Fully First-Order Algorithms for Online Non-Convex Bilevel Optimization

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Abstract

In this work, we study non-convex-strongly-convex online bilevel optimization (OBO) using only first-order oracle. Existing OBO algorithms are mainly based on hypergradient descent, which requires access to a Hessian-vector product (HVP) oracle and potentially incurs high computational costs. By reformulating the original OBO problem as a single-level online problem with inequality constraints and constructing a sequence of Lagrangian function, we eliminate the need for HVPs arising from implicit differentiation. Specifically, we propose a fully first-order algorithm for OBO, and provide theoretical guarantees showing that it achieves regret of $O(1 + V_T + H_{2,T})$ with a total of $O(T \log T)$ iterations, where V_T measures the variation in function values and $H_{2,T}$ characterizes the drift variation of the inner-level optimal solution. We also establish a sublinear regret bound under the single-loop structure by introducing additional gradient-variation terms. Furthermore, we develop an improved variant with an adaptive inner-iteration scheme, which removes the dependence on $H_{2,T}$ and achieves regret of $O(\log T + V_T)$. Finally, under the stochastic OBO setting, we establish the regret bound for the fully first-order algorithm, i.e., $O(T^{2/3}(1 + \sigma^2) + V_T + H_{2,T})$. Numerical experiments demonstrate the feasibility of our algorithm and support our theoretical findings.

1 Introduction

Online bilevel optimization (OBO) has attracted growing attention in recent years. It extends bilevel learning to streaming settings, where the underlying data distribution may drift over time. The goal is to generate a sequence of decisions $\{\mathbf{x}_t\}_{t=1}^T$ that tracks the time-varying optimal solutions $\{\mathbf{x}_t^*\}_{t=1}^T$ over the horizon $t \in [T]$ by solving

$$\min_{\mathbf{x}\in\mathcal{X}} F_t(\mathbf{x}) := f_t\left(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})\right) \quad \text{s.t.} \quad \mathbf{y}_t^*(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{y}\in\mathbb{R}^{d_2}} g_t(\mathbf{x}, \mathbf{y}), \tag{P1}$$

where the upper-level function $f_t : \mathcal{X} \times \mathbb{R}^{d_2} \to \mathbb{R}$ is smooth and possibly non-convex, while the lower-level function $g_t : \mathcal{X} \times \mathbb{R}^{d_2} \to \mathbb{R}$ is smooth and μ_g-strongly convex with respect to $\mathbf{y}$, which ensures the existence of a unique minimizer $\mathbf{y}_t^*(\mathbf{x})$ for all given $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^{d_1}$.

Existing OBO methods [31, 18, 2, 14] build on hypergradient descent, which requires an inner iterative procedure to approximately solve the lower-level problem, and then update the upper decision $\mathbf{x}$ using the hypergradient $\nabla F_t(\mathbf{x})$. Achieving a sufficiently accurate hypergradient typically requires many Hessian-vector products (HVP) to control the approximation error [8, 26]. However, this reliance on HVPs may limit the practical applicability of existing methods, particularly when second-order computations are prohibitively expensive or unavailable.

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Table 1: Comparison of algorithms on bilevel local regret without window averaging as defined in Eq.(2) in terms of gradient queries, required Hessian-vector product (HVP) queries, fully single-loop structure, local regret bounds, and corresponding theorems. V_T and $H_{2,T}$ are defined in Eq.(4). $E_{2,T}$ and P_T are defined in Eq.(9) and Theorem 3.2, respectively.

Algorithm	**Grad. Query**	**HVP Query**	**Fully S-Loop**	**Local Regret Bound**
SOBOW [18]	$O(T)$	$O(T^2)$	✗	$O(1+V_T+H_{2,T})$
OBBO [2]	$O(T\log T)$	$O(T\log T)$	✗	$O(1+V_T+H_{2,T})$
FSOBO [14]	$O(T)$	$O(T)$	✓	$O(1+V_T+H_{2,T}+E_{2,T})$
AOBO [14]	$O(T\log T)$	$O(T\log T)$	✗	$O(1+V_T)$
Algorithm 1	$O(T\log T)$	None	✗	$O(1+V_T+H_{2,T})$
Algorithm 1	$O(T)$	None	✓	$O(T^{1/3}(1+V_T)+T^{2/3}P_T)$
Algorithm 2	$O(T^2+TH_{2,T})$	None	✗	$O(\log T+V_T)$

Table 2: Comparison of stochastic algorithms on bilevel local regret in terms of sample complexities, required HVP term numbers, window-averaged regret form, local regret bounds, and corresponding theorems. In SOGD, $\Delta_T = O(1)+V_T$, $\Psi_T = P_T + E^f_{\mathbf{x},T} + E^g_{\mathbf{xy},T} + E^g_{\mathbf{yy},T}$, where $E^f_{\mathbf{x},T} := \sum_{t=2}^{T}\sup_{\mathbf{z}}\|\nabla_{\mathbf{x}} f_t(\mathbf{z}) - \nabla_{\mathbf{x}} f_{t-1}(\mathbf{z})\|^2$, $E^g_{\mathbf{xy},T} := \sum_{t=2}^{T}\sup_{\mathbf{z}}\|\nabla^2_{\mathbf{xy}} g_t(\mathbf{z}) - \nabla^2_{\mathbf{xy}} g_{t-1}(\mathbf{z})\|^2$.

Algorithm	**Sample Comp.**	**HVP Query**	**WinAvg.**	**Local Regret Bound**
SOBBO [2]	$O(wT)$	$O(T\log w)$	✓	$O(\frac{T}{w}(1+\sigma^2)+V_T+H_{2,T})$
SOGD [24]	$O(T)$	$O(T)$	✗	$O(T^{1/3}(\sigma^2+\Delta_T)+T^{2/3}\Psi_T)$
Algorithm 3	$O(T^{5/3}\log T)$	None	✗	$O(T^{2/3}(1+\sigma^2)+V_T+H_{2,T})$

For example, in large-scale bilevel learning with high-dimensional neural networks, repeated second-order computations and differentiation through long inner optimization trajectories can incur substantial runtime and memory overhead [30]. Similar difficulties arise when the lower-level problem contains millions of variables or when only a limited number of inner updates can be afforded under strict computational budgets [23]. Moreover, in stochastic-policy or reinforcement-learning settings, second-order estimators can also be difficult to implement and may suffer from high variance [28]. Motivated by these challenges, this work studys fully first-order OBO methods that avoid HVPs while maintaining favorable theoretical guarantees.

1.1 Our Contributions

We propose a fully first-order online bilevel optimizer (F^{2}OBO), presented in Algorithm 1. Theoretical analysis shows that it can achieve an $O(1+V_T+H_{2,T})$ upper bound on the standard bilevel local regret, without requiring any second-order information. Moreover, our method supports fully single-loop structure, which performs only one inner-step update per t. By introducing additional regularities of environmental variation, we can still guarantee sublinear regret, i.e., $O(T^{1/3}(1+V_T)+T^{2/3}P_T)$.

Building on the adaptive-iteration strategy in AOBO [14], we further enhance F^{2}OBO in a similar spirit to propose adaptive-iteration fully first-order online bilevel optimizer (AF^{2}OBO), presented in Algorithm 2. Theoretical analysis shows that it eliminates the dependence on the drift variation $H_{2,T}$ of the inner-level optimal solution. Although this comes at the cost of a larger iteration overhead and a weaker regret guarantee, i.e., $O(\log T+V_T)$ with an additional fixed $O(\log T)$ term, our method still ensures sublinear regret even when $\mathbf{y}^*_t(x)$ drifts drastically (e.g., $H_{2,T}=O(T)$ or larger), as shown in Table 1. This robustness is significant compared with static inner-iteration methods.

Finally, we extend F^{2}OBO to the stochastic OBO problem and propose stochastic fully first-order online bilevel optimizer (SF^{2}OBO), presented in Algorithm 3. Compared with SOBBO, we reduce the variance of the biased gradient by appropriately increasing the batch size, thereby eliminating the reliance on window-averaged regret and obtaining $O(T^{2/3}(1+\sigma^2)+V_T+H_{2,T})$. Compared

with SOGD, our result imposes a milder requirement on environmental variation, i.e., when $V_T = H_{2,T} = O(T^{2/3})$, SF^{2}OBO still achieves sublinear regret, whereas SOGD suffers at least $O(T^{4/3})$ superlinear regret. A comparison of the regret bounds is presented in Table 2.

1.2 Related Work

Bilevel Optimization Bilevel optimization (BO) was originally introduced by Bracken and McGill [4]. Existing BO methods mainly include hypergradient-based methods, penalty-based methods, and Hessian- or Jacobian-free methods. Hypergradient-based methods are typically divided into approximate implicit differentiation (AID) [9, 8, 10, 12, 25] and iterative differentiation (ITD) approaches [22, 7, 26, 21, 12]. Penalty-based methods reformulate BO as a constrained single-level problem using inner-level optimality conditions, thereby avoiding implicit-gradient computation and allowing extensions to non-strongly convex inner problems under suitable constraint qualification conditions [27, 19, 11, 1, 20, 5]. Hessian- or Jacobian-free methods further avoid explicit second-order derivatives by estimating hypergradients with zeroth-order approximations [32, 13, 33].

Online Bilevel Optimization. Online bilevel optimization (OBO) has attracted growing attention in recent years. Early studies by Tarzanagh et al. [31] and Lin et al. [18] initiated the investigation of OBO and developed a window-averaged hypergradient framework. Building on this line of work, Bohne et al. [2] employed iterative differentiation (ITD) to compute hypergradients and proposed the OBBO algorithm, together with its stochastic counterpart SOBBO. Later, Nazari et al. [24] introduced a momentum-based search direction and developed SOGD for the stochastic OBO setting. More recently, Jia et al. [14] revisited bilevel local regret without window averaging, proposed the first adaptive-inner-iteration algorithm, and established optimal guarantees. They further introduced a new window-averaging analysis framework that removes the need for sublinear variation assumptions. In addition, Bohne et al. [3] studied non-stationary functional bilevel optimization, which can also be interpreted within the existing OBO framework.

2 Preliminaries

2.1 Notations and Assumptions

We use $\|\cdot\|$ to denote the Euclidean norm. Given the inner-level function g_t in problem (P1), its partial gradients with respect to the upper- and inner-level variables are denoted by $\nabla_{\mathbf{x}} g_t(\mathbf{x}, \mathbf{y})$ and $\nabla_{\mathbf{y}} g_t(\mathbf{x}, \mathbf{y})$. The mixed derivative $\nabla^2_{\mathbf{xy}} g_t(\mathbf{x}, \mathbf{y})$ is defined as the Jacobian of $\nabla_{\mathbf{y}} g_t(\mathbf{x}, \mathbf{y})$ with respect to $\mathbf{x}$, while $\nabla^2_{\mathbf{yy}} g_t(\mathbf{x}, \mathbf{y})$ denotes the Hessian with respect to $\mathbf{y}$. For the composition $f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))$, we use $\nabla f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))$ to denote its total derivative with respect to $\mathbf{x}$. Finally, the condition number of the inner-level objective with respect to $\mathbf{y}$ is denoted by $\kappa_g := L_{g,1}/\mu_g$.

We make the following assumptions on the upper- and inner-level functions. These assumptions are also commonly used in existing OBO studies [18, 24, 2] and are standard in this setting.

Assumption 2.1. For all $t \in [T]$, f_t is $L_{f,0}$-Lipschitz continuous, and $\nabla f_t(\mathbf{x}, \mathbf{y})$ is $L_{f,1}$-Lipschitz continuous. Moreover, there exists a positive constant M such that $|f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))| \le M, \quad \forall \mathbf{x} \in \mathcal{X}$.

Assumption 2.2. For all $t \in [T]$, g_t satisfies the following conditions. For any given $\mathbf{x} \in \mathcal{X}$, $g_t(\mathbf{x}, \mathbf{y})$ is μ_g-strongly convex in $\mathbf{y}$. In addition, $\nabla g_t(\mathbf{x}, \mathbf{y})$ is $L_{g,1}$-Lipschitz continuous, and both $\nabla^2_{\mathbf{xy}} g_t(\mathbf{x}, \mathbf{y})$ and $\nabla^2_{\mathbf{yy}} g_t(\mathbf{x}, \mathbf{y})$ are $L_{g,2}$-Lipschitz continuous.

For stochastic OBO, we additionally give the following assumption to bound the variance of inner- and upper-level first-order gradients, which is common for stochastic regret analysis.

Assumption 2.3. For all $t \in [T]$, the stochastic gradients $\nabla f_t(\mathbf{x}, \mathbf{y}; \xi)$, $\nabla g_t(\mathbf{x}, \mathbf{y}; \zeta)$ are unbiased:

$$\mathbb{E}_\xi \left[\nabla f_t(\mathbf{x}, \mathbf{y}; \xi)\right] = \nabla f_t(\mathbf{x}, \mathbf{y}), \quad \mathbb{E}_\zeta \left[\nabla g_t(\mathbf{x}, \mathbf{y}; \zeta)\right] = \nabla g_t(\mathbf{x}, \mathbf{y}).$$

Moreover, there exists $\sigma > 0$ such that their conditional variances are uniformly bounded:

$$\mathbb{E}\left[\left\|\nabla f_t(\mathbf{x}, \mathbf{y}; \xi) - \nabla f_t(\mathbf{x}, \mathbf{y})\right\|^2\right] \le \sigma^2, \quad \mathbb{E}\left[\left\|\nabla g_t(\mathbf{x}, \mathbf{y}; \zeta) - \nabla g_t(\mathbf{x}, \mathbf{y})\right\|^2\right] \le \sigma^2.$$

2.2 Single-Level Reformulation of OBO Problem

For non-convex-strongly-convex OBO, the uniqueness of the inner-level solution $\mathbf{y}_t^*(\mathbf{x})$ for any $\mathbf{x} \in \mathcal{X}$ yields a unique closed-form expression for the hypergradient of F_t, given by

$$\begin{aligned}\nabla F_t(\mathbf{x}) =& \nabla_{\mathbf{x}} f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) + \nabla \mathbf{y}_t^*(\mathbf{x}) \nabla_{\mathbf{y}} f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) \\ =& \nabla_{\mathbf{x}} f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) - \nabla^2_{\mathbf{xy}} g_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) \left[\nabla^2_{\mathbf{yy}} g_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))\right]^{-1} \nabla_{\mathbf{y}} f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})). \end{aligned} \tag{1}$$

We consider the following local regret notion as the evaluation criterion for the algorithms:

$$\mathrm{BLReg}(T) = \sum_{t=1}^{T} \left\| \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \nabla F_t(\mathbf{x}_t), \gamma \right) \right\|^2. \tag{2}$$

Here, $\mathcal{G}_{\mathcal{X}}$ denotes the gradient mapping with respect to the constraint set $\mathcal{X}$, defined as

$$\mathcal{G}_{\mathcal{X}} \left(\mathbf{x}, \mathbf{g}, \gamma \right) := \frac{1}{\gamma} \left(\mathbf{x} - \operatorname*{argmin}_{\mathbf{u} \in \mathcal{X}} \left\{ \langle \mathbf{g}, \mathbf{u} \rangle + \frac{1}{2\gamma} \|\mathbf{u} - \mathbf{x}\|^2 \right\} \right), \tag{3}$$

for given $\mathbf{g} \in \mathbb{R}^{d_1}$ and $\gamma > 0$. We mainly use the following two variation regularities to establish our regret bound, which are also used in Lin et al. [18], Bohne et al. [2] and Nazari et al. [24]:

$$V_T := \sum_{t=2}^{T} \sup_{\mathbf{x} \in \mathcal{X}} |F_{t-1}(\mathbf{x}) - F_t(\mathbf{x})| \quad \text{and} \quad H_{2,T} := \sum_{t=2}^{T} \sup_{\mathbf{x} \in \mathcal{X}} \left\| \mathbf{y}_{t-1}^*(\mathbf{x}) - \mathbf{y}_t^*(\mathbf{x}) \right\|^2. \tag{4}$$

Following previous works [15, 29, 19, 6], the original problem (P1) can be equivalently written as an inequality constrained single-level problem, shown as:

$$\min_{\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathbb{R}^{d_2}} f_t(\mathbf{x}, \mathbf{y}) \quad \text{s.t.} \quad g_t(\mathbf{x}, \mathbf{y}) - g_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) \leq 0. \tag{P2}$$

By treating the constraint as a penalty term, we construct the following Lagrangian function $\mathcal{L}_t$ for some $\lambda_t > 0$:

$$\mathcal{L}_t(\mathbf{x}, \mathbf{y}, \lambda_t) = f_t(\mathbf{x}, \mathbf{y}) + \lambda_t \left(g_t(\mathbf{x}, \mathbf{y}) - g_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) \right). \tag{5}$$

We can finally approximately transform problem (P2) into an min-max optimization problem for $\mathcal{L}_t^*(\mathbf{x})$ as follows:

$$\min_{\mathbf{x} \in \mathcal{X}} \mathcal{L}_t^*(\mathbf{x}) := \mathcal{L}_t(\mathbf{x}, \mathbf{y}_{\lambda,t}^*(\mathbf{x}), \lambda_t) \quad \text{s.t.} \quad \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}) = \operatorname{argmin}_{\mathbf{y} \in \mathbb{R}^{\mathrm{d}_2}} \mathcal{L}_{\mathrm{t}}(\mathbf{x}, \mathbf{y}, \lambda_{\mathrm{t}}). \tag{P3}$$

Here the multiplier λ_t represents the degree to which the solution to (P3) conforms to the constraint, and we have $\lim_{\lambda_t \to \infty} \mathcal{L}_t^*(\mathbf{x}) = f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))$.

Based on the above, we approximate the OBO problem by an online min-max optimization problem. The ingenuity of this approach lies in its elimination of the requirement for a second-order gradient oracle arising from the implicit term $\nabla \mathbf{y}_t^*(\mathbf{x})$ in Eq.(1). Specifically, we have

$$\begin{aligned}\nabla \mathcal{L}_t^*(\mathbf{x}) =& \nabla_{\mathbf{x}} \mathcal{L}_t(\mathbf{x}, \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}), \lambda_t) + \nabla \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}) \nabla_{\mathbf{y}} \mathcal{L}_t(\mathbf{x}, \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}), \lambda_t) \\ =& \nabla_{\mathbf{x}} f_t(\mathbf{x}, \mathbf{y}_{\lambda_t,t}^*(\mathbf{x})) + \lambda_t \left(\nabla_{\mathbf{x}} g_t(\mathbf{x}, \mathbf{y}_{\lambda_t,t}^*(\mathbf{x})) - \nabla_{\mathbf{x}} g_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) \right), \end{aligned} \tag{6}$$

note that $\nabla_{\mathbf{y}} \mathcal{L}_t(\mathbf{x}, \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}), \lambda_t) = 0$ holds for all $t \in [T]$ and any $\lambda_t > 0$, thus the hypergradient does not require the calculation of $\nabla \mathbf{y}_{\lambda_t,t}^*(\mathbf{x})$.

3 A Fully First-Order Algorithm for Online Bilevel Optimization

This section develops and analyzes a fully first-order algorithm for determistic OBO.

3.1 Fully First-order Online Bilevel Optimizer (F^{2}OBO)

Similar to existing OBO algorithms [31, 18, 2], we still attempt to apply online gradient descent (OGD) to problem (P3). Consider a sequence $\{\lambda_t\}_{t=1}^T$ that governs our approximation, the hypergradient $\nabla \mathcal{L}_t^*(\mathbf{x}_t)$ is calculated at the current iterate $\mathbf{x}_t$ and used to update $\mathbf{x}_{t+1}$ at each time step t. Therefore,

Algorithm 1 Fully First-order Online Bilevel Optimizer (F^{2}OBO)

Input: $\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \{\lambda_t, \beta_t, \gamma_t, K_t\}_{t=1}^T, \alpha$

for $t = 1$ to T **do**
  Output $\mathbf{x}_t$ and $\mathbf{y}_t$, receive f_t and g_t.
  warm start $\mathbf{y}_t^1 \leftarrow \mathbf{y}_t$, $\mathbf{z}_t^1 \leftarrow \mathbf{z}_t$
  for $k = 1$ to K_t **do**
    $\mathbf{z}_t^{k+1} \leftarrow \mathbf{z}_t^k - \alpha \nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{z}_t^k)$
    $\mathbf{y}_t^{k+1} \leftarrow \mathbf{y}_t^k - \beta_t \left(\nabla_{\mathbf{y}} f_t(\mathbf{x}_t, \mathbf{y}_t^k) + \lambda_t \nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{y}_t^k)\right)$
  end for
  update $\mathbf{y}_{t+1} \leftarrow \mathbf{y}_t^{K_t+1}$, $\mathbf{z}_{t+1} \leftarrow \mathbf{z}_t^{K_t+1}$
  calculate $\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) = \nabla_{\mathbf{x}} f_t(\mathbf{x}_t, \mathbf{y}_{t+1}) + \lambda_t \left(\nabla_{\mathbf{x}} g_t(\mathbf{x}_t, \mathbf{y}_{t+1}) - \nabla_{\mathbf{x}} g_t(\mathbf{x}_t, \mathbf{z}_{t+1})\right)$
  $\mathbf{x}_{t+1} \leftarrow \operatorname{argmin}_{\mathbf{u} \in \mathcal{X}} \left\{ \left\langle \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \mathbf{u} \right\rangle + \frac{1}{2\gamma_t} \|\mathbf{u} - \mathbf{x}_t\|^2 \right\}$
  $\lambda_{t+1} \leftarrow \Lambda(t, \lambda_t)$
end for

according to Eq.(6), we need to solve two subproblems to obtain $\mathbf{y}_t^*(\mathbf{x}_t)$ and $\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)$. We can solve both subproblems simultaneously via a nested inner loop of gradient descent, starting from $\mathbf{z}_t^1 = \mathbf{z}_t$ and $\mathbf{y}_t^1 = \mathbf{y}_t$ for K times, as shown below:

$$\mathbf{z}_t^{k+1} \leftarrow \mathbf{z}_t^k - \alpha \nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{z}_t^k) \quad \text{and} \quad \mathbf{y}_t^{k+1} \leftarrow \mathbf{y}_t^k - \beta \nabla_{\mathbf{y}} \mathcal{L}_t(\mathbf{x}_t, \mathbf{y}_t^k, \lambda_t). \tag{7}$$

This simple strategy yields excellent theoretical performance because both g_t and $\mathcal{L}_t$ is strongly convex and smooth with respect to $\mathbf{y}$ under Assumptions 2.1, 2.2 and if set $\lambda_t \geq 2L_{f,1}/\mu_g$ for all $t \in [T]$, which can also be seen in several offline works [15, 6]. Finally with $\mathbf{z}_{t+1} = \mathbf{z}_t^{K+1}$ and $\mathbf{y}_{t+1} = \mathbf{y}_t^{K+1}$, the approximated hypergradient $\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)$ can be calculated by

$$\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) := \nabla_{\mathbf{x}} f_t(\mathbf{x}_t, \mathbf{y}_{t+1}) + \lambda_t \left(\nabla_{\mathbf{x}} g_t(\mathbf{x}_t, \mathbf{y}_{t+1}) - \nabla_{\mathbf{x}} g_t(\mathbf{x}_t, \mathbf{z}_{t+1})\right).$$

The multiplier λ_t must be carefully chosen: a small λ_t may cause the solution of (P3) to deviate from that of (P1), leading to accumulated bias, while a large λ_t may induce overly large gradients in Eq.(7) and destabilize optimization [15]. To balance this trade-off, we update $\lambda_{t+1} = \Lambda(t, \lambda_t)$ via a designed non-decreasing schedule, yielding the final fully first-order online bilevel optimizer (F^{2}OBO) in Algorithm 1. The following theorem gives a regret upper bound of F^{2}OBO.

Theorem 3.1. *Under Assumptions 2.1, 2.2, let* $\Lambda(t, \lambda_t) = \left(1 + \frac{1}{t}\right)^\tau \lambda_t$ *for some* $\tau > 0$, $\lambda_1 > \frac{2L_{f,1}}{\mu_g}$, $\alpha \leq \frac{1}{L_{g,1}}$, $\beta_t \leq \frac{1}{2\lambda_t L_{g,1}}$, $\gamma_t \equiv \gamma \leq \min\{\frac{1}{2L_F}, \frac{1}{16\kappa_g}\sqrt{\frac{1-\rho}{C_\lambda}}\}$ *and* $K_t \equiv K \geq \max\{1 - \frac{\ln 2}{\ln \rho}, \frac{\ln c - 2\tau \ln T}{\ln \rho}\}$ *for some constants* $\rho \in (0, 1)$, C_λ *and* $c > 0$, *Algorithm 1 can guarantee*

$$\mathrm{BLReg}(T) \leq O\Big(\sum_{t=1}^T \frac{1}{t^{2\tau}} + V_T + H_{2,T}\Big).$$

Theorem 3.1 shows that our regret bound mainly depends on two variation regularities, V_T and $H_{2,T}$. Specifically, if we set $\tau > 1/2$, then we achieve a regret of $O(1 + V_T + H_{2,T})$, which is comparable to SOBOW [18] and OBBO [2]. Our algorithm requires a time complexity of $O(T \log T)$, which is on par with OBBO and better than that of SOBOW.

3.2 Single-Loop F^{2}OBO

Although sufficiently many inner-loop iterations can yield favorable theoretical guarantees, it may be difficult to deploy in practice due to the substantial computational overhead, particularly in time-critical settings where performing multiple inner-loop iterations per round may be impractical. Here we proof that F^{2}OBO still achieves sublinear regret under the single-loop structure, i.e., $K_t = 1$. To control the accumulated error aused by inexactly solving the inner subproblems, we introduce the following two variation regularities to further restrict the first-order gradient changes:

$$E_{\mathbf{y},T}^f = \sum_{t=2}^T \sup_{\mathbf{x},\mathbf{y}} \|\nabla_{\mathbf{y}} f_t(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{y}} f_{t-1}(\mathbf{x}, \mathbf{y})\|^2, \quad E_{\mathbf{y},T}^g = \sum_{t=2}^T \sup_{\mathbf{x},\mathbf{y}} \|\nabla_{\mathbf{y}} g_t(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{y}} g_{t-1}(\mathbf{x}, \mathbf{y})\|^2.$$

We believe that these additional terms are necessary in this case. Similar ideas have been adopted in the analyses of existing second-order single-loop methods, such as FSOBO [14] and SOGD [24].

Theorem 3.2. *Under Assumptions 2.1, 2.2, let* $\Lambda(t,\lambda_t)=\left(1+\frac{1}{t}\right)^{\tau}\lambda_t$ *for some* $\tau\in(0,\frac{1}{2})$, $\lambda_1>\frac{2L_{f,1}}{\mu_g}$, $\alpha\le\frac{1}{L_{g,1}}$, $\beta_t=\frac{1}{2\lambda_t L_{g,1}}$, $\gamma_t<\min\{\frac{1}{2L_F},\frac{1}{8\lambda_1 t^{\tau}}\sqrt{\frac{1-\rho}{3C_g}}\}$ *and* $K_t\equiv 1$ *for some constant* C_g, *with* $P_T:=H_{2,T}+E^f_{\mathbf{y},T}+E^g_{\mathbf{y},T}$, *Algorithm 1 can guarantee*

$$\mathrm{BLReg}(T)\le O\left(T^{\tau}(1+V_T)+T^{1-2\tau}+T^{2\tau}P_T\right).$$

Theorem 3.2 shows that SF^{2}OBO enjoys a sublinear regret guarantee under polynomially sublinear environmental variation. In particular, suppose that $V_T=O(T^{\alpha})$ and $P_T=O(T^{\beta})$ for some $\alpha,\beta\in[0,1)$, it suffices to choose τ such that $\tau+\alpha<1$, $2\tau+\beta<1$, and $\tau>0$. Equivalently, any $0<\tau<\min\{1-\alpha,(1-\beta)/2\}$ is valid. Since $\alpha,\beta<1$, this interval is always nonempty. Therefore, for any polynomially sublinear variation rates $V_T=O(T^{\alpha})$ and $P_T=O(T^{\beta})$, there exists a valid choice of τ such that every term in the regret bound is sublinear in T, yielding $\mathrm{BLReg}(T)=o(T)$. In particular, by setting $\tau=1/3$, we obtain

$$\mathrm{BLReg}(T)\le O\left(T^{1/3}(1+V_T)+T^{2/3}P_T\right), \tag{8}$$

a sublinear regret guarantee holds when $V_T=o(T^{2/3})$ and $P_T=o(T^{1/3})$.

Remark 3.3. Compared with the regret bound of FSOBO as shown in Table 1, Eq.(8) incurs an additional T-dependent factor, leading to a weaker guarantee. This degradation stems from the cross-problem error induced by fully first-order oracle feedback. In the single-loop setting, this error is excessively amplified, even with additional variation regularities to control it, we believe it is difficult to eliminate entirely. However, existing second-order single-loop algorithms typically require assuming sublinear variation of the Hessian or Jacobian of g_t. For example, FSOBO assumes

$$E_{2,T}=E^f_{\mathbf{y},T}+E^g_{\mathbf{yy},T},\quad\text{where}\quad E^g_{\mathbf{yy},T}=\sum_{t=2}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla^2_{\mathbf{yy}}g_t(\mathbf{x},\mathbf{y})-\nabla^2_{\mathbf{yy}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|^2, \tag{9}$$

is sublinear. Such assumptions is harder to justify and guarantee. In contrast, our first-order algorithm only requires a sublinear first-order gradient changes, which is easier to verify and more practical.

4 An Adaptive Variant of F^{2}OBO

F^{2}OBO solves the two inner subproblems using a fixed number of iterations at each time step t. This static strategy is inevitably sensitive to changes, which is described by $H_{2,T}$, and may fail to guarantee sublinear regret when the lower-level optimal solutions exhibit linear or even superlinear drift. In this section, we consider improving F^{2}OBO by applying an adaptive inner iteration approach [14] to improve the regret upper bound.

4.1 Adaptive-iteration Fully First-order Online Bilevel Optimizer (AF^{2}OBO)

Considering (P3), $\mathcal{L}_t$ remains $(\lambda_t\mu_g/2)$-strongly convex and $(2\lambda_t L_{g,1})$-smooth if $\lambda_t\ge 2L_{f,1}/\mu_g$ for all $t\in[T]$. The distance between the current iterate and the optimum can be upper bounded by the norm of a computable gradient. Therefore, the tolerance parameters $\delta_{\mathbf{y}}$ and $\delta_{\mathbf{z}}$ can be used as stopping criteria for obtaining sufficiently accurate approximate solutions $\mathbf{y}_{t+1}$ and $\mathbf{z}_{t+1}$:

$$\frac{\lambda_t\mu_g}{2}\left\|\mathbf{y}_{t+1}-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|\le\left\|\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t,\mathbf{y}_{t+1},\lambda_t)\right\|\le\delta_{\mathbf{y}}\quad\text{and} \tag{10}$$

$$\mu_g\left\|\mathbf{z}_{t+1}-\mathbf{y}^*_t(\mathbf{x}_t)\right\|\le\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x}_t,\mathbf{z}_{t+1})\right\|\le\delta_{\mathbf{z}}. \tag{11}$$

We use two adaptive iterative processes to solve for $\mathbf{y}^*_t(\mathbf{x}_t)$ and $\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)$ respectively to constitute our adaptive-iteration fully first-order online bilevel optimizer (AF^{2}OBO), as shown in Algorithm 2. The following theorem provides the regret and computational cost guarantees of AF^{2}OBO.

Theorem 4.1. *Under Assumptions 2.1, 2.2, let* $\Lambda(t,\lambda_t)=\left(1+\frac{1}{t}\right)^{\tau}\lambda_t$ *for some* $\tau>0$, $\lambda_1>\frac{2L_{f,1}}{\mu_g}$, $\alpha\le\frac{1}{L_{g,1}}$, $\beta_t\le\frac{1}{2\lambda_t L_{g,1}}$, $\gamma\le\frac{1}{2L_F}$ *for all* $t\in[T]$, $\delta_{\mathbf{y}}=\frac{1}{\sqrt{T}}$ *and* $\delta_{\mathbf{z}}=\frac{1}{\sqrt{T^{1+2\tau}}}$, *with* $\mathcal{I}_T$ *representing*

Algorithm 2 Adaptive-iteration Fully First-order Online Bilevel Optimizer (AF^{2}OBO)

Input: $\mathbf{x}_1$, $\mathbf{y}_1$, $\mathbf{z}_1$, $\{\lambda_t, \beta_t\}_{t=1}^T$, α, γ, $\delta_{\mathbf{y}}$, $\delta_{\mathbf{z}}$

for $t = 1$ to T **do**
  Output $\mathbf{x}_t$ and $\mathbf{y}_t$, receive f_t and g_t.
  Set $\mathbf{y}_{t+1} \leftarrow \mathbf{y}_t$, $\mathbf{z}_{t+1} \leftarrow \mathbf{z}_t$
  while $\|\nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{z}_{t+1})\| > \delta_{\mathbf{z}}$ **do**
    $\mathbf{z}_{t+1} \leftarrow \mathbf{z}_{t+1} - \alpha \nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{z}_{t+1})$
  end while
  while $\|\nabla_{\mathbf{y}} \mathcal{L}_t(\mathbf{x}_t, \mathbf{y}_{t+1}, \lambda_t)\| > \delta_{\mathbf{y}}$ **do**
    $\mathbf{y}_{t+1} \leftarrow \mathbf{y}_{t+1} - \beta \left(\nabla_{\mathbf{y}} f_t(\mathbf{x}_t, \mathbf{y}_{t+1}) + \lambda_t \nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{y}_{t+1})\right)$
  end while
  calculate $\widetilde{\nabla} \mathcal{L}_t^*(\mathbf{x}_t) = \nabla_{\mathbf{x}} f_t(\mathbf{x}_t, \mathbf{y}_{t+1}) + \lambda_t \left(\nabla_{\mathbf{x}} g_t(\mathbf{x}_t, \mathbf{y}_{t+1}) - \nabla_{\mathbf{x}} g_t(\mathbf{x}_t, \mathbf{z}_{t+1})\right)$
  $\mathbf{x}_{t+1} \leftarrow \operatorname{argmin}_{\mathbf{u} \in \mathcal{X}} \left\{ \left\langle \widetilde{\nabla} \mathcal{L}_t^*(\mathbf{x}_t), \mathbf{u} \right\rangle + \frac{1}{2\gamma} \|\mathbf{u} - \mathbf{x}_t\|^2 \right\}$
  $\lambda_{t+1} \leftarrow \Lambda(t, \lambda_t)$
end for

the total number of inner-loop iterations, Algorithm 2 can guarantee

$$\mathrm{BLReg}(T) \leq O\left(\sum_{t=1}^{T} \frac{1}{t^{2\tau}} + V_T\right) \quad \textit{and} \quad \mathcal{I}_T \leq O\left(T^{1+2\tau} + T^{2\tau} H_{2,T}\right).$$

As shown in Theorem 4.1, the main improvement of our method is that it eliminates the dependency of the regret upper bound on $H_{2,T}$, which describes the drift of the inner optimal solution $\|\mathbf{y}_t^*(\mathbf{x}) - \mathbf{y}_{t-1}^*(\mathbf{x})\|$ for any $\mathbf{x} \in \mathcal{X}$.

4.2 Discussion on the Improved Regret Bound

Note that AOBO [14] achieves the optimal bilevel local regret $\Omega(1 + V_T)$ with $O(T \log T)$ query of gradients and HVPs. Our method achieves the same bound if we set, i.e., $\tau = 1$, which requires $O(T^3)$ first-order gradient evaluations. If we choose $\tau = 1/2$, We can guarantee a sublinear regret of $O(\log T + V_T)$, which does not rely on $H_{2,T}$. In this setting, the total number of gradient quiries is at most $O(T^2 + T H_{2,T})$.

The upper bound in Theorem 4.1 have the advantage when the drift of $\mathbf{y}_t^*(\mathbf{x})$ is more severe. We provide a 1-dimensional example for illustration. Consider a set of $\{f_t, g_t\}_{t=1}^T$ within constraints $\mathcal{X} = \{x \,|\, x \in [-1, 1]\}$ and $y \in \mathbb{R}$ that satisfies $V_T = 0$ and $H_{2,T} = O(T)$:

$$f_t(x, y) \equiv \widetilde{f}(y) = c e^{-y^2} \quad \text{and} \quad g_t(x, y) = \frac{\mu_g}{2} \left(y - (-1)^t x\right)^2.$$

Our construction satisfies Assumptions 2.1, 2.2 when constant c is set appropriately. With $y_t^*(x) = (-1)^t x$, $f_t(x, y_t^*(x)) = \widetilde{f}(y_t^*(x)) \equiv c e^{-x^2}$, it holds that

$$V_T = \sum_{t=2}^{T} \sup_{x \in [-1,1]} \left|\widetilde{f}(y_t^*(x)) - \widetilde{f}(y_{t-1}^*(x))\right| = 0, \quad H_{2,T} = \sum_{t=2}^{T} \sup_{x \in [-1,1]} \left|(-1)^t x - (-1)^{t-1} x\right|^2 = 4T - 4.$$

In the case of oscillatory drift in the inner-level optimal solution, our AF^{2}OBO still guarantees a sublinear regret bound of $O(\log T)$ if $\tau = 1/2$, while the regret guarantees of static inner-iteration methods, such as OBBO and F^{2}OBO, will suffer from linear regret.

5 F^{2}OBO with Stochastic First-Order Gradient

In this section, we study the stochastic OBO problem with our fully first-order algorithm, where the objective functions are formulated as:

$$f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) = \mathbb{E}_{\xi \sim \mathcal{D}_t^f} \left[f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}); \xi)\right], \quad g_t(\mathbf{x}, \mathbf{y}) = \mathbb{E}_{\zeta \sim \mathcal{D}_t^g} \left[g_t(\mathbf{x}, \mathbf{y}; \zeta)\right],$$

where $\{\mathcal{D}_t^f, \mathcal{D}_t^g\}_{t=1}^T$ are the time-varying upper- and inner-level distributions, respectively.

Algorithm 3 Stochastic Fully First-order Online Bilevel Optimizer (SF^{2}OBO)

Input: $\mathbf{x}_1, \mathbf{y}_1, \{\lambda_t, \alpha_t, \beta_t, \gamma_t, K_t, B_t, \widehat{B}_t\}_{t=1}^T$

for $t = 1$ to T **do**
  Output $\mathbf{x}_t$ and $\mathbf{y}_t$, receive f_t and g_t.
  warm start $\mathbf{y}_t^1 \leftarrow \mathbf{y}_t$, $\mathbf{z}_t^1 \leftarrow \mathbf{z}_t$
  for $k = 1$ to K_t **do**
    draw samples $\xi_t^k \sim \mathcal{D}_t^f$ and $\zeta_t^k \sim \mathcal{D}_t^g$ with batch size B_t
    $\mathbf{z}_t^{k+1} \leftarrow \mathbf{z}_t^k - \alpha_t \nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{z}_t^k; \zeta_t^k)$
    $\mathbf{y}_t^{k+1} \leftarrow \mathbf{y}_t^k - \beta_t \left(\nabla_{\mathbf{y}} f_t(\mathbf{x}_t, \mathbf{y}_t^k; \xi_t^k) + \lambda_t \nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{y}_t^k; \zeta_t^k)\right)$
  end for
  update $\mathbf{y}_{t+1} \leftarrow \mathbf{y}_t^{K_t+1}$, $\mathbf{z}_{t+1} \leftarrow \mathbf{z}_t^{K_t+1}$, draw samples $\xi_t \sim \mathcal{D}_t^f$, $\zeta_t \sim \mathcal{D}_t^g$ with batch size $\widehat{B}_t$
  calculate $\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t) = \nabla_{\mathbf{x}} f_t(\mathbf{x}_t, \mathbf{y}_{t+1}; \xi_t) + \lambda_t \left(\nabla_{\mathbf{x}} g_t(\mathbf{x}_t, \mathbf{y}_{t+1}; \zeta_t) - \nabla_{\mathbf{x}} g_t(\mathbf{x}_t, \mathbf{z}_{t+1}; \zeta_t)\right)$
  $\mathbf{x}_{t+1} \leftarrow \operatorname{argmin}_{\mathbf{u} \in \mathcal{X}} \left\{ \left\langle \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \mathbf{u} \right\rangle + \frac{1}{2\gamma_t} \|\mathbf{u} - \mathbf{x}_t\|^2 \right\}$
  $\lambda_{t+1} \leftarrow \Lambda(t, \lambda_t)$
end for

We replace the exact gradients in F^{2}OBO with stochastic gradients, and control the accumulated variance in the regret through the inner- and upper-level batch sizes B_t and $\widehat{B}_t$, respectively, for all $t \in [T]$. This leads to Algorithm 3. The following theorem establishes a regret bound.

Theorem 5.1. *Under Assumption 2.1, 2.2 and 2.3, Let* $\Lambda(t, \lambda_t) = \left(1 + \frac{1}{t}\right)^\tau \lambda_t$ *for some* $\tau \in (0, \frac{1}{2})$, $\lambda_1 > \frac{2L_{f,1}}{\mu_g}$, $\alpha_t = \frac{1}{L_{g,1} t^a}$, $\beta_t = \frac{1}{2\lambda_1 L_{g,1} t^b}$, $\gamma_t \equiv \gamma \le \min\{\frac{1}{2L_F}, \frac{1}{48\kappa_g L_{g,1}} \sqrt{\frac{1-\rho}{7c_\lambda}}\}$ *for some constant* $c_\lambda > 0$, $K_t = 4\kappa_g \max\{t^{b-\tau}, t^a\} \log t$, $\widehat{B}_t = t^{B_1}$ *and* $B_t = t^{B_2}$, *Algorithm 3 can guarantee*

$$\mathbb{E}\left[\mathrm{BLReg}(T)\right] \le O\left(T^{1-2\tau} + \sigma^2 \max\{T^{1+2\tau-B_1}, T^{1+3\tau-b-B_2}, T^{1+2\tau-a-B_2}\} + V_T + H_{2,T}\right).$$

As shown in Theorem 5.1, the three variance terms arise from the updates of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, respectively. The first $T^{1+2\tau-B_1}$ term is the main bottleneck, due to the bias of the gradient mapping and the dependence of $\nabla\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t)$ variance on λ_t. This makes the parameter selection more challenging, shown as $\mathbb{E}[\|\nabla\mathcal{L}_t^*(\mathbf{x}; \mathcal{S}_t) - \nabla\mathcal{L}_t^*(\mathbf{x})\|^2] \le (1 + 2\lambda_t^2)\sigma^2$.

Corollary 5.2. *Under Assumption 2.1, 2.2 and 2.3, with the same parameter setting in Theorem 5.1, let* $\tau = \frac{1}{6}, a = 0, b = \frac{1}{6}, B_1 = B_2 = \frac{2}{3}$, *which means* $K_t = 4\kappa_g \log t$, *Algorithm 3 can guarantee*

$$\mathbb{E}\left[\mathrm{BLReg}(T)\right] \le O\left(T^{2/3}(1 + \sigma^2) + V_T + H_{2,T}\right).$$

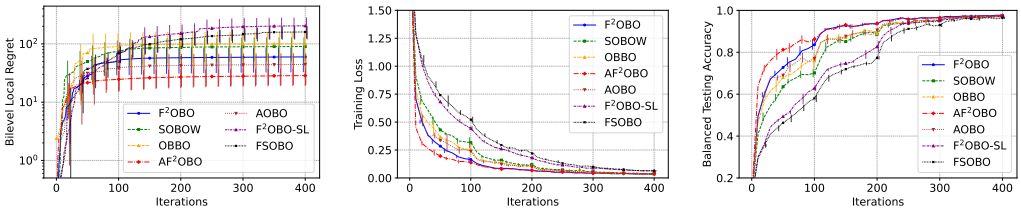
Corollary 5.2 gives a specific result of SF^{2}OBO. SOBBO [2] reduces the upper-level variance by using a window-averaged hypergradient, which yields a variance term of $O(T\sigma^2/w)$. However, this approach inherently relies on the analysis of window-averaged regret, which may not accurately characterize the algorithm's performance in dynamic environments. SOGD [24] establishes results under the standard local regret by employing variance reduction techniques and achieve a $O(T^{1/3}\sigma^2)$. However, its guarantees crucially depend on sublinear environmental variation assumptions as shown in Table 2, and additionally require a bounded variance assumption on stochastic second-order gradients, i.e., $\mathbb{E}\|\nabla_{\mathbf{xy}}^2 g_t(\mathbf{z}; \zeta) - \nabla_{\mathbf{xy}}^2 g_t(\mathbf{z})\|^2 \le \sigma^2$ and $\mathbb{E}\|\nabla_{\mathbf{yy}}^2 g_t(\mathbf{z}; \zeta) - \nabla_{\mathbf{yy}}^2 g_t(\mathbf{z})\|^2 \le \sigma^2$.

6 Experiments

In this section, we provide the empirical studies on the applications of parametric loss tuning for imbalanced data tasks and online hyperparameter optimization.

6.1 Parametric Loss Tuning for Imbalanced Data

Tuning for imbalanced data addresses skewed class distributions, where models, especially high-capacity deep networks, often bias toward the majority class and perform poorly on minority classes. Parametric training loss can be used to address this issue by balancing accuracy and fairness while



(a) regret (b) training loss (c) test accuracy

Figure 1: Performance of different OBO algorithms on the parametric loss tuning for imbalanced data over five runs. F^{2}OBO-SL denotes F^{2}OBO under the single-loop structure, i.e., $K_t \equiv 1$.

Table 3: Performance of different stochastic OBO algorithms on online hyperparameter optimization.

Algorithm	Training loss	Test accuracy	Running time
SOBBO	1.3884 ± 0.0519	0.7979 ± 0.0005	150.80 ± 1.84
SOGD	2.0910 ± 0.0477	0.7607 ± 0.0005	$\mathbf{41.28 \pm 1.51}$
SF^{2}OBO	$\mathbf{1.2914 \pm 0.0664}$	$\mathbf{0.8003 \pm 0.0016}$	103.02 ± 1.77

preventing overfitting, and we use it to optimize the model's performance on the training set $\mathcal{D}_t^{\mathrm{tr}}$ with $\lambda = (\gamma_j, \Delta_j)_{j=1}^J$, which means that the logits adjustment is treated as an upper variable and optimized by validation set $\mathcal{D}_t^{\mathrm{val}}$.

$$\min_{\lambda \in \mathbb{R}^{d_1}} F_t(\lambda) = \frac{1}{|\mathcal{D}_t^{\mathrm{val}}|} \sum_{(x_i,y_i)\in\mathcal{D}_t^{\mathrm{val}}} u_{y_i} \mathcal{L}(f(\theta_t^*(\lambda); x_i), y_i) \quad \text{s.t.} \quad \theta_t^*(\lambda) = \operatorname*{argmin}_{\theta \in \mathbb{R}^{d_2}} g_t(\lambda, \theta),$$

$$\text{where} \quad g_t(\lambda, \theta) = -\frac{1}{|\mathcal{D}_t^{\mathrm{tr}}|} \sum_{(x_i,y_i)\in\mathcal{D}_t^{\mathrm{tr}}} \log \frac{\exp(\gamma_{y_i}[f(\theta; x_i)]_{y_i} + \Delta_{y_i})}{\sum_{j=1}^J \exp(\gamma_j [f(\theta; x_i)]_j + \Delta_j)}.$$

Here u_j is the inverse class frequency (i.e., the reciprocal of the proportion of class-j samples) [17]. We use a lightweight MNIST CNN with four 3×3 conv blocks (Conv-ReLU-2 $\times$ 2 MaxPool–BN), followed by flattening and a 10-way linear classifier.

We evaluated performance over 400 timesteps in four 100-timestep phases, transitioning from 0.6^i to 0.9^i distribution for each class $i = 0, 1, \ldots, 9$. We tune learning rates $\alpha, \beta, \gamma \in \{0.001, 0.005, 0.01, 0.05, 0.1, 0.5\}$, set $\lambda_1 = 5.0$ and $\tau = 1$. In Figure 1, we compare the performance of different algorithms, where AF^{2}OBO achieves the best balanced testing accuracy. SF^{2}OBO achieves better empirical performance than FSOBO. However, it exhibits a larger local regret, which also corroborates our theoretical results.

6.2 Online Hyperparameter Optimization

We formulate online hyperparameter optimization as a bilevel problem, where the inner problem updates the model parameters and the upper problem optimizes hyperparameters according to validation performance. Under a dynamic data stream, both parameters and hyperparameters are updated at each time step to adapt to distribution shifts. At t, the optimizer receives $\mathcal{D}_t = \{\mathcal{D}_t^{\mathrm{tr}}, \mathcal{D}_t^{\mathrm{val}}\}$, model parameters θ_t, and hyperparameters λ_t. We optimize the L2 regularization coefficient with the box constraint $\lambda_t \in \Lambda = [0, 0.1]^{d_1}$ for a linear model, and define the objective as follows:

$$\min_{\lambda \in \Lambda} F_t(\lambda) = \frac{1}{|\mathcal{D}_t^{\mathrm{val}}|} \sum_{(x_i,y_i)\in\mathcal{D}_t^{\mathrm{val}}} \mathcal{L}(f(\theta_t^*(\lambda); x_i), y_i) \quad \text{s.t.} \quad \theta_t^*(\lambda) = \operatorname*{argmin}_{\theta \in \mathbb{R}^{d_2}} g_t(\lambda, \theta),$$

$$\text{where} \quad g_t(\lambda, \theta) = \frac{1}{|\mathcal{D}_t^{\mathrm{tr}}|} \sum_{(x_i,y_i)\in\mathcal{D}_t^{\mathrm{tr}}} \mathcal{L}\left(f(\theta; x_i), y_i\right) + \Omega(\lambda, \theta).$$

Here, $\Omega(\lambda, \theta) = \frac{1}{2} \sum_{i=0}^{d-1} \lambda_i \|\theta_i\|^2$. Given the optimal model parameter $\theta_t^*(\lambda_t)$, we update the hyperparameters using the validation loss. We evaluate stochastic OBO methods on the 20 Newsgroups text classification task [16]. We repeat each experiment five times. Each run lasts 10,000 iterations, where

5%, 10%, and 15% label noise is injected at iterations 2,500, 5,000, and 7,500 to simulate distribution shifts. Table 3 compares the training loss, test accuracy, and running time of different methods. The results demonstrate that SF^{2}OBO achieves the best overall performance with competitive efficiency.

7 Conclusion and Discussion

This work studies first-order algorithms for non-convex-strongly-convex OBO problem. We propose F^{2}OBO, a fully first-order algorithm, and prove that, under suitable parameter choices, it achieves regret and computational complexity comparable to those of second-order algorithms. We also establish a regret upper bound under the more efficient single-loop structure, which involves an additional gradient-variation term. We further develop an adaptive-iteration variant, AF^{2}OBO, which removes the dependence on $H_{2,T}$ by dynamically adjusting the number of inner iterations, at the cost of additional computation. Finally, we extend our analysis to the stochastic OBO setting, propose SF^{2}OBO, and establish an upper bound on its expected regret.

Limitation and future work Our theoretical analysis is still restricted to OBO problems with strongly convex inner-level objectives, which limits the applicability of our algorithm. In future work, we will extend the analysis of first-order algorithms to OBO problems whose inner-level objectives are non-convex but satisfy the PL condition. Developing an appropriate regret notion and designing efficient algorithms for this setting is an interesting and meaningful direction.

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A Experimental Details and Additional Results

In this section, we provide the detailed parameter settings for the experiments in Section 6 and hyperparameter sensitivity experiments for F^{2}OBO.

A.1 Detailed Parameter Settings

In the parametric loss tuning experiment for imbalanced data, all algorithms use an inner learning rate $\alpha = 0.1$ and an outer learning rate $\gamma = 0.001$. For OBBO and F^{2}OBO, we set the number of inner iterations to $K = 5$. For AOBO, we set $\delta = 0.1$; for AF^{2}OBO, we set $\delta_y = 0.5$ and $\delta_z = 0.1$. The number of linear-system solving steps for AOBO and SOBOW is set to 5. For all our first-order algorithms, we use $\lambda_1 = 5$ with $\tau = 1$.

In the online hyperparameter optimization experiment, each algorithm receives a dataset with batch size 256 at every time step. We use an inner learning rate $\alpha = 0.1$ and an outer learning rate $\gamma = 0.001$. For SOGD, we set the momentum coefficient to $\eta = 0.9$ and the linear-system learning rate to $\beta = 0.1$. For SOBBO, we use a window size of $w = 50$ with $K = 5$ inner iterations. For our SF^{2}OBO, we also set $K = 5$ and use $\lambda_1 = 5$ with $\tau = 1$.

A.2 Hyperparameter Sensitivity Experiments for F^{2}OBO

We conduct hyperparameter sensitivity experiments for F^{2}OBO under the parametric loss tuning task for imbalanced data. Specifically, we study the sensitivity with respect to the number of inner iterations K and the penalty-schedule parameter τ in λ_t. All other experimental settings and hyperparameters are identical to those reported in the main paper.

Table 4: Sensitivity of F^{2}OBO to the number of inner iterations K.

Iterations	$K = 1$	$K = 3$	$K = 5$	$K = 10$
100	0.6151 ± 0.0006	0.7471 ± 0.0143	0.8302 ± 0.0215	0.8654 ± 0.0116
200	0.8222 ± 0.0027	0.9227 ± 0.0002	0.9309 ± 0.0098	0.9301 ± 0.0092
300	0.9469 ± 0.0005	0.9600 ± 0.0006	0.9636 ± 0.0010	0.9613 ± 0.0021
400	0.9741 ± 0.0026	0.9767 ± 0.0011	0.9771 ± 0.0023	0.9752 ± 0.0022

Table 5: Sensitivity of F^{2}OBO to the penalty-schedule parameter τ.

Iterations	$\tau = 0$	$\tau = 0.5$	$\tau = 1$	$\tau = 5$
100	0.8365 ± 0.0265	0.8324 ± 0.0208	0.8302 ± 0.0215	0.8483 ± 0.0083
200	0.9307 ± 0.0081	0.9388 ± 0.0031	0.9309 ± 0.0098	0.9336 ± 0.0113
300	0.9637 ± 0.0019	0.9647 ± 0.0007	0.9636 ± 0.0010	0.9623 ± 0.0038
400	0.9759 ± 0.0028	0.9764 ± 0.0019	0.9771 ± 0.0023	0.9749 ± 0.0033

B Preliminary Lemmas

We list all the lemmas used in the proof below.

Lemma B.1 (Lemma 16 in Tarzanagh et al. [31])**.** *Under Assumptions 2.1, 2.2, for all $t \in [T]$, any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$, we have*

$$\|\mathbf{y}_t^*(\mathbf{x}_1) - \mathbf{y}_t^*(\mathbf{x}_2)\| \leq \frac{L_{g,1}}{\mu_g}\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

Lemma B.2 (Lemma 4.1 in Chen et al. [6])**.** *Under Assumptions 2.1 and 2.2, let $\lambda > \frac{2L_{f,1}}{\mu_g}$, for all $t \in [T]$, it holds that*

$$\left|f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) - \mathcal{L}_{\lambda,t}^*(\mathbf{x})\right| \leq \frac{D_0}{\lambda}, \tag{12}$$

$$\left\|\nabla f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x})) - \nabla \mathcal{L}_{\lambda,t}^*(\mathbf{x})\right\| \leq \frac{D_1}{\lambda}, \tag{13}$$

$$\mathcal{L}_{\lambda,t}^*(\mathbf{x}) \text{ is } L_{\mathcal{L},1}\text{-smooth.} \tag{14}$$

Lemma B.3 (Lemma 16 in Tarzanagh et al. [31]). *Under Assumptions 2.1, 2.2, for all $t \in [T]$, any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$, we have*

$$\|\nabla f_t(\mathbf{x}_1, \mathbf{y}_t^*(\mathbf{x}_1)) - \nabla f_t(\mathbf{x}_2, \mathbf{y}_t^*(\mathbf{x}_2))\| \leq L_F \|\mathbf{x}_1 - \mathbf{x}_2\|,$$

where

$$L_F := L_{f,1} + \frac{L_{f,0}L_{g,2}}{\mu_g} + \frac{2L_{g,1}L_{f,1}}{\mu_g} + \frac{L_{g,1}^2 L_{f,1}}{\mu_g^2} + \frac{2L_{f,0}L_{g,1}L_{g,2}}{\mu_g^2} + \frac{L_{f,0}L_{g,1}^2 L_{g,2}}{\mu_g^3}.$$

Lemma B.4 (Lemma B.7 in Jia et al. [14]). *Under Assumption 2.2, we have*

$$\sum_{t=1}^{T} \left(f_t\left(\mathbf{x}_t, \mathbf{y}_t^*(\mathbf{x}_t)\right) - f_t\left(\mathbf{x}_{t+1}, \mathbf{y}_t^*(\mathbf{x}_{t+1})\right)\right) \leq 2M + V_T,$$

where V_T is defined in Eq.(4).

Lemma B.5 (Based on Lemma 3.2 in Kwon et al. [15]). *Let $\lambda > \frac{2L_{f,1}}{\mu_g}$, for all $t \in [T]$, it holds that $\|\mathbf{y}_t^*(\mathbf{x}) - \mathbf{y}_{\lambda,t}^*(\mathbf{x})\| \leq \frac{2L_{f,0}}{\lambda\mu_g}$.*

Proof. Lemma 3.2 [15] shows that for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and $\lambda_2 \geq \lambda_1 \geq \frac{2L_{f,1}}{\mu_g}$, it holds that

$$\left\|\mathbf{y}_{\lambda_1,t}^*(\mathbf{x}_1) - \mathbf{y}_{\lambda_2,t}^*(\mathbf{x}_2)\right\| \leq \frac{2(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}\frac{L_{f,0}}{\mu_g} + 3\kappa_g \left\|\mathbf{x}_2 - \mathbf{x}_1\right\|.$$

Our result can be achieved if $\lambda = \lambda_1$ for any $\lambda \geq \frac{2L_{f,1}}{\mu_g}$, $\lambda_2 \to +\infty$ and $\mathbf{x}_1 = \mathbf{x}_2$. □

Lemma B.6. *Under Assumptions 2.1 and 2.2, for any $\lambda > \frac{2L_{f,1}}{\mu_g}$ and $t \in [T]$, $\mathcal{L}_{\lambda,t}^*(\mathbf{x})$ is $L_{\mathcal{L},0}$-Lipschitz continuous, where*

$$L_{\mathcal{L},0} := L_{f,0} + \frac{2L_{f,0}L_{g,1}}{\mu_g}.$$

Proof. For any $\lambda > \frac{2L_{f,1}}{\mu_g}$, it follows that

$$\begin{aligned}
\|\nabla \mathcal{L}_t^*(\mathbf{x})\| =& \left\|\nabla_{\mathbf{x}} f_t(\mathbf{x}, \mathbf{y}_{\lambda,t}^*(\mathbf{x})) + \lambda\left(\nabla_{\mathbf{x}} g_t(\mathbf{x}, \mathbf{y}_{\lambda,t}^*(\mathbf{x})) - \nabla_{\mathbf{x}} g_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))\right)\right\| \\
\leq& \left\|\nabla_{\mathbf{x}} f_t(\mathbf{x}, \mathbf{y}_{\lambda,t}^*(\mathbf{x}))\right\| + \lambda\left\|\nabla_{\mathbf{x}} g_t(\mathbf{x}, \mathbf{y}_{\lambda,t}^*(\mathbf{x})) - \nabla_{\mathbf{x}} g_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))\right\| \\
\leq& L_{f,0} + \lambda L_{g,1}\left\|\mathbf{y}_{\lambda,t}^*(\mathbf{x})) - \mathbf{y}_t^*(\mathbf{x})\right\| \\
\leq& L_{f,0} + \frac{2L_{f,0}L_{g,1}}{\mu_g} =: L_{\mathcal{L},0},
\end{aligned}$$

where the last inequality comes from the $L_{f,0}$-Lipschitz continuity of f_t under Assumption 2.2 and Lemma B.5. □

C Proof of Section 3

In this section, we provide the regret upper bound for Algorithm 1 and the regret upper bound under the single-loop structure in Sections C.1 and C.2, respectively. The following lemma establishes a preliminary regret upper bound for the fully first-order algorithm applied to the approximation problem (P3) in the deterministic OBO setting.

Lemma C.1. *Under Assumptions 2.1, 2.2, suppose that $\gamma_t \leq \frac{1}{2L_F}$ for all $t \in [T]$. Consider the update*

$$\mathbf{x}_{t+1} \leftarrow \mathbf{x}_t - \gamma_t \mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t(\mathbf{x}_t), \gamma_t\right).$$

Then, we obtain

$$\mathrm{BLReg}(T) \leq \frac{16M}{\gamma_T} + \frac{8}{\gamma_T}V_T + 16D_1^2\sum_{t=1}^{T}\frac{1}{\lambda_t^2} + 16\sum_{t=1}^{T}\left\|\nabla\mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2.$$

Proof. It begins with the smoothness of $F_t(\mathbf{x})$ shown in Lemma B.3, we have

$$\begin{aligned}
&F_t(\mathbf{x}_{t+1}) - F_t(\mathbf{x}_t) \\
\leq& \langle \nabla F_t(\mathbf{x}_t)), \mathbf{x}_{t+1} - \mathbf{x}_t \rangle + \frac{L_F}{2} \|\mathbf{x}_{t+1} - \mathbf{x}_t\|^2 \\
=& -\gamma_t \left\langle \nabla F_t(\mathbf{x}_t)), \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\rangle + \frac{\gamma_t^2 L_F}{2} \left\| \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\|^2 \\
=& -\gamma_t \left\langle \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\rangle \\
&+ \gamma_t \left\langle \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla F_t(\mathbf{x}_t), \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\rangle + \frac{\gamma_t^2 L_F}{2} \left\| \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\|^2,
\end{aligned}$$

then with Young's inequality,

$$\begin{aligned}
F_t(\mathbf{x}_{t+1}) - F_t(\mathbf{x}_t) \leq& -\gamma_t \left\| \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\|^2 + \frac{\gamma_t}{2} \left\| \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\|^2 \\
&+ \frac{\gamma_t}{2} \left\| \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla F_t(\mathbf{x}_t)) \right\|^2 + \frac{\gamma_t^2 L_F}{2} \left\| \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\|^2 \\
\leq& -\frac{\gamma_t}{2}(1 - \gamma_t L_F) \left\| \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\|^2 + \frac{\gamma_t}{2} \left\| \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla F_t(\mathbf{x}_t)) \right\|^2.
\end{aligned} \tag{15}$$

Moreover, by

$$\begin{aligned}
-\left\| \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\|^2 \leq& \left\| \mathcal{G}_{\mathcal{X}}(\mathbf{x}_t, \nabla F_t(\mathbf{x}_t)), \gamma_t) - \mathcal{G}_{\mathcal{X}} \left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t), \gamma_t\right) \right\|^2 \\
&- \frac{1}{2} \|\mathcal{G}_{\mathcal{X}}(\mathbf{x}_t, \nabla F_t(\mathbf{x}_t)), \gamma_t)\|^2,
\end{aligned} \tag{16}$$

substituting Eq.(16) into Eq.(15) and rearranging the terms, it follows that

$$\begin{aligned}
\|\mathcal{G}_{\mathcal{X}}(\mathbf{x}_t, \nabla F_t(\mathbf{x}_t), \gamma_t)\|^2 \leq& \frac{8}{\gamma_t}(F_t(\mathbf{x}_t) - F_t(\mathbf{x}_{t+1})) + 8 \left\| \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla F_t(\mathbf{x}_t) \right\|^2 \\
\leq& \frac{8}{\gamma_t}(F_t(\mathbf{x}_t) - F_t(\mathbf{x}_{t+1})) + 16 \|\nabla F_t(\mathbf{x}_t) - \nabla \mathcal{L}_t^*(\mathbf{x}_t)\|^2 \\
&+ 16 \left\| \nabla \mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) \right\|^2,
\end{aligned}$$

where we choose $\gamma_t \leq \frac{1}{2L_F}$ to ensure that $1 - \gamma_t L_F \geq \frac{1}{2}$. Summing the above inequality over $t = 1, \ldots, T$, and substituting Lemma B.4 and Lemma B.2 equation (13) in it, we obtain

$$\mathrm{BLReg}(T) \leq \frac{16M}{\gamma_t} + \frac{8}{\gamma_t} V_T + 16 D_1^2 \sum_{t=1}^{T} \frac{1}{\lambda_t^2} + 16 \sum_{t=1}^{T} \left\| \nabla \mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) \right\|^2.$$

Thus we finish the proof. □

C.1 Proof of Theorem 3.1

In this section, we give the proof of Theorem 3.1. Lemma C.4 provides an upper bound on the approximate hypergradient error of Algorithm 1 by using the inner-loop iteration error bounds established in Lemmas C.2 and C.3. Substituting this bound into Lemma C.1 yields the proof of Theorem C.5.

Lemma C.2. *Under Assumptions 2.1, 2.2, let* $\beta_t \leq \frac{1}{2\lambda_t L_{g,1}}$ *and* $K_t \equiv K \geq 1 - \frac{\ln 2}{\ln \rho_{\mathbf{y}}}$*, where* $\rho_{\mathbf{y}} = 1 - \frac{\beta_t \lambda_t \mu_g}{2}$*, Algorithm 1 can obtain*

$$\begin{aligned}
&\left\| \mathbf{y}_{t+1} - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t) \right\|^2 \\
\leq& \rho_{\mathbf{y}}^{t-1} \left\| \mathbf{y}_2 - \mathbf{y}_{\lambda,1}^*(\mathbf{x}_1) \right\|^2 + \frac{12 L_{f,0}^2}{\mu_g^2} \sum_{j=0}^{t-2} \frac{1}{\lambda_{t-j}^2} \rho_{\mathbf{y}}^{j+K} + 12 \kappa_g^2 \sum_{j=0}^{t-2} \rho_{\mathbf{y}}^{j+K} \|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\|^2 \\
&+ 12 \sum_{j=0}^{t-2} \rho_{\mathbf{y}}^{j+K} \left\| \mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j}) \right\|^2.
\end{aligned}$$

Proof. It begins with

$$\begin{aligned}
&\left\|\mathbf{y}_t^{k+1}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\\
=&\left\|\mathbf{y}_t^{k}-\beta_t\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t,\mathbf{y}_t^k,\lambda_t)-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\\
=&\left\|\mathbf{y}_t^{k}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2-2\beta_t\left\langle\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t,\mathbf{y}_t^k,\lambda_t),\mathbf{y}_t^k-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\rangle+\beta_t^2\left\|\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t,\mathbf{y}_t^k,\lambda_t)\right\|^2\\
\leq&\left\|\mathbf{y}_t^{k}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2-\frac{\beta_t\lambda_t\mu_g}{2}\left\|\mathbf{y}_t^k-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2-\left(\frac{\beta_t}{2\lambda_tL_{g,1}}-\beta_t^2\right)\left\|\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t,\mathbf{y}_t^k,\lambda_t)\right\|^2\\
=&\rho_{\mathbf{y}}\left\|\mathbf{y}_t^{k}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2,
\end{aligned}\tag{17}$$

where we set $\beta_t\leq\frac{1}{2\lambda_tL_{g,1}}$ and $\rho_{\mathbf{y}}:=1-\frac{\beta_t\lambda_t\mu_g}{2}$. The first inequality follows from the fact that, if $\lambda_t\geq\frac{2L_{f,1}}{\mu_g}$ holds for all $t\in[T]$, then $\mathcal{L}_t$ is $(\lambda_t\mu_g/2)$-strongly convex and $(2\lambda_tL_{g,1})$-smooth, for any $\mathbf{x}\in\mathcal{X}$, $\mathbf{y}\in\mathbb{R}^{d_2}$,

$$\begin{aligned}
\frac{\lambda_t\mu_g}{2}\left\|\mathbf{y}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x})\right\|^2&\leq\left\langle\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x},\mathbf{y},\lambda_t),\mathbf{y}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x})\right\rangle\\
\frac{1}{2\lambda_tL_{g,1}}\left\|\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x},\mathbf{y},\lambda_t)\right\|^2&\leq\left\langle\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x},\mathbf{y},\lambda_t),\mathbf{y}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x})\right\rangle.
\end{aligned}$$

Thus we natually have

$$\begin{aligned}
\left\|\mathbf{y}_{t+1}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\leq&\rho_{\mathbf{y}}^{K_t}\left\|\mathbf{y}_t-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\\
\leq&2\rho_{\mathbf{y}}^{K_t}\left\|\mathbf{y}_t-\mathbf{y}_{\lambda_{t-1},t-1}^*(\mathbf{x}_{t-1})\right\|^2+2\rho_{\mathbf{y}}^{K_t}\left\|\mathbf{y}_{\lambda_{t-1},t-1}^*(\mathbf{x}_{t-1})-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\left\|\mathbf{y}_{\lambda_{t-1},t-1}^*(\mathbf{x}_{t-1})-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\\
\leq&3\left\|\mathbf{y}_{\lambda_{t-1},t-1}^*(\mathbf{x}_{t-1})-\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1})\right\|^2+3\left\|\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1})-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
&+3\left\|\mathbf{y}_t^*(\mathbf{x}_t)-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\\
\leq&\frac{6L_{f,0}^2}{\lambda_t^2\mu_g^2}+6\kappa_g^2\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2+6\left\|\mathbf{y}_{t-1}^*(\mathbf{x}_t)-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2,
\end{aligned}\tag{18}$$

where the last ineqaulity comes from Lemma B.5 and B.1. Let $K_t\equiv K\geq1-\frac{\ln2}{\ln\rho_{\mathbf{y}}}$ such that $2\rho_{\mathbf{y}}^K\leq\rho_{\mathbf{y}}$, we have

$$\begin{aligned}
&\left\|\mathbf{y}_{t+1}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\\
\leq&\rho_{\mathbf{y}}\left\|\mathbf{y}_t-\mathbf{y}_{\lambda_{t-1},t-1}^*(\mathbf{x}_{t-1})\right\|^2+\frac{12L_{f,0}^2}{\lambda_t^2\mu_g^2}\rho_{\mathbf{y}}^K+12\kappa_g^2\rho_{\mathbf{y}}^K\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2\\
&+12\rho_{\mathbf{y}}^K\left\|\mathbf{y}_{t-1}^*(\mathbf{x}_t)-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
\leq&\rho_{\mathbf{y}}^{t-1}\left\|\mathbf{y}_2-\mathbf{y}_{\lambda,1}^*(\mathbf{x}_1)\right\|^2+\frac{12L_{f,0}^2}{\mu_g^2}\sum_{j=0}^{t-2}\frac{1}{\lambda_{t-j}^2}\rho_{\mathbf{y}}^{j+K}+12\kappa_g^2\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^{j+K}\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2\\
&+12\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^{j+K}\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j})-\mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2.
\end{aligned}\tag{19}$$

Thus we finish the proof. □

Lemma C.3. *Under Assumptions 2.1, 2.2, let $\alpha\leq\frac{1}{L_{g,1}}$ and $K_t\equiv K\geq1-\frac{\ln2}{\ln\rho_{\mathbf{z}}}$, where $\rho_{\mathbf{z}}=1-\alpha\mu_g$, Algorithm 1 can obtain*

$$\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\leq\rho_{\mathbf{z}}^{t-1}\left\|\mathbf{z}_2-\mathbf{y}_1^*(\mathbf{x}_1)\right\|^2+4\kappa_g^2\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^{j+K}\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2$$

$$+4\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^{j+K}\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j})-\mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2.$$

Proof. It begins with

$$\begin{aligned}
&\left\|\mathbf{z}_t^{k+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
=&\left\|\mathbf{z}_t^{k}-\alpha\nabla_{\mathbf{y}}g_t(\mathbf{x}_t,\mathbf{z}_t^k)-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
=&\left\|\mathbf{z}_t^{k}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2-2\alpha\left\langle\nabla_{\mathbf{y}}g_t(\mathbf{x}_t,\mathbf{z}_t^k),\mathbf{z}_t^k-\mathbf{y}_t^*(\mathbf{x}_t)\right\rangle+\alpha^2\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x}_t,\mathbf{z}_t^k)\right\|^2\\
\leq&\left\|\mathbf{z}_t^{k}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2-\alpha\mu_g\left\|\mathbf{z}_t^k-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2-\left(\frac{\alpha}{L_{g,1}}-\alpha^2\right)\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x}_t,\mathbf{z}_t^k)\right\|^2\\
=&\rho_{\mathbf{z}}\left\|\mathbf{z}_t^{k}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2,
\end{aligned}$$

where we set $\alpha\leq\frac{1}{L_{g,1}}$ and $\rho_{\mathbf{z}}:=1-\alpha\mu_g$. the first inequality follows from Assumption 2.1 and 2.2. Then we complete the proof by

$$\begin{aligned}
&\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
\leq&\rho_{\mathbf{z}}^{K_t}\left\|\mathbf{z}_t-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
\leq&2\rho_{\mathbf{z}}^{K_t}\left\|\mathbf{z}_t-\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1})\right\|^2+2\rho_{\mathbf{z}}^{K_t}\left\|\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1})-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
\overset{(i)}{\leq}&\rho_{\mathbf{z}}\left\|\mathbf{z}_t-\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1})\right\|^2+4\kappa_g^2\rho_{\mathbf{z}}^{K}\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2+4\rho_{\mathbf{z}}^{K}\left\|\mathbf{y}_{t-1}^*(\mathbf{x}_t)-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
\leq&\rho_{\mathbf{z}}^{t-1}\left\|\mathbf{z}_2-\mathbf{y}_1^*(\mathbf{x}_1)\right\|^2+4\kappa_g^2\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^{j+K}\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2\\
&+4\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^{j+K}\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j})-\mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2,
\end{aligned}$$

where in (i), we set $K_t\equiv K\geq 1-\frac{\ln 2}{\ln\rho_{\mathbf{z}}}$ such that $2\rho_{\mathbf{z}}^K\leq\rho_{\mathbf{z}}$. □

Lemma C.4. *Under Assumptions 2.1 and 2.2, let $\Lambda(t,\lambda_t)=\left(1+\frac{1}{t}\right)^{\tau}\lambda_t$ for some $\tau>0$, $\lambda_1>\frac{2L_{f,1}}{\mu_g}$, $\alpha\leq\frac{1}{L_{g,1}}$, $\beta_t\leq\frac{1}{2\lambda_t L_{g,1}}$, $\gamma_t\equiv\gamma\leq\frac{1}{4\kappa_g}\sqrt{\frac{1-\rho}{C_\lambda}}$ and $K_t\equiv K\geq\max\{1-\frac{\ln 2}{\ln\rho},\frac{\ln c-2\tau\ln T}{\ln\rho}\}$ for some $c>0$, the approximation error of hypergradient in Algorithm 1 can be bounded as follows:*

$$\begin{aligned}
\sum_{t=2}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\leq&2c_\tau\Delta_1+\frac{576c\kappa_g^3L_{f,0}^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+\frac{8\gamma^2C_\lambda\kappa_g^2D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}\\
&+\frac{4\gamma^2C_\lambda\kappa_g^2}{1-\rho}\mathrm{BLReg}(T)+\frac{2C_\lambda}{1-\rho}\sum_{t=2}^{T}\sup_{\mathbf{x}\in\mathbb{R}^{d_1}}\left\|\mathbf{y}_{t-1}^*(\mathbf{x})-\mathbf{y}_t^*(\mathbf{x})\right\|^2.
\end{aligned}$$

where ρ, c, c_τ, C_λ and Δ_1 are some constants.

Proof. It begins with

$$\begin{aligned}
\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\leq&3\left\|\nabla_{\mathbf{x}}f_t(\mathbf{x}_t,\mathbf{y}_{t+1})-\nabla_{\mathbf{x}}f_t(\mathbf{x}_t,\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}))\right\|^2\\
&+3\lambda_t^2\left\|\nabla_{\mathbf{x}}g_t(\mathbf{x}_t,\mathbf{y}_{t+1})-\nabla_{\mathbf{x}}g_t(\mathbf{x}_t,\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t))\right\|^2\\
&+3\lambda_t^2\left\|\nabla_{\mathbf{x}}g_t(\mathbf{x}_t,\mathbf{z}_{t+1})-\nabla_{\mathbf{x}}g_t(\mathbf{x}_t,\mathbf{y}_t^*(\mathbf{x}_t))\right\|^2\\
\overset{(i)}{\leq}&3\left(L_{f,1}^2+\lambda_t^2L_{g,1}^2\right)\left\|\mathbf{y}_{t+1}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2+3\lambda_t^2L_{g,1}^2\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\
\overset{(ii)}{\leq}&6\lambda_t^2L_{g,1}^2\underbrace{\left\|\mathbf{y}_{t+1}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2}_{\mathcal{L}_t\text{ inner iters bound: }(a)}+3\lambda_t^2L_{g,1}^2\underbrace{\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2}_{g_t\text{ inner iters bound: }(b)},
\end{aligned}\tag{20}$$

where (i) comes from the smoothness of f_t and g_t under Assumption 2.2, and (ii) comes from $\lambda_t > \frac{2L_{f,1}}{\mu_g}$, thus $L_{f,1} < \lambda_t L_{g,1}$. Substituting Lemma C.2 and Lemma C.3 into equation (20), we have

$$
\begin{aligned}
&\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \\
\leq& 6\lambda_t^2 L_{g,1}^2 \rho_{\mathbf{y}}^{t-1} \left\|\mathbf{y}_2 - \mathbf{y}_{\lambda,1}^*(\mathbf{x}_1)\right\|^2 + 3\lambda_t^2 L_{g,1}^2 \rho_{\mathbf{z}}^{t-1} \|\mathbf{z}_2 - \mathbf{y}_1^*(\mathbf{x}_1)\|^2 + 72\kappa_g^2 L_{f,0}^2 \sum_{j=0}^{t-2} \frac{\lambda_t^2}{\lambda_{t-j}^2} \rho_{\mathbf{y}}^{j+K} \\
&+ 72\lambda_t^2 \kappa_g^2 L_{g,1}^2 \sum_{j=0}^{t-2} \rho_{\mathbf{y}}^{j+K} \|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\|^2 + 72\lambda_t^2 L_{g,1}^2 \sum_{j=0}^{t-2} \rho_{\mathbf{y}}^{j+K} \left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2 \\
&+ 12\lambda_t^2 \kappa_g^2 L_{g,1}^2 \sum_{j=0}^{t-2} \rho_{\mathbf{z}}^{j+K} \|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\|^2 + 12\lambda_t^2 L_{g,1}^2 \sum_{j=0}^{t-2} \rho_{\mathbf{z}}^{j+K} \|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\|^2.
\end{aligned}
$$

Suppose that, for a sufficiently large K, i.e., $K \geq \frac{\ln c - 2\tau \ln T}{\ln \rho}$, $\lambda_t^2 \rho^K \leq c$ holds for some constant $c > 0$, where $\rho := \max\{\rho_{\mathbf{y}}, \rho_{\mathbf{z}}\}$. Then we have

$$
\begin{aligned}
&\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \\
\leq& 6\lambda_t^2 L_{g,1}^2 \rho_{\mathbf{y}}^{t-1} \left\|\mathbf{y}_2 - \mathbf{y}_{\lambda,1}^*(\mathbf{x}_1)\right\|^2 + 3\lambda_t^2 L_{g,1}^2 \rho_{\mathbf{z}}^{t-1} \|\mathbf{z}_2 - \mathbf{y}_1^*(\mathbf{x}_1)\|^2 + 72c\kappa_g^2 L_{f,0}^2 \sum_{j=0}^{t-2} \frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} \\
&+ 72c\kappa_g^2 L_{g,1}^2 \sum_{j=0}^{t-2} \rho_{\mathbf{y}}^j \|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\|^2 + 72cL_{g,1}^2 \sum_{j=0}^{t-2} \rho_{\mathbf{y}}^j \left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2 \\
&+ 12c\kappa_g^2 L_{g,1}^2 \sum_{j=0}^{t-2} \rho_{\mathbf{z}}^j \|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\|^2 + 12cL_{g,1}^2 \sum_{j=0}^{t-2} \rho_{\mathbf{z}}^j \|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\|^2 \\
\overset{(i)}{\leq}& \lambda_t^2 \rho^{t-1} \Delta_1' + 72c\kappa_g^2 L_{f,0}^2 \sum_{j=0}^{t-2} \frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} + C_\lambda \sum_{j=0}^{t-2} \rho^j \left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2 \\
&+ C_\lambda \kappa_g^2 \sum_{j=0}^{t-2} \rho^j \|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\|^2,
\end{aligned} \tag{21}
$$

where in (i), we define $\Delta_1' := 6L_{g,1}^2 \|\mathbf{y}_2 - \mathbf{y}_{\lambda_1,1}^*(\mathbf{x}_1)\|^2 + 3L_{g,1}^2 \|\mathbf{z}_2 - \mathbf{y}_1^*(\mathbf{x}_1)\|^2$, $C_\lambda := 84cL_{g,1}^2$ and $\rho := \max\{\rho_{\mathbf{y}}, \rho_{\mathbf{z}}\}$. After telescoping equation (21), we have

$$
\begin{aligned}
&\sum_{t=2}^{T} \left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \\
\leq& \Delta_1 \text{`} \sum_{t=2}^{T} \lambda_t^2 \rho^{t-1} + 72c\kappa_g^2 L_{f,0}^2 \sum_{t=2}^{T} \sum_{j=0}^{t-2} \frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} + C_\lambda \sum_{t=2}^{T} \sum_{j=0}^{t-2} \rho^j \left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2 \\
&+ C_\lambda \kappa_g^2 \sum_{t=2}^{T} \sum_{j=0}^{t-2} \rho^j \|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\|^2.
\end{aligned} \tag{22}
$$

Note that, for $\rho = e^{-\theta}$ and $\lambda_t = \prod_{i=1}^{t} (1 + \frac{1}{i})^\tau \lambda_1 = \lambda_1 t^\tau$, we have

$$
\sum_{t=1}^{T} \lambda_t^2 \rho^{t-1} = \lambda_1^2 \sum_{t=1}^{T} t^{2\tau} \rho^{t-1} \leq \frac{\lambda_1^2}{\rho} \sum_{t=1}^{\infty} t^{2\tau} \rho^t = \lambda_1^2 e^\theta \sum_{t=1}^{\infty} t^{2\tau} e^{-\theta t} \leq \lambda_1^2 e^\theta \left(\int_0^\infty x^{2\tau} e^{-\theta x} \, \mathrm{d}x + 1 \right).
$$

Using the standard Gamma integral

$$
\int_0^\infty x^{s-1} e^{-\theta x} \, \mathrm{d}x = \frac{\Gamma(s)}{\theta^s}, \qquad s > 0, \ \theta > 0,
$$

we obtain

$$\int_0^\infty x^{2\tau} e^{-\theta x}\,\mathrm{d}x = \frac{\Gamma(2\tau+1)}{\theta^{2\tau+1}}.$$

Therefore,

$$\sum_{t=1}^{T} \lambda_t^2 \rho^{t-1} \le \lambda_1^2 e^{\theta}\left(\frac{\Gamma(2\tau+1)}{\theta^{2\tau+1}}+1\right) =: c_\tau. \tag{23}$$

Also, if we set $m = t-j$, then

$$\sum_{t=2}^{T}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} < \sum_{t=1}^{T}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} = \sum_{t=1}^{T}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{(t-j)^{2\tau}\lambda_1^2} = \frac{1}{\lambda_1^2}\sum_{m=2}^{T}\frac{1}{m^{2\tau}}\sum_{j=0}^{T-m}\rho_{\mathbf{y}}^j \le \frac{4L_{g,1}}{\lambda_1^2\mu_g}\sum_{m=2}^{T}\frac{1}{m^{2\tau}}.$$

Substituting the above inequality and equation (23) into equation (22), and adopting a fixed step-size policy where $\gamma_t \equiv \gamma$, it holds that

$$\begin{aligned}
&\sum_{t=2}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \\
\le& c_\tau\Delta_1' + \frac{288c\kappa_g^3 L_{f,0}^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + \frac{C_\lambda\kappa_g^2}{1-\rho}\sum_{t=2}^{T}\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2 + \frac{C_\lambda}{1-\rho}\sum_{t=2}^{T}\left\|\mathbf{y}_{t-1}^*(\mathbf{x}_t) - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2 \\
\le& c_\tau\Delta_1' + \frac{288c\kappa_g^3 L_{f,0}^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + \frac{2\gamma^2 C_\lambda\kappa_g^2}{1-\rho}\sum_{t=2}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla F_t(\mathbf{x}_t)\right\|^2 + \frac{2\gamma^2 C_\lambda\kappa_g^2}{1-\rho}\mathrm{BLReg}(T) \\
&+ \frac{C_\lambda}{1-\rho}\sum_{t=2}^{T}\sup_{\mathbf{x}\in\mathbb{R}^{d_1}}\left\|\mathbf{y}_{t-1}^*(\mathbf{x}) - \mathbf{y}_t^*(\mathbf{x})\right\|^2 \\
\le& c_\tau\Delta_1' + \frac{288c\kappa_g^3 L_{f,0}^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + \frac{4\gamma^2 C_\lambda\kappa_g^2 D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + \frac{4\gamma^2 C_\lambda\kappa_g^2}{1-\rho}\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \\
&+ \frac{2\gamma^2 C_\lambda\kappa_g^2}{1-\rho}\mathrm{BLReg}(T) + \frac{C_\lambda}{1-\rho}\sum_{t=2}^{T}\sup_{\mathbf{x}\in\mathbb{R}^{d_1}}\left\|\mathbf{y}_{t-1}^*(\mathbf{x}) - \mathbf{y}_t^*(\mathbf{x})\right\|^2,
\end{aligned}$$

where the first inequality comes from the fact that $\sum_{j=0}^{t-2}\rho^j < \sum_{j=0}^{\infty}\rho^j = \frac{1}{1-\rho}$ for $\forall\rho\in(0,1)$, and the last inequality follows Lemma B.2 Eq.(13). Let $\gamma \le \frac{1}{4\kappa_g}\sqrt{\frac{1-\rho}{C_\lambda}}$ such that $1 - \frac{4\gamma^2 C_\lambda\kappa_g^2}{1-\rho} \ge \frac{1}{2}$, thus we complete the proof by

$$\begin{aligned}
\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \le& 2c_\tau\Delta_1 + \frac{576c\kappa_g^3 L_{f,0}^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + \frac{8\gamma^2 C_\lambda\kappa_g^2 D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} \\
&+ \frac{4\gamma^2 C_\lambda\kappa_g^2}{1-\rho}\mathrm{BLReg}(T) + \frac{2C_\lambda}{1-\rho}\sum_{t=2}^{T}\sup_{\mathbf{x}\in\mathbb{R}^{d_1}}\left\|\mathbf{y}_{t-1}^*(\mathbf{x}) - \mathbf{y}_t^*(\mathbf{x})\right\|^2,
\end{aligned}$$

where $\Delta_1 := \Delta_1' + \|\widetilde{\nabla}\mathcal{L}_1^*(\mathbf{x}_1) - \nabla\mathcal{L}_1^*(\mathbf{x}_1)\|^2$. □

Theorem C.5 (Restatement of Theorem 3.1)**.** *Under Assumptions 2.1, 2.2, let* $\Lambda(t,\lambda_t) = \left(1+\frac{1}{t}\right)^\tau\lambda_t$ *for some* $\tau > 0$, $\lambda_1 > \frac{2L_{f,1}}{\mu_g}$, $\alpha \le \frac{1}{L_{g,1}}$, $\beta_t \le \frac{1}{2\lambda_t L_{g,1}}$, $\gamma_t \equiv \gamma \le \min\{\frac{1}{2L_F}, \frac{1}{16\kappa_g}\sqrt{\frac{1-\rho}{C_\lambda}}\}$ *and* $K_t \equiv K \ge \max\{1 - \frac{\ln 2}{\ln\rho}, \frac{\ln c - 2\tau\ln T}{\ln\rho}\}$ *for some* $c > 0$*, Algorithm 1 can guarantee*

$$\begin{aligned}
\mathrm{BLReg}(T) \le& \frac{32M}{\gamma} + \frac{16}{\gamma}V_T + \frac{32D_1^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + 64c_\tau\Delta_1 + \frac{18432c\kappa_g^3 L_{f,0}^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} \\
&+ \frac{256\gamma^2 C_\lambda\kappa_g^2 D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + \frac{64C_\lambda}{1-\rho}H_{2,T}
\end{aligned}$$

$$=O\left(\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+V_T+H_{2,T}\right),$$

where ρ, c, c_τ, C_λ and Δ_1 are some constants.

Proof. We substitute Lemma C.4 into Lemma C.1 to get

$$\begin{aligned}\mathrm{BLReg}(T)\leq&\frac{16M}{\gamma}+\frac{8}{\gamma}V_T+\frac{16D_1^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+32c_\tau\Delta_1+\frac{9216c\kappa_g^3L_{f,0}^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+\frac{32C_\lambda}{1-\rho}H_{2,T}\\&+\frac{64\gamma^2C_\lambda\kappa_g^2}{1-\rho}\mathrm{BLReg}(T)+\frac{128\gamma^2C_\lambda\kappa_g^2D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}.\end{aligned}$$

Let $\gamma\leq\frac{1}{16\kappa_g}\sqrt{\frac{1-\rho}{C_\lambda}}$ such that $1-\frac{64\gamma^2C_\lambda\kappa_g^2}{1-\rho}\geq\frac{1}{2}$, thus finally have

$$\begin{aligned}\mathrm{BLReg}(T)\leq&\frac{32M}{\gamma}+\frac{16}{\gamma}V_T+\frac{32D_1^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+64c_\tau\Delta_1+\frac{18432c\kappa_g^3L_{f,0}^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}\\&+\frac{256\gamma^2C_\lambda\kappa_g^2D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+\frac{64C_\lambda}{1-\rho}H_{2,T}\\=&O\left(\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+V_T+H_{2,T}\right).\end{aligned}$$

Our final bound depends on the choice of τ:

$$\begin{cases}\mathrm{BLReg}(T)\leq O\left(1+V_T+H_{2,T}\right) & \text{if set } \tau\in(\frac{1}{2},\infty)\\ \mathrm{BLReg}(T)\leq O\left(\log T+V_T+H_{2,T}\right) & \text{if set } \tau=\frac{1}{2}\\ \mathrm{BLReg}(T)\leq O\left(T^{1-\tau}+V_T+H_{2,T}\right) & \text{if set } \tau\in(0,\frac{1}{2})\end{cases}.$$

□

C.2 Proof of Theorem 3.2

In this section, we prove the regret upper bound of F^{2}OBO under $K_t\equiv 1$. We present single-loop counterparts of Lemmas C.2 and C.3 in Lemmas C.6 and C.7, respectively. Then, in Lemma C.8, we establish the final upper bound on the hypergradient approximation error, which completes the proof of Theorem C.9.

Lemma C.6. *Under Assumptions 2.1, 2.2, let $\beta_t=\frac{1}{2\lambda_tL_{g,1}}$ and $K_t\equiv 1$, Algorithm 1 can obtain*

$$\begin{aligned}&\left\|\mathbf{y}_{t+1}-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\\\leq&\rho_{\mathbf{y}}^{t-1}\left\|\mathbf{y}_2-\mathbf{y}^*_{\lambda_1,1}(\mathbf{x}_1)\right\|^2+288\kappa_g^3\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2+\frac{1536L_{f,0}^2\kappa_g^3}{\mu_g^2}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j(\lambda_{t-j}-\lambda_{t-1-j})^2}{\lambda_{t-j}^2\lambda_{t-1-j}^2}\\&+\frac{384\kappa_g}{\mu_g^2}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_{t-j}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1-j}(\mathbf{x},\mathbf{y})\right\|^2\\&+\frac{384\kappa_g}{\mu_g^2}\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_{t-j}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1-j}(\mathbf{x},\mathbf{y})\right\|^2,\end{aligned}$$

where $\rho_{\mathbf{y}}=1-\frac{1}{8\kappa_g}$.

Proof. We begin with equation (17) in Lemma C.2, it follows that

$$\left\|\mathbf{y}_{t+1}-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\leq\left(1-\frac{\beta_t\lambda_t\mu_g}{2}\right)\left\|\mathbf{y}_t-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2$$

$$
\begin{aligned}
&\leq \left(1-\frac{\beta_t\lambda_t\mu_g}{2}\right)(1+\theta)\left\|\mathbf{y}_t-\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})\right\|^2 \\
&\quad+\left(1-\frac{\beta_t\lambda_t\mu_g}{2}\right)\left(1+\frac{1}{\theta}\right)\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 \\
&\overset{(i)}{\leq}\left(1-\frac{1}{8\kappa_g}\right)\left\|\mathbf{y}_t-\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})\right\|^2 \\
&\quad+16\kappa_g\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2, \qquad (24)
\end{aligned}
$$

where in (i), we set $\theta=\frac{\beta_t\lambda_t\mu_g}{4}$ such that $(1-\frac{\beta_t\lambda_t\mu_g}{2})(1+\frac{\beta_t\lambda_t\mu_g}{4})=1-\frac{\beta_t\lambda_t\mu_g}{4}-\frac{\beta_t^2\lambda_t^2\mu_g^2}{8}\leq 1-\frac{\beta_t\lambda_t\mu_g}{4}$, and $\beta_t=\frac{1}{2\lambda_t L_{g,1}}$. Then it begins with

$$
\begin{aligned}
\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 \leq& 2\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})-\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_t)\right\|^2 \\
&+2\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_t)-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 \\
\overset{(i)}{\leq}&18\kappa_g^2\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2+2\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_t)-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2, \qquad (25)
\end{aligned}
$$

where (i) comes from Lemma 3.2 [15]. Remind that $\mathcal{L}_t(\mathbf{x},\mathbf{y},\lambda)$ is $(\lambda_t\mu_g/2)$-strongly convex in $\mathbf{y}$, the following formula can be obtained immediately

$$
\left\|\mathbf{y}^*_{\lambda_t,t}(\mathbf{x})-\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x})\right\|\leq\frac{2}{\lambda_t\mu_g}\left\|\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}),\lambda_t)\right\| \qquad (26)
$$

We also noticed that

$$
\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}),\lambda_t)=\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))+\lambda_t\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x})) \qquad (27)
$$

$$
\nabla_{\mathbf{y}}\mathcal{L}_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}),\lambda_{t-1})=\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))+\lambda_{t-1}\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))=0, \qquad (28)
$$

then substitute Eq.(28) into Eq.(27), we obtain

$$
\begin{aligned}
\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}),\lambda_t)=&\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x})) \\
&+\lambda_t\left(\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))\right) \\
&+(\lambda_t-\lambda_{t-1})\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x})).
\end{aligned}
$$

With

$$
\begin{aligned}
\left\|\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))\right\|=&\left\|\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y}^*_{t-1}(\mathbf{x}))\right\| \\
\leq&L_{g,1}\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x})-\mathbf{y}^*_{t-1}(\mathbf{x})\right\|\leq\frac{2L_{f,0}L_{g,1}}{\lambda_{t-1}\mu_g},
\end{aligned}
$$

it naturally holds that

$$
\begin{aligned}
\left\|\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}),\lambda_t)\right\|\leq&\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))\right\| \\
&+\lambda_t\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))\right\| \\
&+\frac{2L_{f,0}L_{g,1}}{\mu_g}\frac{\lambda_t-\lambda_{t-1}}{\lambda_{t-1}}. \qquad (29)
\end{aligned}
$$

Finally we substitute Eq.(29) into Eq.(26) to get

$$
\begin{aligned}
\left\|\mathbf{y}^*_{\lambda_t,t}(\mathbf{x})-\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x})\right\|\leq&\frac{2}{\lambda_t\mu_g}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y})\right\| \\
&+\frac{2}{\mu_g}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|
\end{aligned}
$$

$$+\frac{4L_{f,0}L_{g,1}}{\mu_g^2}\frac{\lambda_t-\lambda_{t-1}}{\lambda_t\lambda_{t-1}}$$

and natually have

$$\begin{aligned}\left\|\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}))-\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}))\right\|^2 \leq&\frac{12}{\lambda_t^2\mu_g^2}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y})\right\|^2\\&+\frac{12}{\mu_g^2}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|^2\\&+\frac{48L_{f,0}^2L_{g,1}^2}{\mu_g^4}\frac{(\lambda_t-\lambda_{t-1})^2}{\lambda_t^2\lambda_{t-1}^2}.\end{aligned}\tag{30}$$

Substitute Eq.(30) into Eq.(25) to obtain

$$\begin{aligned}&\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\\\leq&18\kappa_g^2\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2+\frac{24}{\lambda_t^2\mu_g^2}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y})\right\|^2\\&+\frac{24}{\mu_g^2}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|^2+\frac{96L_{f,0}^2L_{g,1}^2}{\mu_g^4}\frac{(\lambda_t-\lambda_{t-1})^2}{\lambda_t^2\lambda_{t-1}^2}.\end{aligned}\tag{31}$$

Finally, substitute Eq.(31) into Eq.(24), we finish the proof by

$$\begin{aligned}&\left\|\mathbf{y}_{t+1}-\mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\\\leq&\rho_{\mathbf{y}}\left\|\mathbf{y}_t-\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})\right\|^2+288\kappa_g^3\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2+\frac{1536L_{f,0}^2\kappa_g^3}{\mu_g^2}\frac{(\lambda_t-\lambda_{t-1})^2}{\lambda_t^2\lambda_{t-1}^2}\\&+\frac{384\kappa_g}{\lambda_t^2\mu_g^2}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y})\right\|^2+\frac{384\kappa_g}{\mu_g^2}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|^2\\\leq&\rho_{\mathbf{y}}^{t-1}\left\|\mathbf{y}_2-\mathbf{y}^*_{\lambda_1,1}(\mathbf{x}_1)\right\|^2+288\kappa_g^3\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2+\frac{1536L_{f,0}^2\kappa_g^3}{\mu_g^2}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j(\lambda_{t-j}-\lambda_{t-1-j})^2}{\lambda_{t-j}^2\lambda_{t-1-j}^2}\\&+\frac{384\kappa_g}{\mu_g^2}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_{t-j}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1-j}(\mathbf{x},\mathbf{y})\right\|^2\\&+\frac{384\kappa_g}{\mu_g^2}\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_{t-j}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1-j}(\mathbf{x},\mathbf{y})\right\|^2,\end{aligned}$$

where $\rho_{\mathbf{y}}:=1-\frac{1}{8\kappa_g}$. □

Lemma C.7. *Under Assumptions 2.1, 2.2, let $\alpha\leq\frac{1}{L_{g,1}}$ and $K_t\equiv 1$, Algorithm 1 can obtain*

$$\begin{aligned}\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\leq&\rho_{\mathbf{z}}^{t-1}\left\|\mathbf{z}_2-\mathbf{y}_1^*(\mathbf{x}_1)\right\|^2+\frac{8\kappa_g^2}{\alpha\mu_g}\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2\\&+\frac{8}{\alpha\mu_g}\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\sup_{\mathbf{x}}\left\|\mathbf{y}^*_{t-1-j}(\mathbf{x})-\mathbf{y}^*_{t-j}(\mathbf{x})\right\|^2,\end{aligned}$$

where $\rho_{\mathbf{z}}=1-\frac{\alpha\mu_g}{2}$.

Proof. It follows that

$$\begin{aligned}&\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\leq(1-\alpha\mu_g)\left\|\mathbf{z}_t-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\\\leq&(1-\alpha\mu_g)(1+\theta)\left\|\mathbf{z}_t-\mathbf{y}^*_{t-1}(\mathbf{x}_{t-1})\right\|^2+(1-\alpha\mu_g)\left(1+\frac{1}{\theta}\right)\left\|\mathbf{y}^*_{t-1}(\mathbf{x}_{t-1})-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\frac{\alpha\mu_g}{2}\right)\left\|\mathbf{z}_t-\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1})\right\|^2+\frac{8\kappa_g^2}{\alpha\mu_g}\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2+\frac{8}{\alpha\mu_g}\sup_{\mathbf{x}}\left\|\mathbf{y}_{t-1}^*(\mathbf{x})-\mathbf{y}_t^*(\mathbf{x})\right\|^2\\
&\leq \rho_{\mathbf{z}}^{t-1}\left\|\mathbf{z}_2-\mathbf{y}_1^*(\mathbf{x}_1)\right\|^2+\frac{8\kappa_g^2}{\alpha\mu_g}\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2\\
&\quad+\frac{8}{\alpha\mu_g}\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\sup_{\mathbf{x}}\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x})-\mathbf{y}_{t-j}^*(\mathbf{x})\right\|^2,
\end{aligned}
$$

where in the second inequality, we set $\theta=\frac{\alpha\mu_g}{2}$ such that $(1-\alpha\mu_g)(1+\frac{\alpha\mu_g}{2})\leq 1-\frac{\alpha\mu_g}{2}$, and $\rho_{\mathbf{z}}:=1-\frac{\alpha\mu_g}{2}$. □

Lemma C.8. *Under Assumptions 2.1 and 2.2, let $\Lambda(t,\lambda_t)=\left(1+\frac{1}{t}\right)^{\tau}\lambda_t$ for some $\tau>0$, $\lambda_1>\frac{2L_{f,1}}{\mu_g}$, $\alpha\leq\frac{1}{L_{g,1}}$, $\beta_t=\frac{1}{2\lambda_t L_{g,1}}$, $\gamma_t\leq\frac{1}{\lambda_1 t^{\tau}}\sqrt{\frac{1-\rho}{6C_g}}$ and $K_t\equiv 1$, Algorithm 1 can obtain*

$$
\begin{aligned}
\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\leq&2c_\tau\Delta_1+2c_\lambda C_S+\frac{6c_\gamma C_g}{1-\rho}\mathrm{BLReg}(T)+\frac{6c_\gamma C_g D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}\\
&+\frac{2c_g'\lambda_T^2}{1-\rho_{\mathbf{z}}}H_{2,T}+\frac{2c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E_{\mathbf{y},T}^f+\frac{2c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E_{\mathbf{y},T}^g,
\end{aligned}
$$

where $\rho=\max\{\rho_{\mathbf{y}},\rho_{\mathbf{z}}\}$, $\rho_{\mathbf{y}}=1-\frac{1}{8\kappa_g}$, $\rho_{\mathbf{z}}=1-\frac{\alpha\mu_g}{2}$, Δ_1, c_τ, c_λ C_S, c_g, c_g', C_g are some constants.

Proof. We begin with equation (20) in Lemma C.4, which gives

$$
\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\leq 6\lambda_t^2L_{g,1}^2\left\|\mathbf{y}_{t+1}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2+3\lambda_t^2L_{g,1}^2\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2.
$$

Substituting the bounds from Lemma C.6 and Lemma C.7 into the above inequality yields

$$
\begin{aligned}
\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\leq&6\lambda_t^2\rho_{\mathbf{y}}^{t-1}L_{g,1}^2\left\|\mathbf{y}_2-\mathbf{y}_{\lambda,1}^*(\mathbf{x}_1)\right\|^2+3\lambda_t^2\rho_{\mathbf{z}}^{t-1}L_{g,1}^2\left\|\mathbf{z}_2-\mathbf{y}_1^*(\mathbf{x}_1)\right\|^2\\
&+c_\lambda\lambda_t^2\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j(\lambda_{t-j}-\lambda_{t-1-j})^2}{\lambda_{t-j}^2\lambda_{t-1-j}^2}+C_g\lambda_t^2\sum_{j=0}^{t-2}\rho^j\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2\\
&+c_g\lambda_t^2\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_{t-j}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1-j}(\mathbf{x},\mathbf{y})\right\|^2\\
&+c_g\lambda_t^2\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_{t-j}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1-j}(\mathbf{x},\mathbf{y})\right\|^2\\
&+c_g'\lambda_t^2\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\sup_{\mathbf{x}}\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x})-\mathbf{y}_{t-j}^*(\mathbf{x})\right\|^2,
\end{aligned}
$$

where, for notational simplicity, we define

$$
\begin{aligned}
c_\lambda&:=6L_{g,1}^2\cdot\frac{1536L_{f,0}^2\kappa_g^3}{\mu_g^2},\\
C_g&:=6L_{g,1}^2\cdot 288\kappa_g^3+3L_{g,1}^2\cdot\frac{8\kappa_g^2}{\alpha\mu_g},\\
c_g&:=6L_{g,1}^2\cdot\frac{384\kappa_g}{\mu_g},\\
c_g'&:=3L_{g,1}^2\cdot\frac{8}{\alpha\mu_g},\\
\rho&:=\max\{\rho_{\mathbf{y}},\rho_{\mathbf{z}}\},
\end{aligned}
$$

and have

$$
\begin{aligned}
&\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\\
\leq& c_\tau\Delta_1+c_\lambda\sum_{t=1}^{T}\lambda_t^2\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j(\lambda_{t-j}-\lambda_{t-1-j})^2}{\lambda_{t-j}^2\lambda_{t-1-j}^2}+C_g\sum_{t=1}^{T}\lambda_t^2\sum_{j=0}^{t-2}\rho^j\left\|\mathbf{x}_{t-1-j}-\mathbf{x}_{t-j}\right\|^2\\
&+c_g\sum_{t=1}^{T}\lambda_t^2\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_{t-j}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1-j}(\mathbf{x},\mathbf{y})\right\|^2\\
&+c_g\sum_{t=1}^{T}\lambda_t^2\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_{t-j}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1-j}(\mathbf{x},\mathbf{y})\right\|^2\\
&+c_g'\sum_{t=1}^{T}\lambda_t^2\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\sup_{\mathbf{x}}\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x})-\mathbf{y}_{t-j}^*(\mathbf{x})\right\|^2,
\end{aligned}
\tag{32}
$$

where $\Delta_1:=6L_{g,1}^2\|\mathbf{y}_2-\mathbf{y}_{\lambda_1,1}^*(\mathbf{x}_1)\|^2+3L_{g,1}^2\|\mathbf{z}_2-\mathbf{y}_1^*(\mathbf{x}_1)\|^2+\|\widetilde{\nabla}\mathcal{L}_1^*(\mathbf{x}_1)-\nabla\mathcal{L}_1^*(\mathbf{x}_1)\|^2$. For the second term in Eq.(32), we define $S(T)$ and let $m=t-j$, it follows that

$$
S(T):=\sum_{t=2}^{T}\lambda_t^2\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\frac{(\lambda_{t-j}-\lambda_{t-1-j})^2}{\lambda_{t-j}^2\lambda_{t-1-j}^2}=\sum_{t=2}^{T}\lambda_t^2\sum_{m=2}^{t}\rho_{\mathbf{y}}^{t-m}\frac{(\lambda_m-\lambda_{m-1})^2}{\lambda_m^2\lambda_{m-1}^2},
$$

with $\lambda_m-\lambda_{m-1}=\lambda_1(m^\tau-(m-1)^\tau)=\lambda_1\tau\xi^{\tau-1}\leq\lambda_1\tau(m-1)^{\tau-1}$ for some $\xi\in[m-1,m]$, it holds that

$$
\frac{(\lambda_m-\lambda_{m-1})^2}{\lambda_m^2\lambda_{m-1}^2}\leq\frac{\lambda_1^2\tau^2(m-1)^{2\tau-2}}{\lambda_1^4m^{2\tau}(m-1)^{2\tau}}\leq\frac{\tau^2}{\lambda_1^2}\frac{1}{(m-1)^2m^{2\tau}}\approx\frac{\tau^2}{\lambda_1^2}\frac{1}{m^{2\tau+2}}.
$$

Thus we have

$$
S(T)=\frac{\tau^2}{\lambda_1^2}\sum_{t=2}^{T}\lambda_t^2\sum_{m=2}^{t}\frac{\rho_{\mathbf{y}}^{t-m}}{m^{2\tau+2}}=\tau^2\sum_{t=2}^{T}t^{2\tau}\sum_{m=2}^{t}\frac{\rho_{\mathbf{y}}^{t-m}}{m^{2\tau+2}}=\tau^2\sum_{t=2}^{T}t^{2\tau}\sum_{j=0}^{t-2}\frac{\rho^j}{(t-j)^{2\tau+2}},
$$

for $0\leq j\leq\lfloor\frac{t}{2}\rfloor$,

$$
\sum_{j=0}^{\lfloor\frac{t}{2}\rfloor}\frac{\rho_{\mathbf{y}}^j}{(t-j)^{2\tau+2}}\leq\left(\frac{2}{t}\right)^{2\tau+2}\sum_{j=0}^{\infty}\rho_{\mathbf{y}}^j\leq\left(\frac{2}{t}\right)^{2\tau+2}\frac{1}{1-\rho_{\mathbf{y}}},
$$

for $\lfloor\frac{t}{2}\rfloor+1\leq j\leq t-2$,

$$
\sum_{j=\lfloor\frac{t}{2}\rfloor+1}^{t-2}\frac{\rho_{\mathbf{y}}^j}{(t-j)^{2\tau+2}}\leq\frac{1}{2^{2\tau+2}}\sum_{j=\lfloor\frac{t}{2}\rfloor+1}^{t-2}\rho_{\mathbf{y}}^j\leq\frac{1}{2^{2\tau+2}}\cdot\frac{\rho_{\mathbf{y}}^{\lfloor\frac{t}{2}\rfloor+1}}{1-\rho_{\mathbf{y}}}.
$$

Finally, we have

$$
S(T)\leq\tau^2\sum_{t=2}^{T}\left(\frac{2^{2\tau+2}}{1-\rho_{\mathbf{y}}}\cdot\frac{1}{t^2}+\frac{\rho_{\mathbf{y}}^{\lfloor\frac{t}{2}\rfloor+1}t^{2\tau}}{2^{2\tau+2}(1-\rho_{\mathbf{y}})}\right)=:C_S,
$$

for any $\tau>0$, $C_S=O(1)$. Thus it holds that

$$
\begin{aligned}
&\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\\
\leq& c_\tau\Delta_1+c_\lambda C_S+\frac{C_g}{1-\rho}\sum_{t=1}^{T}\lambda_t^2\left\|\mathbf{x}_{t-1}-\mathbf{x}_t\right\|^2+\frac{c_g'\lambda_T^2}{1-\rho_{\mathbf{z}}}\sum_{t=1}^{T}\sup_{\mathbf{x}\in\mathcal{X}}\left\|\mathbf{y}_{t-1}^*(\mathbf{x})-\mathbf{y}_t^*(\mathbf{x})\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}\sum_{t=1}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y})\right\|^2\\
&+\frac{c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}\sum_{t=1}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|^2.\\
\leq& c_\tau\Delta_1+c_\lambda C_S+\frac{C_g}{1-\rho}\sum_{t=1}^{T}\lambda_t^2\gamma_t^2\left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t,\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t),\gamma_t\right)\right\|^2\\
&+\frac{c_g'\lambda_T^2}{1-\rho_{\mathbf{z}}}\sum_{t=1}^{T}\sup_{\mathbf{x}\in\mathcal{X}}\left\|\mathbf{y}_{t-1}^*(\mathbf{x})-\mathbf{y}_t^*(\mathbf{x})\right\|^2\\
&+\frac{c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}\sum_{t=1}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y})\right\|^2\\
&+\frac{c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}\sum_{t=1}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|^2\\
\leq& c_\tau\Delta_1+c_\lambda C_S+\frac{C_g}{1-\rho}\sum_{t=1}^{T}\lambda_t^2\gamma_t^2\Big(3\left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t,\nabla F_t(\mathbf{x}_t),\gamma_t\right)\right\|^2+3\left\|\nabla F_t(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\\
&+\left\|\nabla\mathcal{L}_t^*(\mathbf{x}_t)-\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\Big)+\frac{c_g'\lambda_T^2}{1-\rho_{\mathbf{z}}}\sum_{t=1}^{T}\sup_{\mathbf{x}\in\mathcal{X}}\left\|\mathbf{y}_{t-1}^*(\mathbf{x})-\mathbf{y}_t^*(\mathbf{x})\right\|^2\\
&+\frac{c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}\sum_{t=1}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y})\right\|^2\\
&+\frac{c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}\sum_{t=1}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|^2.
\end{aligned}
$$

Now suppose that for sufficiently small γ_t such that $\lambda_t^2\gamma_t^2\leq c_\gamma$ for some constant $c_\gamma>0$, we obtain

$$
\begin{aligned}
&\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\\
\leq& c_\tau\Delta_1+c_\lambda C_S+\frac{3c_\gamma C_g}{1-\rho}\mathrm{BLReg}(T)+\frac{3c_\gamma C_g D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+\frac{3c_\gamma C_g}{1-\rho}\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\\
&+\frac{c_g'\lambda_T^2}{1-\rho_{\mathbf{z}}}H_{2,T}+\frac{c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E_{\mathbf{y},T}^f+\frac{c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E_{\mathbf{y},T}^g.
\end{aligned}
$$

Let $1-\frac{3c_\gamma C_g}{1-\rho}\geq\frac{1}{2}$, which means

$$
\begin{aligned}
c_\gamma\leq&\frac{1-\rho}{6C_g}\\
\gamma\leq&\sqrt{\frac{1-\rho}{6\lambda_t^2C_g}}=\frac{1}{\lambda_1 t^\tau}\sqrt{\frac{1-\rho}{6C_g}},
\end{aligned}
$$

finally we finish the proof that

$$
\begin{aligned}
\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\leq&2c_\tau\Delta_1+2c_\lambda C_S+\frac{6c_\gamma C_g}{1-\rho}\mathrm{BLReg}(T)+\frac{6c_\gamma C_g D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}\\
&+\frac{2c_g'\lambda_T^2}{1-\rho_{\mathbf{z}}}H_{2,T}+\frac{2c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E_{\mathbf{y},T}^f+\frac{2c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E_{\mathbf{y},T}^g.
\end{aligned}
$$

□

Theorem C.9 (Restatement of Theorem 3.2)**.** *Under Assumptions 2.1, 2.2, let* $\Lambda(t,\lambda_t)=\left(1+\frac{1}{t}\right)^{\tau}\lambda_t$ *for some* $\tau\in(0,\frac{1}{2})$, $\lambda_1>\frac{2L_{f,1}}{\mu_g}$, $\alpha\le\frac{1}{L_{g,1}}$, $\beta_t=\frac{1}{2\lambda_t L_{g,1}}$, $\gamma_t<\min\{\frac{1}{2L_F},\frac{1}{8\lambda_1 t^{\tau}}\sqrt{\frac{1-\rho}{3C_g}}\}$ *and* $K_t\equiv 1$*, Algorithm 1 can guarantee*

$$\mathrm{BLReg}(T)\le O\left(T^{\tau}(1+V_T)+T^{1-2\tau}+T^{2\tau}(H_{2,T}+E^f_{\mathbf{y},T}+E^g_{\mathbf{y},T})\right),$$

where ρ*,* C_g *are some constants and*

$$E^f_{\mathbf{y},T}=\sum_{t=2}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}f_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}f_{t-1}(\mathbf{x},\mathbf{y})\right\|^2,$$
$$E^g_{\mathbf{y},T}=\sum_{t=2}^{T}\sup_{\mathbf{x},\mathbf{y}}\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{y}}g_{t-1}(\mathbf{x},\mathbf{y})\right\|^2.$$

Proof. After substituting Lemma C.8 into Lemma C.1, we obtain

$$\begin{aligned}\mathrm{BLReg}(T)\le&\frac{16M}{\gamma_T}+\frac{8}{\gamma_T}V_T+16D_1^2\sum_{t=1}^{T}\frac{1}{\lambda_t^2}+32c_\tau\Delta_1+32c_\lambda C_S+\frac{96c_\gamma C_g}{1-\rho}\mathrm{BLReg}(T)\\&+\frac{96c_\gamma C_g D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+\frac{32c'_g\lambda_T^2}{1-\rho_{\mathbf{z}}}H_{2,T}+\frac{32c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E^f_{\mathbf{y},T}+\frac{32c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E^g_{\mathbf{y},T}.\end{aligned}$$

Let $1-\frac{96c_\gamma C_g}{1-\rho}\ge\frac{1}{2}$, $c_\gamma\le\frac{1-\rho}{192C_g}$, which means

$$\begin{aligned}\lambda_t^2\gamma_t^2\le&\frac{1-\rho}{192C_g}\\ \gamma_t\le&\frac{1}{8\lambda_1 t^{\tau}}\sqrt{\frac{1-\rho}{3C_g}}.\end{aligned}$$

Thus we finally have

$$\begin{aligned}\mathrm{BLReg}(T)\le&\frac{16M}{\gamma_T}+\frac{8}{\gamma_T}V_T+\frac{16D_1^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}+32c_\tau\Delta_1+32c_\lambda C_S+\frac{96c_\gamma C_g D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}\frac{1}{t^{2\tau}}\\&+\frac{32c'_g\lambda_T^2}{1-\rho_{\mathbf{z}}}H_{2,T}+\frac{32c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E^f_{\mathbf{y},T}+\frac{32c_g\lambda_T^2}{1-\rho_{\mathbf{y}}}E^g_{\mathbf{y},T}\\=&O\left(T^{\tau}(1+V_T)+T^{1-2\tau}+T^{2\tau}(H_{2,T}+E^f_{\mathbf{y},T}+E^g_{\mathbf{y},T})\right).\end{aligned}$$

□

D Proof of Section 4

In this section, we present a regret upper bound proof for AF^{2}OBO. AF^{2}OBO improves the upper bound of the hypergradient error for the approximation problem (P3) by using a dynamic inner iteration technique, which eliminates the dependence on $H_{2,T}$, as shown in Lemma D.1 below. Finally, following the same proof process as F^{2}OBO, we give the proof in Theorem D.2.

Lemma D.1. *Under Assumptions 2.1 and 2.2, let* $\Lambda(t,\lambda_t)=\left(1+\frac{1}{t}\right)^{\tau}\lambda_t$ *for some* $\tau>0$, $\lambda_1>\frac{2L_{f,1}}{\mu_g}$, $\delta_{\mathbf{y}}=\frac{1}{\sqrt{T}}$ *and* $\delta_{\mathbf{z}}=\frac{1}{\sqrt{T^{1+2\tau}}}$*, the approximation error of hypergradient in Algorithm 2 can be bounded as*

$$\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\le 24\kappa_g^2+3\lambda_1^2\kappa_g^2.$$

Proof. We first bound the hypergradient approximation error as follows:

$$\begin{aligned}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 &\leq 6\lambda_t^2 L_{g,1}^2 \left\|\mathbf{y}_{t+1} - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2 + 3\lambda_t^2 L_{g,1}^2 \left\|\mathbf{z}_{t+1} - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2 \\ &\leq 6\lambda_t^2 L_{g,1}^2 \frac{4\delta_{\mathbf{y}}^2}{\lambda_t^2\mu_g^2} + 3\lambda_t^2 L_{g,1}^2 \frac{\delta_{\mathbf{z}}^2}{\mu_g^2} \\ &= 24\kappa_g^2\delta_{\mathbf{y}}^2 + 3\lambda_t^2\kappa_g^2\delta_{\mathbf{z}}^2. \end{aligned} \tag{33}$$

The second inequality follows from the error criteria

$$\frac{\lambda_t\mu_g}{2}\left\|\mathbf{y}_{t+1} - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\| \leq \left\|\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t, \mathbf{y}_{t+1}, \lambda_t)\right\| \leq \delta_{\mathbf{y}},$$

and

$$\mu_g\left\|\mathbf{z}_{t+1} - \mathbf{y}_t^*(\mathbf{x}_t)\right\| \leq \left\|\nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{z}_{t+1})\right\| \leq \delta_{\mathbf{z}},$$

which are implied by the μ_g-strong convexity of g_t and the $(\lambda_t\mu_g/2)$-strong convexity of $\mathcal{L}_t$, respectively. Therefore, summing over $t = 1, \ldots, T$ gives

$$\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \leq 24\kappa_g^2\delta_{\mathbf{y}}^2 T + 3\lambda_T^2\kappa_g^2\delta_{\mathbf{z}}^2 T.$$

By choosing $\delta_{\mathbf{y}} = \frac{1}{\sqrt{T}}$ and $\delta_{\mathbf{z}} = \frac{1}{\sqrt{T^{1+2\tau}}}$, we obtain

$$\sum_{t=1}^{T}\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \leq 24\kappa_g^2 + 3\lambda_1^2\kappa_g^2.$$

□

Theorem D.2 (Restatement of Theorem 4.1)**.** *Under Assumptions 2.1, 2.2, let* $\Lambda(t, \lambda_t) = \left(1 + \frac{1}{t}\right)^{\tau}\lambda_t$ *for some* $\tau > 0$*,* $\lambda_1 > \frac{2L_{f,1}}{\mu_g}$*,* $\alpha \leq \frac{1}{L_{g,1}}$*,* $\beta_t \leq \frac{1}{2\lambda_t L_{g,1}}$*,* $\gamma \leq \frac{1}{2L_F}$ *for all* $t \in [T]$*,* $\delta_{\mathbf{y}} = \frac{1}{\sqrt{T}}$ *and* $\delta_{\mathbf{z}} = \frac{1}{\sqrt{T^{1+2\tau}}}$*, Algorithm 2 can guarantee*

$$\mathrm{BLReg}(T) \leq \frac{16M}{\gamma} + \frac{8}{\gamma}V_T + \frac{16D_1^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + 384\kappa_g^2 + 48\lambda_1^2\kappa_g^2 = O\left(\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + V_T\right).$$

The total number of inner iterations $\mathcal{I}_T$ *satisfies*

$$\mathcal{I}_T \leq O\left(T^{1+2\tau} + T^{2\tau}H_{2,T}\right).$$

Proof. We begins with Lemma C.1 and substitute Lemma D.1 into it to obtain

$$\mathrm{BLReg}(T) \leq \frac{16M}{\gamma} + \frac{8}{\gamma}V_T + \frac{16D_1^2}{\lambda_1^2}\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + 384\kappa_g^2 + 48\lambda_1^2\kappa_g^2 = O\left(\sum_{t=1}^{T}\frac{1}{t^{2\tau}} + V_T\right),$$

where $\gamma_t \equiv \gamma \leq \frac{1}{2L_F}$, $\delta_{\mathbf{y}} = \frac{1}{\sqrt{T}}$ and $\delta_{\mathbf{z}} = \frac{1}{\sqrt{T^{1+2\tau}}}$. Now consider the two inner-loop iteration process. The sequences $\{\mathbf{y}_t^k\}_{k=1}^{K_t}$ and $\{\mathbf{z}_t^m\}_{m=1}^{M_t}$ are generated by

$$\begin{aligned}\mathbf{y}_t^{k+1} &\leftarrow \mathbf{y}_t^k - \beta_t\left(\nabla_{\mathbf{y}} f_t(\mathbf{x}_t, \mathbf{y}_t^k) + \lambda_t\nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{y}_t^k)\right), \quad \text{and} \\ \mathbf{z}_t^{m+1} &\leftarrow \mathbf{z}_t^m - \alpha\nabla_{\mathbf{y}} g_t(\mathbf{x}_t, \mathbf{z}_t^m),\end{aligned}$$

respectively. For $\{\mathbf{y}_t^k\}_{k=1}^{K_t}$, it holds that

$$\frac{\delta_{\mathbf{y}}^2}{4\lambda_t^2 L_{g,1}^2} \leq \left\|\mathbf{y}_t^{K_t} - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2 \leq \rho_{\mathbf{y}}^{K_t - 1}\left\|\mathbf{y}_t^1 - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2, \tag{34}$$

where $\rho_{\mathbf{y}} \in (0, 1)$. The second inequality follows from equation (17) in Lemma C.2 under the stepsize condition $\beta_t \leq 1/(2\lambda_t L_{g,1})$ for all $t \in [T]$. The first inequality follows from the fact that $\mathcal{L}_t(\mathbf{x}_t, \cdot, \lambda_t)$ is $(2\lambda_t L_{g,1})$-smooth with respect to $\mathbf{y}$. Hence,

$$2\lambda_t L_{g,1}\left\|\mathbf{y}_t^{K_t} - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\| \geq \left\|\nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t, \mathbf{y}_t^{K_t}, \lambda_t)\right\| > \delta_{\mathbf{y}}.$$

After rearranging Eq.(34), we have

$$\begin{aligned}\rho_{\mathbf{y}}^{K_t-1} \geq & \frac{\delta_{\mathbf{y}}^2}{4\lambda_t^2 L_{g,1}^2 \left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2} \\ K_t \leq & 1 + \frac{4\lambda_t^2 L_{g,1}^2}{\ln \rho_{\mathbf{y}}^{-1}} \left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 + \frac{2\ln \delta_{\mathbf{y}}}{\ln \rho_{\mathbf{y}}}.\end{aligned}$$

Let $\mathcal{I}_T^{\mathbf{y}}$ denotes the total iteration number of $\mathbf{y}$ and $\mathcal{I}_T^{\mathbf{z}}$ denote the total iteration number of $\mathbf{z}$, it immediately follows that

$$\mathcal{I}_T^{\mathbf{y}} := \sum_{t=1}^{T} K_t \leq \left(1 + \frac{2\ln \delta_{\mathbf{y}}}{\ln \rho_{\mathbf{y}}}\right) T + \frac{4L_{g,1}^2}{\ln \rho_{\mathbf{y}}^{-1}} \sum_{t=1}^{T} \lambda_t^2 \left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2. \tag{35}$$

Then it begins with

$$\begin{aligned}& \sum_{t=1}^{T} \lambda_t^2 \left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 \\ \leq & 2\sum_{t=1}^{T} \lambda_t^2 \left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})\right\|^2 + 2\sum_{t=1}^{T} \lambda_t^2 \left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1}) - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 \\ \leq & 2\sum_{t=1}^{T} \lambda_t^2 \frac{4\delta_{\mathbf{y}}^2}{\lambda_t^2 \mu_g^2} + 2\sum_{t=1}^{T} \lambda_t^2 \left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1}) - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2.\end{aligned}$$

Remind of equation (18), we have

$$\begin{aligned}\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1}) - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 \leq & \frac{6L_{f,0}^2}{\lambda_t^2 \mu_g^2} + 6\kappa_g^2 \left\|\mathbf{x}_{t-1} - \mathbf{x}_t\right\|^2 + 6\left\|\mathbf{y}^*_{t-1}(\mathbf{x}_t) - \mathbf{y}^*_t(\mathbf{x}_t)\right\|^2 \\ \leq & \frac{6L_{f,0}^2}{\lambda_t^2 \mu_g^2} + 6\gamma^2 \kappa_g^2 L_{\mathcal{L},0}^2 + 6\left\|\mathbf{y}^*_{t-1}(\mathbf{x}_t) - \mathbf{y}^*_t(\mathbf{x}_t)\right\|^2,\end{aligned}$$

where the last inequality comes from Lemma B.6. Thus we obtain

$$\sum_{t=1}^{T} \lambda_t^2 \left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 \leq \frac{8\delta_{\mathbf{y}}^2}{\mu_g^2} T + \frac{12L_{f,0}^2}{\mu_g^2} T + 12\gamma^2 \lambda_1^2 \kappa_g^2 L_{\mathcal{L},0}^2 T^{1+2\tau} + 6\lambda_1^2 T^{2\tau} H_{2,T}.$$

After substituting it into Eq.(35), we finally obtain

$$\begin{aligned}\mathcal{I}_T^{\mathbf{z}} \leq & \left(1 + \frac{2\ln \delta_{\mathbf{y}}}{\ln \rho_{\mathbf{y}}}\right) T + \frac{4L_{g,1}^2}{\ln \rho_{\mathbf{y}}^{-1}} \left(\frac{8\delta_{\mathbf{y}}^2}{\mu_g^2} T + \frac{12L_{f,0}^2}{\mu_g^2} T + 12\gamma^2 \lambda_1^2 \kappa_g^2 L_{\mathcal{L},0}^2 T^{1+2\tau} + 6\lambda_1^2 T^{2\tau} H_{2,T}\right) \\ = & O\left(T^{1+2\tau} + T^{2\tau} H_{2,T}\right).\end{aligned}$$

Based on the similar analysis process for $\mathbf{y}$, we have the following inequality

$$\begin{aligned}\rho_{\mathbf{z}}^{M_t-1} \geq & \frac{\delta_{\mathbf{z}}^2}{L_{g,1}^2 \left\|\mathbf{z}_t - \mathbf{y}^*_t(\mathbf{x}_t)\right\|^2} \\ M_t \leq & 1 + \frac{L_{g,1}^2}{\ln \rho_{\mathbf{z}}^{-1}} \left\|\mathbf{z}_t - \mathbf{y}^*_t(\mathbf{x}_t)\right\|^2 + \frac{2\ln \delta_{\mathbf{z}}}{\ln \rho_{\mathbf{z}}}.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathcal{I}_T^{\mathbf{z}} \leq & \left(1 + \frac{2\ln \delta_{\mathbf{z}}}{\ln \rho_{\mathbf{z}}}\right) T + \frac{L_{g,1}^2}{\ln \rho_{\mathbf{z}}^{-1}} \sum_{t=1}^{T} \left\|\mathbf{z}_t - \mathbf{y}^*_t(\mathbf{x}_t)\right\|^2 \\ \leq & \left(1 + \frac{2\ln \delta_{\mathbf{z}}}{\ln \rho_{\mathbf{z}}}\right) T + \frac{2L_{g,1}^2}{\ln \rho_{\mathbf{z}}^{-1}} \left(\sum_{t=1}^{T} \left\|\mathbf{z}_t - \mathbf{y}^*_{t-1}(\mathbf{x}_{t-1})\right\|^2 + \sum_{t=1}^{T} \left\|\mathbf{y}^*_{t-1}(\mathbf{x}_{t-1}) - \mathbf{y}^*_t(\mathbf{x}_t)\right\|^2\right)\end{aligned}$$

$$\leq \left(1 + \frac{2\ln \delta_{\mathbf{z}}}{\ln \rho_{\mathbf{z}}}\right) T + \frac{2\kappa_g^2 \delta_{\mathbf{z}}^2}{\ln \rho_{\mathbf{z}}^{-1}} T + \frac{4\gamma^2 \kappa_g^2 L_{\mathcal{L},0}^2 L_{g,1}^2}{\ln \rho_{\mathbf{z}}^{-1}} T + \frac{4L_{g,1}^2}{\ln \rho_{\mathbf{z}}^{-1}} H_{2,T} = O\left(T \ln T + H_{2,T}\right).$$

It is true that

$$\mathcal{I}_T = \mathcal{I}_T^{\mathbf{y}} + \mathcal{I}_T^{\mathbf{z}} \leq O\left(T^{1+2\tau} + T^{2\tau} H_{2,T}\right).$$

□

E Proof of Section 5

We first present Lemma E.1, which establishes a preliminary regret upper bound for the fully first-order algorithm applied to the approximation problem equation (P3) in the stochastic OBO setting. Lemmas E.2 and E.3 provide the inner-loop iteration error bounds under stochastic noise, respectively. These bounds further lead to the approximate stochastic hypergradient error bound in Lemma E.4, which ultimately yields the proof of Theorem C.9.

Lemma E.1. *Under Assumptions 2.1, 2.2, suppose that $\gamma_t \leq \frac{1}{2L_F}$ for all $t \in [T]$. Consider the update*

$$\mathbf{x}_{t+1} \leftarrow \mathbf{x}_t - \gamma_t \mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right).$$

Then, we obtain

$$\begin{aligned}\mathbb{E}\left[\mathrm{BLReg}(T)\right] \leq & \sum_{t=1}^{T} \frac{8}{\gamma_t} \mathbb{E}\left[F_t(\mathbf{x}_t) - F_t(\mathbf{x}_{t+1})\right] + 24 D_1^2 \sum_{t=1}^{T} \frac{1}{\lambda_t^2} + 72\sigma^2 \sum_{t=1}^{T} \frac{\lambda_t^2}{\widehat{B}_t} \\ & + 24 \sum_{t=1}^{T} \mathbb{E}\left[\left\|\nabla \mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\right].\end{aligned}$$

Proof. We begin with the L_F-smoothness of F_t. By Lemma B.3, we have

$$\begin{aligned}& F_t(\mathbf{x}_{t+1}) - F_t(\mathbf{x}_t) \\ \leq & \left\langle \nabla F_t(\mathbf{x}_t), \mathbf{x}_{t+1} - \mathbf{x}_t \right\rangle + \frac{L_F}{2} \left\|\mathbf{x}_{t+1} - \mathbf{x}_t\right\|^2 \\ = & -\gamma_t \left\langle \nabla F_t(\mathbf{x}_t), \mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\rangle + \frac{\gamma_t^2 L_F}{2} \left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\|^2 \\ = & -\gamma_t \left\langle \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\rangle + \frac{\gamma_t^2 L_F}{2} \left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\|^2 \\ & - \gamma_t \left\langle \nabla F_t(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\rangle \\ \leq & -\gamma_t \left(1 - \frac{\gamma_t L_F}{2}\right) \left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\|^2 + \frac{\gamma_t}{2} \left\|\nabla F_t(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t)\right\|^2 \\ & + \frac{\gamma_t}{2} \left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\|^2 \\ = & -\frac{\gamma_t}{2}\left(1 - \gamma_t L_F\right) \left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\|^2 + \frac{\gamma_t}{2} \left\|\nabla F_t(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t)\right\|^2.\end{aligned}$$

Let $\gamma_t \leq \frac{1}{2L_F}$, rearranging the above inequality gives

$$\left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\|^2 \leq \frac{4}{\gamma_t}\left(F_t(\mathbf{x}_t) - F_t(\mathbf{x}_{t+1})\right) + 2\left\|\nabla F_t(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t)\right\|^2.$$

Moreover, by

$$\frac{1}{2}\left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \nabla F_t(\mathbf{x}_t), \gamma_t\right)\right\|^2 \leq \left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t), \gamma_t\right)\right\|^2 + \left\|\nabla F_t(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t)\right\|^2,$$

it follows that

$$\left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \nabla F_t(\mathbf{x}_t), \gamma_t\right)\right\|^2 \leq \frac{8}{\gamma_t}\left(F_t(\mathbf{x}_t) - F_t(\mathbf{x}_{t+1})\right) + 8\left\|\nabla F_t(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t)\right\|^2$$

$$\leq \frac{8}{\gamma_t}\left(F_t(\mathbf{x}_t) - F_t(\mathbf{x}_{t+1})\right) + 24\left\|\nabla F_t(\mathbf{x}_t) - \nabla \mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 + 24\left\|\nabla \mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 + 24\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t;\mathcal{S}_t)\right\|^2. \tag{36}$$

Note that, under Assumption 2.3, we naturally have

$$\begin{aligned}
\mathbb{E}_{\mathcal{S}_t}\left[\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t;\mathcal{S}_t)\right\|^2\right] =& \mathbb{E}_{\mathcal{S}_t}\left[\left\|\nabla_{\mathbf{x}} f_t(\mathbf{x}_t,\mathbf{y}_{t+1}) - \nabla_{\mathbf{x}} f_t(\mathbf{x}_t,\mathbf{y}_{t+1};\xi_t)\right.\right. \\
&+\lambda_t\left(\nabla_{\mathbf{x}} g_t(\mathbf{x}_t,\mathbf{y}_{t+1}) - \nabla_{\mathbf{x}} g_t(\mathbf{x}_t,\mathbf{y}_{t+1};\zeta_t)\right) \\
&\left.\left.+\lambda_t\left(\nabla_{\mathbf{x}} g_t(\mathbf{x}_t,\mathbf{z}_{t+1}) - \nabla_{\mathbf{x}} g_t(\mathbf{x}_t,\mathbf{z}_{t+1})\right)\right\|^2\right] \\
=& \mathbb{E}_{\xi_t}\left[\left\|\nabla_{\mathbf{x}} f_t(\mathbf{x}_t,\mathbf{y}_{t+1}) - \nabla_{\mathbf{x}} f_t(\mathbf{x}_t,\mathbf{y}_{t+1};\xi_t)\right\|^2\right] \\
&+\lambda_t^2\mathbb{E}_{\zeta_t}\left[\left\|\nabla_{\mathbf{x}} g_t(\mathbf{x}_t,\mathbf{y}_{t+1}) - \nabla_{\mathbf{x}} g_t(\mathbf{x}_t,\mathbf{y}_{t+1};\zeta_t)\right\|^2\right] \\
&+\lambda_t^2\mathbb{E}_{\zeta_t}\left[\left\|\nabla_{\mathbf{x}} g_t(\mathbf{x}_t,\mathbf{z}_{t+1}) - \nabla_{\mathbf{x}} g_t(\mathbf{x}_t,\mathbf{z}_{t+1};\zeta_t)\right\|^2\right] \\
\leq& (1+2\lambda_t^2)\frac{\sigma^2}{\widehat{B}_t} \leq \frac{3\lambda_t^2\sigma^2}{\widehat{B}_t},
\end{aligned}$$

and substituting the above inequality into Eq.(36), we complete the proof by

$$\begin{aligned}
\mathbb{E}\left[\mathrm{BLReg}(T)\right] \leq& \sum_{t=1}^{T}\frac{8}{\gamma_t}\mathbb{E}\left[F_t(\mathbf{x}_t) - F_t(\mathbf{x}_{t+1})\right] + 24D_1^2\sum_{t=1}^{T}\frac{1}{\lambda_t^2} + 72\sigma^2\sum_{t=1}^{T}\frac{\lambda_t^2}{\widehat{B}_t} \\
&+ 24\sum_{t=1}^{T}\mathbb{E}\left[\left\|\nabla\mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\right].
\end{aligned}$$

□

Lemma E.2. *Under Assumptions 2.1, 2.2 and 2.3, let $\beta_t \leq \frac{1}{2\lambda_t L_{g,1}}$, Algorithm 3 can obtain*

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathbf{y}_{t+1} - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\right] \\
\leq& \rho_{\mathbf{y}}^{t-1}\mathbb{E}\left[\left\|\mathbf{y}_2 - \mathbf{y}_{\lambda_1,1}^*(\mathbf{x}_1)\right\|^2\right] + \frac{4\sigma^2}{\mu_g}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j\beta_{t-j}\lambda_{t-j}}{B_{t-j}} + \frac{12L_{f,0}^2 q_t^{K_t}}{\mu_g^2}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}} \\
&+ 12\kappa_g^2 q_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\mathbb{E}\left[\left\|\mathbf{x}_{t-j} - \mathbf{x}_{t-1-j}\right\|^2\right] + 12q_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\mathbb{E}\left[\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2\right],
\end{aligned}$$

where $q_t = 1 - \frac{\beta_t\lambda_t\mu_g}{2}$. For each $t \in [T]$, we choose $K_t > 0$ such that $2q_t^{K_t} \leq \rho_{\mathbf{y}}$ for some constant $\rho_{\mathbf{y}} > 0$.

Proof. Let $\widetilde{\mathbf{g}}_{\mathbf{y}} := \nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t,\mathbf{y}_t^k,\lambda_t;\mathcal{S}_t)$, $\mathbf{g}_{\mathbf{y}} := \nabla_{\mathbf{y}}\mathcal{L}_t(\mathbf{x}_t,\mathbf{y}_t^k,\lambda_t)$. With the update rule

$$\mathbf{y}_t^{k+1} \leftarrow \mathbf{y}_t^k - \beta_t\widetilde{\mathbf{g}}_{\mathbf{y}},$$

we have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{S}_t^k}\left[\left\|\mathbf{y}_t^{k+1} - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\right] \\
=& \mathbb{E}_{\mathcal{S}_t^k}\left[\left\|\mathbf{y}_t^k - \beta_t\widetilde{\mathbf{g}}_{\mathbf{y}} - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\right] \\
\leq& \left\|\mathbf{y}_t^k - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2 - 2\beta_t\mathbb{E}_{\mathcal{S}_t^k}\left[\left\langle\widetilde{\mathbf{g}}_{\mathbf{y}}, \mathbf{y}_t^k - \mathbf{y}_{\lambda_t,t}^*((\mathbf{x}_t)\right\rangle\right] + \beta_t^2\mathbb{E}_{\mathcal{S}_t^k}\left[\left\|\widetilde{\mathbf{g}}_{\mathbf{y}}\right\|^2\right] \\
\overset{(i)}{=}& \left\|\mathbf{y}_t^k - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2 - 2\beta_t\left\langle\mathbf{g}_{\mathbf{y}}, \mathbf{y}_t^k - \mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\rangle + \beta_t^2\left\|\mathbf{g}_{\mathbf{y}}\right\|^2 + \beta_t^2(1+\lambda_t^2)\frac{\sigma^2}{B_t}
\end{aligned}$$

$$\overset{(ii)}{\leq} \left\|\mathbf{y}_t^k - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 - \beta_t\frac{\lambda_t\mu_g}{2}\left\|\mathbf{y}_t^k - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 - \left(\frac{\beta_t}{2\lambda_t L_{g,1}} - \beta_t^2\right)\left\|\mathbf{g}_{\mathbf{y}}\right\|^2 + \beta_t^2(1+\lambda_t^2)\frac{\sigma^2}{B_t}$$
$$\leq q_t\left\|\mathbf{y}_t^k - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2 + 2\beta_t^2\lambda_t^2\frac{\sigma^2}{B_t}$$

where (i) comes from the fact that

$$\mathbb{E}_{\mathcal{S}_t^k}\left[\left\|\mathbf{g}_{\mathbf{y}} - \widetilde{\mathbf{g}}_{\mathbf{y}}\right\|^2\right] = \mathbb{E}_{\mathcal{S}_t^k}\left[\left\|\nabla_{\mathbf{y}} f_t(\mathbf{x}_t,\mathbf{y}_t^k) - \nabla_{\mathbf{y}} f_t(\mathbf{x}_t,\mathbf{y}_t^k;\xi_t^k)\right.\right.$$
$$\left.\left. + \lambda_t\left(\nabla_{\mathbf{y}} g_t(\mathbf{x}_t,\mathbf{y}_t^k) - \nabla_{\mathbf{y}} g_t(\mathbf{x}_t,\mathbf{y}_t^k;\zeta_t^k)\right)\right\|^2\right] \leq (1+\lambda_t^2)\frac{\sigma^2}{B_t},$$

and (ii) comes from the $(\lambda_t\mu_g/2)$-strongly convexity and $(2\lambda_t L_{g,1})$-smoothness of $\mathcal{L}_t$ w.r.t. $\mathbf{y}$, and the last inequality comes from $\beta_t \leq \frac{1}{2\lambda_t L_{g,1}}$, let $q_t := 1 - \frac{\beta_t\lambda_t\mu_g}{2}$, we then obtain

$$\mathbb{E}\left[\left\|\mathbf{y}_{t+1} - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\right] \leq q_t^{K_t}\mathbb{E}\left[\left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\right] + 2\beta_t^2\lambda_t^2\frac{\sigma^2}{B_t}\sum_{j=0}^{K_t-1} q_t^j$$
$$\leq q_t^{K_t}\mathbb{E}\left[\left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\right] + \frac{4\beta_t\lambda_t\sigma^2}{\mu_g B_t}. \tag{37}$$

Recall from equation (18) that we also have

$$\left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2$$
$$\leq 2\left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})\right\|^2 + 2\left\|\mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1}) - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2$$
$$\leq 2\left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})\right\|^2 + \frac{12L_{f,0}^2}{\lambda_t^2\mu_g^2} + 12\kappa_g^2\left\|\mathbf{x}_{t-1} - \mathbf{x}_t\right\|^2 + 12\left\|\mathbf{y}^*_{t-1}(\mathbf{x}_t) - \mathbf{y}^*_t(\mathbf{x}_t)\right\|^2,$$

and substituting the above inequality into equation (37), we obtain

$$\mathbb{E}\left[\left\|\mathbf{y}_{t+1} - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\right] \leq 2q_t^{K_t}\mathbb{E}\left[\left\|\mathbf{y}_t - \mathbf{y}^*_{\lambda_{t-1},t-1}(\mathbf{x}_{t-1})\right\|^2\right] + \frac{4\beta_t\lambda_t\sigma^2}{\mu_g B_t} + \frac{12q_t^{K_t}L_{f,0}^2}{\lambda_t^2\mu_g^2}$$
$$+ 12q_t^{K_t}\kappa_g^2\mathbb{E}\left[\left\|\mathbf{x}_{t-1} - \mathbf{x}_t\right\|^2\right] + 12q_t^{K_t}\mathbb{E}\left[\left\|\mathbf{y}^*_{t-1}(\mathbf{x}_t) - \mathbf{y}^*_t(\mathbf{x}_t)\right\|^2\right].$$

Telescoping the above inequality, we complete the proof by

$$\mathbb{E}\left[\left\|\mathbf{y}_{t+1} - \mathbf{y}^*_{\lambda_t,t}(\mathbf{x}_t)\right\|^2\right]$$
$$\leq \mathbb{E}\left[\left\|\mathbf{y}_2 - \mathbf{y}^*_{\lambda_1,1}(\mathbf{x}_1)\right\|^2\right]\prod_{j=0}^{t-2} 2q_j^{K_j} + \frac{4\sigma^2}{\mu_g}\sum_{j=0}^{t-2}\frac{\beta_{t-j}\lambda_{t-j}}{B_{t-j}}\prod_{m=1}^{j} 2q_{t+1-m}^{K_{t+1-m}}$$
$$+ \frac{6L_{f,0}^2}{\mu_g^2}\sum_{j=0}^{t-2}\frac{1}{\lambda_{t-j}^2}\prod_{m=0}^{j} 2q_{t-m}^{K_{t-m}} + 6\kappa_g^2\sum_{j=0}^{t-2}\mathbb{E}\left[\left\|\mathbf{x}_{t-j} - \mathbf{x}_{t-1-j}\right\|^2\right]\prod_{m=0}^{j} 2q_{t-m}^{K_{t-m}}$$
$$+ 6\sum_{j=0}^{t-2}\mathbb{E}\left[\left\|\mathbf{y}^*_{t-1-j}(\mathbf{x}_{t-j}) - \mathbf{y}^*_{t-j}(\mathbf{x}_{t-j})\right\|^2\right]\prod_{m=0}^{j} 2q_{t-m}^{K_{t-m}}$$
$$\leq \rho_{\mathbf{y}}^{t-1}\mathbb{E}\left[\left\|\mathbf{y}_2 - \mathbf{y}^*_{\lambda_1,1}(\mathbf{x}_1)\right\|^2\right] + \frac{4\sigma^2}{\mu_g}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j\beta_{t-j}\lambda_{t-j}}{B_{t-j}} + \frac{12L_{f,0}^2 q_t^{K_t}}{\mu_g^2}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2}$$
$$+ 12\kappa_g^2 q_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\mathbb{E}\left[\left\|\mathbf{x}_{t-j} - \mathbf{x}_{t-1-j}\right\|^2\right] + 12q_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\mathbb{E}\left[\left\|\mathbf{y}^*_{t-1-j}(\mathbf{x}_{t-j}) - \mathbf{y}^*_{t-j}(\mathbf{x}_{t-j})\right\|^2\right],$$

where the last inequality follows by assuming that, for all $t \in [T]$, K_t is sufficiently large such that $2q_t^{K_t} \leq \rho_{\mathbf{y}}$ for some constant $\rho_{\mathbf{y}}$. □

Lemma E.3. *Under Assumptions 2.1, 2.2 and 2.3, let $\alpha_t \leq \frac{1}{L_{g,1}}$, Algorithm 3 can obtain*

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathbf{z}_{t+1} - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right] \\
\leq&\rho_{\mathbf{z}}^{t-1}\mathbb{E}\left[\left\|\mathbf{z}_2 - \mathbf{y}_1^*(\mathbf{x}_1)\right\|^2\right] + \frac{\sigma^2}{\mu_g}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{z}}^j\alpha_{t-j}}{B_{t-j}} + 4p_t^{K_t}\kappa_g^2\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\mathbb{E}\left[\left\|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\right\|^2\right] \\
&+ 4p_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\mathbb{E}\left[\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2\right],
\end{aligned}$$

where $p_t = 1 - \alpha_t\mu_g$. For each $t \in [T]$, we choose $K_t > 0$ such that $2p_t^{K_t} \leq \rho_{\mathbf{z}}$ for some constant $\rho_{\mathbf{z}} > 0$.

Proof. By applying a derivation similar to that in Lemma E.2, we have

$$\begin{aligned}
&\mathbb{E}_{\zeta_t^k}\left[\left\|\mathbf{z}_t^{k+1} - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right] \\
=&\mathbb{E}_{\zeta_t^k}\left[\left\|\mathbf{z}_t^k - \alpha_t\nabla_{\mathbf{y}}g_t(\mathbf{x}_t, \mathbf{z}_t^k; \zeta_t^k) - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right] \\
\leq&\left\|\mathbf{z}_t^k - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2 - 2\alpha_t\left\langle\nabla_{\mathbf{y}}g_t(\mathbf{x}_t, \mathbf{z}_t^k), \mathbf{z}_t^k - \mathbf{y}_t^*(\mathbf{x}_t)\right\rangle + \alpha_t^2\mathbb{E}_{\zeta_t^k}\left[\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x}_t, \mathbf{z}_t^k; \zeta_t^k)\right\|^2\right] \\
=&\left\|\mathbf{z}_t^k - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2 - 2\alpha_t\left\langle\nabla_{\mathbf{y}}g_t(\mathbf{x}_t, \mathbf{z}_t^k), \mathbf{z}_t^k - \mathbf{y}_t^*(\mathbf{x}_t)\right\rangle + \alpha_t^2\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x}_t, \mathbf{z}_t^k)\right\|^2 + \frac{\alpha_t^2\sigma^2}{B_t} \\
\leq&\left\|\mathbf{z}_t^k - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2 - \alpha_t\mu_g\left\|\mathbf{z}_t^k - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2 - \left(\frac{\alpha_t}{L_{g,1}} - \alpha_t^2\right)\left\|\nabla_{\mathbf{y}}g_t(\mathbf{x}_t, \mathbf{z}_t^k)\right\|^2 + \frac{\alpha_t^2\sigma^2}{B_t} \\
\leq&p_t\left\|\mathbf{z}_t^k - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2 + \frac{\alpha_t^2\sigma^2}{B_t},
\end{aligned}$$

where $p_t := 1 - \alpha_t\mu_g$. Then, after telescoping over K_t iterations, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathbf{z}_{t+1} - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right] \\
\leq&p_t^{K_t}\mathbb{E}\left[\left\|\mathbf{z}_t - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right] + \frac{\alpha_t^2\sigma^2}{B_t}\sum_{j=0}^{K_t-1}p_t^j \\
\leq&2p_t^{K_t}\mathbb{E}\left[\left\|\mathbf{z}_t - \mathbf{y}_{t-1}^*(\mathbf{x}_{t-1})\right\|^2\right] + 2p_t^{K_t}\mathbb{E}\left[\left\|\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1}) - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right] + \frac{\alpha_t\sigma^2}{\mu_g B_t} \\
\leq&2p_t^{K_t}\mathbb{E}\left[\left\|\mathbf{z}_t - \mathbf{y}_{t-1}^*(\mathbf{x}_{t-1})\right\|^2\right] + 4p_t^{K_t}\kappa_g^2\mathbb{E}\left[\left\|\mathbf{x}_{t-1} - \mathbf{x}_t\right\|^2\right] \\
&+ 4p_t^{K_t}\mathbb{E}\left[\left\|\mathbf{y}_{t-1}^*(\mathbf{x}_t) - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right] + \frac{\alpha_t\sigma^2}{\mu_g B_t}.
\end{aligned}$$

Finally, telescoping the above inequality yields

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathbf{z}_{t+1} - \mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right] \\
\leq&\rho_{\mathbf{z}}^{t-1}\mathbb{E}\left[\left\|\mathbf{z}_2 - \mathbf{y}_1^*(\mathbf{x}_1)\right\|^2\right] + \frac{\sigma^2}{\mu_g}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{z}}^j\alpha_{t-j}}{B_{t-j}} + 4p_t^{K_t}\kappa_g^2\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\mathbb{E}\left[\left\|\mathbf{x}_{t-1-j} - \mathbf{x}_{t-j}\right\|^2\right] \\
&+ 4p_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\mathbb{E}\left[\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2\right],
\end{aligned}$$

where the last inequality follows by assuming that, for all $t \in [T]$, K_t is sufficiently large such that $2p_t^{K_t} \leq \rho_{\mathbf{z}}$ for some constant $\rho_{\mathbf{z}}$. □

Lemma E.4. *Under Assumptions 2.1, 2.2 and 2.3, let* $\alpha_t \leq \frac{1}{L_{g,1}}$, $\beta_t \leq \frac{1}{2\lambda_t L_{g,1}}$ *and* $\gamma_t \equiv \gamma \leq \frac{1}{8\kappa_g L_{g,1}}\sqrt{\frac{1-\rho}{21c_\lambda}}$, *Algorithm 3 can obtain*

$$\begin{aligned}
&\sum_{t=1}^{T}\mathbb{E}\left[\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\right]\\
\leq& 2c_\tau\Delta_1+\frac{54\lambda_T^2\kappa_g L_{g,1}\sigma^2}{1-\rho}\sum_{t=1}^{T}\frac{\beta_t\lambda_t+\alpha_t}{B_t}+\frac{144\kappa_g^2L_{f,0}^2c_\lambda}{1-\rho}\sum_{t=1}^{T}\frac{1}{\lambda_t^2}+\frac{168L_{g,1}^2c_\lambda}{1-\rho}H_{2,T}\\
&+\frac{672\gamma^2\kappa_g^2L_{g,1}^2c_\lambda}{1-\rho}\mathbb{E}\left[\mathrm{BLReg}(T)\right]+\frac{672\gamma^2\kappa_g^2L_{g,1}^2c_\lambda D_1^2}{1-\rho}\sum_{t=1}^{T}\frac{1}{\lambda_t^2}+\frac{2016\gamma^2\kappa_g^2L_{g,1}^2c_\lambda\sigma^2}{1-\rho}\sum_{t=1}^{T}\frac{\lambda_t^2}{\widehat{B}_t},
\end{aligned}$$

where c_τ, Δ_1 *are some constants. For each* $t\in[T]$, *we choose* $K_t>0$ *such that* $\lambda_t^2q_t^{K_t}\leq c_\lambda$, $\lambda_t^2p_t^{K_t}\leq c_\lambda$ *for some constant* c_λ.

Proof. Recall from equation (20) in Lemma C.4 that

$$\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\leq 6\lambda_t^2L_{g,1}^2\left\|\mathbf{y}_{t+1}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2+3\lambda_t^2L_{g,1}^2\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2.$$

Then, substituting the results of Lemma E.2 and Lemma E.3 into the above inequality, we obtain

$$\begin{aligned}
&\sum_{t=2}^{T}\mathbb{E}\left[\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\right]\\
\leq& 6L_{g,1}^2\sum_{t=2}^{T}\lambda_t^2\mathbb{E}\left[\left\|\mathbf{y}_{t+1}-\mathbf{y}_{\lambda_t,t}^*(\mathbf{x}_t)\right\|^2\right]+3L_{g,1}^2\sum_{t=2}^{T}\lambda_t^2\mathbb{E}\left[\left\|\mathbf{z}_{t+1}-\mathbf{y}_t^*(\mathbf{x}_t)\right\|^2\right]\\
\leq& 6L_{g,1}^2\mathbb{E}\left[\left\|\mathbf{y}_2-\mathbf{y}_{\lambda_1,1}^*(\mathbf{x}_1)\right\|^2\right]\sum_{t=2}^{T}\lambda_t^2\rho_{\mathbf{y}}^{t-1}+24\kappa_gL_{g,1}\sigma^2\sum_{t=2}^{T}\lambda_t^2\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j\beta_{t-j}\lambda_{t-j}}{B_{t-j}}\\
&+72\kappa_g^2L_{f,0}^2\sum_{t=2}^{T}\lambda_t^2q_t^{K_t}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2}+72\kappa_g^2L_{g,1}^2\sum_{t=2}^{T}\lambda_t^2q_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\mathbb{E}\left[\left\|\mathbf{x}_{t-j}-\mathbf{x}_{t-1-j}\right\|^2\right]\\
&+72L_{g,1}^2\sum_{t=2}^{T}\lambda_t^2q_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{y}}^j\mathbb{E}\left[\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j})-\mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2\right]\\
&+3L_{g,1}^2\mathbb{E}\left[\left\|\mathbf{z}_2-\mathbf{y}_1^*(\mathbf{x}_1)\right\|^2\right]\sum_{t=2}^{T}\lambda_t^2\rho_{\mathbf{z}}^{t-1}+3\kappa_gL_{g,1}\sigma^2\sum_{t=2}^{T}\lambda_t^2\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{z}}^j\alpha_{t-j}}{B_{t-j}}\\
&+12\kappa_g^2L_{g,1}^2\sum_{t=2}^{T}\lambda_t^2p_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\mathbb{E}\left[\left\|\mathbf{x}_{t-j}-\mathbf{x}_{t-1-j}\right\|^2\right]\\
&+12L_{g,1}^2\sum_{t=2}^{T}\lambda_t^2p_t^{K_t}\sum_{j=0}^{t-2}\rho_{\mathbf{z}}^j\mathbb{E}\left[\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j})-\mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2\right].
\end{aligned}$$

Now, assume that, for a sufficiently large K_t, $\lambda_t^2q_t^{K_t}\leq c_\lambda$ and $\lambda_t^2p_t^{K_t}\leq c_\lambda$. For simplicity, define $\rho:=\max\{\rho_{\mathbf{y}},\rho_{\mathbf{z}}\}$. Then, remind of equation (23), it holds that

$$\begin{aligned}
&\sum_{t=1}^{T}\mathbb{E}\left[\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)-\nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\right]\\
\leq& c_\tau\Delta_1+27\kappa_gL_{g,1}\sigma^2\sum_{t=2}^{T}\lambda_t^2\sum_{j=0}^{t-2}\frac{\rho^j}{B_{t-j}}(\beta_{t-j}\lambda_{t-j}+\alpha_{t-j})+72\kappa_g^2L_{f,0}^2c_\lambda\sum_{t=2}^{T}\sum_{j=0}^{t-2}\frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2}
\end{aligned}$$

$$
\begin{aligned}
&+ 84L_{g,1}^2 c_\lambda \sum_{t=2}^{T}\sum_{j=0}^{t-2} \rho^j \mathbb{E}\left[\left\|\mathbf{y}_{t-1-j}^*(\mathbf{x}_{t-j}) - \mathbf{y}_{t-j}^*(\mathbf{x}_{t-j})\right\|^2\right] \\
&+ 84\kappa_g^2 L_{g,1}^2 c_\lambda \sum_{t=2}^{T}\sum_{j=0}^{t-2} \rho^j \mathbb{E}\left[\left\|\mathbf{x}_{t-j} - \mathbf{x}_{t-1-j}\right\|^2\right] \\
\leq & c_\tau \Delta_1 + 27\kappa_g L_{g,1}\sigma^2 \sum_{t=2}^{T} \lambda_t^2 \sum_{j=0}^{t-2} \frac{\rho^j}{B_{t-j}}(\beta_{t-j}\lambda_{t-j} + \alpha_{t-j}) + 72\kappa_g^2 L_{f,0}^2 c_\lambda \sum_{t=2}^{T}\sum_{j=0}^{t-2} \frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} \\
&+ \frac{84\kappa_g^2 L_{g,1}^2 c_\lambda}{1-\rho} \sum_{t=2}^{T} \mathbb{E}\left[\left\|\mathbf{x}_t - \mathbf{x}_{t-1}\right\|^2\right] + \frac{84L_{g,1}^2 c_\lambda}{1-\rho} H_{2,T},
\end{aligned}
$$

where $\Delta_1 := 6L_{g,1}^2 \mathbb{E}[\|\mathbf{y}_2 - \mathbf{y}_{\lambda_1,1}^*(\mathbf{x}_1)\|^2] + 3L_{g,1}^2 \mathbb{E}[\|\mathbf{z}_2 - \mathbf{y}_1^*(\mathbf{x}_1)\|^2]$. Then we have

$$
\begin{aligned}
&\sum_{t=1}^{T} \mathbb{E}\left[\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\right] \\
\leq & c_\tau \Delta_1 + 27\kappa_g L_{g,1}\sigma^2 \sum_{t=2}^{T} \lambda_t^2 \sum_{j=0}^{t-2} \frac{\rho^j}{B_{t-j}}(\beta_{t-j}\lambda_{t-j} + \alpha_{t-j}) + 72\kappa_g^2 L_{f,0}^2 c_\lambda \sum_{t=2}^{T}\sum_{j=0}^{t-2} \frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} \\
&+ \frac{336\kappa_g^2 L_{g,1}^2 c_\lambda}{1-\rho} \sum_{t=1}^{T} \gamma_t^2 \mathbb{E}\Big[\left\|\mathcal{G}_{\mathcal{X}}\left(\mathbf{x}_t, \nabla F_t(\mathbf{x}_t), \gamma_t\right)\right\|^2 + \left\|\nabla F_t(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 \\
&+ \left\|\nabla\mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2 + \left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t; \mathcal{S}_t)\right\|^2 \Big] + \frac{84L_{g,1}^2 c_\lambda}{1-\rho} H_{2,T}.
\end{aligned}
$$

Let $\gamma_t \equiv \gamma \leq \frac{1}{8\kappa_g L_{g,1}} \sqrt{\frac{1-\rho}{21 c_\lambda}}$, we can gaurantee that $1 - \frac{336\kappa_g^2 L_{g,1}^2 c_\lambda}{1-\rho} \geq \frac{1}{2}$, thus it holds that

$$
\begin{aligned}
&\sum_{t=1}^{T} \mathbb{E}\left[\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\right] \\
\leq & 2c_\tau \Delta_1 + 54\kappa_g L_{g,1}\sigma^2 \sum_{t=2}^{T} \lambda_t^2 \sum_{j=0}^{t-2} \frac{\rho^j}{B_{t-j}}(\beta_{t-j}\lambda_{t-j} + \alpha_{t-j}) + 144\kappa_g^2 L_{f,0}^2 c_\lambda \sum_{t=2}^{T}\sum_{j=0}^{t-2} \frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} \\
&+ \frac{672\gamma^2\kappa_g^2 L_{g,1}^2 c_\lambda}{1-\rho} \mathbb{E}\left[\mathrm{BLReg}(T)\right] + \frac{672\gamma^2\kappa_g^2 L_{g,1}^2 c_\lambda D_1^2}{C_1(1-\rho)} \sum_{t=1}^{T} \frac{1}{\lambda_t^2} + \frac{2016\gamma^2\kappa_g^2 L_{g,1}^2 c_\lambda \sigma^2}{1-\rho} \sum_{t=1}^{T} \frac{\lambda_t^2}{\widehat{B}_t} \\
&+ \frac{168L_{g,1}^2 c_\lambda}{1-\rho} H_{2,T}. \qquad (38)
\end{aligned}
$$

Consider the second term. Letting $m = t - j$, we have

$$
\sum_{t=2}^{T} \lambda_t^2 \sum_{j=0}^{t-2} \frac{\rho^j}{B_{t-j}}(\beta_{t-j}\lambda_{t-j} + \alpha_{t-j}) = \sum_{m=2}^{T} \frac{\beta_m\lambda_m + \alpha_m}{B_m} \sum_{t=m}^{T} \lambda_t^2 \rho^{t-m} \leq \frac{\lambda_T^2}{1-\rho} \sum_{t=1}^{T} \frac{\beta_t\lambda_t + \alpha_t}{B_t}.
$$

For the third term, we similarly have

$$
\sum_{t=2}^{T}\sum_{j=0}^{t-2} \frac{\rho_{\mathbf{y}}^j}{\lambda_{t-j}^2} = \sum_{m=2}^{T} \frac{1}{\lambda_m^2} \sum_{t=m}^{T} \rho_{\mathbf{y}}^{t-m} \leq \frac{1}{1-\rho} \sum_{t=1}^{T} \frac{1}{\lambda_t^2}.
$$

Substituting the above two inequalities into equation (38), we complete the proof by

$$
\begin{aligned}
&\sum_{t=1}^{T} \mathbb{E}\left[\left\|\widetilde{\nabla}\mathcal{L}_t^*(\mathbf{x}_t) - \nabla\mathcal{L}_t^*(\mathbf{x}_t)\right\|^2\right] \\
\leq & 2c_\tau \Delta_1 + \frac{54\lambda_T^2\kappa_g L_{g,1}\sigma^2}{1-\rho} \sum_{t=1}^{T} \frac{\beta_t\lambda_t + \alpha_t}{B_t} + \frac{144\kappa_g^2 L_{f,0}^2 c_\lambda}{1-\rho} \sum_{t=1}^{T} \frac{1}{\lambda_t^2} + \frac{168L_{g,1}^2 c_\lambda}{1-\rho} H_{2,T}
\end{aligned}
$$

$$+\frac{672\gamma^2\kappa_g^2L_{g,1}^2c_\lambda}{1-\rho}\mathbb{E}\left[\mathrm{BLReg}(T)\right]+\frac{672\gamma^2\kappa_g^2L_{g,1}^2c_\lambda D_1^2}{1-\rho}\sum_{t=1}^{T}\frac{1}{\lambda_t^2}+\frac{2016\gamma^2\kappa_g^2L_{g,1}^2c_\lambda\sigma^2}{1-\rho}\sum_{t=1}^{T}\frac{\lambda_t^2}{\widehat{B}_t}.$$

□

Theorem E.5 (Restatement of Theorem 5.1)**.** *Under Assumption 2.1, 2.2 and 2.3, Let* $\Lambda(t,\lambda_t)=\left(1+\frac{1}{t}\right)^\tau\lambda_t$ *for some* $\tau\in(0,\frac{1}{2})$, $\lambda_1>\frac{2L_{f,1}}{\mu_g}$, $\alpha_t=\frac{1}{L_{g,1}t^a}$, $\beta_t=\frac{1}{2\lambda_1L_{g,1}t^b}$, $\gamma_t\equiv\gamma\le\min\{\frac{1}{2L_F},\frac{1}{48\kappa_gL_{g,1}}\sqrt{\frac{1-\rho}{7c_\lambda}}\}$, $K_t=4\kappa_g\max\{t^{b-\tau},t^a\}\log t$, $\widehat{B}_t=t^{B_1}$ *and* $B_t=t^{B_2}$, *Algorithm 3 can guarantee*

$$\mathbb{E}\left[\mathrm{BLReg}(T)\right]\le O\left(T^{1-2\tau}+\sigma^2\max\{T^{1+2\tau-B_1},T^{1+3\tau-b-B_2},T^{1+2\tau-a-B_2}\}+V_T+H_{2,T}\right),$$

where ρ, c_λ *are some constants.*

Proof. Substituting Lemma E.4 into Lemma E.1, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{BLReg}(T)\right]\\
\le&\sum_{t=1}^{T}\frac{8}{\gamma_t}\left(F_t(\mathbf{x}_t)-F_t(\mathbf{x}_{t+1})\right)+24D_1^2\sum_{t=1}^{T}\frac{1}{\lambda_t^2}+72\sigma^2\sum_{t=1}^{T}\frac{\lambda_t^2}{\widehat{B}_t}+\frac{3456\kappa_g^2L_{f,0}^2c_\lambda}{1-\rho}\sum_{t=1}^{T}\frac{1}{\lambda_t^2}\\
&+48c_\tau\Delta_1+\frac{1296\lambda_T^2\kappa_gL_{g,1}\sigma^2}{1-\rho}\sum_{t=1}^{T}\frac{\beta_t\lambda_t+\alpha_t}{B_t}+\frac{16128\gamma^2\kappa_g^2L_{g,1}^2c_\lambda}{1-\rho}\mathbb{E}\left[\mathrm{BLReg}(T)\right]\\
&+\frac{16128\gamma^2\kappa_g^2L_{g,1}^2c_\lambda D_1^2}{1-\rho}\sum_{t=1}^{T}\frac{1}{\lambda_t^2}+\frac{48384\gamma^2\kappa_g^2L_{g,1}^2c_\lambda\sigma^2}{1-\rho}\sum_{t=1}^{T}\frac{\lambda_t^2}{\widehat{B}_t}+\frac{4032L_{g,1}^2c_\lambda}{1-\rho}H_{2,T}.
\end{aligned}$$

Choosing $\gamma\le\frac{1}{48\kappa_gL_{g,1}}\sqrt{\frac{1-\rho}{7c_\lambda}}$ ensures that $1-\frac{4032\gamma^2\kappa_g^2L_{g,1}^2c_\lambda}{1-\rho}\ge\frac{1}{2}$. Moreover, with

$$\lambda_t=\lambda_1t^\tau,\quad\tau\in\left(0,\frac{1}{2}\right),\quad\beta_t=\frac{1}{2\lambda_1L_{g,1}t^b},\quad\alpha_t=\frac{1}{L_{g,1}t^a},\quad\widehat{B}_t=t^{B_1},\quad B_t=t^{B_2},$$

we finally have

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{BLReg}(T)\right]\\
\le&\frac{32M}{\gamma}+\frac{16}{\gamma}V_T+\frac{48D_1^2}{\lambda_1^2}\sum_{t=1}^{T}t^{-2\tau}+144\lambda_1^2\sigma^2\sum_{t=1}^{T}t^{2\tau-B_1}+\frac{6912\kappa_g^2L_{f,0}^2c_\lambda}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}t^{-2\tau}\\
&+\frac{2592\lambda_1^2\kappa_g\sigma^2T^{2\tau}}{1-\rho}\sum_{t=1}^{T}(t^{\tau-b-B_2}+t^{-a-B_2})+\frac{32256\gamma^2\kappa_g^2L_{g,1}^2c_\lambda D_1^2}{\lambda_1^2(1-\rho)}\sum_{t=1}^{T}t^{-2\tau}\\
&+96c_\tau\Delta_1+\frac{96768\gamma^2\lambda_1^2\kappa_g^2L_{g,1}^2c_\lambda\sigma^2}{1-\rho}\sum_{t=1}^{T}t^{2\tau-B_1}+\frac{8064L_{g,1}^2c_\lambda}{1-\rho}H_{2,T}\\
=&O\left(T^{1-2\tau}+\sigma^2\max\{T^{1+2\tau-B_1},T^{1+3\tau-b-B_2},T^{1+2\tau-a-B_2}\}+V_T+H_{2,T}\right),
\end{aligned}$$

which completes the proof. □

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