning requirements, and initial prices are uniform at $100/m$. Both dynamics use the same instances, separately tuned fixed stepsizes selected through pilot runs, and an iteration limit of $50{,}000$. We compare averaged iteration counts at five market-clearing residual tolerances, with all runs reaching the smallest tolerance, 0.001. Surprisingly, the preliminary results in Fig. 6 show that Multiplicative Tâtonnement and Relative Tâtonnement have similar performance in terms of iteration counts on this family of instances.

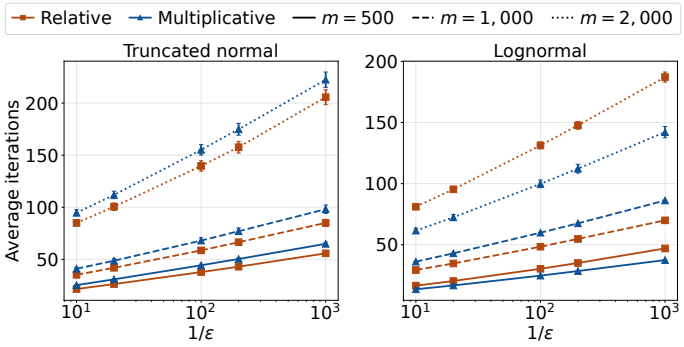

Figure 6: Preliminary experiments on synthetic instances with $\|A_i\mathbf{x}_i\|_2$ disutilities

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