THE POWER OF KNOWING HORIZONS IN ONE-WAY TRADING: A UNIFIED FRAMEWORK FOR OPTIMAL COMPETITIVE RATIOS

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ABSTRACT

This paper studies the One-way Trading Problem (OTP) with different forms of horizon information. We first show that knowing only the earliest horizon in OTP yields no algorithmic improvement. We then propose three new algorithms for the other forms of horizon information: (i) the Latest Horizon Algorithm (LHA), an optimal solution that strictly improves upon the original bound in OTP when the latest horizon is known; (ii) the Earliest-and-Latest Horizon Algorithm (ErLHA), an asymptotically optimal solution when both the earliest and the latest horizons are known; (iii) the Exact Horizon Algorithm (ExHA), an optimal solution and also a special form of ErLHA when the earliest and latest horizons coincide and become the exact horizon. Notably, ErLHA attains asymptotic optimality when both the earliest and latest horizons are known, recovering the optimality in all other settings, and thereby standing as the first unified and optimal framework for OTP with different horizon information.

1 INTRODUCTION

The classical One-way Trading Problem (OTP), which is also called the Online Conversion Problem (OCP), models a fundamental online decision-making scenario in which a decision-maker seeks to convert a total budget of one dollar into yen over time. A sequence of N exchange rates arrives online, one at a time, over N time slots (i.e., the trading horizon), where each exchange rate is positive and bounded within $[L, U]$. Upon the arrival of an exchange rate, the decision-maker knows its value and must irrevocably decide how many dollars to exchange into yen at that exchange rate. The trading horizon N is revealed only after the exchange-rate sequence terminates. The objective is to maximize the total amount of yen obtained, subject to the budget constraint. OTP and its closely related variant, the online knapsack problem (OKP) (Cao et al., 2020), serve as canonical models for a wide range of resource allocation settings, with applications in online advertisement allocation (Lin et al., 2022), cloud resource management (Zhang et al., 2017), and beyond.

OTP assumes that the trading horizon N may end at any time. Under this assumption, the optimal competitive ratio is $1 + \ln(U/L)$, which is known to be optimal (Cao et al., 2020). However, the absence of any horizon information allows the adversary to fully exploit this uncertainty, either by generating arbitrarily long sequences of exchange rates with vanishingly small increases, or by terminating the trading at an arbitrarily early time. The optimal strategy for OTP is fundamentally shaped by these two extremes, and hence can be improved when the horizon information is available.

Consistent with this theoretical perspective, real-world markets often use mandatory disclosure and procedural rules to ensure transparency about the trading horizon, in order to mitigate information asymmetry, adverse selection, and deteriorating market liquidity (Akerlof, 1978; Glosten & Milgrom, 1985; Kyle, 1985). Such disclosure of the trading horizon typically involves several different forms of horizon information: (i) **an earliest horizon**, before which trading cannot terminate (Refinitiv US SEF LLC, 2019; Torcasio & Vise, 2023); (ii) **a latest horizon**, by which trading must terminate (U.S. Securities and Exchange Commission, n.d.); (iii) **an earliest-and-latest horizon window**, within which trading terminates (Chen, 2026); (iv) **an exact horizon**, at which trading terminates (Financial Industry Regulatory Authority (FINRA), 2026). Please refer to Appendix A for more details of real-world examples on OTP with different horizon information.

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Both theoretically and practically motivated, we investigate the impact of horizon information on online decision-making in OTP and address three fundamental questions: whether, how, and to what extent horizon information can improve online decision-making in OTP. Although it is natural to expect that additional horizon information always reduces uncertainty and improves online performance, our results show that knowing only the earliest horizon provides no improvement (Section 3), whereas knowing only the latest horizon can lead to a clear improvement (Section 4). Moreover, the earliest horizon can now demonstrate its benefit in the presence of a latest horizon (Section 5). We will formulate OTP under various forms of horizon information in Section 2, and then unfold our results in the following order:

(i) OTP with Earliest Horizon (OTP-Er). In Section 3, we show that when only the earliest horizon N^- is known ($N^- \leq N$), the best achievable competitive ratio is still $1 + \ln(U/L)$ (Theorem 1).
(ii) OTP with Latest Horizon (OTP-L). In Section 4, we discuss OTP-L with only knowing the latest horizon N^+ ($N^+ \geq N$). We propose the *Latest Horizon Algorithm (LHA)* (Alg. 1). LHA achieves a tight and optimal competitive ratio (Theorems 2–4), which strictly improves upon $1 + \ln(U/L)$ (Corollary 1) and converges to it when N^+ approaches infinity (Corollary 2).
(iii) OTP with Earliest and Latest Horizons (OTP-ErL). In Section 5, we discuss the OTP-ErL case where both the earliest horizon N^- and the latest horizon N^+ are known ($N^- \leq N \leq N^+$). We propose the *Earliest-and-Latest Horizon Algorithm (ErLHA)* (Alg. 2). ErLHA achieves a tight competitive ratio (Theorems 5–6), and attains the best achievable result under its algorithm structure (Theorem 7), recovers the optimal competitive ratio in the OTP with no horizon information, OTP-Er, and OTP-L (Theorem 8), and is asymptotically optimal in OTP-ErL (Theorem 9).
(iv) OTP with Exact Horizon (OTP-Ex). Also in Section 5, we discuss OTP-EX and propose the *Exact Horizon Algorithm (ExHA)* as a special form of ErLHA when the earliest and latest horizons coincide and become the exact horizon N (Alg. 3), which achieves the optimal competitive ratio $\alpha^*_{\text{Ex}}(N) = 1 + \Phi(N, \omega)$, where Φ is defined through a hypergeometric function (Ritelli et al., 2021) and ω is a constant (Theorem 10).

In addition, we also provide a detailed comparison with the related literature in Appendix B, together with a broader overview of online problems from the perspective of horizon information.

2 PRELIMINARIES

The OTP with No Horizon Information (OTP-NH). For clarity, we refer to the OTP with no horizon information as OTP-NH. In OTP-NH, a decision-maker is trading 1 "dollar" into "yen". In time slots $i = 1,\ 2,\ \ldots,\ N$, exchange rates $p_1, p_2, \ldots, p_N$ arrive sequentially, one at a time. There are N time slots in total, where the knowledge of N, either exactly or within a range, constitutes horizon information. The number of time slots N is also the number of exchange rates, and it represents the trading horizon. In OTP-NH, the decision-maker does not know the trading horizon N in advance, and learns N only when the exchange-rate sequence ends and no further exchange rates arrive. The decision-maker does not know the value of each exchange rate p_i before it arrives, but only the lower and upper bounds $[L, U]$ of each exchange rate, such that $0 < L \leq p_i \leq U$ for all i. Without loss of generality, we assume that the upper bound U is strictly larger than the lower bound L, such that $L < U$. In the i-th time slot, the exchange rate p_i arrives. The decision-maker knows its value and must decide immediately and irrevocably the amount of dollars to trade at this exchange rate. If the decision-maker trades x_i dollars at the exchange rate p_i, they gain $p_i x_i$ yen. The decision-maker's goal is to maximize the amount of yen obtained after processing the N exchange rates ($\sum_{i=1}^{N} p_i x_i$) within the 1-dollar budget ($\sum_{i=1}^{N} x_i \leq 1$). OTP-NH can be characterized by:

$$\text{OTP-NH}: \max_{\{x_i\}_{\forall i \in [N]}} \sum_{i=1}^{N} p_i x_i \quad \text{s.t.} \sum_{i=1}^{N} x_i \leq 1, \quad x_i \geq 0, \forall i \in [N], \tag{1}$$

where the first constraint is the budget constraint, and the second constraint defines the decision space.

Competitive Ratio. We use I to denote a sequence of exchange rates, defined by the number of exchange rates N and the value of each exchange rate $\{p_i\}_{i \in [N]}$. We denote the total amount of yen obtained by an online algorithm ALG by $\text{ALG}(I)$, and denote as $\text{OPT}(I)$ the optimal amount of yen obtained by the offline optimal algorithm OPT, where I is fully known to OPT in advance, including

the number of exchange rates N and the value of each exchange rate p_i. To evaluate the performance of an online algorithm ALG, we use the competitive ratio and say ALG is α-competitive if

$$\sup_{\forall I} \text{OPT}(I)/\text{ALG}(I) \leq \alpha, \tag{2}$$

which is a standard performance metric in the one-way trading problem (El-Yaniv et al., 2001; Zhang et al., 2017; Cao et al., 2020).

OTP with Horizon Information. We consider that the decision-maker in OTP has access to horizon information, which provides knowledge of the value or the range of the trading horizon N. Such horizon information can arise in four distinct forms, each inducing a corresponding variant of OTP. **(i) OTP with Earliest Horizon (OTP-Er)**: The decision-maker knows an earliest horizon N^-, before which the sequence of exchange rates cannot terminate ($N \geq N^-$ for all exchange-rate sequences I). **(ii) OTP with Latest Horizon (OTP-L)**: The decision-maker knows a latest horizon N^+, by which the sequence of exchange rates must terminate, i.e., $N \leq N^+$ for all exchange-rate sequences I. **(iii) OTP with Earliest and Latest Horizon (OTP-ErL)**: The decision-maker knows an earliest and a latest horizon, between which the sequence of exchange rates terminates, i.e., $N^- \leq N \leq N^+$ for all exchange-rate sequences I. **(iv) OTP with Exact Horizon (OTP-Ex)**: OTP-Ex is a special case of OTP-ErL, where the two horizons coincide ($N^- = N^+$). In OTP-Ex, the decision-maker knows the exact number of time slots N (the exact horizon), which is the same across all exchange-rate sequences I.

3 OTP-Er: Optimality of $1 + \ln(U/L)$

In this section, we discuss our results on OTP-Er, where the decision-maker knows the earliest horizon. Although it is natural to expect that knowing the earliest horizon alone reduces uncertainty and should improve online performance, Theorem 1 shows that it yields no improvement.

Theorem 1. *(Optimal Result for OTP-Er) Given L, U, and N^-, $1 + \ln(U/L)$ is the best achievable competitive ratio in OTP-Er.*

Please refer to Appendix C for the complete proof of Theorem 1. Theorem 1 establishes the optimal competitive ratio for OTP-Er based on a key observation that any algorithm designed for OTP-Er could be slightly altered to achieve no worse performance in OTP-NH than in OTP-Er. We denote the best achievable competitive ratio in OTP-NH by α_0^*, such that $\alpha_0^* = 1 + \ln(U/L)$ (Cao et al., 2020). Let α_{Er}^* be the best achievable competitive ratio in OTP-Er. Then Theorem 1 directly implies

$$\alpha_{\text{Er}}^* = \alpha_0^* = 1 + \ln(U/L) \tag{3}$$

and that the knowledge of the earliest horizon cannot improve the best achievable competitive ratio for OTP-Er over that for OTP-NH. Indeed, though somewhat counterintuitive, any exchange-rate sequence in OTP-NH can be transformed into an instance of OTP-Er by prepending sufficiently many rates equal to L, and it is this structural correspondence that embeds the classical lower bound into OTP-Er, thereby nullifying the informational value of knowing only the earliest horizon. However, it remains a question whether the informational value of the earliest horizon in OTP-Er is dominated by the uncertainty of the infinitely distant latest horizon, instead of being trivial by its nature. We will investigate this question in the rest of this paper.

4 OTP-L: Improvement over $1 + \ln(U/L)$

In this section, we study OTP-L, and we focus on the nontrivial case $N^+ \geq 2$. We begin in Section 4.1 by introducing a parameterized design of LHA, then identify the best achievable competitive ratio for LHA in Section 4.2, and finally we establish the optimality of LHA and hence the intrinsic hardness of OTP-L in Section 4.3. In addition, in Section 4.3, we also analyze how the hardness of OTP-L varies with the latest horizon, offering a deeper understanding of the informational power of the latest horizon N^+ in OTP.

4.1 The Latest Horizon Algorithm (LHA): A Primal-Dual Approach

In this section, we develop a parameterized design of LHA using a primal-dual approach. LHA is inspired by, but fundamentally differs from, the traditional threshold-based solution to OTP-NH (Cao

Algorithm 1 Latest Horizon Algorithm (LHA)

1: **Initialize:** $q_0 \leftarrow 0$, $\beta(N^+) \leftarrow \left[1 + (N^+ - 1)(1 - (L/U)^{\frac{1}{N^+ - 1}})\right]^{-1}$;
2: **for** the i-th time slot, if p_i arrives: **do**
3: $q_i \leftarrow \max\{q_{i-1}, p_i\}$,
4: $\Delta q_i \leftarrow q_i - q_{i-1}$.
5: $x_i \leftarrow \beta(N^+)\Delta q_i / q_i$.
6: Convert x_i dollars into yen at rate p_i.
7: $D_i \leftarrow q_i$.
8: **end for**

et al., 2020), as we no longer rely on a threshold function. This is because the knowledge of the latest horizon fundamentally constrains exchange-rate dynamics, rendering threshold-based designs overly conservative: (i) Infinitesimal increases in exchange rates keep the ratio between the maximum and minimum exchange rates close to 1, narrowing the gap between online and offline performance. (ii) Conversely, covering the full range from L to U before N^+ requires substantial increases in exchange rates, creating opportunities for online algorithms to outperform threshold-based designs.

To capture these merits brought by the knowledge of the latest horizon, we employ the classic primal-dual approach, where we focus on the primal problem stated in Eq. (1) and the corresponding dual problems incurred at every online time slot $i \in [N]$:

$$\text{Dual}_i: \quad \min_{y_i} \quad y_i \quad \text{s.t.} \quad y_i \geq p_j, \forall j \in [i], \tag{4}$$

where y_i is the dual variable updated after processing the i-th exchange rate, and p_j is the j-th exchange rate (Buchbinder & Naor, 2009; Cao et al., 2020). Primal–dual theory will guide our approach to OTP-L via Lemma 1, with the detailed proof provided in Appendix D.

Lemma 1. *(Primal-Dual Sufficient Condition) For an online algorithm ALG with output $\{x_i\}$, let D_i denote the dual variable updated after processing the i-th exchange rate. Define $P_k = \sum_{i=1}^{k} x_i p_i$, $\forall k \in [N]$. ALG achieves an α-competitive ratio if* ***(i) (Feasibility Condition)*** *$\{x_i\}_{i\in[N]}$ are feasible solutions to the primal problem in Eq. (1); $\{D_i\}_{i\in[N]}$ are feasible solutions to the dual problems in Eq. (4);* ***(ii) (Initial Condition)*** *$\exists k \in [N]$, such that $P_k \geq \frac{1}{\alpha} D_k$; and* ***(iii) (Incremental Condition)*** *$\forall i \in \{k+1, \ldots, N\}$, there is $P_i - P_{i-1} \geq \frac{1}{\alpha}(D_i - D_{i-1})$.*

By Lemma 1, we design and summarize in Alg. 1 the Latest Horizon Algorithm (LHA), which is parameterized by $\beta(N^+)$ (Line 1). LHA maintains q_i, the maximum exchange rate observed up to time slot i (Line 3), initialized to $q_0 = 0$ (Line 1). We use $\Delta q_i := q_i - q_{i-1}$ to record the increment when a new maximum exchange rate is reached (Line 4). In each time slot, the algorithm converts $x_i = \beta(N^+)\Delta q_i / q_i$ dollars into yen at the exchange rate p_i (Lines 5–6). It then sets the dual variable $D_i \leftarrow q_i$ and proceeds to the next time slot (Line 7). Parameter $\beta(N^+)$ is the most critical component of LHA, balancing online performance against budget consumption. Its design is closely tied to the optimality of LHA, and we will derive and justify this choice in the next section.

4.2 The Competitive Ratio for LHA

In this section, we derive the competitive ratio for LHA (Alg. 1), along with the corresponding optimality of our parameter choice for $\beta(N^+)$. We first state the competitive ratio of LHA in Theorem 2, with the detailed proof provided in Appendix E.

Theorem 2. *(LHA Competitive Ratio) Given L, U, and N^+, the best competitive ratio for LHA is $\alpha_L^*(N^+) = 1 + (N^+ - 1)\left[1 - (L/U)^{1/(N^+-1)}\right]$, which is achieved when $\beta(N^+) = 1/\alpha_L^*(N^+)$.*

Theorem 2 establishes that $\alpha_{\text{L}}^*(N^+)$ is the best achievable competitive ratio for LHA, attained by setting $\beta(N^+) = 1/\alpha_{\text{L}}^*(N^+)$. This optimal choice of $\beta(N^+)$ is obtained by characterizing budget consumption under the most budget-demanding exchange-rate sequence, while maintaining consistent online performance across all exchange-rate sequences. Furthermore, we are also interested in how precisely this competitive ratio describes LHA's performance. Specifically, we denote by $\text{LHA}^*(I)$ the online performance of LHA on exchange-rate sequence I with our choice of $\beta(N^+)$,

consider the definition of competitive ratio (Eq. (2)), and then we also want to know whether there is any exchange-rate sequence I on which the ratio between $\mathrm{OPT}(I)$ and $\mathrm{LHA}^*(I)$ is exactly $\alpha_{\mathrm{L}}^*(N^+)$. We show the existence of such an exchange-rate sequence in Theorem 3.

Theorem 3. *(LHA Tightness) Given L, U, N^+, there exists an exchange-rate sequence I where $OPT(I)$ is exactly $\alpha_L^*(N^+)$ times $LHA^*(I)$.*

Please refer to Appendix F for the proof of Theorem 3. Theorem 3 directly shows that $\alpha_{\mathrm{L}}^*(N^+)$ can be tightly achieved by LHA on certain exchange-rate sequences, establishing that $\alpha_{\mathrm{L}}^*(N^+)$ is not only the best competitive ratio for LHA, but also the ***best achievable*** one.

4.3 The Optimal Result in OTP-L and the Power of the Latest Horizon

In this section, we first show the optimality of LHA in OTP-L, then we discuss the power of the knowledge of the latest horizon, and finally show that LHA also recovers the optimal result in OTP-NH. First, we show that LHA achieves the optimal competitive ratio in OTP-L in Theorem 4, with the detailed proof provided in Appendix G.

Theorem 4. *(LHA Optimality) Given L, U, and N^+, no algorithm can achieve a competitive ratio smaller than $\alpha_L^*(N^+)$.*

Theorem 4 implies the optimality of LHA and the intrinsic hardness of OTP-L. The optimality of LHA is established through the design of $\beta(N^+)$, which explicitly captures the trade-off between performance and budget usage on worst-case exchange-rate sequences. Next, we confirm the improvement brought by the latest horizon in Corollary 1, and investigate its behavior with an arbitrarily large latest horizon in Corollary 2. Please refer to Appendix H and Appendix I for the proofs of Corollaries 1 and 2, respectively.

Corollary 1. *(Latest Horizon Improvement) $\alpha_L^*(N^+) < \alpha_0^*$ and $\alpha_0^* - \alpha_L^*(N^+) = \ln(U/L) - (N^+ - 1)[1 - (L/U)^{1/(N^+-1)}]$.*

Corollary 2. *(Infinite Latest Horizon) When the latest horizon N^+ becomes arbitrarily large, we have $\lim_{N^+\to\infty} \alpha_L^*(N^+) = 1 + \ln(U/L)$.*

Corollary 1 confirms that knowing the latest horizon improves the best achievable competitive ratio for OTP-L, and thereby softens its hardness, while Corollary 2 shows that, as the latest horizon N^+ increases, the best achievable competitive ratio, and hence the hardness, for OTP-L ($\alpha_{\mathrm{L}}^*(N^+)$) approaches that of OTP-NH (α_0^*). These results are consistent with the intuition that the informational value of a latest horizon stems from the non-negligible granularity of exchange-rate increases, though it vanishes when the latest horizon is extremely distant.

5 OTP-ErL: Unification through One Framework

In this section, we study OTP-ErL and its special case OTP-Ex. We introduce the Earliest-and-Latest Horizon Algorithm (ErLHA) in Section 5.1, and we discuss ErLHA's competitive ratio in Section 5.2. Section 5.3 will show that ErLHA is optimal for OTP-NH, OTP-Er, and OTP-L, and is asymptotically optimal for OTP-ErL. In Section 5.4, we study OTP-Ex, and we propose the Exact Horizon Algorithm (ExHA) and discuss its optimality in OTP-Ex (El-Yaniv et al., 2001), which concludes the unification of ErLHA over OTPs with different horizon information. We further present numerical results for ErLHA in Section 5.5, illustrating the relationships among OTP variants with different forms of horizon information.

5.1 ErLHA: Exploiting the Earliest Horizon

Despite the negative results of Theorem 1, the earliest horizon can become valuable when combined with knowledge of the latest horizon in two ways: (i) First, given knowledge of the latest horizon, we could adaptively decide an amount $Y(q_{N^-}) \in [0, 1]$ of dollars to convert into yen at the earliest horizon N^- after observing the highest exchange rate q_{N^-} up to time slot N^-, which ensures a gain of $Y(q_{N^-})L$ yen and guarantees a competitive ratio even if no additional dollars are converted into yen. We call the budget $Y(q_{N^-})$ the ***adaptive early budget***, and the remaining budget $1 - Y(q_{N^-})$ the ***adaptive late budget***. (ii) Second, the guaranteed gain of $Y(q_{N^-})L$ allows ErLHA to maintain a

Algorithm 2 Earliest-and-Latest Horizon Algorithm (ErLHA)

1: **Input:** N^-, N^+, L, U.
2: **Initialize:** Set $q_0 \leftarrow q_0^*$ by Eq. (5), set $\beta_1 \leftarrow \beta_1^*$ by Eq. (9).
3: **for** time slot $i = 1, 2, \ldots, N^-$ **do**
4: $\quad q_i \leftarrow \max\{q_{i-1}, p_i\}$,
5: $\quad \Delta q_i \leftarrow q_i - q_{i-1}$.
6: $\quad x_i \leftarrow \frac{\beta_1 \Delta q_i}{q_i - L}$ if $\Delta q_i > 0$; otherwise $x_i \leftarrow 0$.
7: $\quad$ **if** $i = N^-$ **then**
8: $\quad\quad$ Set $Y(q_{N^-}) \leftarrow Y^*(q_{N^-})$ by Eq. (10).
9: $\quad\quad x_i \leftarrow x_i + (Y(q_{N^-}) - \sum_{j=1}^{N^-} x_j)$.
10: $\quad$ **end if**
11: $\quad$ Convert x_i amount of dollars at rate p_i.
12: **end for**
13: Set $\beta_2(q_{N^-}) \leftarrow \beta_2^*(q_{N^-})$ by Eq. (11).
14: **for** time slot $i = N^- + 1, ...N$ **do**
15: $\quad q_i \leftarrow \max\{q_{i-1}, p_i\}$,
16: $\quad \Delta q_i \leftarrow q_i - q_{i-1}$.
17: $\quad x_i \leftarrow \beta_2(q_{N^-}) \Delta q_i / q_i$.
18: $\quad$ Convert x_i amount of dollars at rate p_i.
19: **end for**

target competitive ratio α without making additional conversions at rates below $\alpha Y(q_{N^-})L$. Therefore, to achieve a competitive ratio α, we can safely ignore the exchange rates lower than a certain value q_0, while only converting dollars at higher exchange rates. We call the value q_0 the ***trading entry threshold***.

Based on the above strategies, we propose our Earliest-and-Latest Horizon Algorithm (ErLHA) in Alg. 2. ErLHA is parameterized by $q_0 \in [L, U]$, $\beta_1 \geq 0$, $Y(q_{N^-}) \in [0, 1]$, and $\beta_2(q_{N^-}) \geq 0$, where q_0 and β_1 are initialized before the first time slot (Line 2), and $Y(q_{N^-})$ and $\beta_2(q_{N^-})$ are initialized after observing the exchange rate p_{N^-} (and hence q_{N^-}) of time slot N^- (Lines 8 and 13).

ErLHA is summarized in Alg. 2. Before the first time slot, we set q_0 to q_0^* in Line 2 by

$$q_0^* := \begin{cases} L, & \text{when } N^+ > N^- \text{ and } 1 - \frac{1}{N^-} \leq \frac{Lf_M(L)}{U-L}, \\ \min\Big\{ q \in (L, U) \;\Big|\; \dfrac{L(U - Lf_M(q) - q)}{Lf_M(q) + q} & \text{when } N^+ > N^- \text{ and } 1 - \frac{1}{N^-} > \frac{Lf_M(L)}{U-L}, \\ \qquad = \inf\limits_{q_{N^-} \in [q, U]} X_M(q, q_{N^-}, N^-) \Big\}, & \\ \text{the unique solution to } \frac{L}{q} G(q, U, N^-) = 1, & \text{when } N^+ = N^-, \end{cases} \tag{5}$$

where $M := N^+ - N^-$ and $f_M(\cdot)$, $G(\cdot)$, and $X_M(\cdot)$ are functions defined as

$$f_M(x) := \begin{cases} M[1 - (x/U)^{\frac{1}{M}}], & \forall x \in [L, U], \text{ if } M > 0, \\ 0, & \forall x \in [L, U], \text{ if } M = 0, \end{cases}, \tag{6}$$

$$G(x, y, z) := z[1 - ((x - L)/(y - L))^{\frac{1}{z}}], \forall x \in [L, U], y \in [x, U], z \geq 1, \tag{7}$$

$$X_M(x, y, z) := (U - x)y / [G(x, y, z)(Lf_M(y) + y) + f_M(y)(y - x)]. \tag{8}$$

Note that we use the convention $X_M(q, q, N^-) = +\infty$, so including the boundary point $q_{N^-} = q$ in Eq. (5) does not affect the infimum. In Line 2, we also set β_1 to β_1^*, where

$$\beta_1^* := \begin{cases} \frac{UL}{(U - q_0^*)(Lf_M(q_0^*) + q_0^*)} - \frac{L}{U - q_0^*}, & \text{when } N^+ > N^-, \\ \frac{L}{q_0^*}, & \text{when } N^+ = N^-. \end{cases} \tag{9}$$

We use q_i to record the highest exchange rate up to p_i (Line 4), but only consider the exchange rates no less than q_0, such that $q_i := \max\{q_0, p_1, p_2, \ldots, p_i\}$. We use Δq_i to denote the increments in q_i, such that $\Delta q_i := q_i - q_{i-1}$ (Line 5). At each time slot before the earliest horizon N^-, we convert $x_i = \beta_1 \Delta q_i / (q_i - L)$ dollars into yen if $\Delta q_i > 0$; otherwise we make no conversion (Lines 6 and 11). Note that we do not convert any dollars before an exchange rate higher than q_0 arrives. At the earliest horizon N^- (Lines 7–10), we observe p_{N^-} and know the highest exchange rate q_{N^-} up to time slot N^-, then we set the adaptive early budget $Y(q_{N^-})$ to $Y^*(q_{N^-})$ in Line 8, where

$$Y^*(q_{N^-}) := \begin{cases} 1, \text{ when } N^+ = N^- \text{ or } q_{N^-} = U, \\ \frac{q_{N^-} - f_M(q_{N^-})\beta_1^*(q_{N^-} - q_0)}{Lf_M(q_{N^-}) + q_{N^-}}, \text{ when } N^+ > N^- \text{ and } q_{N^-} < U. \end{cases} \tag{10}$$

Any unused portion of budget $Y^*(q_{N^-})$ will be converted to yen at the exchange rate p_{N^-} (Line 9). After the earliest horizon N^- (Line 13), we set $\beta_2(q_{N^-})$ to $\beta_2^*(q_{N^-})$, where

$$\beta_2^*(q_{N^-}) = \begin{cases} 0, \text{ when } N^+ = N^- \text{ or } q_{N^-} = U, \\ (1 - Y^*(q_{N^-}))/f_M(q_{N^-}), \text{ when } N^+ > N^- \text{ and } q_{N^-} < U, \end{cases} \tag{11}$$

and we convert $x_i = \beta_2(q_{N^-})\Delta q_i/q_i$ dollars into yen in every time slot from $N^- + 1$ to $N \leq N^+$. Please refer to Appendix J for a detailed discussion of the complexity of solving for q_0^*.

5.2 The Best Achievable Competitive Ratio for ErLHA

In this section, we will discuss the competitive ratio of ErLHA with the parameter choices q_0^*, β_1^*, $Y^*(q_{N^-})$, and $\beta_2^*(q_{N^-})$, as well as its tightness and optimality within the design framework. We first provide the competitive ratio of ErLHA in Theorem 5, with the proof provided in Appendix K.

Theorem 5. *(Competitive Ratio of ErLHA) With the parameters $q_0 = q_0^*$, $\beta_1 = \beta_1^*$, $Y(q_{N^-}) = Y^*(q_{N^-})$, and $\beta_2(q_{N^-}) = \beta_2^*(q_{N^-})$, ErLHA produces feasible solutions to OTP-ErL and achieves a competitive ratio of*

$$\alpha_{ErL}^*(N^-, N^+) = \begin{cases} 1/\beta_1^*, \textit{ when } N^+ = N^-, \\ (Lf_M(q_0^*) + q_0^*)/L, \textit{ when } N^+ > N^-. \end{cases} \tag{12}$$

Theorem 5 provides a lower upperbound on the competitive ratio of ErLHA. Next, let ErLHA$^*(I)$ denote the amount of yen obtained by ErLHA on exchange-rate sequence I, we show in Theorem 6 the tightness of ErLHA and in Theorem 7 ErLHA's optimality within its design framework.

Theorem 6. *(ErLHA Competitive Ratio Tightness) There exist exchange-rate sequences I, such that ErLHA$^*(I) = OPT(I)/\alpha_{ErL}^*(N^-, N^+)$.*

Theorem 7. *(Best Achievable Competitive Ratio of ErLHA) $\alpha_{ErL}^*(N^-, N^+)$ is the best achievable competitive ratio for ErLHA under any parameter choices of q_0, β_1, $Y(q_{N^-})$, and $\beta_2(q_{N^-})$.*

Please refer to Appendices L and M for the proofs of Theorems 6 and 7, respectively. Theorem 6 shows that there exist exchange-rate sequences I on which the ratio between OPT(I) and ErLHA$^*(I)$ is exactly $\alpha_{\text{ErL}}^*(N^-, N^+)$, while Theorem 7 confirms that, under our parameter choices, ErLHA's competitive ratio $\alpha_{\text{ErL}}^*(N^-, N^+)$ is the best achievable competitive ratio under its two-phase design with adaptive budget allocation. In what follows, we show that ErLHA also stands as a unification of the optimal solutions for OTP with different forms of horizon information.

5.3 A Unified Framework with Optimality

In this section, we will show that ErLHA unifies the optimal solutions for these problems. To begin with, we first show that ErLHA recovers the best achievable competitive ratio in OTP-NH, OTP-Er, and OTP-L, as stated in Theorem 8, with the formal proof deferred to Appendix N.1

Theorem 8. *(ErLHA Optimality) ErLHA achieves the optimal competitive ratio in OTP-L, OTP-NH, and OTP-Er. Specifically, given L and U, we have (i) $\alpha_{ErL}^*(1, N^+) = \alpha_L^*(N^+)$ and (ii) $\lim_{N^+ \to +\infty} \alpha_{ErL}^*(1, N^+) = \alpha_0^* = 1 + \ln(U/L)$*

It is worth noting that, though ErLHA requires both the earliest horizon N^- and the latest horizon N^+ as inputs, it accepts $N^- = 1$ and a large N^+, thereby OTP-NH, OTP-Er, and OTP-L can be solved by ErLHA. Theorem 8 confirms that the ErLHA achieves the best achievable results in OTP-NH, OTP-Er, and OTP-L. Meanwhile, ErLHA achieves the optimal competitive ratio whenever the earliest horizon N^- approaches 1 in the OTP-ErL, thereby satisfying the asymptotic optimality in the OTP-ErL problem, as stated in Theorem 9, with the formal proof deferred to Appendix N.2.

Theorem 9. *(ErLHA Asymptotic Optimality) The competitive ratio $\alpha_{ErL}^*(N^-, N^+)$ is asymptotically optimal for OTP-ErL.*

Theorem 8 and Theorem 9 show the optimality of ErLHA in OTP-NH, OTP-Er, and OTP-L, as well as its converging behavior to the optimal solution in OTP-ErL. ErLHA therefore provides a unified framework for these problems. In addition, we also found that a less distant N^+ results in a no-worse competitive ratio of ErLHA, as stated in Lemma 2, with the formal proof deferred to Appendix N.3.

Algorithm 3 Exact Horizon Algorithm (ExHA)

1: **Input:** L, U, N.
2: **Initialize:** $q_0 \leftarrow q_0^*$ by Lemma 3, $\beta_1 \leftarrow L/q_0^*, Y \leftarrow 1$;
3: **for** time slot $i = 1, 2, \ldots, N$ **do**
4: $\quad q_i \leftarrow \max\{q_{i-1}, p_i\}$,
5: $\quad \Delta q_i \leftarrow q_i - q_{i-1}$.
6: $\quad x_i \leftarrow \beta_1 \Delta q_i/(q_i - L)$.
7: $\quad$ **if** $i = N$ **then**
8: $\quad\quad x_i \leftarrow x_i + (Y - \sum_{j=1}^{N} x_j)$.
9: $\quad$ **end if**
10: $\quad$ Convert x_i amount of dollars at rate p_i.
11: **end for**

Lemma 2. *(Impact of Latest Horizon) Given L, U, and N^-, we have $\alpha^*_{ErL}(N^-, N^+) \leq \alpha^*_{ErL}(N^-, N^+ + 1)$.*

Next, we show that ErLHA also unifies the optimal solution to OTP-Ex.

5.4 OTP-Ex: A Special Case of OTP-ErL

In this section, we discuss our results for the OTP-Ex problem. We focus on the nontrivial case $N \geq 2$. OTP-Ex is a special case of OTP-ErL in which the earliest and latest horizons coincide, i.e., $N^- = N = N^+$, so every exchange-rate sequence contains exactly N time slots. Consequently, ErLHA naturally yields feasible solutions for OTP-Ex. Indeed, our definitions of ErLHA's parameter choices q_0^*, β_1^*, $Y^*(q_{N^-})$, and $\beta_2^*(q_{N^-})$ cover the case $N^- = N^+$ (Eqs. (5)–(11)). We summarize these values in Lemma 3, and provide the detailed proof in Appendix O.

Lemma 3. *(ErLHA behavior in OTP-Ex) When $N^+ = N^- = N$, the optimal parameter choices for ErLHA are $Y^*(q_{N^-}) = 1$ and $\beta_2^*(q_{N^-}) = 0$ for all q_{N^-}, $\beta_1^* = \frac{L}{q_0^*}$, and $q_0^* = L(1 + \Phi(N, \omega))$, with $\Phi(N, \omega) := \frac{(N-1)(\phi(N,\omega))^N}{\omega}$, $\omega := (N-1)\left[N\left(\frac{L}{U-L}\right)^{\frac{1}{N}}\right]^{\frac{-N}{N-1}}$, and*

$$\phi(N, \omega) := \omega \cdot {}_{N-1}F_{N-2}\left(\begin{matrix} \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N} \\ \frac{2}{N-1}, \cdots, \frac{N-2}{N-1}, \frac{N}{N-1} \end{matrix} \middle| -\frac{N^N}{(N-1)^{N-1}}\,\omega^{N-1} \right), \tag{13}$$

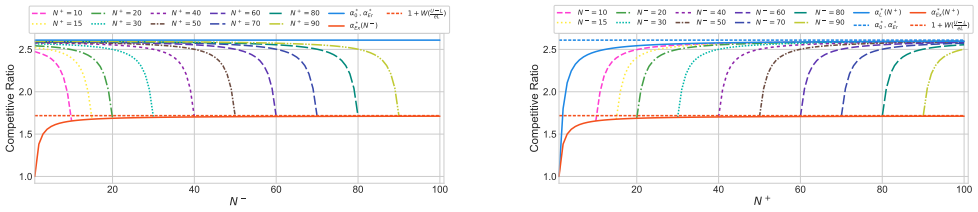
where ${}_{N-1}F_{N-2}$ is the hypergeometric function (Ritelli et al., 2021).

Lemma 3 specifies the optimal parameter choices of ErLHA when the earliest and latest horizons coincide. Based on Lemma 3, we provide in Alg. 3 the special form of ErLHA when the earliest and latest horizons coincide, and name it the Exact Horizon Algorithm (ExHA). By Lemma 3, we may, without loss of generality, omit the parameters β_2 and $Y(q_{N^-})$ and parameterize ErLHA solely by q_0, β_1, and Y. Before the first time slot (Line 2), we initialize q_0, β_1, and Y to $q_0 = q_0^*$, $\beta_1 = \frac{L}{q_0^*}$, and $Y = 1$, with $q_0^* = L(1 + \Phi(N, \omega))$ by Lemma 3. In the ExHA algorithm, we use q_i to record the highest exchange rate up to time slot i, but only consider those no less than q_0, such that $q_i = \max\{q_0, p_1, p_2, \ldots, p_i\}$ (Line 4). We use Δq_i to denote the increments in q_i, such that $\Delta q_i = q_i - q_{i-1}$ (Line 5). In each time slot $i = 1, \ldots N-1$, we convert $x_i = \frac{\beta_1 \Delta q_i}{q_i - L}$ dollars into yen (Lines 6 and 10). Note that we do not convert any dollars before an exchange rate higher than q_0 arrives. In the last time slot $i = N$, any remaining budget will be converted to yen at the exchange rate p_N (Line 8). Below we formulate in Theorem 10 the competitive ratio of ExHA, which is denoted by $\alpha^*_{\text{Ex}}(N)$. Please refer to Appendix P for the proof of Theorem 10.

Theorem 10. *(ExHA Competitive Ratio) The ExHA algorithm provides feasible solutions for OTP-Ex and achieves a competitive ratio of $\alpha^*_{Ex}(N) = 1 + \Phi(N, \omega)$ in the OTP-Ex problem.*

Theorem 10 gives the competitive ratio achieved by ExHA in OTP-Ex. Meanwhile, by Lemma 3 and Theorem 10, we notice that the competitive ratio $\alpha^*_{\text{Ex}}(N)$ is the unique solution to $\alpha = N\left[1 - [(L\alpha - L)/(U - L)]^{\frac{1}{N}}\right]$, which successfully recovers the optimal result given by El-Yaniv et al. (2001), directly implying the optimality of ExHA for OTP-Ex in Theorem 11.

Theorem 11. *(ExHA Optimality) Given L, U, and N, no online algorithm can achieve a competitive ratio smaller than $\alpha^*_{Ex}(N)$ in the OTP-Ex problem.*



(a) ErLHA's competitive ratio as N^- increases. (b) ErLHA's competitive ratio as N^+ increases.

Figure 1: The Competitive ratio of ErLHA approaches $\alpha^*_{\text{Ex}}(N)$. $U/L = 5$.

Please refer to Appendix Q for the proof of Theorem 11. Since the ExHA algorithm is a special form of ErLHA, Theorem 11 successfully extends the optimality of ErLHA to OTP-Ex beyond Theorem 8. Theorem 11 also quantifies the intrinsic hardness of OTP when an exact horizon is known. Next, we investigate the behavior of the optimal competitive ratio $\alpha^*_{\text{Ex}}(N)$ as the exact horizon grows arbitrarily large in Lemma 4, as well as the relationship between the competitive ratios $\alpha^*_{\text{ErL}}(N^-, N^+)$ and $\alpha^*_{\text{Ex}}(N)$ in Corollary 3.

Lemma 4. *(Infinite Exact Horizon) Given L and U, when N becomes arbitrarily large, we have $\lim_{N\to\infty} \alpha^*_{Ex}(N) = 1+W\left((U-L)/(eL)\right) < 1+\ln(U/L)$, where $W(\cdot)$ is the Lambert-W function.*

Corollary 3. *Given L and U, let N_1, N_2, and N_3 be three integers that satisfy $1 \le N_1 = N_2 \le N_3$, we have $\alpha^*_{Ex}(N_1) \le \alpha^*_{ErL}(N_2, N_3)$.*

Please refer to Appendices R and S for proofs of Lemma 4 and Corollary 3, respectively. Lemma 4 shows that an arbitrarily distant exact horizon can diminish the benefit of earliest-horizon information. Corollary 3 verifies that OTP-ErL becomes easier as the two horizons converge, which matches the intuition that OTP-Ex is a special case of OTP-ErL, and it completes a unified theoretical picture of the power conferred by different forms of horizon information in OTP.

Taken together, these results show that ErLHA matches the optimal behavior across all forms of horizon information and attains optimal results in each corresponding OTP setting. As such, ErLHA provides, to our knowledge, the first unified framework that addresses OTP under varying availability of horizon information.

5.5 Numerical Results

In addition, we present the values and trends of competitive ratios α^*_{ErL}, α^*_0, α^*_{Er}, α^*_{L}, and α^*_{Ex} under different earliest and latest horizons in Figure 1. Figure 1, together with our optimality analysis in Theorems 1, 4, 9, and 11, reveals the underlying connection among different forms of horizon information by discovering the consistent pattern in how they interact with the uncertainty and hardness of OTP. When the latest horizon N^+ is extremely distant, it dominates the hardness of the problem, so that knowing only the earliest horizon brings no improvement. As N^+ decreases, the remaining uncertainty becomes manageable by the knowledge of an earliest horizon N^-. As the two horizons converge, the uncertainty and hardness of OTP decrease, reaching their minimum when the horizons coincide. Please refer to Appendices U and T for more numerical results and more details on the experiment setup and explanation.

6 Conclusion and Future Work

In this work, we investigate the power of horizon information in OTP. The results show that an extremely distant latest horizon dominates the hardness of OTP, and it is only when the latest horizon decreases that the remaining uncertainty becomes manageable by the knowledge of an earliest horizon. We propose a unified algorithm, ErLHA, that achieves asymptotically optimal performance that recovers the optimal results in OTP-NH, OTP-Er, OTP-L, and OTP-Ex. Our numerical results further illustrate that the competitive ratio of ErLHA consistently improves as the horizon window narrows. Collectively, these findings establish a unified understanding of the intrinsic power of horizon information in OTP and unveil the underlying connection among different forms of horizon.

AI USE STATEMENT

In this work, we used generative AI tools to assist with translation. We have not used generative AI tools for helping develop theoretical models or conceptual frameworks, formulating mathematical claims, providing critical ingredients for proving mathematical claims, assisting in the writing of proofs, designing or providing feedback on research methodology or experiments, implementing methods, and interpreting results, while generating synthetic data sets, proposing or refining hypotheses, cleaning or reformatting datasets, or supporting qualitative and thematic data analysis are not applicable to this work. Additionally, we used generative AI tools for editing a research paper to improve readability and for proposing a title or keywords for a research paper. We have reviewed all AI-assisted work. We manually checked and confirmed all the AI-suggested rewording, rephrasing, and corrections for grammar and spelling. We take responsibility for the final content of this work, including text, claims, or artifacts produced with the aid of generative AI.

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A Real-world Scenarios

In addition to Section 1, we provide additional details on the real-world motivation for studying OTP under different forms of horizon information in this section. The disclosure and regulation of the trading horizon are widely used as necessary protection for investors in practice. From the information economics perspective, if the information of the trading horizon is opaque or effectively known only to a subset of market participants, it exacerbates information asymmetry, such that disadvantaged traders cannot properly assess the timing or the risk of trading, which can further trigger adverse selection and deteriorate market liquidity (Akerlof, 1978; Glosten & Milgrom, 1985; Kyle, 1985). Therefore, real-world markets often rely on mandatory disclosure and procedural rules to maintain the transparency of the trading horizon in order to strengthen the protection for investors. The disclosure of the trading horizon typically involves several different forms of deadline information:

(i) An **earliest** deadline, before which trading cannot terminate. In practice, the earliest deadline arises from a minimum initial period guaranteed by contracts with the auto-renewal clause. For example, a foreign-exchange dealing contract with an auto-renewal clause may specify a minimum contract duration, before which trading is guaranteed to remain open (Refinitiv US SEF LLC, 2019; Torcasio & Vise, 2023).

(ii) A **latest** deadline, by which trading must terminate. The latest deadline is typically imposed by regular session closing times together with volatility-control mechanisms in the financial market. For instance, intraday trading in the U.S. equity market normally ends at the scheduled market close, although extreme conditions may trigger an earlier halt (U.S. Securities and Exchange Commission, n.d.).

(iii) An an **earliest-and-latest** deadline window, within which trading terminates. Such a deadline window is naturally induced by callable bonds with call protection. For example, for a callable bond, the issuer can buy back the bond anytime after the first-call date, regulated by the call protection, but no later than the maturity date (Chen, 2026).

(iii) An **exact** deadline, at which trading terminates, corresponding to the special case where the earliest and latest deadlines coincide. A canonical example is the maturity date for the non-callable bond specified by the contract. At maturity, the issuer must repay the investor, at which point the security is extinguished and the trading ends sharply (Financial Industry Regulatory Authority (FINRA), 2026).

B Related Works

The one-way trading problem with horizon information is connected to the topics of the one-way trading problem, the online knapsack problem, and learning-augmented online algorithms. However, none of the existing work within these frameworks provides a unified solution to leverage different horizon information in OTP.

B.1 One-way Trading Problem

The One-way Trading Problem (OTP) is also called the Online Conversion Problem, which was pioneered by El-Yaniv et al. (2001), where several variants of OTP were investigated, including the setting with the exact horizon known to the decision-maker. When the exact horizon N is known, they achieved an optimal competitive ratio α characterized as the unique solution to $\alpha = N\left(1 - \left(\frac{\alpha L - L}{U - L}\right)^{1/N}\right)$. For the OTP without horizon information, Lin et al. (2019); Zhang

et al. (2017); Zhou et al. (2008); Cao et al. (2020) achieved the optimal $1 + \ln(U/L)$ competitive ratio. Wang et al. (2025b) recovered the optimal guarantees for OTP both without horizon information and with an exact horizon, by studying delayed reveal and imperfect knowledge of the exact horizon. However, their work is orthogonal to ours. Our work fundamentally differs in that we systematically investigate different forms of horizon information (earliest, latest, earliest-latest horizon window, and exact), whereas they focus on timing and accuracy of a single form of horizon (the exact horizon). Furthermore, our algorithms depart from the traditional threshold-based design adopted in their work, as we showed in Section 4.1.

Other extensions of OTP also consider prediction or query of future exchange rates, fairness, and buyable budget (Elenter et al., 2024; Cao et al.; Hu & Zhang, 2025; Tan et al., 2025). However, none of these works study the intrinsic value of different horizon information in OTP or the relationships among different horizon information.

B.2 Online Knapsack Problem

The online knapsack problem (OKP) is closely related to the one-way trading problem, and Cao et al. (2020) proposed a unified algorithm that achieves an optimal result of $(1 + \ln(U/L))$ for both OTP and OKP. In OKP, the decision maker selects from a sequence of items to maximize the total reward without exceeding the cost budget. The reward and cost of each item are revealed to the decision-maker when the item arrives. Recent work by Wu et al. (2025) considered the variant where the total weight information is known, recovering the optimal result of El-Yaniv et al. (2001) in the context of OKP. The work closest to ours is Balseiro et al. (2023), in which the authors studied an earliest–latest horizon window in a stochastic OKP model. However, our work fundamentally differs in that we consider the exchange rates arranged by an adversary and investigate connections among different horizon information beyond optimal solution to a single case, whereas they consider item values drawn from distributions and focus primarily on deriving optimal results for one specific setting. Furthermore, our horizon-window algorithm ErLHA uses a one-shot computation of fixed parameters, differing from, and yielding lower per-time-slot complexity than, the dual mirror-descent approach in Balseiro et al. (2023).

Other extensions of OKP also considered multi-knapsack, multi-dimensional knapsack, item departure, and delayed decision (Sun et al., 2020; Zhang et al., 2017; Yang et al., 2021; Sun et al., 2022; Burjons & Gehnen, 2025). For a comprehensive overview of recent developments in OKP, we refer the reader to the recent survey by Böckenhauer et al. (2026). However, none of these works consider the impact of horizon information on online decision-making, and they are thereby orthogonal to ours.

B.3 Horizon in Online Problems

Traditionally, online problems are studied under default horizon assumptions inherited from their canonical formulations, with limited attention paid to how different forms of horizon information shape the decision process. From the perspective of horizon information, existing work can be broadly organized into three categories: ex-ante known horizons, dynamic horizons, and adversarial horizons.

Ex ante known horizons. The first category consists of settings in which the horizon information is **fully specified at the beginning** of the process, mainly consisting of three typical assumptions. One of them is the infinite horizon, which is widely adopted in online learning (Cesa-Bianchi et al., 1997) and reinforcement learning, including infinite-horizon, time-average, and discounted-sum Markov decision process (MDP) formulations. For example, time-average Markov decision processes (MDPs) seek an optimal control policy over an infinite horizon (Bai et al., 2024), and time-average partially observable MDPs (POMDPs) provide a strict generalization (Russo et al., 2025). A second assumption is the known finite horizon, as in finite-horizon online learning (Wang et al., 2025a) and bandit problems (XIONG et al., 2022), where the horizon length can be directly exploited in algorithm design and parameter tuning. There are also variants of MDPs and POMDPs considering known finite horizons, which aim to maximize discounted cumulative reward (Guin & Bhatnagar, 2023; Barenboim & Indelman, 2023). A third assumption considers random-horizon models, in which the horizon length is drawn from a known distribution, allowing performance anal-

ysis to explicitly leverage distributional information despite uncertainty in the realized termination time (Luo & Schapire, 2014).

Dynamic horizons. A second line of work studies settings in which the termination time **emerges endogenously from the decision process**. One representative example is resource-driven termination, where the process ends once a budget, inventory, or other limited resource is exhausted, as in bandits with knapsacks (Immorlica et al., 2022), budget-constrained online optimization problems (Balseiro et al., 2023), and the wide scope of online Bayesian optimization (Feldman et al., 2021). Another class arises when termination is a deliberate choice of the decision maker, as in optimal stopping problems (Rubinstein, 2016). More generally, dynamic horizons may also be stochastic and action-dependent, as in MDPs or partially observable MDPs with absorbing or failure states whose occurrence depends on the actions taken (Dufour & Prieto-Rumeau, 2024; Kurniawati, 2022).

Adversarial horizons. The third category encompasses settings in which the termination time is **neither known in advance nor controllable by the decision maker**. In such adversarial or unknown-horizon models, algorithms are required to perform well regardless of when the process ends. This perspective underlies anytime algorithms in online learning and online optimization (Besson & Kaufmann, 2018). Representative examples include classical ski rental and its extensions (Wu et al., 2021; Meyerson, 2005), as well as online trading problems such as OTP (El-Yaniv et al., 2001), online knapsack problems (OKP) (Cao et al., 2020), and magician's problem (Alaei et al., 2013). In this work, we study OTP under the adversarial horizon setting, and further investigate how partial horizon information—specifically, earliest and latest horizons—affects algorithmic performance.

It is worth noting that even for the same underlying problem, different assumptions on horizon information can lead to substantial technical differences. For example, in online learning and bandit problem, some works assume that the horizon length T is known in advance (Wang et al., 2025a; XIONG et al., 2022), while others adopt an infinite or adversarial horizon assumption (Cesa-Bianchi et al., 1997; Besson & Kaufmann, 2018). In the latter setting, techniques such as the doubling trick are typically employed to obtain anytime performance guarantees (Cesa-Bianchi et al., 1997; Besson & Kaufmann, 2018). Similarly, the traditional bandit problems assume a known and finite horizon (XIONG et al., 2022), while the bandit with knapsack variants has its termination time determined by the knapsack (resource) usage (Immorlica et al., 2022). In the latter setting, repeated Lagrangian game is adopted as the major technique. These examples highlight the broad interest within the community in studying classical theoretical problems under different horizon assumptions, and also underscore the necessity of new techniques to address existing problems under alternative horizon models.

B.4 Learning-augmented Online Algorithm

Learning-augmented online algorithms leverage the value of imperfect predictions of problem properties to improve the online decision-making (Purohit et al., 2018; Lykouris & Vassilvitskii, 2021). Studies related to our work include OTP with predictions of the maximum exchange rate (Elenter et al., 2024), OKP with frequency prediction (Im et al., 2021); online resource allocation with predictions of demand (Hwang et al., 2021); airline fare class allocation with predictions of customer distribution (Balseiro et al., 2025), and OKP with predictions of total item weight (Wu et al., 2025). However, none of these works examine the informational value of horizon knowledge itself, which is the focus of our study. Wang et al. (2025b) studied the OTP with predictions of the exact deadline, but it is orthogonal to ours in terms of both the problem setting and algorithm design, as we detailed in Section B.1. To the best of our knowledge, we are the first to investigate and evaluate the informational value of different horizon knowledge via a unified framework in OTP.

C Proofs of Theorem 1

(Optimal Result for OTP-Er) Given L, U, and N^-, $1 + \ln(U/L)$ is the best achievable competitive ratio in OTP-Er.

Algorithm 4 Classic Algorithm (CALG)

1: **Initialize:** $\phi(y)$, $y_0 = 0$;
2: **for** the i-th time slot, if p_i arrives: **do**
3: $x_i \leftarrow \arg\max_{x \in \mathbb{R}_+} p_i x - \int_{y_{i-1}}^{y_{i-1}+x} \phi(\delta) d\delta$
4: Update $y_i \leftarrow y_{i-1} + x_i$.
5: If $y_i > 1, x_i \leftarrow 0, y_i \leftarrow y_{i-1}$.
6: Convert x_i dollars into yen.
7: **end for**

Proof. To prove this theorem, we basically show that the algorithm introduced by Cao et al. (2020) achieves the best achievable competitive ratio in OTP-Er. First, we provide the algorithm introduced by Cao et al. (2020) as Classic Algorithm (CALG) in Alg. 4, where $\phi(y)$ is a threshold function defined as

$$\phi(y) = \begin{cases} L, & y \in [0, \frac{1}{1+\ln(U/L)}], \\ \frac{L}{e} e^{(1+\ln(U/L))y}, & y \in [\frac{1}{1+\ln(U/L)}, 1]. \end{cases} \tag{14}$$

Next, let's recall some conclusions from Cao et al. (2020):

Lemma 5. *(Reproduced from Theorem 3 in Cao et al. (2020)) Given L and U, the best competitive ratio that can be achieved by CALG in OTP-NH is $1 + \ln(U/L)$.*

Lemma 6. *(Reproduced from Theorem 4 in Cao et al. (2020)) Given L and U, $1 + \ln(U/L)$ is the lowest possible competitive ratio for OTP-NH.*

Note that Lemma 5 shows that CALG achieves the competitive ratio $1 + \ln(U/L)$ in OTP-NH, and Lemma 6 shows that $1 + \ln(U/L)$ is optimal in this case. It is straightforward that, given L, U, and N^-, if an exchange-rate sequence $\{p_i\}_{i \in [N]}$ is an exchange-rate sequence of the OTP-Er problem, it is also an exchange-rate sequence of OTP-NH. Together with the above analysis, Lemma 5 leads to the following results of CALG in OTP-Er:

Lemma 7. *Given L, U, and N^-, the best competitive ratio that can be achieved by CALG in OTP-Er is $1 + \ln(U/L)$.*

Proof. Given L, U, and N^-, let $\mathcal{I}$ denote the set of exchange-rate sequences of OTP-NH, and let $\mathcal{I}'$ denote the set of exchange-rate sequences of OTP-Er. By Lemma 5, we have

$$\sup_{I \in \mathcal{I}} \frac{\text{OPT}(I)}{\text{CALG}(I)} \leq 1 + \ln(U/L). \tag{15}$$

Since $\mathcal{I}' \subseteq \mathcal{I}$, we have

$$\sup_{I \in \mathcal{I}'} \frac{\text{OPT}(I)}{\text{CALG}(I)} \leq \sup_{I \in \mathcal{I}} \frac{\text{OPT}(I)}{\text{CALG}(I)} \leq 1 + \ln(U/L). \tag{16}$$

□

Lemma 7 shows that CALG also achieves the $1+\ln(U/L)$ competitive ratio in OTP-Er. Furthermore, we establish the relationship between the competitive ratios of OTP-NH and OTP-Er, as stated in the following lemma:

Lemma 8. *Given L and U, N^-, let $\mathcal{I}$ be the set of exchange-rate sequences of OTP-NH, and let $\mathcal{I}'$ be the set of exchange-rate sequences of OTP-Er with the earliest horizon N^-. For every $I \in \mathcal{I}$ and every online algorithm ALG for OTP-Er, there exists $I' \in \mathcal{I}'$ and an online algorithm ALG$'$ for OTP-NH, such that*

$$\frac{OPT(I')}{ALG(I')} \geq \frac{OPT(I)}{ALG'(I)}. \tag{17}$$

Proof. For any exchange-rate sequence $I = \{p_i\}_{i\in[n]} \in \mathcal{I}$, there exists $I' = \{p'_i\}_{i\in[N^-+n]} \in \mathcal{I}'$ for OTP-Er, where

$$\begin{cases} p'_i = L, \forall i \in [N^-], \\ p'_i = p_{i-N^-}, \forall i \in [N^- + 1, N^- + n]. \end{cases} \tag{18}$$

Then, for any online algorithm ALG for OTP-Er that converts x_i dollars at exchange rate p'_i, we can easily modify it and create an algorithm ALG$'$ for OTP-NH, where ALG$'$ converts x'_i dollars at exchange rate p_i, such that

$$\begin{cases} x'_1 = \sum_{j=1}^{N^-+1} x_j, \\ x'_i = x_{N^-+i}, \forall i \in [2, n]. \end{cases} \tag{19}$$

Obviously, we have OPT(I)=OPT(I'). The algorithm variant ALG$'$ could be created by virtually running ALG on the exchange rates sequence $\{p'_i\}_{i\in[N^-+1]}$ when the first exchange rate p_1 arrives in I, and then converting an amount of $\sum_{j=1}^{N^-+1} x_j$ dollars at p_1. Since $p_1 \geq L = p'_i, \forall i \in [N^-]$, it implies that (i) the total yen obtained by ALG on I' in OTP-Er is

$$\mathrm{ALG}(I') = L\sum_{j=1}^{N^-} x_j + p_1 x_{N^-+1} + \sum_{i=2}^{n} p_i x_{N^-+i}; \tag{20}$$

(ii) by construction, the total yen obtained by ALG$'$ on I in OTP-NH is

$$\mathrm{ALG}'(I) = p_1 \sum_{j=1}^{N^-+1} x_j + \sum_{i=2}^{n} p_i x_{N^-+i}. \tag{21}$$

Since $p_1 \geq L$, we have $\mathrm{ALG}'(I) \geq \mathrm{ALG}(I')$, and thereby

$$\frac{\mathrm{OPT}(I')}{\mathrm{ALG}(I')} \geq \frac{\mathrm{OPT}(I)}{\mathrm{ALG}'(I)}. \tag{22}$$

□

Lemma 8 shows that, for any online algorithm ALG that is designed for OTP-Er, there exists a counterpart ALG$'$ for OTP-NH achieving a no-worse competitive ratio than ALG in OTP-Er. In other words, the optimal competitive ratio for OTP-Er is no better than that for OTP-NH. Now that CALG achieves a competitive ratio of $1 + \ln(U/L)$ in OTP-Er, and $1 + \ln(U/L)$ is optimal in OTP-NH, it is also the best achievable result in OTP-Er. This theorem is proved. □

D Proof of Lemma 1

(Primal-Dual Sufficient Condition) For an online algorithm ALG with output $\{x_i\}$, let D_i denote the dual variable updated after processing the i-th exchange rate. Define $P_k = \sum_{i=1}^{k} x_i p_i, \forall k \in [N]$. ALG achieves an α-competitive ratio if:

(i) (Feasibility Condition) $\{x_i\}_{i\in[N]}$ are feasible solutions to the primal problem in Eqs. (1; $\{D_i\}_{i\in[N]}$ are feasible solutions to the dual problems in Eq. (4).

(ii) (Initial Condition) $\exists k \in [N]$, such that $P_k \geq \frac{1}{\alpha} D_k$.

(iii) (Incremental Condition) $\forall i \in \{k+1, \ldots, N\}$, there is $P_i - P_{i-1} \geq \frac{1}{\alpha}(D_i - D_{i-1})$.

Proof. The feasibility condition is necessary to satisfy the inherent constraint of OTP-L, so we do not provide further discussion. We show how the initial and incremental conditions lead to a competitive ratio of α.

Since $P_N = \sum_{i=1}^{N} x_i p_i$, so for an arbitrary exchange-rate sequence $I = \{p_i\}_{i\in[N]}$, the online performance of ALG is

$$\mathrm{ALG}(I) = P_N. \tag{23}$$

For P_N, we have

$$P_N = (P_N - P_{N-1}) + (P_{N-1} - P_{N-2}) + \cdots + (P_{k+1} - P_k) + P_k \tag{24}$$

$$= P_k + \sum_{i=k+1}^{N} (P_i - P_{i-1}). \tag{25}$$

Similarly, for D_N, we have

$$D_N = D_k + \sum_{i=k+1}^{N} (D_i - D_{i-1}). \tag{26}$$

By the initial and incremental conditions and Eqs. (25)–(26), we have

$$P_N \geq \frac{1}{\alpha} D_N. \tag{27}$$

By the feasibility condition, D_N is a feasible solution to the dual problem at time slot N (Eq. (4)). We denote the optimal value of the dual problem at time slot N as D^*, then it implies

$$D_N \geq D^*. \tag{28}$$

Since the optimal solution to the primal problem is $\text{OPT}(I)$, by the weak duality, we have

$$D^* \geq \text{OPT}(I). \tag{29}$$

By the definition of $\text{ALG}(I)$ and P_N, they naturally implies $\text{ALG}(I) = P_N$. Therefore, we reach the following chain inequality:

$$\text{ALG}(I) = P_N \geq \frac{1}{\alpha} D_N \geq \frac{1}{\alpha} D^* \geq \frac{1}{\alpha} \text{OPT}(I). \tag{30}$$

□

E Proof for Theorem 2

Before we can prove Theorem 2, we need to first state and prove Lemma 9, as we will show in Appendix E.1. After that, we will show the proof of Theorem 2 in Appendix E.2.

E.1 Lemma 9 and its Proof

Lemma 9. *(LHA Competitive Ratio Sufficient Condition) The LHA algorithm is α-competitive if: (i) $\sum_{i=1}^{N} x_i \leq 1$ and (ii) $\beta(N^+) \geq \frac{1}{\alpha} > 0$.*

Proof. We first show that the design of the LHA algorithm, along with $\sum_{i=1}^{N} x_i \leq 1$ and $\beta(N^+) \geq \frac{1}{\alpha}$, satisfies the feasibility condition. By $q_0 = 0$, $p_1 \geq L > 0$, $q_i = \max\{q_{i-1}, p_i\}$, $\Delta q_i = q_i - q_{i-1}$, $\beta(N^+) > 0$, and $x_i = \frac{\beta(N^+)\Delta q_i}{q_i}$, we have

$$x_i = \frac{\beta(N^+)\Delta q_i}{q_i}. \tag{31}$$

Since $\beta(N^+) > 0$, $\Delta q_i \geq 0$, and $q_i \geq L > 0$, we have $x_i \geq 0$. Then, Eq. (31) and condition (i) guarantee that $\{x_i\}$ are feasible solutions to the primal problem. Moreover, since $D_i = q_i = \max_{j \in [i]} p_j$, we have $D_i \geq p_j$ for all $j \in [i]$ and all $i \in [N]$. Hence, each D_i is feasible for the dual problem Dual_i at time slot i. The feasibility condition is satisfied.

Next, we show that LHA satisfies the initial and incremental conditions with $\beta(N^+) \geq \frac{1}{\alpha} > 0$. At $i = 1$, we have $q_1 = \max\{0, p_1\}$, so $D_1 = q_1 = p_1$. We also have $\Delta q_1 = q_1 - 0 = q_1$. Therefore, we have:

$$P_1 = x_1 p_1 = x_1 q_1 = \beta(N^+)\Delta q_1 = \beta(N^+) q_1 \geq \frac{q_1}{\alpha} = \frac{1}{\alpha} D_1, \tag{32}$$

and the initial condition is satisfied at the first time slot.

For the incremental inequality, we consider an arbitrary time slot $i \in [2, N]$, and there are two cases. (i) When $\Delta q_i = 0$, we have $q_i - q_{i-1} = 0$, and hence $D_i - D_{i-1} = 0$, $x_i = 0$, and $P_i - P_{i-1} = 0$. Therefore, the incremental condition is satisfied in this case. (ii) When $\Delta q_i > 0$, we have $p_i > q_{i-1}$, and hence $p_i = q_i$, $\Delta q_i = q_i - q_{i-1} = D_i - D_{i-1}$, and

$$x_i = \frac{\beta(N^+)\Delta q_i}{q_i} = \frac{\beta(N^+)(D_i - D_{i-1})}{p_i} \tag{33}$$

$$\text{and } P_i - P_{i-1} = x_i p_i = \beta(N^+)(D_i - D_{i-1}) \geq \frac{1}{\alpha}(D_i - D_{i-1}). \tag{34}$$

Therefore, the incremental condition is also satisfied in the second case.

In conclusion, we proved that $\sum_{i=1}^{N} x_i \leq 1$ and $\beta(N^+) \geq \frac{1}{\alpha} > 0$ guarantees that LHA satisfies the feasibility condition, initial condition, and the incremental condition. By Lemma 1, LHA achieves an α competitive ratio. □

Lemma 9 states sufficient conditions for the parameter choice in LHA to obtain a provable competitive ratio, which provides two concrete guidelines for selecting the parameter $\beta(N^+)$ in LHA. In particular, it reduces the choice of the parameter $\beta(N^+)$ to an optimization problem over $\beta(N^+)$ under the one-dollar budget constraint. This approach allows us to characterize the best achievable competitive ratio for LHA and to identify the optimal $\beta(N^+)$, as stated in Theorem 2.

E.2 Proof for Theorem 2

(LHA Competitive Ratio) Given L, U, and N^+, the competitive ratio of LHA is

$$\alpha_{\mathrm{L}}^*(N^+) = 1 + (N^+ - 1)\left(1 - \left(\frac{L}{U}\right)^{\frac{1}{N^+ - 1}}\right), \tag{35}$$

which is achieved when $\beta(N^+) = \frac{1}{\alpha_{\mathrm{L}}^*(N^+)}$.

Proof. By Lemma 9, the best competitive ratio that can be achieved by LHA can be found by minimizing the competitive ratio α, subject to $\beta(N^+) \geq \frac{1}{\alpha} > 0$ and $\sum_{i=1}^{N} x_i \leq 1$. Since, by the design of LHA, we have $x_i = \frac{\beta(N^+)\Delta q_i}{q_i}$, $\sum_{i=1}^{N} x_i \leq 1$ implies $\beta(N^+)\sum_{i=1}^{N} \frac{\Delta q_i}{q_i} \leq 1$. Therefore, the best competitive ratio that LHA can achieve is captured by the following minimization problem:

$$\min_{\beta(N^+),\alpha} \quad \alpha \tag{36}$$

$$\text{s.t.} \quad \beta(N^+)\sum_{i=1}^{N} \frac{\Delta q_i}{q_i} \leq 1, \tag{37}$$

$$\beta(N^+) \geq \frac{1}{\alpha}, \tag{38}$$

$$\frac{1}{\alpha} \geq 0. \tag{39}$$

Define $\lambda \triangleq \frac{1}{\beta(N^+)}$, and the minimization problem in Eqs. (36)–(39) is equivalent to

$$\min_{\lambda,\alpha} \quad \alpha \tag{40}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \frac{\Delta q_i}{q_i} \leq \lambda, \tag{41}$$

$$\alpha \geq \lambda, \tag{42}$$

$$\alpha > 0. \tag{43}$$

Since $\Delta q_1 = q_1 - q_0 \geq L > 0$, constraints (41) and (42) directly implies constraint (43). Also, it is straightforward that the minimum value of α is always obtained at $\alpha = \lambda$. Therefore, the optimal α can be obtained by setting $\alpha = \lambda$ and solving the following equivalent form:

$$\min_{\alpha} \quad \alpha \tag{44}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \frac{\Delta q_i}{q_i} \leq \alpha. \tag{45}$$

What remains is to find the minimal α that satisfies constraint (45). Without loss of generality, for the exchange rate sequence where $N < N^+$, we define $q_i = q_N, \forall i \in [N+1, N^+]$. This definition does not affect any previous derivation, and it aligns with the purpose of q_i, which is to record the highest exchange rate up to time slot i. For the left-hand side (LHS) of (45), we have:

$$\sum_{i=1}^{N} \frac{\Delta q_i}{q_i} \leq \sum_{i=1}^{N^+} \frac{\Delta q_i}{q_i} \tag{46}$$

$$= \sum_{i=1}^{N^+} \frac{q_i - q_{i-1}}{q_i} \tag{47}$$

$$= \frac{q_1}{q_1} + \sum_{i=2}^{N^+} \frac{q_i - q_{i-1}}{q_i} \tag{48}$$

$$= 1 + \sum_{i=2}^{N^+} \left(1 - \frac{q_{i-1}}{q_i}\right) \tag{49}$$

$$= 1 + (N^+ - 1) - \sum_{i=2}^{N^+} \frac{q_{i-1}}{q_i} \tag{50}$$

$$= N^+ - (N^+ - 1) \cdot \frac{1}{N^+ - 1} \sum_{i=2}^{N^+} \frac{q_{i-1}}{q_i} \tag{51}$$

$$\leq N^+ - (N^+ - 1) \left(\prod_{i=2}^{N^+} \frac{q_{i-1}}{q_i}\right)^{\frac{1}{N^+ - 1}} \tag{52}$$

$$= N^+ - (N^+ - 1) \left(\frac{q_1}{q_{N^+}}\right)^{\frac{1}{N^+ - 1}} \tag{53}$$

$$\leq N^+ - (N^+ - 1) \left(\frac{L}{U}\right)^{\frac{1}{N^+ - 1}} \tag{54}$$

$$= 1 + (N^+ - 1) \left(1 - \left(\frac{L}{U}\right)^{\frac{1}{N^+ - 1}}\right). \tag{55}$$

where inequality (46) comes from the latest horizon $N \leq N^+$, and the equality is attained when $N = N^+$; Eq. (48) comes from the LHA parameter setting $q_0 = 0$; the inequality (52) follows the AM-GM inequality, and the equality is taken when $\frac{q_{i-1}}{q_i} = \frac{q_{i'-1}}{q_{i'}}, \forall i, i' \in [2, N^+]$; the inequality (54) comes from the definition of q_i, where $q_i = \max_{j \in [i]} p_i, \forall i \in [N^+]$, and from the fact that $p_i \in [L, U]$. The equality in (54) is taken when $p_1 = L$ and $p_N = U$.

(46)–(55) implies that, to satisfy constraint (45), it suffices to guarantee $\alpha \geq 1 + (N^+ - 1)\left(1 - \left(\frac{L}{U}\right)^{\frac{1}{N^+ - 1}}\right)$. As a result, we define $\alpha_{\mathrm{L}}^*(N^+) \triangleq 1 + (N^+ - 1)\left(1 - \left(\frac{L}{U}\right)^{\frac{1}{N^+ - 1}}\right)$. Then, $\beta(N^+) = \frac{1}{\lambda} = \frac{1}{\alpha_{\mathrm{L}}^*(N^+)}$ and $\alpha = \alpha_{\mathrm{L}}^*(N^+)$ form a feasible solution to the minimization problem (36)–(39), which yields a competitive ratio of $\alpha_{\mathrm{L}}^*(N^+)$. Besides, $\alpha_{\mathrm{L}}^*(N^+)$ is the optimal value to (36)–(39) when (i) $N = N^+$ (inequality (46)), (ii) $\frac{q_{i-1}}{q_i} = \frac{q_{i'-1}}{q_{i'}}, \forall i, i' \in [2, N^+]$ (inequality (52)), and (iii) $p_1 = L$ and $p_N = U$ (inequality (55)). □

F Proof of Theorem 3

(LHA Tightness) Given L, U, N^+, there exists an exchange-rate sequence I where $\text{OPT}(I)$ is exactly $\alpha_{\text{L}}^*(N^+)$ times $\text{LHA}^*(I)$.

Proof. We show that, given L, U, N^+, we have $\text{OPT}(I) = \alpha_{\text{L}}^*(N^+)\text{LHA}^*(I)$ on every exchange rate sequence I. Specifically, on every exchange rate sequence, we have

$$\text{OPT}(I) = q_N, \tag{56}$$

since the offline optimal algorithm will always convert all the dollars into yen at the highest exchange rate. On the other hand, by the design of LHA, we have

$$\begin{aligned}
\text{LHA}^*(I) =& \sum_{i=1}^{N} x_i p_i && (57)\\
=& \sum_{i=1}^{N} \frac{\beta(N^+)\Delta q_i}{q_i} p_i && (58)\\
=& \sum_{i=1}^{N} \frac{\beta(N^+)\Delta q_i}{q_i} q_i \quad \text{(since } \Delta q_i > 0 \text{ only when } p_i = q_i > q_{i-1}\text{)} && (59)\\
=& \beta(N^+) \sum_{i=1}^{N} \Delta q_i && (60)\\
=& \frac{q_N}{\alpha_{\text{L}}^*(N^+)} \quad \text{(by the definition of } \Delta q_i \text{ and the relation between } \beta(N^+) \text{ and } \alpha_{\text{L}}^*(N^+)\text{)} && (61)\\
=& \frac{\text{OPT}(I)}{\alpha_{\text{L}}^*(N^+)} \quad \text{(by Eq. (56)),} && (62)
\end{aligned}$$

which proves this lemma. □

G Proof for Theorem 4

(LHA Optimality) Given L, U, and N^+, no algorithm can achieve a competitive ratio smaller than $\alpha_{\text{L}}^*(N^+)$.

Sketch of Proof. We focus on the most budget-demanding exchange-rate sequence that increases geometrically from $p_1 = L$ to $p_{N^+} = U$, and discuss the potential behavior of alternative algorithms. We show that LHA provides the highest possible performance ratio against OTP: (i) if an algorithm trades fewer dollars than LHA at an exchange rate p_i, the adversary can immediately terminate the exchange-rate sequence to induce a worse performance ratio; (ii) if an algorithm wants to achieve a higher performance ratio by trading more dollars at an exchange rate, the remaining budget will be insufficient for it to keep such a higher performance ratio. □

Proof. To prove this theorem, we first construct a set of worst-case exchange-rate sequences and prove that no online algorithm can achieve a competitive ratio smaller than $\alpha_{\text{L}}^*(N^+)$ on them.

Given L, U, and N^+, we define exchange rate sequence $I_{N^+}^*$ as $I_{N^+}^* \triangleq \{p_i^*\}_{i\in[N^+]}$, where

$$p_i^* = L\left(\frac{U}{L}\right)^{\frac{i-1}{N^+-1}}, \forall i \in [N^+]. \tag{63}$$

Next, we define $I_N^* \triangleq \{p_i^*\}_{i\in[N]}$, and define the set of worst-case exchange rate sequence as $\mathcal{I}^* \triangleq \{I_N^*\}_{N\in[N^+]}$. The worst-case exchange rate sequence describes a strategy of the adversary, where it presents exchange rates p_1^*, p_2^*, …, $p_{N^+}^*$ in order, while it may terminate the exchange sequence anytime.

We denote the decision of LHA on exchange rate sequence I_N^* as $X_N^* \triangleq \{x_i^*\}_{i\in[N]}$, where, by the definition of LHA, we have

$$\begin{cases} x_1^* = \frac{1}{\alpha_{\mathrm{L}}^*(N^+)}, \\ x_i^* = \frac{p_i^* - p_{i-1}^*}{\alpha_{\mathrm{L}}^*(N^+)p_i^*}, \forall i \in [2, N^+]. \end{cases} \tag{64}$$

Next, we prove that there does not exist an online algorithm achieving a smaller competitive ratio of exchange-rate sequences $\mathcal{I}^*$ by contradiction.

We consider the decision of an online algorithm ALG on the exchange rate sequence $I_{N^+}^*$, which is denoted by $X'_{N^+} \triangleq \{x'_i\}_{i\in[N^+]}$. Then, since ALG is an online algorithm, its decision on the exchange rate sequence I_N^* is $X'_N \triangleq \{x'_i\}_{i\in[N]}$, for all $N \in [N^+]$. We first show that, for this online algorithm to achieve a better (smaller) competitive ratio than that of LHA, it has to satisfy two conditions, which, however, imply that ALG breaks the one-dollar budget constraint. We denote the optimal offline performance on the exchange rate sequence I_N^* as $\mathrm{OPT}(I_N^*)$, and it is straightforward that $\mathrm{OPT}(I_N^*) = p_N^*$. We denote the performance of LHA on the exchange rate sequence I_N^* as $\mathrm{LHA}(I_N^*)$, and we have $\mathrm{LHA}(I_N^*) = \sum_{i=1}^N x_i^* p_i^*$. We denote the performance of ALG on I_N^* as $\mathrm{ALG}(I_N^*)$, and we have $\mathrm{ALG}(I_N^*) = \sum_{i=1}^N x'_i p_i^*$

We first show that, to achieve a smaller competitive ratio than that of LHA, the ALG's solutions $\{x'_i\}$ need to satisfy Condition 1:

Condition 1. *There exists $j \in [1, N^+]$, such that $x'_i > x_i^*$ for all $i \in [1, j]$.*

We prove the necessity of this condition by contradiction. If $\{x'_i\}$ does not satisfy this condition, then, for all $j \in [1, N^+]$, there exists $i \in [1, j]$ such that $x'_i \le x_i^*$. This implies that, for $j = 1$, there will be $x'_1 \le x_1^*$, such that $x'_1 p_1^* \le x_1^* p_1^*$ and

$$\frac{p_1^*}{x'_1 p_1^*} \ge \frac{p_1^*}{x_1^* p_1^*} \tag{65}$$

$$\Rightarrow \frac{\mathrm{OPT}(I_1^*)}{\mathrm{ALG}(I_1^*)} \ge \frac{\mathrm{OPT}(I_1^*)}{\mathrm{LHA}(I_1^*)}, \tag{66}$$

which implies that ALG's competitive ratio is no less than that of LHA, leading to a contradiction.

Next, suppose that ALG satisfies Condition 1, we show that to achieve a smaller competitive ratio than that of LHA, ALG still needs to satisfy Condition 2:

Condition 2. *Define $f(k) \triangleq \sum_{i=1}^k (x'_i - x_i^*)$. For all $k \in [N^+]$, $f(k) > 0$.*

We prove the necessity of this condition by contradiction. If Condition 2 is not satisfied, it implies that, there exists $k \in [N^+]$, such that $f(k) \le 0$. Denote $k^* \triangleq \min\{k \in [N^+] | f(k) \le 0\}$ and $j^* \triangleq \max\{j \in [N^+] | x'_i > x_i^*, \forall i \in [j]\}$. Then we have $k^* > j^*$ and $f(k^* - 1) > 0$ by their definition. Therefore, we can construct a sequence $\{x''_i\}_{i\in[k^*]}$, such that

$$\begin{cases} x''_i = x_i^*, & \forall i \in [k^* - 1], \\ x''_{k^*} = x'_{k^*} + f(k^* - 1). \end{cases}, \tag{67}$$

which implies that

$$\sum_{i=1}^{k^*} x''_i p_i^* = \sum_{i=1}^{k^*-1} x_i^* p_i^* + x'_{k^*} p_{k^*}^* + p_{k^*}^* f(k^* - 1) \tag{68}$$

$$= \sum_{i=1}^{k^*} x'_i p_i^* - \sum_{i=1}^{k^*-1} (x'_i - x_i^*) p_i^* + p_{k^*}^* f(k^* - 1) \tag{69}$$

$$= \sum_{i=1}^{k^*} x'_i p_i^* + p_{k^*}^* f(k^* - 1) - \sum_{i=2}^{k^*-1} (f(i) - f(i-1)) p_i^* - f(1) p_1^* \tag{70}$$

$$= \sum_{i=1}^{k^*} x'_i p_i^* + p_{k^*}^* f(k^* - 1) + \sum_{i=2}^{k^*-1} f(i-1) p_i^* - \sum_{i=1}^{k^*-1} f(i) p_i^* \tag{71}$$

$$= \sum_{i=1}^{k^*} x'_i p_i^* + p_{k^*}^* f(k^* - 1) + \sum_{i=1}^{k^*-2} f(i) p_{i+1}^* - \sum_{i=1}^{k^*-1} f(i) p_i^* \tag{72}$$

$$= \sum_{i=1}^{k^*} x'_i p_i^* + \sum_{i=1}^{k^*-1} f(i)(p_{i+1}^* - p_i^*) \tag{73}$$

$$\geq \sum_{i=1}^{k^*} x'_i p_i^*. \tag{74}$$

However, by the construction of $\{x''_i\}$, it also implies that

$$x''_{k^*} = x'_{k^*} + f(k^* - 1) \tag{75}$$

$$= x'_{k^*} - x_{k^*}^* + \sum_{i=1}^{k^*-1} (x'_i - x_i^*) + x_{k^*}^* \tag{76}$$

$$= \sum_{i=1}^{k^*} (x'_i - x_i^*) + x_{k^*}^* \tag{77}$$

$$= f(k^*) + x_{k^*}^* \tag{78}$$

$$\leq x_{k^*}^*, \tag{79}$$

where inequality (79) comes from the definition of k^*, and further implies that

$$\sum_{i=1}^{k^*} x''_i p_i^* \leq \sum_{i=1}^{k^*} x_i^* p_i^*. \tag{80}$$

Eqs. (68)–(74) and (80) together implies that

$$\sum_{i=1}^{k^*} x'_i p_i^* \leq \sum_{i=1}^{k^*} x_i^* p_i^* \tag{81}$$

$$\Rightarrow \frac{\text{OPT}(I_{k^*}^*)}{\text{ALG}(I_{k^*}^*)} \geq \frac{\text{OPT}(I_{k^*}^*)}{\text{LHA}(I_{k^*}^*)}, \tag{82}$$

which implies that ALG's competitive ratio is no less than that of LHA, leading to a contradiction. The necessity of Condition 2 is proved.

Finally, we show that satisfying Condition 2 implies that ALG breaks the budget constraint on exchange-rate sequence $I_{N^+}^*$. First, for LHA, there is:

$$\sum_{i=1}^{N^+} x_i^* = \frac{1}{\alpha_{\text{L}}^*(N^+)} + \frac{1}{\alpha_{\text{L}}^*(N^+)} \sum_{i=2}^{N^+} \left(1 - \frac{p_{i-1}^*}{p_i^*}\right) \tag{83}$$

$$= \frac{1}{\alpha_{\text{L}}^*(N^+)} \left[1 + (N^+ - 1) - \sum_{i=2}^{N^+} \left(\frac{L}{U}\right)^{\frac{1}{N^+-1}}\right] \tag{84}$$

$$= \frac{1}{\alpha_{\text{L}}^*(N^+)} \left[1 + (N^+ - 1)\left(1 - \left(\frac{L}{U}\right)^{\frac{1}{N^+-1}}\right)\right] \tag{85}$$

$$= 1. \tag{86}$$

However, to satisfy Condition 2, it holds for ALG that $f(N^+) > 0$, such that $\sum_{i=1}^{N^+}(x'_i - x_i^*) > 0$, and hence $\sum_{i=1}^{N^+} x'_i > 1$, and ALG does not provide feasible solution. Therefore, the online algorithm that achieves a competitive ratio smaller than $\alpha_{\text{L}}^*(N^+)$ does not exist. Theorem 4 is proved. □

H PROOF OF COROLLARY 1

(Latest Horizon Improvement) $\alpha_{\mathrm{L}}^*(N^+) < \alpha_0^*$ and $\alpha_0^* - \alpha_{\mathrm{L}}^*(N^+) = \ln(U/L) - (N^+ - 1)\left(1 - \left(\frac{L}{U}\right)^{\frac{1}{N^+-1}}\right)$.

Proof. By the discussion of Eqs. (46)–(55), the optimal competitive ratio $\alpha_{\mathrm{L}}^*(N^+)$ is the optimal solution to

$$\min_{\alpha} \quad \alpha \tag{87}$$

$$\text{s.t.} \quad \sum_{i=1}^{N^+} \frac{\Delta q_i}{q_i} \leq \alpha, \tag{88}$$

when $q_1 = L$ and $q_i = L\left(\frac{U}{L}\right)^{\frac{i-1}{N^+-1}}, \forall i \in [N^+]$. By Eq. (49), we have

$$\sum_{i=1}^{N^+} \frac{\Delta q_i}{q_i} = 1 + \sum_{i=2}^{N^+} \frac{q_i - q_{i-1}}{q_i}. \tag{89}$$

For all $i \in [2, N^+]$, we have

$$\frac{q_i - q_{i-1}}{q_i} = \frac{1}{q_i}(q_i - q_{i-1}) \tag{90}$$

$$< \int_{q_{i-1}}^{q_i} \frac{1}{t} dt \tag{91}$$

$$= \ln(q_i/q_{i-1}). \tag{92}$$

Therefore, we have

$$\sum_{i=1}^{N^+} \frac{\Delta q_i}{q_i} = 1 + \sum_{i=2}^{N^+} \frac{q_i - q_{i-1}}{q_i} \tag{93}$$

$$< 1 + \sum_{i=2}^{N^+} \ln(q_i/q_{i-1}) \tag{94}$$

$$= 1 + \ln(U/L). \tag{95}$$

In contrast, when $q_1 = L$ and $q_i = L\left(\frac{U}{L}\right)^{\frac{i-1}{N^+-1}}, \forall i \in [N^+]$, we have $\alpha_{\mathrm{L}}^*(N^+) = 1 + (N^+ - 1)\left(1 - \left(\frac{L}{U}\right)^{\frac{1}{N^+-1}}\right) = \sum_{i=1}^{N^+} \frac{\Delta q_i}{q_i} < 1 + \ln(U/L)$.

Finally, by Eq. (3) and Theorem 2, we have $\alpha_0^* - \alpha_{\mathrm{L}}^*(N^+) = \ln(U/L) - (N^+ - 1)\left(1 - \left(\frac{L}{U}\right)^{\frac{1}{N^+-1}}\right)$. This corollary is proved. □

I PROOF OF COROLLARY 2

(Infinite Latest Horizon) When the latest horizon N^+ becomes arbitrarily large, we have:

$$\lim_{N^+ \to \infty} \alpha_{\mathrm{L}}^*(N^+) = 1 + \ln(U/L). \tag{96}$$

Proof. By Theorem 2, set $z \triangleq N^+ - 1$, it suffices to prove

$$\lim_{z \to \infty} z(1 - (L/U)^{\frac{1}{z}}) = \ln(U/L). \tag{97}$$

Set $a = \ln(U/L)$, we have

$$(L/U)^{\frac{1}{z}} = (U/L)^{-\frac{1}{z}} \tag{98}$$

$$=(e^{\ln(U/L)})^{-\frac{1}{z}} \tag{99}$$

$$=e^{-\frac{a}{z}}. \tag{100}$$

Then it suffices to prove that $\lim_{z\to\infty} z(1-e^{-\frac{a}{z}}) = a$.

Set $t = \frac{1}{z}$, then we have

$$z(1-e^{-\frac{a}{z}}) = \frac{1-e^{-at}}{t} \tag{101}$$

$$= a\frac{1-e^{-at}}{at}. \tag{102}$$

We further set $s = at$, then we have

$$a\frac{1-e^{-at}}{at} = a\frac{1-e^{-s}}{s}. \tag{103}$$

When z approaches ∞, t approaches 0 from above, and so does s. It suffices to prove that $\lim_{s\to 0^+} \frac{1-e^{-s}}{s} = 1$.

Define function $f(s) \triangleq e^{-s}$. By the Mean Value Theorem, there exists $r \in (0, s)$, such that $\frac{f(s)-f(0)}{s-0} = f'(r)$, where f' is the first order derivative of f. It implies

$$\frac{e^{-s}-1}{s} = -e^{-r} \tag{104}$$

$$\Rightarrow \frac{1-e^{-s}}{s} = e^{-r}. \tag{105}$$

As $s \to 0^+$, we have $r \to 0^+$, and hence $e^{-r} \to 1$. Therefore

$$\lim_{s\to 0^+} \frac{1-e^{-s}}{s} = 1, \tag{106}$$

which concludes our proof. □

J THE COMPLEXITY OF ERLHA

In this section, we discuss the complexity of ErLHA, and only focus on the case where $M \geq 1$, since $M = 0$ leads to the special case of ExHA. The complexity of ErLHA in different online time slots is different, while the most significant computation comes from the calculation of $Y^*(q_{N^-})$ and $\beta_2^*(q_{N^-})$, which are fortunately $O(1)$. However, the calculation of β_1^*, $Y^*(q_{N^-})$, and β_2^* is all dependent on the value of q_0^*, which is calculated in an offline manner before the online process begins. Since the parameter choice q_0^* cannot be presented in a closed-form expression and it requires solving the equality in Eq. (5), it is critical to know the computational complexity of q_0^*, which is provided in lemma 10.

Lemma 10. *(ErLHA Complexity) The complexity of solving q_0^* is $O(\frac{\mathcal{L}_X(U-L)}{\epsilon_g}\log(\frac{U-L}{\epsilon_b}))$, where $\mathcal{L}_X$ is the Lipschitz constant of function $X_M(q, q_{N^-}, N^-)$, ϵ_g is the error bound of grid search minimization Lipschitz smoothness, and ϵ_b is the error bound of bisection method.*

To prove Lemma 10, we need to first show the monotonicity of the function $X_M(q, q_{N^-}, N^-)$, as discussed in Appendix J.1, and to show that function $X_M(q, q_{N^-}, N^-)$ is Lipschitz continuous with respect to q_{N^-} in Appendix J.2. After that, we will prove Lemma 10 in Appendix J.3.

J.1 MONOTONICITY OF $X_M(q, q_{N^-}, N^-)$

Our proof of monotonicity of $X_M(q, q_{N^-}, N^-)$ is divided into 3 steps. In the first step (Lemma 11), we show that, if a function $g(y)$ is convex and $g(0) = 0$, then $\frac{g(y)}{y}$ is non-decreasing in y; in the second step (Lemma 12) shows the convexity of a family of functions $g_0(y)$; in the last step (Lemma 13), we show that $X_M(q, q_{N^-}, N^-)$ is non-decreasing in q by using Lemmas 11–12.

Lemma 11. *For convex function $g(y)$ with $g(0) = 0$, we have $\frac{g(y)}{y}$ being non-decreasing in $y > 0$.*

Proof. By convexity, we have, $\forall y, y'$ and $\forall \lambda \in [0, 1]$:

$$g(\lambda y + (1-\lambda)y') \leq \lambda g(y) + (1-\lambda)g(y'). \tag{107}$$

For $y' = 0$, it becomes,

$$g(\lambda y) \leq \lambda g(y). \tag{108}$$

Define $s := \lambda y$, and divide LHS and RHS by s, we then have

$$\frac{g(s)}{s} \leq \frac{g(y)}{y}, \forall s \in (0, y]. \tag{109}$$

Such that $\frac{g(y)}{y}$ is non-decreasing in y. This lemma is proved. □

Lemma 12. *Define function $g_0(y) = 1 - (1 - \frac{y}{A})^{\frac{1}{B}}$, with $A \geq y$ and $B \geq 1$ being two real constants. It holds true that $g_0(y)$ is convex in y when $y \geq 0$.*

Proof. For the function $g_0(y)$, its first and second order derivatives are

$$\frac{dg_0}{dy} = \frac{1}{AB}(1 - \frac{y}{A})^{\frac{1}{B}-1}, \tag{110}$$

$$\frac{d^2 g_0}{dy^2} = \frac{B-1}{A^2 B^2}(1 - \frac{y}{A})^{\frac{1}{B}-2} \tag{111}$$

.

Since $B \geq 1$, we have $\frac{B-1}{A^2B^2} \geq 0$. Since $A \geq y$, we have $1 - \frac{y}{A} \geq 0$ and $(1 - \frac{y}{A})^{\frac{1}{B}-2}$, and hence $(1 - \frac{y}{A})^{\frac{1}{B}-2} \geq 0$ and

$$\frac{d^2 g_0}{dy^2} \geq 0. \tag{112}$$

Therefore, the function $g_0(y)$ is convex, and this lemma is proved. □

Lemma 13. *The function $X_M(q, q_{N^-}, N^-)$ is non-decreasing in q.*

Proof. Recall that by the definition of the function X_M in Eq. (8), we have

$$X_M(q, q_{N^-}, N^-) := \frac{(U-q)q_{N^-}}{G(q, q_{N^-}, N^-)(Lf_M(q_{N^-}) + q_{N^-}) + f_M(q_{N^-})(q_{N^-} - q)}. \tag{113}$$

Fix $q_{N^-} \in [L, U]$, define $y(q) := q_{N^-} - q, \forall q \in [L, q_{N^-}]$. Define $b = U - q_{N^-}$, and $g_1(y(q)) = 1 - (1 - \frac{y(q)}{q_{N^-} - L})^{\frac{1}{N^-}}$, such that

$$G(q, q_{N^-}, N^-) = N^-[1 - (\frac{q-L}{q_{N^-} - L})^{\frac{1}{N^-}}] \tag{114}$$

$$= N^-[1 - (1 - \frac{q_{N^-} - q}{q_{N^-} - L})^{\frac{1}{N^-}}] \tag{115}$$

$$= N^- g_1(y(q)). \tag{116}$$

By Lemma 12, since $q_{N^-} - q \leq q_{N^-} - L$, $N^- \geq 1$, and $q \leq q_{N^-}$, we have that $y(q) \geq 0$ and $g_1(y)$ is convex in y.

We define the denominator of $X_M(q, q_{N^-}, N^-)$ as $D(y)$, such that

$$D(y) := N^- g_1(y)(Lf_M(q_{N^-}) + q_{N^-}) + f_M(q_{N^-})y, \tag{117}$$

and hence

$$X_M(q, q_{N^-}, N^-) = \frac{U-q}{y} \cdot \frac{y}{D(y)} \cdot q_{N^-} \tag{118}$$

$$=\frac{b+y}{y}\cdot\frac{y}{D(y)}\cdot q_{N^-} \tag{119}$$

Notice that

$$\frac{D(y)}{y}=N^-(Lf_M(q_{N^-})+q_{N^-})\frac{g_1(y)}{y}+f_M(q_{N^-}), \tag{120}$$

which, since $\frac{g_1(y)}{y}$ is non-decreasing in y by Lemma 11, is also non-decreasing in y, so that $\frac{y}{D(y)}$ is non-increasing in y and non-decreasing in q. Also, $\frac{b+y}{y}=1+\frac{b}{y}$ is non-increasing in y and non-decreasing in q. Therefore, $X_M(q,q_{N^-},N^-)$ is non-decreasing in q. □

Corollary 4. *The function* $\inf_{q_{N^-}\in[q,U]} X_M(q,q_{N^-},N^-)$ *is non-decreasing in* $q\in[L,U)$.

Proof. For any q and q' that $L\le q\le q'<U$, by Lemma 13, we have

$$\inf_{q_{N^-}\in[q',U]} X_M(q',q_{N^-},N^-)\ge \inf_{q_{N^-}\in[q,U]} X_M(q',q_{N^-},N^-) \tag{121}$$

$$\ge \inf_{q_{N^-}\in[q,U]} X_M(q,q_{N^-},N^-). \tag{122}$$

□

J.2 Lipschitz Continuity of $X_M(q,q_{N^-},N^-)$

Lemma 14. *Fix any* $q\in(L,U)$, $\delta>0$, *and* $M\ge 1$ *such that* $q+\delta<U-\delta$, *the function* $X_M(q,q_{N^-},N^-)$ *is Lipschitz continuous on* $q_{N^-}\in[q+\delta,U-\delta]$ *with bounded gradient* $\mathcal{L}_X$, *such that* $\left|X_M(q,q_{N^-},N^-)-X_M(q,q'_{N^-},N^-)\right|\le\mathcal{L}_X|q_{N^-}-q'_{N^-}|$, $\forall q_{N^-},q'_{N^-}\in[q+\delta,U-\delta]$, *where* $\delta>0$ *and*

$$\mathcal{L}_X=\frac{(U-q)}{d_{\min}(q,N^-,L,\delta)}+\frac{U(U-q)d'_{\max}(q,U,L,N^-,N^+)}{d^2_{\min}(q,N^-,L,\delta)}, \tag{123}$$

$d_{\min}(q,N^-,L,\delta):=qN^-[1-(\frac{q-L}{q+\delta-L})^{\frac{1}{N^-}}]$, *and*

$$d'_{\max}(q,U,L,N^-,N^+):=(q-L)^{-1}(LM[1-(\frac{q}{U})^{\frac{1}{M}}]+U) \tag{124}$$

$$+N^-[1-(\frac{q-L}{U-L})^{\frac{1}{N^-}}](1-\frac{L}{U}) \tag{125}$$

$$+U^{-\frac{1}{M}}q^{\frac{1}{M}-1}(U-q)+M[1-(\frac{q}{U})^{\frac{1}{M}}]. \tag{126}$$

Proof. Define $D(q_{N^-})$ as the denominator of $X_M(q,q_{N^-},N^-)$, such that

$$D(q_{N^-}):=G(q,q_{N^-},N^-)(Lf_M(q_{N^-})+q_{N^-})+f_M(q_{N^-})(q_{N^-}-q), \tag{127}$$

and hence $X_M(q,q_{N^-},N^-)=\frac{(U-q)q_{N^-}}{D(q_{N^-})}$. The first order derivative of $X_M(q,q_{N^-},N^-)$ with respect to q_{N^-} is then

$$\frac{dX_M(q,q_{N^-},N^-)}{dq_{N^-}}=\frac{(U-q)[D(q_{N^-})-q_{N^-}D'(q_{N^-})]}{D^2(q_{N^-})}, \tag{128}$$

such that

$$\left|\frac{dX_M(q,q_{N^-},N^-)}{dq_{N^-}}\right|=\frac{(U-q)|D(q_{N^-})-q_{N^-}D'(q_{N^-})|}{D^2(q_{N^-})} \tag{129}$$

$$\le\frac{(U-q)|D(q_{N^-})|+(U-q)|q_{N^-}D'(q_{N^-})|}{D^2(q_{N^-})} \tag{130}$$

$$=\frac{(U-q)D(q_{N^-})+(U-q)q_{N^-}|D'(q_{N^-})|}{D^2(q_{N^-})} \tag{131}$$

$$=\frac{(U-q)}{D(q_{N^-})}+\frac{q_{N^-}(U-q)|D'(q_{N^-})|}{D^2(q_{N^-})}, \tag{132}$$

where Eq. (131) comes from $D(q_{N^-}) \geq 0$ and $q_{N^-} > 0$. What remains is to get an upper bound of $|D'(q_{N^-})|$ and an lower bound on $D(q_{N^-})$.

First, we analyze the lower bound of $D(q_{N^-})$. Since $f_M(q_{N^-}) = M[1-(q_{N^-}/U)^{\frac{1}{M}}]$ is decreasing in q_{N^-}, we have

$$f_M(q_{N^-}) \geq f_M(U-\delta) \geq f_M(U) = 0, \tag{133}$$

and hence

$$D(q_{N^-}) \geq q_{N^-}G(q, q_{N^-}, N^-) \tag{134}$$

$$\geq qG(q, q_{N^-}, N^-). \tag{135}$$

Since $G(q, q_{N^-}, N^-) = N^-[1-(\frac{q-L}{q_{N^-}-L})^{\frac{1}{N^-}}]$ is increasing in q_{N^-}, we have

$$D(q_{N^-}) \geq qG(q, q_{N^-}, N^-) \tag{136}$$

$$\geq qG(q, q+\delta, N^-) \tag{137}$$

$$=qN^-[1-(\frac{q-L}{q+\delta-L})^{\frac{1}{N^-}}]. \tag{138}$$

Next, we find the upper bound for $|D'(q_{N^-})|$. Notice that

$$\begin{aligned} D'(q_{N^-}) =& \frac{dG(q, q_{N^-}, N^-)}{dq_{N^-}}(Lf_M(q_{N^-})+q_{N^-}) \\ &+ G(q, q_{N^-}, N^-)[Lf'_M(q_{N^-})+1] + f'_M(q_{N^-})(q_{N^-}-q) + f_M(q_{N^-}), \end{aligned} \tag{139}$$

such that

$$|D'(q_{N^-})| \leq |\frac{dG(q, q_{N^-}, N^-)}{dq_{N^-}}|(Lf_M(q_{N^-})+q_{N^-}) \tag{140}$$

$$+ G(q, q_{N^-}, N^-)|Lf'_M(q_{N^-})+1| + |f'_M(q_{N^-})|(q_{N^-}-q) + f_M(q_{N^-}). \tag{141}$$

Since $f_M(q_{N^-})$ is decreasing in q_{N^-}, we have

$$f_M(q_{N^-}) \leq f_M(q+\delta) \leq f_M(q) = M[1-(\frac{q}{U})^{\frac{1}{M}}]. \tag{142}$$

Since $G(q, q_{N^-}, N^-)$ is increasing in q_{N^-}, we have

$$G(q, q_{N^-}, N^-) \leq G(q, U-\delta, N^-) \leq G(q, U, N^-) = N^-[1-(\frac{q-L}{U-L})^{\frac{1}{N^-}}]. \tag{143}$$

By Eq. (139), we have

$$|D'(q_{N^-})| \leq |\frac{dG(q, q_{N^-}, N^-)}{dq_{N^-}}|(LM[1-(\frac{q}{U})^{\frac{1}{M}}]+q_{N^-}) \tag{144}$$

$$+ N^-[1-(\frac{q-L}{U-L})^{\frac{1}{N^-}}]|Lf'_M(q_{N^-})+1| \tag{145}$$

$$+ |f'_M(q_{N^-})|(q_{N^-}-q) + M[1-(\frac{q}{U})^{\frac{1}{M}}]. \tag{146}$$

What remains is to get the upper bounds of $|\frac{dG(q,q_{N^-},N^-)}{dq_{N^-}}|$, $|Lf'_M(q_{N^-})+1|$, and $|f'_M(q_{N^-})|$.

For $f'_M(q_{N^-})$, we have

$$f'_M(q_{N^-}) = M(-\frac{1}{M})(\frac{q_{N^-}}{U})^{\frac{1}{M}-1}\frac{1}{U} \tag{147}$$

$$= -U^{-\frac{1}{M}}q_{N^-}^{\frac{1}{M}-1}, \tag{148}$$

which is non-decreasing in q_{N^-}, such that

$$\begin{aligned} f'_M(q_{N^-}) \geq & f'_M(q+\delta) && (149)\\ \geq & f'_M(q) && (150)\\ = & -U^{-\frac{1}{M}} q^{\frac{1}{M}-1} && (151)\end{aligned}$$

Also, we have

$$\begin{aligned} f'_M(q_{N^-}) \leq & f'_M(U-\delta) && (152)\\ \leq & f'_M(U) && (153)\\ = & -U^{-1} && (154)\\ < & 0. && (155)\end{aligned}$$

In conclusion, we have

$$|f'_M(q_{N^-})| \leq U^{-\frac{1}{M}} q^{\frac{1}{M}-1}. \tag{156}$$

Meanwhile, we have $Lf'_M(q_{N^-}) \in [-LU^{-\frac{1}{M}} q^{\frac{1}{M}-1}, -\frac{L}{U}]$, and

$$Lf'_M(q_{N^-}) + 1 \in [1 - LU^{-\frac{1}{M}} q^{\frac{1}{M}-1}, 1 - \frac{L}{U}], \tag{157}$$

where $1 - \frac{L}{U} > 0$, and

$$\begin{aligned} 1 - LU^{-\frac{1}{M}} q^{\frac{1}{M}-1} \geq & 1 - Lq^{-\frac{1}{M}} q^{\frac{1}{M}-1} && (158)\\ = & 1 - \frac{L}{q} && (159)\\ > & 0, && (160)\end{aligned}$$

which implies

$$|Lf'_M(q_{N^-}) + 1| \leq 1 - \frac{L}{U}. \tag{161}$$

On the other hand, for $G(q, q_{N^-}, N^-)$, we have

$$\begin{aligned} \frac{dG(q, q_{N^-}, N^-)}{dq_{N^-}} = & N^-[-\frac{1}{N^-}(\frac{q-L}{q_{N^-}-L})^{\frac{1}{N^-}-1}(q-L)(-1)(q_{N^-}-L)^{-2}] && (162)\\ = & (q-L)^{\frac{1}{N^-}}(q_{N^-}-L)^{-\frac{1}{N^-}-1}, && (163)\end{aligned}$$

which is decreasing in q_{N^-}. As a result,

$$\begin{aligned} \frac{dG(q, q_{N^-}, N^-)}{dq_{N^-}} \leq & (q-L)^{\frac{1}{N^-}}(q-L)^{-\frac{1}{N^-}-1} && (164)\\ = & (q-L)^{-1}. && (165)\end{aligned}$$

Considering the above bounds, for $D'(q_{N^-})$, we have

$$\begin{aligned} |D'(q_{N^-})| \leq & |\frac{dG(q, q_{N^-}, N^-)}{dq_{N^-}}|(LM[1-(\frac{q}{U})^{\frac{1}{M}}] + q_{N^-}) && (166)\\ & + N^-[1-(\frac{q-L}{U-L})^{\frac{1}{N^-}}]|Lf'_M(q_{N^-}) + 1| && (167)\\ & + |f'_M(q_{N^-})|(q_{N^-} - q) + M[1-(\frac{q}{U})^{\frac{1}{M}}] \quad \text{(By Eq. (146))} && (168)\\ & \leq (q-L)^{-1}(LM[1-(\frac{q}{U})^{\frac{1}{M}}] + U) \quad \text{(By Eq. (165))} && (169)\\ & + N^-[1-(\frac{q-L}{U-L})^{\frac{1}{N^-}}](1-\frac{L}{U}) \quad \text{(By Eq. (161))} && (170)\end{aligned}$$

$$+ U^{-\frac{1}{M}} q^{\frac{1}{M}-1}(U-q) + M[1-(\frac{q}{U})^{\frac{1}{M}}] \quad \text{(By Eq. (156)).} \tag{171}$$

Recall that, by Eq. (132), we have

$$|\frac{dX_M(q,q_{N^-},N^-)}{dq_{N^-}}| \leq \frac{(U-q)}{D(q_{N^-})} + \frac{q_{N^-}(U-q)|D'(q_{N^-})|}{D^2(q_{N^-})} \tag{172}$$

$$\leq \frac{(U-q)}{D(q_{N^-})} + \frac{U(U-q)|D'(q_{N^-})|}{D^2(q_{N^-})}. \tag{173}$$

Set $d_{\min}(q,N^-,L,\delta) := qN^-[1-(\frac{q-L}{q+\delta-L})^{\frac{1}{N^-}}]$, and

$$d'_{\max}(q,U,L,N^-,N^+) := (q-L)^{-1}(LM[1-(\frac{q}{U})^{\frac{1}{M}}]+U) \tag{174}$$

$$+ N^-[1-(\frac{q-L}{U-L})^{\frac{1}{N^-}}](1-\frac{L}{U}) \tag{175}$$

$$+ U^{-\frac{1}{M}} q^{\frac{1}{M}-1}(U-q) + M[1-(\frac{q}{U})^{\frac{1}{M}}], \tag{176}$$

then we have

$$|\frac{dX_M(q,q_{N^-},N^-)}{dq_{N^-}}| \leq \frac{(U-q)}{D(q_{N^-})} + \frac{q_{N^-}(U-q)|D'(q_{N^-})|}{D^2(q_{N^-})} \tag{177}$$

$$\leq \frac{(U-q)}{d_{\min}(q,N^-,L,\delta)} + \frac{U(U-q)d'_{\max}(q,U,L,N^-,N^+)}{d^2_{\min}(q,N^-,L,\delta)} \tag{178}$$

$$= \mathcal{L}_X, \tag{179}$$

which proves this lemma. □

J.3 Proof of Lemma 10

Below we provide the proof of Lemma 10,

Proof of Lemma 10. Recall that $f_M(q) = M[1-(\frac{q}{U})^{\frac{1}{M}}]$, so that $\frac{L(U-Lf_M(q)-q)}{(Lf_M(q)+q)}$ is decreasing in q. By Corollary 4, $\inf_{q_{N^-}\in[q,U]} X_M(q,q_{N^-},N^-)$ is non-decreasing in q. Therefore, we can use the bisection method on the interval $q \in (L,U)$ to find the q that achieves

$$\frac{L(U-Lf_M(q)-q)}{Lf_M(q)+q} = \inf_{q_{N^-}\in[q,U]} X_M(q,q_{N^-},N^-), \tag{180}$$

and the number of bisection method iterations is $O(\log(\frac{U-L}{\epsilon_b}))$, where ϵ_b is the error of the bisection method.

Within each bisection iteration, we need to evaluate the value of the left-hand side (LHS) and the right-hand side (RHS), where LHS requires $O(1)$ complexity to calculate. For RHS, we need to solve

$$\inf \quad X_M(q,q_{N^-},N^-) = \frac{(U-q)q_{N^-}}{G(q,q_{N^-},N^-)(Lf_M(q_{N^-})+q_{N^-}) + f_M(q_{N^-})(q_{N^-}-q)} \tag{181}$$

$$\text{s.t.} \quad q_{N^-} \in [q,U]. \tag{182}$$

By the Lipschitz constant $\mathcal{L}_X$ for $X_M(q,q_{N^-},N^-)$ (Lemma 14), the inner minimization problem $\inf_{q_{N^-}} X_M(q,q_{N^-},N^-)$ can be solved by grid search with complexity $O\Big(\frac{(U-L)\mathcal{L}_X}{\epsilon_g}\Big)$[1]. Therefore, the total complexity to calculate would be $O(\frac{\mathcal{L}_X\,(U-L)}{\epsilon_g}\log(\frac{U-L}{\epsilon_b}))$. □

[1] https://people.eecs.berkeley.edu/~jiantao/227c2022spring/scribe/227C_Lecture_02.pdf

K COMPETITIVE RATIO OF ERLHA

In this section, we will investigate the feasibility and optimality of the parameter choices q_0^*, β_1^*, $Y^*(q_{N^-})$, and $\beta_2^*(q_{N^-})$ within the design structure of ErLHA, and finally provide a formal proof of Theorem 5. We will first derive the feasibility condition for choosing the parameters for ErLHA and a lower bound of ErLHA's online performance in Appendix K.1, derive the sufficient conditions for not violating the resource budget in Appendix K.2, formulate the competitive ratio of ErLHA in Appendix K.3, derive the optimal parameter choices for ErLHA in Appendix K.4, prove the feasibility of ErLHA's parameter choices in Appendix K.5, and finally provide the proof to Theorem 5 in Appendix K.6.

K.1 ERLHA PERFORMANCE LOWER BOUND

To begin with, we start with the feasibility condition for choosing the parameters for ErLHA and a lower bound of ErLHA's online performance.

Lemma 15. *(ErLHA Performance Lower Bound) The total yen obtained by ErLHA has a tight lower bound $Y(q_{N^-})L+\beta_1(q_{N^-}-q_0)+\beta_2(q_{N^-})(q_N-q_{N^-})$, when it satisfies the following constraints:*

(i) (Valid Parameter Condition) $\beta_1 \geq 0$; $q_0 \in [L,U]$; $\beta_2(q_{N^-}) \geq 0$, $Y(q_{N^-}) \in [0,1], \forall q_{N^-} \in [q_0,U]$.

(ii) (Condition for Adaptive Early Budget Sufficiency) $\sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} \leq Y(q_{N^-})$, $\forall q_{N^-} \in [q_0,U]$.

(iii) (Condition for Adaptive Late Budget Sufficiency) $\sum_{i=N^-+1}^{N} \frac{\beta_2(q_{N^-})\Delta q_i}{q_i} \leq (1-Y(q_{N^-}))$, $\forall q_{N^-} \in [q_0,U]$ and $q_N \in [q_{N^-},U]$.

Sketch of Proof. The valid parameter condition ensures that the amount of dollars converted in every time slot is non-negative. The condition for adaptive early budget sufficiency ensures that the amount of dollars converted at the earliest horizon N^- is non-negative. The condition for adaptive late budget sufficiency ensures that the 1 dollar budget constraint is not violated. The performance lower bound comes directly from the design structure of ErLHA. □

Proof. Condition (i) implies the decisions x_i are non-negative values and the trading entry threshold q_0 is within the range of the exchange rates; condition (ii) implies the early budget is $Y(q_{N^-})$ sufficient for decisions before and in time slot N^-; condition (iii) implies that the late budget $1-Y(q_{N^-})$ is sufficient for decisions in and after time slot N^-+1.

When these three conditions are satisfied, the total yens obtained by ErLHA is

$$\sum_{i=1}^{N^-} x_i p_i + \sum_{i=N^-+1}^{N} x_i p_i = \sum_{i=1}^{N^--1} x_i p_i + x_{N^-}p_{N^-} + \sum_{i=N^-+1}^{N} x_i p_i \tag{183}$$

$$= \sum_{i\in[N^--1]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} p_i + \mathbf{1}_{\{\Delta q_{N^-}>0\}} \frac{\beta_1 \Delta q_{N^-}}{q_{N^-} - L} p_{N^-}$$

$$+ \left(Y(q_{N^-}) - \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} \right) p_{N^-} + \sum_{i=N^-+1}^{N} x_i q_i \tag{184}$$

$$= \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} q_i + \left(Y(q_{N^-}) - \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} \right) p_{N^-} \tag{185}$$

$$+ \sum_{i=N^-+1}^{N} x_i q_i \tag{186}$$

$$\geq \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} q_i + \left(Y(q_{N^-}) - \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} \right) L \tag{187}$$

$$+ \sum_{i=N^-+1}^{N} x_i q_i \quad \text{(tight when } p_{N^-} = L) \tag{188}$$

$$= Y(q_{N^-})L + \beta_1(q_{N^-} - q_0) + \beta_2(q_{N^-}) \cdot (q_N - q_{N^-}). \tag{189}$$

This lemma is proved. □

Lemma 15 identifies three conditions that should be considered when choosing the parameters of ErLHA to ensure feasible solutions and establishes a performance lower bound when these conditions are satisfied.

K.2 Sufficient Conditions for Budget Sufficiency

Since the budget consumption of ErLHA is closely related to the parameters q_0, β_1, and $\beta_2(q_{N^-})$, it is necessary to translate the conditions for adaptive early and late budget sufficiency into explicit instructions on the feasible ranges for these parameters, which leads to the following lemma:

Lemma 16. *(Sufficient Conditions for Budget Sufficiency) A sufficient condition for adaptive early budget sufficiency is*

$$\beta_1 G(q_0, q_{N^-}, N^-) \leq Y(q_{N^-}), \forall q_{N^-} \in [q_0, U]. \tag{190}$$

A sufficient condition for adaptive late budget sufficiency is

$$\beta_2(q_{N^-}) M \left[1 - \left(\frac{q_{N^-}}{q_N} \right)^{\frac{1}{M}} \right] \leq 1 - Y(q_{N^-}), \forall q_{N^-} \in [q_0, U], q_N \in [q_{N^-}, U]. \tag{191}$$

Sketch of Proof. This lemma comes from the analysis of Lemma 15 within the context of ErLHA. As long as ErLHA satisfies the inequalities (190)–(191), it satisfies the conditions for adaptive early and late budget sufficiency. □

Proof. We first prove that the first inequality implies the condition for adaptive early budget sufficiency, then show that the second inequality implies the condition for adaptive late budget sufficiency.

Recall that the condition for adaptive early budget sufficiency requires that

$$\sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} \leq Y(q_{N^-}), \forall q_{N^-} \in [q_0, U]. \tag{192}$$

Note that when $\beta_1 = 0$, the condition for adaptive early budget sufficiency is always satisfied since $Y(q_{N^-}) \geq 0$, so below we focus on the case with $\beta_1 > 0$. We first analyze the LHS of the inequality (192):

$$\sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1 \Delta q_i}{q_i - L} = \sum_{i\in[N^-]:\Delta q_i>0} x_i \tag{193}$$

$$= \sum_{i=1}^{N^-} x_i \quad \text{(since } x_i = 0 \text{ if } \Delta q_i = 0) \tag{194}$$

$$= \beta_1 \sum_{i=1}^{N^-} \frac{x_i}{\beta_1} \tag{195}$$

$$= \beta_1 \sum_{i=1}^{N^-} [1 - \frac{\beta_1 - x_i}{\beta_1}] \tag{196}$$

$$=\beta_1[N^- - \sum_{i=1}^{N^-} \frac{\beta_1 - x_i}{\beta_1}] \tag{197}$$

$$\leq \beta_1 \left[N^- - N^- \left(\prod_{i=1}^{N^-} \frac{\beta_1 - x_i}{\beta_1} \right)^{\frac{1}{N^-}} \right] \tag{198}$$

$$=\beta_1 \left[N^- - N^- \left(\prod_{i=1}^{N^-} \frac{q_{i-1} - L}{q_i - L} \right)^{\frac{1}{N^-}} \right] \tag{199}$$

$$=\beta_1 N^- \left[1 - \left(\frac{q_0 - L}{q_{N^-} - L} \right)^{\frac{1}{N^-}} \right] \tag{200}$$

$$=\beta_1 G(q_0, q_{N^-}, N^-), \tag{201}$$

where Eq. (199) comes from the fact that, when $\Delta q_i = 0$, we have $x_i = 0$, $(\beta_1 - x_i)/\beta_1 = 1 = (q_{i-1} - L)/(q_i - L)$. Therefore, inequality (190) implies the condition for adaptive early budget sufficiency.

Next, we show that inequality (191) implies the condition for late budget sufficiency. Recall that the late budget sufficiency condition requires that

$$\sum_{i=N^-+1}^{N} \frac{\beta_2(q_{N^-})\Delta q_i}{q_i} \leq (1 - Y(q_{N^-})). \tag{202}$$

WLOG, we define $q_i = q_N$ for all $i = N+1, \ldots, N^+$. We first analyze the LHS of inequality (202), where we have:

$$\sum_{i=N^-+1}^{N} \frac{\Delta q_i}{q_i} \leq \sum_{i=N^-+1}^{N^+} \frac{\Delta q_i}{q_i} \tag{203}$$

$$= \sum_{i=N^-+1}^{N^+} \frac{q_i - q_{i-1}}{q_i} \tag{204}$$

$$=(N^+ - N^-) - \sum_{i=N^-+1}^{N^+} \frac{q_{i-1}}{q_i} \tag{205}$$

$$\leq (N^+ - N^-) - (N^+ - N^-) \left(\prod_{i=N^-+1}^{N^+} \frac{q_{i-1}}{q_i} \right)^{\frac{1}{N^+ - N^-}} \tag{206}$$

$$=(N^+ - N^-) - (N^+ - N^-) \left(\frac{q_{N^-}}{q_{N^+}} \right)^{\frac{1}{N^+ - N^-}} \tag{207}$$

$$=(N^+ - N^-) \left[1 - \left(\frac{q_{N^-}}{q_N} \right)^{\frac{1}{N^+ - N^-}} \right]. \tag{208}$$

where inequality (203) comes from $N \leq N^+$; inequality (206) comes from the AM-GM inequality. Inequality (208) implies that inequality (191) is sufficient to satisfy the condition for late budget sufficiency, which concludes our proof. □

Lemma 16 states the sufficient conditions for the parameter choice in ErLHA to generate feasible solutions, which provides two concrete guidelines for determining the parameters in ErLHA.

K.3 ErLHA Competitive Ratio Formulation

While lemma 16 provides two concrete guidelines for determining the parameters in ErLHA, another critical factor in choosing the parameters of ErLHA is the resulting competitive ratio. Recall that

Lemma 15 establishes a performance lower bound for ErLHA, while it is straightforward that the offline optimal performance is no more than q_N. Thus, the competitive ratio of ErLHA can be formulated as the ratio between these two bounds. Consequently, the optimal parameter choice for ErLHA is obtained by maximizing this competitive ratio subject to the three conditions identified in Lemma 15, as stated in the following lemma:

Lemma 17. *(ErLHA Competitive Ratio Formulation) The best competitive ratio that ErLHA can achieve is the optimal value for the following min-max problem, which is denoted as* $\mathtt{P}_0$*:*

$$\min_{\beta_1,\beta_2(\cdot),Y(\cdot),q_0} \quad \max_{q_{N^-},q_N} \left[(\frac{Y(q_{N^-})L}{q_0}-\beta_1)\frac{q_0}{q_N}+(\beta_1-\beta_2(q_{N^-}))\frac{q_{N^-}}{q_N}+\beta_2(q_{N^-})\right]^{-1} \tag{209}$$

$$\textit{s.t.} \quad \beta_1 N^- \left[1-\left(\frac{q_0-L}{q_{N^-}-L}\right)^{\frac{1}{N^-}}\right] \le Y(q_{N^-}), \forall q_{N^-} \in [q_0,U] \tag{210}$$

$$\beta_2(q_{N^-})M\left[1-\left(\frac{q_{N^-}}{q_N}\right)^{\frac{1}{M}}\right] \le 1-Y(q_{N^-}), \forall q_{N^-} \in [q_0,U], q_N \in [q_{N^-},U] \tag{211}$$

$$q_0 \in [L,U], \tag{212}$$

$$\beta_1 \ge 0, \tag{213}$$

$$\beta_2(q_{N^-}) \ge 0, \forall q_{N^-} \in [q_0,U], \tag{214}$$

$$Y(q_{N^-}) \in [0,1], \forall q_{N^-} \in [q_0,U]. \tag{215}$$

Proof. By the calculation of q_i that $q_i \leftarrow \max\{q_{i-1}, p_i\}$ (Line 4 in Alg.2), it is straightforward that

$$q_N = \max\{q_0, \max_{i\in[1,N]} p_i\} \ge \max_{i\in[1,N]} p_i = \text{OPT}(I), \tag{216}$$

where the last equality comes from the fact that the offline optimal solution OPT always converts all its dollars into yen at the highest exchange rate in an exchange-rate sequence. Meanwhile, by Lemma 15, if the valid parameter condition, the condition for adaptive early budget sufficiency, and the condition for adaptive late budget sufficiency are satisfied, the online performance of ErLHA has a lower bound of $Y(q_{N^-})L+\beta_1(q_{N^-}-q_0)+\beta_2(q_{N^-})(q_N-q_{N^-})$. In this case, the competitive ratio of ErLHA satisfies:

$$\max_{\forall I}\frac{\text{OPT}(I)}{\text{ALG}(I)} \le \max_{q_{N^-},q_N} \quad \frac{q_N}{Y(q_{N^-})L+\beta_1(q_{N^-}-q_0)+\beta_2(q_{N^-})(q_N-q_{N^-})} \tag{217}$$

$$= \max_{q_{N^-},q_N}\left[\frac{Y(q_{N^-})L}{q_N}+\beta_1(\frac{q_{N^-}}{q_N}-\frac{q_0}{q_N})+\beta_2(q_{N^-})(1-\frac{q_{N^-}}{q_N})\right]^{-1} \tag{218}$$

$$= \max_{q_{N^-},q_N}\left[\frac{Y(q_{N^-})L}{q_0}\frac{q_0}{q_N}-\beta_1\frac{q_0}{q_N}+(\beta_1-\beta_2(q_{N^-}))\frac{q_{N^-}}{q_N}+\beta_2(q_{N^-})\right]^{-1} \tag{219}$$

$$= \max_{q_{N^-},q_N}\left[(\frac{Y(q_{N^-})L}{q_0}-\beta_1)\frac{q_0}{q_N}+(\beta_1-\beta_2(q_{N^-}))\frac{q_{N^-}}{q_N}+\beta_2(q_{N^-})\right]^{-1}. \tag{220}$$

The minimization of such competitive ratio under the three conditions specified in Lemma 15 hence can be formulated as the optimization problem $\mathtt{P}_0$ (Eq. (209)), where the constraints (210)–(211) come from the sufficient conditions (Lemma 16) for adaptive early budget sufficiency and adaptive late budget sufficiency (Lemma 15), constraints (212)–(215) come from the valid parameter condition in Lemma 15. □

Lemma 17 summarizes our problem of determining the best parameters for ErLHA as an optimization problem, which also characterizes the best achievable competitive ratio by ErLHA within its design structure.

K.4 Optimal Parameters

By Lemma 17, if we can obtain the optimal solution to $\mathtt{P}_0$, we can also attain the best performance of ErLHA within its design structure. Indeed, $q_0 = q_0^*$, $\beta_1 = \beta_1^*$, $Y(q_{N^-}) = Y^*(q_{N^-})$, and

$\beta_2(q_{N^-}) = \beta_2^*(q_{N^-})$ are the optimal, and therefore feasible, solutions to the optimization problem $\mathtt{P}_0$, and we formally state these results in Lemma 18 and Corollary 8.

Lemma 18. *(Optimal Parameters) The parameters $q_0 = q_0^*$, $\beta_1 = \beta_1^*$, $Y(q_{N^-}) = Y^*(q_{N^-})$, and $\beta_2(q_{N^-}) = \beta_2^*(q_{N^-})$ are optimal solutions to problem $\mathtt{P}_0$.*

Sketch of Proof. This lemma comes from direct analyses of the problem $\mathtt{P}_0$, and the proof can be divided into five stages. First, we find an equivalent form $\mathtt{P}_1$ of problem $\mathtt{P}_0$ and derive the optimal $\beta_2(\cdot)$ (Appendix K.4.1). Second, we put the optimal $\beta_2(\cdot)$ back into $\mathtt{P}_1$, find an equivalent form $\mathtt{P}_2$ of $\mathtt{P}_1$, and solve the optimal $Y(\cdot)$ (in the form of $\beta_Y(\cdot)$, defined later) in Appendix K.4.2. Next, we put the optimal $Y(\cdot)$ back into $\mathtt{P}_2$, find equivalent forms $\mathtt{P}_3$ and $\mathtt{P}_4$ of $\mathtt{P}_2$. In addition, we identify a critical equation involving the optimal β_1 and optimal q_0, transform $\mathtt{P}_4$ into an optimization problem $\mathtt{P}_5$ solely dependent on q_0 (Appendix K.4.3). Finally, we solve the optimal q_0 from $\mathtt{P}_5$ (Appendix K.4.3) and complete the proof (Appendix K.4.4). □

K.4.1 Optimal Choice for $\beta_2(\cdot)$

In this subsection, we discuss the optimal choice for the parameter $\beta_2(q_{N^-})$ for ErLHA. We first provide an equivalent form of the optimization problem $\mathtt{P}_0$ in Lemma 19, and then show the optimal choice of $\beta_2(q_{N^-})$ in Lemma 20. To begin with, we define $\beta_Y(q_{N^-}) := Y(q_{N^-})L/q_0$, and start with Lemma 19. Please note that when $M = 0$, the late-budget constraint is vacuous and is interpreted as $0 \le 1 - \beta_Y(q_{N^-})q_0/L$.

Lemma 19. *To solve the optimal value and optimal solutions to problem $\mathtt{P}_0$ is equivalent to solving the following optimization problem $\mathtt{P}_1$:*

$$\max_{\beta_1,\beta_2(\cdot),\beta_Y(\cdot),q_0} \quad \min_{q_{N^-},q_N} (\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_N} + (\beta_1 - \beta_2(q_{N^-}))\frac{q_{N^-}}{q_N} + \beta_2(q_{N^-}) \tag{221}$$

$$\text{s.t.} \quad \beta_1 N^- \left[1 - \left(\frac{q_0 - L}{q_{N^-} - L}\right)^{\frac{1}{N^-}}\right] \le \frac{\beta_Y(q_{N^-})q_0}{L}, \forall q_{N^-} \in [q_0, U], \tag{222}$$

$$\beta_2(q_{N^-})(N^+ - N^-)\left[1 - \left(\frac{q_{N^-}}{q_N}\right)^{\frac{1}{N^+ - N^-}}\right] \le 1 - \frac{\beta_Y(q_{N^-})q_0}{L}, \quad \forall q_{N^-} \in [q_0, U], q_N \in [q_{N^-}, U], \tag{223}$$

$$q_0 \in [L, U], \tag{224}$$

$$\beta_1 \ge 0, \tag{225}$$

$$\beta_2(q_{N^-}) \ge 0, \forall q_{N^-} \in [q_0, U], \tag{226}$$

$$\beta_Y(q_{N^-}) \in [0, \frac{L}{q_0}], \forall q_{N^-} \in [q_0, U], \tag{227}$$

Proof. First, by the definition $\beta_Y(q_{N^-}) := Y(q_{N^-})L/q_0$, we have $Y(q_{N^-}) = \frac{\beta_Y(q_{N^-})q_0}{L}$, and hence the optimization problem $\mathtt{P}_0$ is equivalent to

$$\mathtt{P}_1: \quad \min_{\beta_1,\beta_2(\cdot),\beta_Y(\cdot),q_0} \quad \max_{q_{N^-},q_N} \left[(\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_N} + (\beta_1 - \beta_2(q_{N^-}))\frac{q_{N^-}}{q_N} + \beta_2(q_{N^-})\right]^{-1} \tag{228}$$

$$\text{s.t.} \quad \beta_1 N^- \left[1 - \left(\frac{q_0 - L}{q_{N^-} - L}\right)^{\frac{1}{N^-}}\right] \le \frac{\beta_Y(q_{N^-})q_0}{L}, \forall q_{N^-} \in [q_0, U], \tag{229}$$

$$\beta_2(q_{N^-})(N^+ - N^-)\left[1 - \left(\frac{q_{N^-}}{q_N}\right)^{\frac{1}{N^+ - N^-}}\right] \le 1 - \frac{\beta_Y(q_{N^-})q_0}{L}, \quad \forall q_{N^-} \in [q_0, U], q_N \in [q_{N^-}, U], \tag{230}$$

$$q_0 \in [L, U], \tag{231}$$

$$\beta_1 \ge 0, \tag{232}$$

$$\beta_2(q_{N^-}) \ge 0, \forall q_{N^-} \in [q_0, U], \tag{233}$$

$$\beta_Y(q_{N^-}) \in [0, \frac{L}{q_0}], \forall q_{N^-} \in [q_0, U]. \tag{234}$$

For a clearer presentation, we further transform the above problem into an equivalent max–min form:

$$\max_{\beta_1, \beta_2(\cdot), \beta_Y(\cdot), q_0} \quad \min_{q_{N^-}, q_N} (\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_N} + (\beta_1 - \beta_2(q_{N^-}))\frac{q_{N^-}}{q_N} + \beta_2(q_{N^-}) \tag{235}$$

$$\text{s.t.} \quad \beta_1 N^- \left[1 - \left(\frac{q_0 - L}{q_{N^-} - L}\right)^{\frac{1}{N^-}}\right] \leq \frac{\beta_Y(q_{N^-})q_0}{L}, \forall q_{N^-} \in [q_0, U], \tag{236}$$

$$\beta_2(q_{N^-})(N^+ - N^-)\left[1 - \left(\frac{q_{N^-}}{q_N}\right)^{\frac{1}{N^+ - N^-}}\right] \leq 1 - \frac{\beta_Y(q_{N^-})q_0}{L}, \quad \forall q_{N^-} \in [q_0, U], q_N \in [q_{N^-}, U], \tag{237}$$

$$q_0 \in [L, U], \tag{238}$$

$$\beta_1 \geq 0, \tag{239}$$

$$\beta_2(q_{N^-}) \geq 0, \forall q_{N^-} \in [q_0, U], \tag{240}$$

$$\beta_Y(q_{N^-}) \in [0, \frac{L}{q_0}], \forall q_{N^-} \in [q_0, U], \tag{241}$$

which proves this lemma. □

Lemma 20. *Given L, U, N^-, M, for any feasible $\beta_1, \beta_Y(q_{N^-}), q_0$, and q_{N^-}, the optimal choice of $\beta_2(q_{N^-})$ to problem $\mathtt{P}_1$ is $\frac{1-\frac{\beta_Y(q_{N^-})q_0}{L}}{f_M(q_{N^-})}$ if $f_M(q_{N^-}) > 0$; the optimal choice of $\beta_2(q_{N^-})$ is 0 if $f_M(q_{N^-}) = 0$.*

Proof. We prove this lemma by separately discussing three cases, with respect to the combination of the values of M and q_{N^-}.

Case 1: $q_{N^-} = U$. In this case, we have $q_{N^-} = q_N = U$, which implies $f_M(q_{N^-}) = 0$, and the objective function of $\mathtt{P}_1$ becomes

$$(\beta_Y(U) - \beta_1)\frac{q_0}{U} + \beta_1, \tag{242}$$

which is independent from β_2. Therefore, the optimal solution allows $\beta_2(q_{N^-}) = 0$ when $q_{N^-} = U$.

Case 2: $q_{N^-} < U$ **while** $M > 0$. In this case, we have $f_M(q_{N^-}) > 0$. For the objective function of $\mathtt{P}_1$ in Eq. (221), we have

$$\min_{q_{N^-}, q_N} (\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_N} + (\beta_1 - \beta_2(q_{N^-}))\frac{q_{N^-}}{q_N} + \beta_2(q_{N^-}) \tag{243}$$

$$= \min_{q_{N^-}, q_N} \frac{1}{q_N}[(\beta_Y(q_{N^-}) - \beta_1)q_0 + (\beta_1 - \beta_2(q_{N^-}))q_{N^-}] + \beta_2(q_{N^-}). \tag{244}$$

Given $\beta_1, \beta_2(q_{N^-}), \beta_Y(q_{N^-}), q_0$, and q_{N^-}, $(\beta_Y(q_{N^-}) - \beta_1)q_0 + (\beta_1 - \beta_2(q_{N^-}))q_{N^-}$ is a constant regarding q_N. When $M > 0$, we have $q_N \geq q_{N^-}$, so the minimum value of the objective function is either at $q_N = q_{N^-}$ or $q_N = U$. Therefore, we have

$$\min_{q_{N^-}, q_N} (\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_N} + (\beta_1 - \beta_2(q_{N^-}))\frac{q_{N^-}}{q_N} + \beta_2(q_{N^-}) \tag{245}$$

$$= \min_{q_{N^-} \in [q_0, U)} \min\{(\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_{N^-}} + \beta_1, \tag{246}$$

$$\frac{1}{U}[(\beta_Y(q_{N^-}) - \beta_1)q_0 + (\beta_1 - \beta_2(q_{N^-}))q_{N^-}] + \beta_2(q_{N^-})\}, \tag{247}$$

which indicates that the objective function of $\mathtt{P}_1$ equals

$$\min_{q_{N^-} \in [q_0, U)} \min\{(\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_{N^-}} + \beta_1,$$

$$\frac{1}{U}[(\beta_Y(q_{N^-})-\beta_1)q_0+(\beta_1-\beta_2(q_{N^-}))q_{N^-}]+\beta_2(q_{N^-})\} \tag{248}$$

For Eq. (247), the value of $\beta_2(q_{N^-})$ does not affect the first term $(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_{N^-}}+\beta_1$. For the second term, we have

$$\frac{1}{U}[(\beta_Y(q_{N^-})-\beta_1)q_0+(\beta_1-\beta_2(q_{N^-}))q_{N^-}]+\beta_2(q_{N^-}) \tag{249}$$

$$=\frac{1}{U}[(\beta_Y(q_{N^-})-\beta_1)q_0+\beta_1 q_{N^-}]+\beta_2(q_{N^-})(1-\frac{q_{N^-}}{U}) \tag{250}$$

where, since $q_{N^-}<U$, higher $\beta_2(q_{N^-})$ leads to higher value of this term. Meanwhile, note that the maximum value of $\beta_2(q_{N^-})$ is limited by the constraint (223), so it must satisfy

$$\beta_2(q_{N^-})\leq\frac{1-\frac{\beta_Y(q_{N^-})q_0}{L}}{(N^+-N^-)\left[1-\left(\frac{q_{N^-}}{q_N}\right)^{\frac{1}{N^+-N^-}}\right]},\forall q_N\in[q_{N^-},U] \tag{251}$$

$$\Rightarrow\quad \beta_2(q_{N^-})\leq\frac{1-\frac{\beta_Y(q_{N^-})q_0}{L}}{M\left[1-\left(\frac{q_{N^-}}{U}\right)^{\frac{1}{M}}\right]} \tag{252}$$

$$=\frac{1-\frac{\beta_Y(q_{N^-})q_0}{L}}{f_M(q_{N^-})}. \tag{253}$$

Therefore, the optimal $\beta_2(q_{N^-})$ satisfies

$$\beta_2(q_{N^-})=\frac{1-\frac{\beta_Y(q_{N^-})q_0}{L}}{f_M(q_{N^-})}. \tag{254}$$

Note that, in this case, we have $M>0$ and $q_{N^-}<U$, and hence $f_M(q_{N^-})>0$, so that $\beta_2(q_{N^-})$ is well defined.

Case 3: $q_{N^-}<U$ **while** $M=0$**.** In this case, we have $f_M(q_{N^-})=0$. For any $\beta_1,\beta_Y(q_{N^-}),q_0,q_{N^-}$, if $M=0$, it leads to $f_M(q_{N^-})=0$ and $q_N=q_{N^-}$, such that the objective function of $\mathtt{P}_1$ equals

$$\min_{q_{N^-}\in[q_0,U]}(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_{N^-}}+\beta_1, \tag{255}$$

which is independent from β_2, and hence the optimal solution allows $\beta_2(q_{N^-})=0,\forall q_{N^-}\in[q_0,U]$.

In conclusion, Case 1 covers the optimal choice of $\beta_2(q_{N^-})$ when $q_{N^-}=U$, Case 2 and 3 cover the optimal choice of $\beta_2(q_{N^-})$ when $q_{N^-}<U$. Also, in all cases, the value of $\beta_2(q_{N^-})$ satisfies the constraints (223) and (226), so that this constraint is no longer needed. This lemma is proved. □

K.4.2 Optimal Choice for $\beta_Y(\cdot)$

In this subsection, we first show an equivalent form of the optimization problem $\mathtt{P}_1$ in Lemma 21 by replacing $\beta_2(q_{N^-})$ by the optimal choices that we obtained in Lemma 20, then we prove the optimal choice of $\beta_Y(q_{N^-})$ in Lemma 22.

Lemma 21. *When the decision variable $\beta_2(q_{N^-})$ takes the following values:*

$$\beta_2(q_{N^-})=\begin{cases}\frac{1-\beta_Y(q_{N^-})q_0/L}{f_M(q_{N^-})}, & \text{if } f_M(q_{N^-})>0,\\ 0, & \text{if } f_M(q_{N^-})=0,\end{cases} \tag{256}$$

the optimal value and optimal solutions to the optimization problem $\mathtt{P}_1$ can be obtained by solving the following optimization problem $\mathtt{P}_2$:

$$\mathtt{P}_2:\quad \max_{\beta_1,\beta_Y(\cdot),q_0}\quad \min\left\{(\beta_Y(U)-\beta_1)\frac{q_0}{U}+\beta_1, \right. \tag{257}$$

$$\min_{q_{N^-}\in[q_0,U)} \min\Big\{(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_{N^-}}+\beta_1, \tag{258}$$

$$\beta_1(\frac{q_{N^-}-q_0}{U})+\frac{1-\frac{q_{N^-}}{U}}{f_M(q_{N^-})}+\frac{\beta_Y(q_{N^-})q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-})+q_{N^-}-U]\Big\}\Big\} \tag{259}$$

$$\textit{s.t.}\quad \beta_1 G(q_0,q_{N^-},N^-)\le\frac{\beta_Y(q_{N^-})q_0}{L},\forall q_{N^-}\in[q_0,U], \tag{260}$$

$$q_0\in[L,U], \tag{261}$$

$$\beta_1\ge 0, \tag{262}$$

$$\beta_Y(q_{N^-})\in[0,\frac{L}{q_0}],\forall q_{N^-}\in[q_0,U]. \tag{263}$$

Proof. We notice that for the objective function Eq. (221) of problem $\mathtt{P}_1$, it can be simplified to different equivalent forms under different value of q_N and q_{N^-}.

For any β_1, $\beta_Y(q_{N^-})$, and q_0, when $q_{N^-}=U$, it directly implies $q_N=q_{N^-}=U$. In this case, we set $\beta_2(q_{N^-})$ to 0 as in Eq. (256), so that

$$(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_N}+(\beta_1-\beta_2(q_{N^-}))\frac{q_{N^-}}{q_N}+\beta_2(q_{N^-})=(\beta_Y(U)-\beta_1)\frac{q_0}{U}+\beta_1. \tag{264}$$

For any β_1, $\beta_Y(q_{N^-})$, and q_0, when $q_{N^-}<U$ and $M>0$, it implies $f_M(q_{N^-})>0$. In this case, $\beta_2(q_{N^-})$ is set to $\frac{1-\beta_Y(q_{N^-})q_0/L}{f_M(q_{N^-})}$. Then, by the same analysis of Case 2 of Lemma 20 (Eqs. (244)–(247)), we have

$$\begin{aligned}&\min_{q_{N^-},q_N}(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_N}+(\beta_1-\beta_2(q_{N^-}))\frac{q_{N^-}}{q_N}+\beta_2(q_{N^-})\\ =&\min_{q_{N^-}\in[q_0,U)}\min\Big\{(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_{N^-}}+\beta_1,\end{aligned} \tag{265}$$

$$\beta_1(\frac{q_{N^-}-q_0}{U})+\frac{1-\frac{q_{N^-}}{U}}{f_M(q_{N^-})}+\frac{\beta_Y(q_{N^-})q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-})+q_{N^-}-U]\Big\}.$$

When $q_{N^-}<U$ and $M=0$, the objective function of $\mathtt{P}_1$ equals

$$\min_{q_{N^-}\in[q_0,U]}(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_{N^-}}+\beta_1, \tag{266}$$

which is independent from the value of $\beta_2(q_{N^-})$ and is subsumed by Eqs. (257)–(258). Therefore, by Eqs. (264)–(265), it is clear that, when setting $\beta_2(q_{N^-})$ by Eq. (256), we have

$$\begin{aligned}&\min_{q_{N^-},q_N}(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_N}+(\beta_1-\beta_2(q_{N^-}))\frac{q_{N^-}}{q_N}+\beta_2(q_{N^-})\\ =&\min\Bigg\{(\beta_Y(U)-\beta_1)\frac{q_0}{U}+\beta_1,\end{aligned} \tag{267}$$

$$\min_{q_{N^-}\in[q_0,U)}\min\Big\{(\beta_Y(q_{N^-})-\beta_1)\frac{q_0}{q_{N^-}}+\beta_1, \tag{268}$$

$$\beta_1(\frac{q_{N^-}-q_0}{U})+\frac{1-\frac{q_{N^-}}{U}}{f_M(q_{N^-})}+\frac{\beta_Y(q_{N^-})q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-})+q_{N^-}-U]\Big\}\Bigg\}, \tag{269}$$

which recovers the objective function of problem $\mathtt{P}_2$, thereby proving this lemma. Please note that, as shown in the proof, the second term in the inner minimum of Eq. (259) describes, and hence only concerns, the exchange rate with $M>0$, so it is well defined. □

Lemma 22. *Given L, U, N^-, N^+, and M, for all feasible β_1 and q_0, the optimal choice for $\beta_Y(q_{N^-})$ to the optimization problem $\mathtt{P}_2$ is*

$$\beta_Y(q_{N^-})=\begin{cases}\frac{L}{q_0},\ \textit{when } q_{N^-}=U\\ \frac{L(q_{N^-}-f_M(q_{N^-})\beta_1(q_{N^-}-q_0))}{q_0(Lf_M(q_{N^-})+q_{N^-})},\ \textit{when } q_{N^-}<U.\end{cases} \tag{270}$$

Proof. By Lemma 21, when $q_{N^-} = U$, the value of the objective function of $\mathtt{P}_2$ is $(\beta_Y(U) - \beta_1)\frac{q_0}{U} + \beta_1$, which increases as $\beta_Y(q_{N^-})$ increases, so the optimal $\beta_Y(q_{N^-})$ for this case is $\frac{L}{q_0}$.

When $q_{N^-} < U$ and $M > 0$, we have $f_M(q_{N^-}) > 0$. By Lemma 21, the value of the objective function of the problem $\mathtt{P}_2$ is

$$\min\Big\{(\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_{N^-}} + \beta_1, \\ \beta_1(\frac{q_{N^-} - q_0}{U}) + \frac{1 - \frac{q_{N^-}}{U}}{f_M(q_{N^-})} + \frac{\beta_Y(q_{N^-})q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-}) + q_{N^-} - U]\Big\}, \tag{271}$$

where the first term in Eq. (271) increases as $\beta_Y(q_{N^-})$ increase. Next, we investigate the monotonicity of the second term with respect to $\beta_Y(q_{N^-})$. We notice that the weight of $\beta_Y(q_{N^-})$ in the second term is $\frac{q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-}) + q_{N^-} - U]$. Note that $f_M(q_{N^-}) = M\left[1 - \left(\frac{q_{N^-}}{U}\right)^{\frac{1}{M}}\right]$ by its definition. It will be helpful to recall the result of Corollaries 1–2, which implies the following chain inequality of $f_M(q_{N^-})$:

$$f_M(q_{N^-}) = M\left[1 - \left(\frac{q_{N^-}}{U}\right)^{\frac{1}{M}}\right] \le \ln(\frac{U}{q_{N^-}}) \le \frac{U}{q_{N^-}} - 1, \tag{272}$$

and hence

$$\begin{aligned}
\frac{q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-}) + q_{N^-} - U] \le& \frac{q_0}{ULf_M(q_{N^-})}[L(\frac{U}{q_{N^-}} - 1) + q_{N^-} - U] && (273)\\
=& \frac{q_0}{ULf_M(q_{N^-})}[\frac{L}{q_{N^-}}(U - q_{N^-}) + q_{N^-} - U] && (274)\\
=& \frac{q_0}{ULf_M(q_{N^-})}(\frac{L}{q_{N^-}} - 1)(U - q_{N^-}) && (275)\\
\le& 0 \text{ (Since } q_{N^-} \ge q_0 \ge L) \,. && (276)
\end{aligned}$$

This is to say, the second term of Eq. (271) decreases as $\beta_Y(q_{N^-})$ increases, while the first term increases as $\beta_Y(q_{N^-})$ increases. Therefore, in this case, for all possible $q_{N^-} \in [q_0, U)$, to optimize the objective function for problem $\mathtt{P}_2$, we need to consider the relation of three points: 0 (the smallest possible value of $\beta_Y(q_{N^-})$ required by the constraint Eq. (263)), the intersection of the first and second terms of Eq. (271), and $\frac{L}{q_0}$ (the largest possible value of $\beta_Y(q_{N^-})$ required by the constraint Eq. (263)). We first calculate the value of $\beta_Y(q_{N^-})$ that makes the two terms in Eq. (271) same:

$$\begin{aligned}
&(\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_{N^-}} + \beta_1, \\
=&\beta_1(\frac{q_{N^-} - q_0}{U}) + \frac{1 - \frac{q_{N^-}}{U}}{f_M(q_{N^-})} + \frac{\beta_Y(q_{N^-})q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-}) + q_{N^-} - U] && (277)\\
\Rightarrow\quad & \beta_Y(q_{N^-})\frac{q_0}{q_{N^-}} + \beta_1(1 - \frac{q_0}{q_{N^-}}) && (278)\\
=&\frac{q_{N^-}}{U}\beta_1(1 - \frac{q_0}{q_{N^-}}) + \frac{1 - \frac{q_{N^-}}{U}}{f_M(q_{N^-})} + \beta_Y(q_{N^-})\frac{q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-}) + q_{N^-} - U] && (279)\\
\Rightarrow\quad & \beta_Y(q_{N^-})\{\frac{q_0}{q_{N^-}} - \frac{q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-}) + q_{N^-} - U]\} && (280)\\
=&\frac{1 - \frac{q_{N^-}}{U}}{f_M(q_{N^-})} + \beta_1(1 - \frac{q_0}{q_{N^-}})(\frac{q_{N^-}}{U} - 1) && (281)\\
\Rightarrow\quad & \beta_Y(q_{N^-})q_0\{\frac{1}{q_{N^-}} - \frac{1}{U} - \frac{q_{N^-} - U}{ULf_M(q_{N^-})}\} = \frac{U - q_{N^-}}{Uf_M(q_{N^-})} + \beta_1(1 - \frac{q_0}{q_{N^-}})\frac{q_{N^-} - U}{U} && (282)\\
\Rightarrow\quad & \beta_Y(q_{N^-})\frac{q_0}{q_{N^-}ULf_M(q_{N^-})}[ULf_M(q_{N^-}) - q_{N^-}Lf_M(q_{N^-}) - q_{N^-}(q_{N^-} - U)] && (283)\\
=&\frac{U - q_{N^-}}{Uf_M(q_{N^-})} + \frac{\beta_1}{q_{N^-}U}(q_{N^-} - q_0)(q_{N^-} - U) && (284)\\
\Rightarrow\quad & \beta_Y(q_{N^-})\frac{q_0}{q_{N^-}ULf_M(q_{N^-})}[Lf_M(q_{N^-})(U - q_{N^-}) + q_{N^-}(U - q_{N^-})] && (285)
\end{aligned}$$

$$= \frac{1}{q_{N^-} U f_M(q_{N^-})} [q_{N^-}(U - q_{N^-}) + f_M(q_{N^-})\beta_1(q_{N^-} - q_0)(q_{N^-} - U)] \tag{286}$$

$$\Rightarrow \quad \beta_Y(q_{N^-}) \frac{q_0(U - q_{N^-})}{q_{N^-} U L f_M(q_{N^-})} [L f_M(q_{N^-}) + q_{N^-}] \tag{287}$$

$$= \frac{U - q_{N^-}}{q_{N^-} U f_M(q_{N^-})} [q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0)] \tag{288}$$

$$\Rightarrow \quad \beta_Y(q_{N^-}) \frac{q_0}{L} [L f_M(q_{N^-}) + q_{N^-}] = q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0) \tag{289}$$

$$\Rightarrow \quad \beta_Y(q_{N^-}) = \frac{L(q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0))}{q_0(L f_M(q_{N^-}) + q_{N^-})}, \tag{290}$$

where Eq. (290) gives the intersection where the two terms of Eq. (271) are equal.

Next, we show that the RHS of Eq. (290) is smaller than the largest possible value of $\beta_Y(q_{N^-})$ required by Eq. (263), where

$$\frac{L(q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0))}{q_0(L f_M(q_{N^-}) + q_{N^-})} \leq \frac{L q_{N^-}}{q_0(L f_M(q_{N^-}) + q_{N^-})} \tag{291}$$

$$\leq \frac{L q_{N^-}}{q_0 q_{N^-}} \tag{292}$$

$$\leq \frac{L}{q_0}, \tag{293}$$

satisfying the upper bound of the constraint (263).

On the other hand, we show that any feasible choice of β_1 makes the RHS of Eq. (290) also meet the lower bound of the constraint (263) by contradiction. Suppose that the β_1 is set to a value that makes

$$\frac{L(q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0))}{q_0(L f_M(q_{N^-}) + q_{N^-})} \leq 0, \tag{294}$$

then the optimal $\beta_Y = 0$ and β_1 needs to satisfy the constraint Eq. (260), such that $\beta_1 \leq 0$, so such feasible β_1 can only be 0. However, $\beta_1 = 0$ leads to

$$\frac{L(q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0))}{q_0(L f_M(q_{N^-}) + q_{N^-})} = \frac{L q_{N^-}}{q_0(L f_M(q_{N^-}) + q_{N^-})} > 0, \tag{295}$$

which contradicts the assumption. Therefore, as long as β_1 satisfies the constraint, we have $\frac{L(q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0))}{q_0(L f_M(q_{N^-}) + q_{N^-})} > 0$ and it is the optimal value for $\beta_Y(q_{N^-})$ in this case.

Finally, when $q_{N^-} < U$ and $M = 0$, it implies $q_{N^-} = q_N$. In this case, by Lemma 21, the value of the objective function of problem $\mathtt{P}_2$ is $(\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_{N^-}} + \beta_1$, which also increases as $\beta_Y(q_{N^-})$ increases, so the optimal $\beta_Y(q_{N^-})$ for this case is $\frac{L}{q_0}$. Meanwhile, we notice that $\frac{L}{q_0}$ equals the value of $\frac{L(q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0))}{q_0(L f_M(q_{N^-}) + q_{N^-})}$, so it is equivalent to set $\beta_Y(q_{N^-})$ by Eq. (290) in this case. This conclusion completes the proof of this lemma. □

K.4.3 Optimal Choice for β_1 and q_0

Lemma 23. *When the decision variable $\beta_Y(q_{N^-})$ takes the following values:*

$$\beta_Y(q_{N^-}) = \begin{cases} \frac{L}{q_0}, \text{ when } q_{N^-} = U \\ \frac{L(q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0))}{q_0(L f_M(q_{N^-}) + q_{N^-})}, \text{ when } q_{N^-} < U, \end{cases} \tag{296}$$

the optimal value and optimal solutions to the optimization problem $\mathtt{P}_2$ can be obtained by solving the following optimization problem $\mathtt{P}_3$:

$$\mathtt{P}_3: \quad \max_{\beta_1, q_0} \quad \min \left\{ (\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1, \min_{q_{N^-} \in [q_0, U)} \frac{L + \beta_1(q_{N^-} - q_0)}{L f_M(q_{N^-}) + q_{N^-}} \right\} \tag{297}$$

$$\textit{s.t.} \quad \beta_1 \le \frac{q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0)}{G(q_0, q_{N^-}, N^-)(Lf_M(q_{N^-}) + q_{N^-})}, \forall q_{N^-} \in [q_0, U], \tag{298}$$

$$\beta_1 G(q_0, U, N^-) \le 1, \tag{299}$$

$$q_0 \in [L, U], \tag{300}$$

$$\beta_1 \ge 0. \tag{301}$$

Proof. We notice that for the objective function Eqs. (257)–(259), of problem $\mathtt{P}_2$, it equals different forms under different values of q_N and q_{N^-}, when $\beta_Y(q_{N^-})$ is set by Eq. (296).

By Lemmas (21)–(22), when $q_{N^-} = U$, the value of the objective function of $\mathtt{P}_2$ equals $(\beta_Y(U) - \beta_1)\frac{q_0}{U} + \beta_1$. We set $\beta_Y(q_{N^-}) = \frac{L}{q_0}$, i.e., $\beta_Y(U) = \frac{L}{q_0}$, and then the objective function of $\mathtt{P}_2$ further equals $(\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1$, which recovers the first term of the objective function (297).

Next we discuss the case where $q_{N^-} < U$. By setting $\beta_Y(q_{N^-})$ to $\frac{L(q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0))}{q_0(Lf_M(q_{N^-}) + q_{N^-})}$, the objective function Eqs. (257)–(259) of $\mathtt{P}_2$ equals

$$\min_{q_{N^-} \in [q_0, U)} \min \Big\{ (\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_{N^-}} + \beta_1, \tag{302}$$

$$\beta_1(\frac{q_{N^-} - q_0}{U}) + \frac{1 - \frac{q_{N^-}}{U}}{f_M(q_{N^-})} + \frac{\beta_Y(q_{N^-})q_0}{ULf_M(q_{N^-})}[Lf_M(q_{N^-}) + q_{N^-} - U] \Big\} \tag{303}$$

$$= \min_{q_{N^-} \in [q_0, U)} (\beta_Y(q_{N^-}) - \beta_1)\frac{q_0}{q_{N^-}} + \beta_1 \quad \text{(Eq. (296) makes the two terms equal)} \tag{304}$$

$$= \min_{q_{N^-} \in [q_0, U)} \frac{L + \beta_1(q_{N^-} - q_0)}{Lf_M(q_{N^-}) + q_{N^-}}, \tag{305}$$

which recovers the second term of the objective function (297). The constraint (298) comes from replacing $\beta_Y(q_{N^-})$ by Eq. (296), and constraint. (299) serves as the special case of constraint (298) where $q_{N^-} = U$. This lemma is proved. □

Lemma 24. *For any feasible β_1 and q_0, we have*

$$\min \left\{ (\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1, \min_{q_{N^-} \in [q_0, U)} \frac{L + \beta_1(q_{N^-} - q_0)}{Lf_M(q_{N^-}) + q_{N^-}} \right\} \tag{306}$$

$$= \min \left\{ (\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1, \frac{L}{Lf_M(q_0) + q_0} \right\} \tag{307}$$

Proof. Define $R(x) = \frac{L + \beta_1(x - q_0)}{Lf_M(x) + x}$. We first show that $\lim_{x \to U} R(x) = (\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1$. Specifically,

$$\lim_{x \to U} R(x) = \frac{L + \beta_1(U - q_0)}{L \times 0 + U} \tag{308}$$

$$= \frac{L}{U} + \beta_1(1 - \frac{q_0}{U}) \tag{309}$$

$$= (\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1. \tag{310}$$

Then, to prove this lemma, it is sufficient to prove that $R(x) \ge \min\{R(q_0), \lim_{x \to U} R(x)\}, \forall x \in [q_0, U)$.

We set the denominator of $R(x)$ as $h(x) := Lf_M(x) + x = x + LM - LM(\frac{x}{U})^{1/M}$, and set its numerator as $H(x) := L + \beta_1(x - q_0)$. We first show that $h(x)$ is convex on $[q_0, U)$. Specifically, we calculate the first and second order derivatives of $h(x)$:

$$h'(x) = 1 - \frac{L}{U}(\frac{x}{U})^{\frac{1}{M} - 1}, \tag{311}$$

$$h''(x) = -\frac{L}{U^2}(\frac{1}{M} - 1)(\frac{x}{U})^{\frac{1}{M}-2}. \tag{312}$$

For the case where $M \geq 1$, hence $\frac{1}{M} \in (0, 1]$, we have $h''(x) \geq 0$, such that $h(x)$ is convex on $[q_0, U)$. Notice also $f_M(x) \geq 0$ when $x \in [q_0, U)$, such that $h(x) > 0$ on $x \in [q_0, U)$. Next we show that the first order derivative of $R(x)$ is non increasing on $[q_0, U)$ in this case. We have

$$R'(x) = \frac{H'(x)h(x) - H(x)h'(x)}{h^2(x)} \tag{313}$$

$$= \frac{\beta_1 h(x) - H(x)h'(x)}{h^2(x)}. \tag{314}$$

For the numerator of $R'(x)$, we have

$$\frac{d(\beta_1 h(x) - H(x)h'(x))}{dx} = \beta_1 h'(x) - \beta_1 h'(x) - H(x)h''(x) \tag{315}$$

$$= -H(x)h''(x) \tag{316}$$

$$\leq 0 \quad (\text{since } H(x) > 0 \text{ and } h''(x) \geq 0). \tag{317}$$

Therefore, the numerator of $R(x)$ is non-increasing on $[q_0, U)$. Since $h(x) > 0$, the sign of $R'(x)$ is the same as its numerator, and there are three cases: (i) always positive, then the minimum value of $R(x)$ is achieved at q_0; (ii) always negative, then the minimum value is achieved at U; (iii) positive then negative, then the minimum value is achieved at either q_0 or U. As a result, for the case where $M \geq 1$, we have

$$\min_{q_{N^-} \in [q_0, U)} R(q_{N^-}) = \min\left\{(\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1, \frac{L}{Lf_M(q_0) + q_0}\right\}. \tag{318}$$

For the case where $M = 0$, we have

$$R(q_{N^-}) = \frac{L + \beta_1(q_{N^-} - q_0)}{q_{N^-}}, \tag{319}$$

and

$$R'(q_{N^-}) = \frac{\beta_1 q_0 - L}{q_{N^-}^2}, \tag{320}$$

where $R'(q_{N^-})$ is either non-negative or non-positive, depending on the value of $\beta_1 q_0 - L$, and the minimum value of $R(q_{N^-})$ is achieved either at q_0 or U. Note that, in this case, we have

$$\lim_{q_{N^-} \to U} R(q_{N^-}) = (\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1, \tag{321}$$

and

$$R(q_0) = \frac{L}{q_0} = \frac{L}{Lf_M(q_0) + q_0} \quad (\text{since } f_M(q_0) = 0 \text{ when } M = 0), \tag{322}$$

so that Eq. (318) also holds true in this case. Therefore, this lemma is proved. □

Corollary 5. *The optimal value and optimal solutions to the optimization problem* $\mathtt{P}_3$ *can be obtained by solving the following optimization problem* $\mathtt{P}_4$*:*

$$\mathtt{P}_4: \quad \max_{\beta_1, q_0} \quad \min\left\{(\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1, \frac{L}{Lf_M(q_0) + q_0}\right\} \tag{323}$$

$$\textit{s.t.} \quad \beta_1 \leq \frac{q_{N^-} - f_M(q_{N^-})\beta_1(q_{N^-} - q_0)}{G(q_0, q_{N^-}, N^-)(Lf_M(q_{N^-}) + q_{N^-})}, \forall q_{N^-} \in [q_0, U], \tag{324}$$

$$\beta_1 G(q_0, U, N^-) \leq 1, \tag{325}$$

$$q_0 \in [L, U], \tag{326}$$

$$\beta_1 \geq 0. \tag{327}$$

Proof. This corollary comes directly from Lemmas (23)–(24). □

Lemma 25. *There exists optimal solution β_1^* and q_0^* to the optimization problem $\mathtt{P}_4$ that satisfy*

$$(\frac{L}{q_0^*} - \beta_1^*)\frac{q_0^*}{U} + \beta_1^* = \frac{L}{Lf_M(q_0^*) + q_0^*}. \tag{328}$$

Proof. We first go through a series of simplification of certain terms that appear in the optimization problem $\mathtt{P}_4$, in order to make sure our proof looks clean and neat.

We define $T_1(\beta_1, q_0) := \frac{L}{U} + (1 - \frac{q_0}{U})\beta_1$, and $T_2(q_0) = \frac{L}{Lf_M(q_0)+q_0}$. Then the following claim holds:

Claim 1. *To prove Lemma 25, it suffices to prove that there exists optimal solution β_1^* and q_0^* to the optimization problem $\mathtt{P}_4$ that satisfy*

$$T_1(\beta_1^*, q_0^*) = T_2(q_0^*). \tag{329}$$

With the definition of T_1 and T_2, the objective function of $\mathtt{P}_4$ becomes

$$\max_{\beta_1, q_0} \quad \min\left\{(\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1, \frac{L}{Lf_M(q_0) + q_0}\right\} \tag{330}$$

$$= \max_{q_0} \quad \max_{\beta_1} \quad \min\{T_1(\beta_1, q_0), T_2(q_0)\}. \tag{331}$$

Define

$$B_M(q_0, q_{N^-}, N^-) = \frac{q_{N^-}}{G(q_0, q_{N^-}, N^-)(Lf_M(q_{N^-}) + q_{N^-}) + f_M(q_{N^-})(q_{N^-} - q_0)}, \tag{332}$$

then the first constraint (324) for the problem $\mathtt{P}_4$ is equivalent to

$$\beta_1 \le B_M(q_0, q_{N^-}, N^-), \forall q_{N^-} \in [q_0, U]. \tag{333}$$

Recall the definition $T_1(\beta_1, q_0) := \frac{L}{U} + \frac{(U-q_0)\beta_1}{U}$, and define

$$x = (U - q_0)\beta_1, \tag{334}$$

such that constraints (324)–(325) implies

$$x \le (U - q_0)B_M(q_0, q_{N^-}, N^-), \forall q_{N^-} \in [q_0, U] \quad \text{(by inequality (333))}, \tag{335}$$

$$x \le \frac{U - q_0}{G(q_0, U, N^-)}. \tag{336}$$

Define $T_3(x) := \frac{L}{U} + \frac{x}{U}$ and $Y(q_0, N^-) := \frac{U-q_0}{G(q_0,U,N^-)}$, recall the definition of the function X_M in Eq. (8), we have

$$X_M(q, q_{N^-}, N^-) := \frac{(U - q)q_{N^-}}{G(q, q_{N^-}, N^-)(Lf_M(q_{N^-}) + q_{N^-}) + f_M(q_{N^-})(q_{N^-} - q)}, \tag{337}$$

which also implies that $X_M(q_0, q_{N^-}, N^-) = (U - q_0)B_M(q_0, q_{N^-}, N^-)$. Then $\mathtt{P}_4$ is equivalent to the following problem $\mathtt{P}_4'$

$$\mathtt{P}_4': \quad \max_{q_0} \quad \max_{x} \quad \min\{T_3(x), T_2(q_0)\} \tag{338}$$

$$\text{s.t.} \quad x \le X_M(q_0, q_{N^-}, N^-), \forall q_{N^-} \in [q_0, U], \tag{339}$$

$$x \le Y(q_0, N^-), \tag{340}$$

$$q_0 \in [L, U], \tag{341}$$

$$x \ge 0, \tag{342}$$

and makes the following claim true:

Claim 2. *To prove that there exists optimal solution β_1^* and q_0^* to the optimization problem $\mathtt{P}_4$ that satisfy $T_1(\beta_1^*, q_0^*) = T_2(q_0^*)$, it suffices to prove that there exists optimal solution x^* and q_0^* to problem $\mathtt{P}_4'$ such that $T_3(x^*) = T_2(q_0^*)$.*

Notice that, for any fixed q_0, $T_2(q_0)$ is constant. Then, since $T_3(x)$ is non-decreasing in x, we have

$$\max_x \quad \min\{T_3(x), T_2(q_0)\} = \min\{\max_x T_3(x), T_2(q_0)\}, \tag{343}$$

while it worth noting that an optimal x for the LHS of Eq. (343) is not necessary an optimal x for the RHS of Eq. (343). Accordingly, we can construct an optimization problem $\mathtt{P}_4''$, such that

$$\begin{aligned}
\mathtt{P}_4'': \quad \max_{q_0} \quad & \min\{\max_x T_3(x), T_2(q_0)\} && (344)\\
\text{s.t.} \quad & x \le X_M(q_0, q_{N^-}, N^-), \forall q_{N^-} \in [q_0, U], && (345)\\
& x \le Y(q_0, N^-), && (346)\\
& q_0 \in [L, U], && (347)\\
& x \ge 0, && (348)
\end{aligned}$$

which has the same optimal value and optimal choice of q_0 as problem $\mathtt{P}_4'$, but not necessarily the same optimal x. Specifically, with the same optimal q_0^* to both $\mathtt{P}_4'$ and $\mathtt{P}_4''$, we define

$$x^*(q_0^*) := \arg\max_x \min\{T_3(x), T_2(q_0^*)\}, \tag{349}$$

$$\hat{x}^*(q_0^*) := \arg\max_x T_3(x), \tag{350}$$

where $x^*(q_0^*)$ is the optimal x to problem $\mathtt{P}_4'$, and $\hat{x}^*(q_0^*)$ is the optimal x to problem $\mathtt{P}_4''$. Then, we have $T_3(\hat{x}^*(q_0^*)) \ge T_3(x^*(q_0^*))$. According to the monotonicity of $T_3(x) := \frac{L}{U} + \frac{x}{U}$, which is non-decreasing in x, we also have $\hat{x}^*(q_0^*) \ge x^*(q_0)$.

Please note that if we can prove that $T_3(x(q_0^*))$ increases from some value below $T_2(q_0^*)$ to some value above it as $x(q_0^*)$ increases on it feasible range prescribed by q_0^* and β_1, then we can prove that there exists an intersection of $T_3(x(q_0^*))$ and $T_2(q_0^*)$, and this intersection is a maximizer of problem $\mathtt{P}_4'$ that makes $T_3(x^*(q_0^*)) = T_2(q_0^*)$. Specifically, this reasoning leads to the following claim:

Claim 3. *To prove that there exists optimal solution x^* and q_0^* to problem $\mathtt{P}_4'$ such that $T_3(x^*) = T_2(q_0^*)$, it suffices to prove that there exists optimal $q*_0$ to problems $\mathtt{P}_4'$ and $\mathtt{P}_4''$ that satisfies $T_3(\hat{x}^*(q_0^*)) \ge T_2(q_0^*)$ and $\min_x T_3(x) \le T_2(q_0^*)$.*

Step 1: $\min_x T_3(x) \le T_2(q_0^*)$. For any q_0, $x = 0$ is a feasible solution to both problems $\mathtt{P}_4'$ and $\mathtt{P}_4''$ with $T_3(0) = \frac{L}{U}$. Now we look at the monotonicity of $T_2(q_0)$. When $M = 0$, its denominator satisfies $Lf_M(q_0) + q_0 = q_0$, which is non-decreasing in q_0. When $M > 0$, we have

$$\begin{aligned}
\frac{d(Lf_M(q_0) + q_0)}{dq_0} =& \frac{d(q_0 - LM(\frac{q_0}{U})^{\frac{1}{M}})}{dq_0} && (351)\\
=& 1 - LM\frac{1}{M}(\frac{q_0}{U})^{\frac{1}{M}-1}\frac{1}{U} && (352)\\
=& 1 - \frac{L}{U}(\frac{q_0}{U})^{\frac{1}{M}-1} && (353)\\
\ge& 1 - \frac{L}{U}(\frac{L}{U})^{\frac{1}{M}-1} \quad \text{(by } M \ge 1 \text{ and the monotonicity of } (\cdot)^{\frac{1}{M}-1}\text{)} && (354)\\
=& 1 - (\frac{L}{U})^{\frac{1}{M}} && (355)\\
\ge& 0 \quad \text{(by } L \le U\text{)}, && (356)
\end{aligned}$$

which concludes that $Lf_M(q_0) + q_0$ is non-decreasing in q_0, and $T_2(q_0)$ is non-increasing in q_0. Therefore, we further have

$$T_2(q_0) \ge \lim_{q_0 \to U} T_2(q_0) = \frac{L}{U} = T_3(0), \forall q_0 \in [L, U]. \tag{357}$$

Therefore, for any q_0, $\min_x T_3(x) \le T_2(q_0)$, which implies $\min_x T_3(x) \le T_2(q_0^*)$.

Step 2: another transform. When $q_0 = U$, we have $T_1(\beta_1, U) = \frac{L}{U} + \frac{(U-q_0)\beta_1}{U} = \frac{L}{U} + \frac{0\beta_1}{U} = \frac{L}{U}$ and also $T_2(U) = \frac{L}{U}$. Notice that T_3 is increasing in x, then according to the definition, the value of

$\hat{x}^*(q_0)$ equals the upper bound of either the constraint (345) or the constraint (346), depending on which one is lower. Specifically, we have

$$\hat{x}^*(q_0) := \min\{Y(q_0, N^-), \inf_{q_{N^-} \in [q_0, U]} X_M(q_0, q_{N^-}, N^-)\}, \tag{358}$$

and it is straightforward that $\hat{x}^*(q_0) \geq 0$, otherwise problem $\mathtt{P}_4''$ would have no feasible solution and would be meaningless. Then, we define $T_4(q_0) := T_3(\hat{x}^*(q_0))$, the optimal q_0 to problem $\mathtt{P}_4''$ can be obtained by solving the following problem $\hat{\mathtt{P}}_4$

$$\hat{\mathtt{P}}_4: \quad \max_{q_0} \quad \min\{T_4(q_0), T_2(q_0)\} \tag{359}$$

$$\text{s.t.} \quad q_0 \in [L, U]. \tag{360}$$

We denote the optimal q_0 to $\hat{\mathtt{P}}_4$ by $\hat{q}_0^*$. Then the following claim holds:

Claim 4. *To prove that there exists optimal $q*_0$ to problems $\mathtt{P}_4'$ and $\mathtt{P}_4''$ that satisfies $T_3(\hat{x}^*(q_0^*)) \geq T_2(q_0^*)$, it suffices to prove that there exists optimal solution $\hat{q}_0^* \in [L, U]$ to $\hat{\mathtt{P}}_4$ such that $T_4(\hat{q}_0^*) \geq T_2(\hat{q}_0^*)$.*

Step 3: Monotonicity of $T_4(q_0)$. We prove that $T_4(q_0)$ is non-decreasing in q_0. Since $T_4(q_0) = T_3(\hat{x}^*(q_0))$, and T_3 is increasing in $\hat{x}^*(q_0)$, it suffices to prove that $\hat{x}^*(q_0)$ is non-decreasing in q_0, where, by the definition of $\hat{x}^*(q_0)$, it suffices to prove that (i) $Y(q_0, N^-)$ is non-decreasing in q_0; (ii) $X_M(q_0, q_{N^-}, N^-)$ is non-decreasing in q_0, for all $q_{N^-} \in [q_0, U)$. We note that (ii) holds true by Lemma (13). What remains is to prove that $Y(q_0, N^-)$ is non-decreasing in q_0.

Step 3.1: Monotonicity of $Y(q_0, N^-)$. Let $y(q_0) := U - q_0$, and $g(y) := 1 - (1 - \frac{y(q_0)}{U-L})^{\frac{1}{N^-}}$, then we have

$$Y(q_0, N^-) = \frac{U - q_0}{G(q_0, U, N^-)} \tag{361}$$

$$= \frac{y(q_0)}{N^-[1 - (\frac{q_0 - L}{U-L})^{\frac{1}{N^-}}]} \tag{362}$$

$$= \frac{y(q_0)}{N^- g(y(q_0))}. \tag{363}$$

For function $g(y)$, its first and second order derivatives are

$$\frac{dg}{dy} = \frac{1}{N^-(U-L)}(1 - \frac{y}{U-L})^{\frac{1}{N^-}-1}, \tag{364}$$

$$\frac{d^2g}{dy^2} = \frac{N^- - 1}{(N^-)^2(U-L)^2}(1 - \frac{y}{U-L})^{\frac{1}{N^-}-2}. \tag{365}$$

Since $N^- \geq 1$, we have $\frac{N^- - 1}{(N^-)^2(U-L)^2} \geq 0$. Since $y = U - q_0 \leq U - L$, we have $1 - \frac{y}{U-L} \geq 0$ and $(1 - \frac{y}{U-L})^{\frac{1}{N^-}-2} \geq 0$, and hence

$$\frac{d^2g}{dy^2} \geq 0, \tag{366}$$

such that $g(y)$ is convex in y. Note that $g(0) = 0$, so by Lemma (11), it implies that $\frac{g(y)}{y}$ is non-decreasing in y, and $\frac{y}{g(y)}$ is non-increasing in y. Since $y = U - q_0$ is non-increasing in q_0, $\frac{y(q_0)}{g(y(q_0))}$ is non-decreasing in q_0, and hence $Y(q_0, N^-) = \frac{y(q_0)}{N^- g(y(q_0))}$ is non-decreasing in q_0.

Step 3.2: Monotonicity of $\hat{x}^*(q_0)$. For any q_0 and q_0' that $L \leq q_0 \leq q_0' < U$, (i) by **Step 3.1**, we have $Y(q_0', N^-) \geq Y(q_0, N^-)$; (ii) by Corollary 4, we have

$$\inf_{q_{N^-} \in [q_0', U)} X_M(q_0', q_{N^-}, N^-) \geq \inf_{q_{N^-} \in [q_0, U)} X_M(q_0, q_{N^-}, N^-). \tag{367}$$

Therefore,

$$\hat{x}^*(q_0') = \min\{Y(q_0', N^-), \inf_{q_{N^-} \in [q_0', U)} X_M(q_0', q_{N^-}, N^-)\} \tag{368}$$

$$\geq \min\{Y(q_0, N^-), \inf_{q_{N^-} \in [q_0, U)} X_M(q_0, q_{N^-}, N^-)\} \tag{369}$$

$$= \hat{x}^*(q_0), \tag{370}$$

so that $\hat{x}^*(q_0)$ is non-decreasing in q_0, and hence $T_4(q_0)$ is non-decreasing in q_0.

Step 4: We now prove that there exists optimal solution $\hat{q}_0^* \in [L, U)$ to $\hat{\mathtt{P}}_4$ such that $T_4(\hat{q}_0^*) \geq T_2(\hat{q}_0^*)$. We first prove that $\lim_{q_0 \to U} T_4(q_0) > \lim_{q_0 \to U} T_2(q_0)$. It is straightforward that

$$\lim_{q_0 \to U} T_2(q_0) = \lim_{q_0 \to U} \frac{L}{L f_M(q_0) + q_0} \tag{371}$$

$$= \lim_{q_0 \to U} \frac{L}{LM[1 - (\frac{q_0}{U})^{\frac{1}{M}}] + q_0} \tag{372}$$

$$= \frac{L}{U}. \tag{373}$$

What's left is to show the limit $\lim_{q_0 \to U} T_4(q_0)$, which, since $T_4(q_0) := T_3(\hat{x}^*(q_0))$, will be established upon the limit of $\hat{x}^*(q_0))$. Because the value of $\hat{x}^*(q_0))$ depends on $Y(q_0, N^-)$ and $\inf_{q_{N^-} \in [q_0, U)} X_M(q_0, q_{N^-}, N^-)$, we will analyze their limits one by one.

Step 4.1: the limit of $Y(q_0, N^-)$. Set $\epsilon := U - q_0$, such that ϵ approaches 0^+ as q_0 approaches U. Set $a := U - L$, then we have

$$\frac{q_0 - L}{U - L} = \frac{a - \epsilon}{a} = 1 - \frac{\epsilon}{a}. \tag{374}$$

By Taylor expansion, we have, as ϵ approaches 0:

$$(1 - \frac{\epsilon}{a})^{\frac{1}{N^-}} = 1 - \frac{1}{N^-}\frac{\epsilon}{a} + o(\epsilon^2), \tag{375}$$

$$1 - (1 - \frac{\epsilon}{a})^{\frac{1}{N^-}} = \frac{\epsilon}{N^- a} + o(\epsilon^2). \tag{376}$$

Therefore,

$$\lim_{q_0 \to U} Y(q_0, N^-) = \lim_{q_0 \to U} \frac{U - q_0}{G(q_0, U, N^-)} \tag{377}$$

$$= \lim_{q_0 \to U} \frac{U - q_0}{N^-[1 - (\frac{q_0 - L}{U - L})^{\frac{1}{N^-}}]} \tag{378}$$

$$= \lim_{\epsilon \to 0^+} \frac{\epsilon}{N^-[1 - (1 - \frac{\epsilon}{a})^{\frac{1}{N^-}}]} \tag{379}$$

$$= \lim_{\epsilon \to 0^+} \frac{\epsilon}{N^-(\frac{\epsilon}{N^- a} + o(\epsilon^2))} \tag{380}$$

$$= \lim_{\epsilon \to 0^+} \frac{\epsilon}{\frac{\epsilon}{a} + o(\epsilon^2)} \tag{381}$$

$$= \lim_{\epsilon \to 0^+} \frac{1}{\frac{1}{a} + \frac{o(\epsilon^2)}{\epsilon}} \tag{382}$$

$$= \lim_{\epsilon \to 0^+} \frac{1}{\frac{1}{a} + o(\epsilon)} \tag{383}$$

$$= a \tag{384}$$

$$= U - L. \tag{385}$$

Step 4.2: The limit of $X_M(q_0, q_{N^-}, N^-)$ Define $\phi(x) = x^{\frac{1}{M}}$ on $x \in [\frac{q_{N^-}}{U}, 1]$. By mean value theorem, these exists $\xi \in [\frac{q_{N^-}}{U}, 1]$, such that

$$1 - (\frac{q_{N^-}}{U})^{\frac{1}{M}} = \phi'(\xi)(1 - \frac{q_{N^-}}{U}) = \frac{1}{M}\xi^{\frac{1}{M}-1}\frac{U - q_{N^-}}{U}, \tag{386}$$

and hence,

$$f_M(q_{N^-}) = M[1 - (\frac{q_{N^-}}{U})^{\frac{1}{M}}] = \xi^{\frac{1}{M}-1}\frac{U - q_{N^-}}{U}. \tag{387}$$

Since $\xi \in [\frac{q_{N^-}}{U}, 1]$, $\xi \geq \frac{q_0}{U}$. Since $M \geq 1$, $\xi^{\frac{1}{M}-1} \leq (\frac{q_0}{U})^{\frac{1}{M}-1}$. Define $C_1(q_0) := (\frac{q_0}{U})^{\frac{1}{M}-1} = (\frac{U}{q_0})^{\frac{M-1}{M}}$, then we have

$$\begin{aligned} f_M(q_{N^-}) \leq & (\frac{q_0}{U})^{\frac{1}{M}-1}\frac{U - q_{N^-}}{U} && (388)\\ \leq & \frac{C_1(q_0)(U - q_0)}{U}. && (389) \end{aligned}$$

Similarly, for $G(q_0, q_{N^-}, N^-)$, define $\psi(x) = x^{\frac{1}{N^-}}$ on $[\frac{q_0 - L}{q_{N^-} - L}, 1]$, then there exists $\eta \in [\frac{q_0 - L}{q_{N^-} - L}, 1]$, such that

$$1 - (\frac{q_0 - L}{q_{N^-} - L})^{\frac{1}{N^-}} = \psi'(\eta)(1 - \frac{q_0 - L}{q_{N^-} - L}) = \frac{1}{N^-}\eta^{\frac{1}{N^-}-1}\frac{q_{N^-} - q_0}{q_{N^-} - L}. \tag{390}$$

Define $C_2(q_0) := (\frac{q_0 - L}{U - L})^{\frac{1-N^-}{N^-}}$, we have

$$\begin{aligned} G(q_0, q_{N^-}, N^-) = & \eta^{\frac{1}{N^-}-1}\frac{q_{N^-} - q_0}{q_{N^-} - L} && (391)\\ \leq & (\frac{q_0 - L}{q_{N^-} - L})^{\frac{1}{N^-}-1}\frac{q_{N^-} - q_0}{q_{N^-} - L} && (392)\\ = & (\frac{q_{N^-} - L}{q_0 - L})^{1-\frac{1}{N^-}}\frac{q_{N^-} - q_0}{q_{N^-} - L} && (393)\\ \leq & (\frac{U - L}{q_0 - L})^{1-\frac{1}{N^-}}\frac{q_{N^-} - q_0}{q_{N^-} - L} && (394)\\ \leq & C_2(q_0)\frac{U - q_0}{q_{N^-} - L}. && (395) \end{aligned}$$

With the upper bounds on $f_M(q_{N^-})$ and $G(q_0, q_{N^-}, N^-)$, define the denominator of $X_M(q_0, q_{N^-}, N^-)$ by $D_M(q_0, q_{N^-}, N^-)$, such that

$$\begin{aligned} D_M(q_0, q_{N^-}, N^-) := & G(q_0, q_{N^-}, N^-)(Lf_M(q_{N^-}) + q_{N^-}) + f_M(q_{N^-})(q_{N^-} - q_0) && (396)\\ \leq & C_2(q_0)\frac{U - q_0}{q_{N^-} - L}(L\frac{C_1(q_0)(U - q_0)}{U} + q_{N^-}) + \frac{C_1(q_0)(U - q_0)}{U}(U - q_0) && (397)\\ = & \frac{LC_1(q_0)C_2(q_0)(U - q_0)^2}{U(q_{N^-} - L)} + \frac{C_2(q_0)(U - q_0)q_{N^-}}{q_{N^-} - L} + \frac{C_1(q_0)(U - q_0)^2}{U} && (398)\\ \leq & \frac{LC_1(q_0)C_2(q_0)(U - q_0)^2}{U(q_0 - L)} + \frac{C_2(q_0)(U - q_0)q_{N^-}}{q_0 - L} + \frac{C_1(q_0)(U - q_0)^2}{U}, && (399) \end{aligned}$$

and hence

$$\begin{aligned} X_M(q_0, q_{N^-}, N^-) = & \frac{(U - q_0)q_{N^-}}{D_M(q_0, q_{N^-}, N^-)} && (400)\\ \geq & \frac{q_{N^-}}{\frac{C_2(q_0)q_{N^-}}{q_0 - L} + \frac{LC_1(q_0)C_2(q_0)(U - q_0)}{U(q_0 - L)} + \frac{C_1(q_0)(U - q_0)}{U}}, \forall q_{N^-} \in [q_0, U). && (401) \end{aligned}$$

Define $C_3(q_0) := \frac{LC_1(q_0)C_2(q_0)}{U(q_0 - L)} + \frac{C_1(q_0)}{U}$, we have

$$\inf_{q_{N^-} \in [q_0, U]} X_M(q_0, q_{N^-}, N^-) \geq \inf_{q_{N^-} \in [q_0, U]} \frac{q_{N^-}}{\frac{C_2(q_0)q_{N^-}}{q_0 - L} + (U - q_0)C_3(q_0)}. \tag{402}$$

As q_0 approaches U, $C_1(q_0) = (\frac{U}{q_0})^{\frac{M-1}{M}}$ approaches 1, $C_2(q_0) := (\frac{q_0-L}{U-L})^{\frac{1-N^-}{N^-}}$ approaches 1, $C_3(q_0) := \frac{LC_1(q_0)C_2(q_0)}{U(q_0-L)} + \frac{C_1(q_0)}{U}$ approaches $\frac{L}{U(U-L)} + \frac{1}{U}$, and $(U-q_0)C_3(q_0)$ approaches 0. Therefore

$$\lim_{q_0\to U} \inf_{q_{N^-}\in[q_0,U]} X_M(q_0, q_{N^-}, N^-) \geq \lim_{q_0\to U} \inf_{q_{N^-}\in[q_0,U]} \frac{q_{N^-}}{\frac{C_2(q_0)q_{N^-}}{q_0-L} + (U-q_0)C_3(q_0)} \tag{403}$$

$$= \lim_{q_0\to U} \inf_{q_{N^-}\in[q_0,U]} \frac{q_{N^-}}{\frac{q_{N^-}}{U-L}} \tag{404}$$

$$= U - L. \tag{405}$$

Step 4.3: Limit of $\lim_{q_0\to U} T_4(q_0)$. By **Step 4.1** and **Step 4.2**, $\lim_{q_0\to U} \hat{x}^*(q_0) \geq U - L$, such that

$$\lim_{q_0\to U} T_4(q_0) = \lim_{q_0\to U} T_3(\hat{x}^*(q_0)) \tag{406}$$

$$= \lim_{q_0\to U} \frac{L}{U} + \frac{\hat{x}^*(q_0)}{U} \tag{407}$$

$$\geq \frac{L}{U} + \frac{U-L}{U} \tag{408}$$

$$= 1 \tag{409}$$

$$\geq \frac{L}{U} \tag{410}$$

$$= \lim_{q_0\to U} T_2(q_0) \quad \text{(by Eq. (357))}. \tag{411}$$

Recall that $T_4(q_0)$ is non-decreasing in q_0 (Step 3) and $T_2(q_0)$ is non-increasing in q_0 (Step 1), Eq. (357) concludes that

$$\max_{q_0\in[L,U]} T_4(q_0) \geq \max_{q_0\in[L,U]} T_2(q_0). \tag{412}$$

By the monotonicity of $T_4(q_0)$ and $T_2(q_0)$, we discuss 2 cases:

Case 1: if $T_4(L) < T_2(L)$. In this case, functions $T_4(q_0)$ and $T_2(q_0)$ has in intersection in $q_0 \in [L, U]$, and this intersection serves as the optimal $\hat{q}_0^*$ for problem $\hat{\mathrm{P}}_4$, where $T_4(\hat{q}_0^*) = T_2(\hat{q}_0^*)$ and $\hat{q}_0^* \leq U$, such that there exists optimal solution $\hat{q}_0^* \in [L, U]$ to $\hat{\mathrm{P}}_4$ such that $T_4(\hat{q}_0^*) \geq T_2(\hat{q}_0^*)$.

Case 2: if $T_4(L) \geq T_2(L)$. In this case, the optimal $\hat{q}_0^*$ for problem $\hat{\mathrm{P}}_4$ is at $\hat{q}_0^* = L$ and it is straightforward that $\hat{q}_0^* \leq U$, such that there exists optimal solution $\hat{q}_0^* \in [L, U]$ to $\hat{\mathrm{P}}_4$ such that $T_4(\hat{q}_0^*) \geq T_2(\hat{q}_0^*)$.

In both Cases 1 and 2, it holds true that there exists optimal solution $\hat{q}_0^* \in [L, U]$ to $\hat{\mathrm{P}}_4$ such that $T_4(\hat{q}_0^*) \geq T_2(\hat{q}_0^*)$. By Claim 4, Step 1, and Claims 1–3, this lemma is proved. □

Corollary 6. *Given L, U, N^-, N^+, and M, for all q_0, the optimal choice for β_1 to the problem P_4 is*

$$\beta_1 = \begin{cases} \frac{L(U-Lf_M(q_0)-q_0)}{(U-q_0)(Lf_M(q_0)+q_0)}, & \text{if } q_0 < U, \\ 0, & \text{if } q_0 = U. \end{cases} \tag{413}$$

Proof. This corollary directly comes from Lemma 25. We note that when $q_0 = U$, the equation $(\frac{L}{q_0} - \beta_1)\frac{q_0}{U} + \beta_1 = \frac{L}{Lf_M(q_0)+q_0}$ holds regardless of the value of β_1, so we can set it to 0 to satisfy the constraints (324), (325), and (327). □

Lemma 26. *When β_1 is set by*

$$\beta_1 = \begin{cases} \frac{L(U-Lf_M(q_0)-q_0)}{(U-q_0)(Lf_M(q_0)+q_0)}, & \text{if } q_0 < U, \\ 0, & \text{if } q_0 = U, \end{cases} \tag{414}$$

the optimal value and optimal solution to the problem $\mathtt{P}_4$ *(Eq. (323)) can be obtained by solving the following problem* $\mathtt{P}_5$*:*

$$\mathtt{P}_5 \quad \max_{q_0 \in [L,U]} \quad \frac{L}{Lf_M(q_0)+q_0} \tag{415}$$

$$s.t. \frac{L(U-Lf_M(q_0)-q_0)}{(Lf_M(q_0)+q_0)} \leq X_M(q_0, q_{N^-}, N^-), \forall q_{N^-} \in [q_0, U], \text{ if } q_0 < U. \tag{416}$$

Proof. By Lemma (25), when β_1 and q_0 take values that satisfy Eq. (414) the objective function of $\mathtt{P}_4$ equals to $\frac{L}{Lf_M(q_0)+q_0}$.

Next, we notice that when $q_0 = U$, $\beta_1 = 0 \geq 0$ satisfies the constraint (327). When $q_0 < U$, we have

$$\beta_1 = \frac{L(U-Lf_M(q_0)-q_0)}{(U-q_0)(Lf_M(q_0)+q_0)}, \tag{417}$$

and its denominator is no less than 0. For its numerator, we have

$$\begin{aligned}
L(U-Lf_M(q_0)-q_0) \geq & L(U-q_0-Lf_M(q_0)) && (418)\\
\geq & L[U-q_0-L\ln(\frac{U}{q_0})] \quad \text{(by Corollaries (1) and (2))} && (419)\\
\geq & L[U-q_0-L(\frac{U}{q_0}-1)] \quad \text{(by } x \geq \ln(x)+1) && (420)\\
= & L(U-q_0)(1-\frac{L}{q_0}) && (421)\\
\geq & 0 \quad \text{(by } L \leq q_0 \leq U), && (422)
\end{aligned}$$

which indicates that

$$\beta_1 = \frac{L(U-Lf_M(q_0)-q_0)}{(U-q_0)(Lf_M(q_0)+q_0)} \geq 0 \tag{423}$$

also satisfies the constraint (327). Therefore, the constraint (327) holds for all q_0 and could be omitted.

In addition, we notice that the constraint (325) is subsumed by the constraint (324) when $q_0 < U$, and, since $\beta_1 = 0$ when $q_0 = U$, constraint (325) holds when $q_0 = U$, so the constraint (325) can also be omitted.

Finally, we put $\beta_1 = \frac{L(U-Lf_M(q_0)-q_0)}{(U-q_0)(Lf_M(q_0)+q_0)}$ into the constraint (324), it becomes

$$\frac{L(U-Lf_M(q_0)-q_0)}{(U-q_0)(Lf_M(q_0)+q_0)} \tag{424}$$

$$\leq \frac{q_{N^-}}{G(q_0, q_{N^-}, N^-)(Lf_M(q_{N^-})+q_{N^-}) + f_M(q_{N^-})(q_{N^-}-q_0)}, \forall q_{N^-} \in [q_0, U], \text{ if } q_0 < U, \tag{425}$$

which is equivalent to

$$\frac{L(U-Lf_M(q_0)-q_0)}{(Lf_M(q_0)+q_0)}$$

$$\leq \frac{(U-q_0)q_{N^-}}{G(q_0, q_{N^-}, N^-)(Lf_M(q_{N^-})+q_{N^-}) + f_M(q_{N^-})(q_{N^-}-q_0)}, \forall q_{N^-} \in [q_0, U], \text{ if } q_0 < U. \tag{426}$$

Note that the RHS of the inequality (426) equals $X_M(q_0, q_{N^-}, N^-)$, so that the inequality (426) recovers the constraint (416) of the problem $\mathtt{P}_5$. This lemma is proved. □

Lemma 27. *The optimal choice of q_0 to problem $\mathtt{P}_5$ is*

$$q_0 := \begin{cases} L, & \text{when } N^+ > N^- \text{ and } 1 - \frac{1}{N^-} \leq \frac{Lf_M(L)}{U-L}, \\ \min\Big\{q \in (L,U) \;\Big|\; \dfrac{L(U - Lf_M(q) - q)}{Lf_M(q) + q} = \inf\limits_{q_{N^-} \in [q,U]} X_M(q, q_{N^-}, N^-)\Big\}, & \text{when } N^+ > N^- \text{ and } 1 - \frac{1}{N^-} > \frac{Lf_M(L)}{U-L}, \\ \text{the unique solution to } \frac{L}{q} G(q, U, N^-) = 1, & \text{when } N^+ = N^-. \end{cases} \tag{427}$$

Proof. We first discuss the case where $M = 0$. In this case, we have $f_M(q_0) = 0$ for all q_0, so that the objective function of problem $\mathtt{P}_5$ becomes

$$\max_{q_0 \in [L,U]} \frac{L}{q_0}, \tag{428}$$

and constraint (416) becomes

$$\frac{L}{q_0} \leq \frac{1}{G(q_0, q_{N^-}, N^-)}, \ \forall q_{N^-} \in [q_0, U], \text{ if } q_0 < U. \tag{429}$$

By the definition of function G, it is straight forward that G is increasing in q_{N^-}, so that $\frac{1}{G(q_0, q_{N^-}, N^-)}$ is decreasing in q_{N^-}, and constraint (416) is further equivalent to

$$\frac{L}{q_0} \leq \frac{1}{G(q_0, U, N^-)}, \text{ if } q_0 < U. \tag{430}$$

Note that the LHS of inequality (430) decreases in q_0 and the RHS of it increases in q_0, where $\frac{L}{U} < \lim_{q_0 \to U} \frac{1}{G(q_0, U, N^-)} = \infty$ and $\frac{L}{L} = 1 \geq \frac{1}{G(L, U, N^-)} = \frac{1}{N^-}$, there exists an intersection for the LHS and RHS of inequality (430), which satisfies this equality and maximizes the objective function. Therefore, when $M = 0$, i.e., $N^+ = N^-$ the optimal choice for q_0 is the unique solution of

$$\frac{L}{q} G(q, U, N^-) = 1. \tag{431}$$

When $M > 0$, i.e., $N^+ > N^-$, the constraint (416) is equivalent to

$$\frac{L(U - Lf_M(q_0) - q_0)}{(Lf_M(q_0) + q_0)} \leq \inf_{q_{N^-} \in [q_0, U]} X_M(q_0, q_{N^-}, N^-), \text{ if } q_0 < U. \tag{432}$$

Note that by corollary 4, the RHS of the inequality (432) is non-decreasing in q_0. Also, by Eqs. (351)–(356), we know that $Lf_M(q_0) + q_0$ is non-decreasing in q_0, so that the LHS of inequality (432)

$$\frac{L(U - Lf_M(q_0) - q_0)}{(Lf_M(q_0) + q_0)} = \frac{LU}{(Lf_M(q_0) + q_0)} - L \tag{433}$$

is non-increasing in q_0. In addition, it worth noting that the objective function of problem $\mathtt{P}_5$ is also non-increasing in q_0. We now need to analyze the left and right endpoints of both sides of inequality (432) to discuss the optimal choice of q_0.

When $q_0 \to U$, by Eqs. (403)–(405), we have

$$\lim_{q_0 \to U} \inf_{q_{N^-} \in [q_0, U]} X_M(q_0, q_{N^-}, N^-) \geq U - L. \tag{434}$$

On the other hand, for the LHS of inequality (432), we have

$$\frac{LU}{(Lf_M(U) + U)} - L = 0 < \lim_{q_0 \to U} \inf_{q_{N^-} \in [q_0, U]} X_M(q_0, q_{N^-}, N^-). \tag{435}$$

When $q_0 = L$, for the RHS of inequality (432), we have

$$\inf_{q_{N^-} \in [q_0, U]} X_M(L, q_{N^-}, N^-)$$

$$= \inf_{q_{N^-} \in [q_0, U]} \frac{(U-L)q_{N^-}}{G(L, q_{N^-}, N^-)(Lf_M(q_{N^-}) + q_{N^-}) + f_M(q_{N^-})(q_{N^-} - L)} \tag{436}$$

$$= \inf_{q_{N^-} \in [q_0, U]} \frac{(U-L)q_{N^-}}{N^-(Lf_M(q_{N^-}) + q_{N^-}) + f_M(q_{N^-})(q_{N^-} - L)} \tag{437}$$

$$= \inf_{q_{N^-} \in [q_0, U]} \frac{U-L}{\frac{f_M(q_{N^-})}{q_{N^-}} L(N^- - 1) + N^- + f_M(q_{N^-})}. \tag{438}$$

Recall the definition $f_M(q_{N^-}) := M[1 - (\frac{q_{N^-}}{U})^{\frac{1}{M}}]$, which is decreasing in q_{N^-}, so that the denominator of $X_M(L, q_{N^-}, N^-)$ is decreasing in q_{N^-}, and $X_M(L, q_{N^-}, N^-)$ is increasing in q_{N^-}. As a result, we have

$$\inf_{q_{N^-} \in [L, U]} X_M(L, q_{N^-}, N^-) = X_M(L, L, N^-) = \frac{U-L}{N^-(f_M(L)+1)}. \tag{439}$$

On the other hand, for the LHS of inequality (432), we have

$$\frac{LU}{(Lf_M(q_0) + q_0)} - L = \frac{U}{f_M(L)+1} - L. \tag{440}$$

Now that Eq. (435) shows that the right endpoint of the LHS (non-increasing) of inequality (432) is lower than that of the RHS (non-decreasing), if the left endpoint of the LHS of inequality (432) is also lower than that of the RHS, the minimum q_0 satisfying inequality (432) is $q_0 = L$, which maximizes the objective function. A sufficient condition for this to happen is

$$\frac{U}{(f_M(L)+1)} - L \leq \frac{U-L}{N^-(f_M(L)+1)} \tag{441}$$

$$\Rightarrow 1 - \frac{1}{N^-} \leq \frac{Lf_M(L)}{U-L}. \tag{442}$$

Otherwise, an intersection exists for the LHS and RHS of inequality (432) between (L, U), which serves as the maximizer of problem $\mathtt{P}_5$, and it should be solved by

$$\min\Big\{q \in (L, U) \;\Big|\; \frac{L(U - Lf_M(q) - q)}{Lf_M(q) + q} = \inf_{q_{N^-} \in [q, U]} X_M(q, q_{N^-}, N^-)\Big\}, \tag{443}$$

which concludes the proof. □

Corollary 7. $q_0^* < U$.

Proof. We can notice that q_0^* (Eq. (5)) has the same value of the q_0 set by Eq. (427). By the analyses of lemma 27, when $N^+ > N^-$ and $1 - \frac{1}{N^-} \leq \frac{Lf_M(L)}{U-L}$, $q_0^* = L < U$; otherwise, q_0^* is the intersection of the LHS and RHS of the constraint (416), which is strictly lower than U. This corollary is proved. □

K.4.4 Proof of Lemma 18

(Optimal Parameters) The parameters $q_0 = q_0^*$, $\beta_1 = \beta_1^*$, $Y(q_{N^-}) = Y^*(q_{N^-})$, and $\beta_2(q_{N^-}) = \beta_2^*(q_{N^-})$ are optimal solutions to problem $\mathtt{P}_0$.

Proof. (i) Lemma 27 shows that q_0^* is the optimal choice of q_0 for problem $\mathtt{P}_5$; (ii) by Lemma 26, q_0^* is the optimal choice of q_0 for problem $\mathtt{P}_4$; (iii) by Corollary 6 and 7, the optimal choice of β_1 to problem $\mathtt{P}_4$ when $N^+ > N^-$ is

$$\beta_1 = \frac{UL}{(U - q_0^*)(Lf_M(q_0^*) + q_0^*)} - \frac{L}{U - q_0^*}, \tag{444}$$

and when $N^+ = N^-$ it is

$$\beta_1 = \frac{L}{q_0^*}, \tag{445}$$

so that β_1^* and q_0^* are optimal to problem $\mathtt{P}_4$; (iv) by Corollary 5, β_1^* and q_0^* are optimal to problem $\mathtt{P}_3$; (v) by Lemma 23, β_1^* and q_0^* are optimal to problem $\mathtt{P}_2$; (vi) by Lemma 22, the optimal choice of $\beta_Y(q_{N^-})$ to problem $\mathtt{P}_2$ is

$$\beta_Y^*(q_{N^-}) = \begin{cases} \frac{L}{q_0^*}, \text{ when } q_{N^-} = U, \\ \frac{L(q_{N^-} - f_M(q_{N^-})\beta_1^*(q_{N^-} - q_0^*))}{q_0^*(Lf_M(q_{N^-}) + q_{N^-})}, \text{ when } q_{N^-} < U, \end{cases} \tag{446}$$

$$= \begin{cases} \frac{L}{q_0^*}, \text{ when } N^+ = N^- \text{ or } q_{N^-} = U, \\ \frac{L(q_{N^-} - f_M(q_{N^-})\beta_1^*(q_{N^-} - q_0^*))}{q_0^*(Lf_M(q_{N^-}) + q_{N^-})}, \text{ when } N^+ > N^- \text{ and } q_{N^-} < U, \end{cases} \tag{447}$$

which is equivalent to $\beta_Y^*(q_{N^-}) = \frac{Y^*(q_{N^-})L}{q_0^*}$; (vii) by Lemma 21, β_1^*, q_0^*, and $\beta_Y^*(\cdot)$ are optimal to problem $\mathtt{P}_1$; (viii) by Lemma 20, $\beta_2^*(\cdot)$ is optimal to problem $\mathtt{P}_1$; (ix) by Lemma 19 and the definition of decision variable $\beta_Y(\cdot)$, β_1^*, q_0^*, $Y^*(\cdot)$, and $\beta_2^*(\cdot)$ are optimal to problem $\mathtt{P}_0$. □

K.5 ErLHA Parameter Feasibility

We also formally state the feasibility result of ErLHA's parameter choices in Corollary 8.

Corollary 8. *(ErLHA Parameter Feasibility) With the parameters $q_0 = q_0^*$, $\beta_1 = \beta_1^*$, $Y(q_{N^-}) = Y^*(q_{N^-})$, and $\beta_2(q_{N^-}) = \beta_2^*(q_{N^-})$, ErLHA produces feasible solutions for the OTP-ErL problem.*

Proof. This corollary directly comes from Lemmas 15–18. □

K.6 Proof of Theorem 5

(Competitive Ratio of ErLHA) With the parameters $q_0 = q_0^*$, $\beta_1 = \beta_1^*$, $Y(q_{N^-}) = Y^*(q_{N^-})$, and $\beta_2(q_{N^-}) = \beta_2^*(q_{N^-})$, ErLHA produces feasible solutions to OTP-ErL and achieves a competitive ratio of

$$\alpha_{\text{ErL}}^*(N^-, N^+) = \begin{cases} \frac{1}{\beta_1^*}, \text{ when } N^+ = N^-, \\ \frac{Lf_M(q_0^*) + q_0^*}{L}, \text{ when } N^+ > N^-. \end{cases} \tag{448}$$

Proof. The Feasibility result is proved by Corollary 8. Then, by Lemma 17 and 26, the competitive ratio achieved by ErLHA is

$$\max \frac{\text{OPT}(I)}{\text{ALG}(I)} \leq \frac{Lf_M(q_0^*) + q_0^*}{L}. \tag{449}$$

When $N^+ = N^-$, its value equals $\frac{q_0^*}{L} = \frac{1}{\beta_1^*}$, which matches the definition of $\alpha_{\text{ErL}}^*(N^-, N^+)$. This theorem is proved. □

L Proof of Theorem 6

(ErLHA Competitive Ratio Tightness) There exists exchange-rate sequences I, such that $\text{ErLHA}^*(I) = \frac{\text{OPT}(I)}{\alpha_{\text{ErL}}^*(N^-, N^+)}$.

Proof. We focus on the exchange rate sequence I with $p_{N^-} = L$ and $q_{N^-} \in \{q_0^*, U\}$. By the design of ErLHA, with the parameters q_0^*, β_1^*, $Y^*(q_{N^-})$, and $\beta_2^*(q_{N^-})$, the online performance of ErLHA is

$$\text{ErLHA}^*(I) = \sum_{i=1}^{N^-} x_i p_i + \sum_{i=N^-+1}^{N} x_i p_i \tag{450}$$

$$= \sum_{i \in [N^- - 1]: \Delta q_i > 0} x_i p_i + x_{N^-} p_{N^-} + \sum_{i=N^-+1}^{N} x_i p_i \tag{451}$$

$$= \sum_{i\in[N^- -1]:\Delta q_i>0} \frac{\beta_1^*\Delta q_i}{q_i - L} p_i + \mathbf{1}_{\{\Delta q_{N^-}>0\}} \frac{\beta_1^*\Delta q_{N^-}}{q_{N^-} - L} p_{N^-}$$

$$+ (Y^*(q_{N^-}) - \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1^*\Delta q_i}{q_i - L}) p_{N^-} + \sum_{i=N^-+1}^{N} x_i q_i \tag{452}$$

$$= \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1^*\Delta q_i}{q_i - L} q_i + (Y^*(q_{N^-}) - \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1^*\Delta q_i}{q_i - L}) p_{N^-} + \sum_{i=N^-+1}^{N} x_i q_i \tag{453}$$

$$= \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1^*\Delta q_i}{q_i - L} q_i + (Y^*(q_{N^-}) - \sum_{i\in[N^-]:\Delta q_i>0} \frac{\beta_1^*\Delta q_i}{q_i - L}) L + \sum_{i=N^-+1}^{N} x_i q_i \tag{454}$$

$$= Y^*(q_{N^-})L + \beta_1^*(q_{N^-} - q_0^*) + \beta_2^*(q_{N^-})(q_N - q_{N^-}). \tag{455}$$

When $N^+ = N^-$, we have $q_N = q_{N^-}$, and

$$\text{ErLHA}^*(I) = Y^*(q_{N^-})L + \beta_1^*(q_{N^-} - q_0^*), \tag{456}$$

where $Y^*(q_{N^-}) = 1$ since $N^- = N^+$ and we have $q_0^* = \frac{L}{\beta_1^*}$. Therefore, when $N^- = N^+$, we have

$$\text{ErLHA}^*(I) = Y^*(q_{N^-})L + \beta_1^*(q_N - q_0^*) \tag{457}$$

$$= L + \beta_1^*(q_N - \frac{L}{\beta_1^*}) \tag{458}$$

$$= \beta_1^* q_N. \tag{459}$$

In this case, we have $\text{OPT}(I) = q_N$, and $\alpha^*_{\text{ErL}}(N^-, N^+) = \frac{1}{\beta_1^*}$ thereby

$$\text{ErLHA}^*(I) = \beta_1^* q_N = \beta_1^* \text{OPT}(I) = \frac{1}{\alpha^*_{\text{ErL}}(N^-, N^+)} \text{OPT}(I). \tag{460}$$

When $N^- < N^+$ and $q_{N^-} = U$, we have $Y^*(U) = 1$ and $\beta_2^*(U) = 0$, thereby

$$\text{ErLHA}^*(I) = L + \beta_1^*(U - q_0^*), \tag{461}$$

where

$$\beta_1^* = \frac{UL}{(U - q_0^*)(Lf_M(q_0^*) + q_0^*)} - \frac{L}{U - q_0^*}, \tag{462}$$

such that

$$\text{ErLHA}^*(I) = L + (\frac{UL}{(U - q_0^*)(Lf_M(q_0^*) + q_0^*)} - \frac{L}{U - q_0^*})(U - q_0^*) \tag{463}$$

$$= \frac{UL}{Lf_M(q_0^*) + q_0^*}. \tag{464}$$

In this case, the offline optimal performance is $\text{OPT}(I) = U$, and the competitive ratio is $\alpha^* = \frac{Lf_M(q_0^*) + q_0^*}{L}$, such that

$$\text{ErLHA}^*(I) = \frac{UL}{Lf_M(q_0^*) + q_0^*} = \frac{1}{\alpha^*} U = \frac{1}{\alpha^*} \text{OPT}(I). \tag{465}$$

When $N^- < N^+$ and $q_{N^-} = q_0^*$, we have $Y^*(q_{N^-}) = \frac{q_{N^-} - f_M(q_{N^-})\beta_1^*(q_{N^-} - q_0^*)}{Lf_M(q_{N^-}) + q_{N^-}}$ and $\beta_2^*(q_{N^-}) = \frac{1 - Y^*(q_{N^-})}{f_M(q_{N^-})}$, thereby

$$\text{ErLHA}^*(I) = Y^*(q_0^*)L + \beta_2^*(q_0^*)(q_N - q_0^*) \tag{466}$$

$$=Y^*(q_0^*)L+\frac{L-Y^*(q_0^*)L}{Lf_M(q_0^*)}(q_N-q_0^*) \tag{467}$$

$$=Y^*(q_0^*)(L-\frac{q_N-q_0^*}{f_M(q_0^*)})+\frac{q_N-q_0^*}{f_M(q_0^*)} \tag{468}$$

$$=\frac{q_0^*}{Lf_M(q_0^*)+q_0^*}(L-\frac{q_N-q_0^*}{f_M(q_0^*)})+\frac{q_N-q_0^*}{f_M(q_0^*)} \tag{469}$$

$$=\frac{q_0^*}{Lf_M(q_0^*)+q_0^*}(\frac{Lf_M(q_0^*)-q_N+q_0^*}{f_M(q_0^*)})+\frac{q_N-q_0^*}{f_M(q_0^*)} \tag{470}$$

$$=\frac{q_0^*(Lf_M(q_0^*)+q_0^*)-q_Nq_0^*}{f_M(q_0^*)(Lf_M(q_0^*)+q_0^*)}+\frac{q_N-q_0^*}{f_M(q_0^*)} \tag{471}$$

$$=\frac{q_N}{f_M(q_0^*)}-\frac{q_0^*q_N}{f_M(q_0^*)(Lf_M(q_0^*)+q_0^*)} \tag{472}$$

$$=\frac{q_N(Lf_M(q_0^*)+q_0^*)-q_0^*q_N}{f_M(q_0^*)(Lf_M(q_0^*)+q_0^*)} \tag{473}$$

$$=\frac{q_NL}{Lf_M(q_0^*)+q_0^*}. \tag{474}$$

In this case, the offline optimal performance is $\text{OPT}(I)=q_N$, and the competitive ratio is $\alpha^*=\frac{Lf_M(q_0^*)+q_0^*}{L}$, such that

$$\text{ErLHA}^*(I)=\frac{q_NL}{Lf_M(q_0^*)+q_0^*}=\frac{1}{\alpha^*}q_N=\frac{1}{\alpha^*}\text{OPT}(I). \tag{475}$$

The tightness is proved. □

M Proof of Theorem 7

(Best Achievable Competitive Ratio of ErLHA) $\alpha^*_{\text{ErL}}(N^-,N^+)$ is the best achievable competitive ratio for ErLHA under any parameter choices of q_0, β_1, $Y(q_{N^-})$, and $\beta_2(q_{N^-})$.

Proof. Note that on any exchange-rate sequence I' that $\max_{i\in[1,N]}p_i\geq q_0$ and $p_{N^-}=L$, we have

$$q_N=\max\{q_0,\max_{i\in[1,N]}p_i\}=\max_{i\in[1,N]}p_i=\text{OPT}(I'), \tag{476}$$

and

$$\text{ALG}(I')=\sum_{i=1}^{N^-}x_ip_i+\sum_{i=N^-+1}^{N}x_ip_i \tag{477}$$

$$=\sum_{i\in[N^--1]:\Delta q_i>0}x_ip_i+x_{N^-}p_{N^-}+\sum_{i=N^-+1}^{N}x_ip_i \tag{478}$$

$$=\sum_{i\in[N^--1]:\Delta q_i>0}\frac{\beta_1\Delta q_i}{q_i-L}p_i+\mathbf{1}_{\{\Delta q_{N^-}>0\}}\frac{\beta_1\Delta q_{N^-}}{q_{N^-}-L}p_{N^-}$$

$$+(Y(q_{N^-})-\sum_{i\in[N^-]:\Delta q_i>0}\frac{\beta_1\Delta q_i}{q_i-L})p_{N^-}+\sum_{i=N^-+1}^{N}x_iq_i \tag{479}$$

$$=\sum_{i\in[N^-]:\Delta q_i>0}\frac{\beta_1\Delta q_i}{q_i-L}q_i+(Y(q_{N^-})-\sum_{i\in[N^-]:\Delta q_i>0}\frac{\beta_1\Delta q_i}{q_i-L})p_{N^-}+\sum_{i=N^-+1}^{N}x_iq_i \tag{480}$$

$$=\sum_{i\in[N^-]:\Delta q_i>0}\frac{\beta_1\Delta q_i}{q_i-L}q_i+(Y(q_{N^-})-\sum_{i\in[N^-]:\Delta q_i>0}\frac{\beta_1\Delta q_i}{q_i-L})L+\sum_{i=N^-+1}^{N}x_iq_i \tag{481}$$

$$=Y(q_{N^-})L+\beta_1(q_{N^-}-q_0)+\beta_2(q_{N^-})\cdot(q_N-q_{N^-}), \tag{482}$$

where Eq. (481) comes from the property of the exchange-rate sequence I' that $p_{N^-}=L$. Therefore, by Eq. (220), the competitive ratio of ErLHA on this subset of exchange-rate sequences is

$$\max_{\forall I'}\frac{\mathrm{OPT}(I')}{\mathrm{ALG}(I')}=\max_{q_{N^-},q_N}\quad\frac{q_N}{Y(q_{N^-})L+\beta_1(q_{N^-}-q_0)+\beta_2(q_{N^-})(q_N-q_{N^-})} \tag{483}$$

$$=\max_{q_{N^-},q_N}\left[(\frac{Y(q_{N^-})L}{q_0}-\beta_1)\frac{q_0}{q_N}+(\beta_1-\beta_2(q_{N^-}))\frac{q_{N^-}}{q_N}+\beta_2(q_{N^-})\right]^{-1}, \tag{484}$$

which is exactly the objective function of problem $\mathtt{P}_0$, and no other parameter choices can achieve a competitive ratio better (smaller) than $\alpha^*(N^-,N^+)$ on this family of exchange-rate sequences by Lemma 18. This theorem is proved. □

N Proofs for Section 5.3

N.1 Proof of Theorem 8

(ErLHA Optimality) ErLHA achieves the optimal competitive ratio in OTP-L, OTP-NH, and OTP-Er. Specifically, given L and U, we have (i) $\alpha^*_{\mathrm{ErL}}(1,N^+)=\alpha^*_{\mathrm{L}}(N^+)$ and (ii) $\lim_{N^+\to+\infty}\alpha^*_{\mathrm{ErL}}(1,N^+)=\alpha_0^*=1+\ln(U/L)$

Proof. When $N^-=1$ we have $1-1/N^-=0\le Lf_M(L)/(U-L)$, and hence, by the definition of $\alpha^*_{\mathrm{ErL}}(N^-,N^+)$, β_1^*, and q_0^*, we have $q_0^*=L$ and $\alpha^*_{\mathrm{ErL}}(1,N^+)=f_M(L)+1=(N^+-1)[1-(\frac{L}{U})^{\frac{1}{N^+-1}}]+1=\alpha^*_L(N^+)$, which proves (i). By Theorem 1 and Corollary 2, (ii) also holds true. □

N.2 Proof of Theorem 9

(ErLHA Asymptotic Optimality) The competitive ratio $\alpha^*_{\mathrm{ErL}}(N^-,N^+)$ is asymptotically optimal for OTP-ErL.

Proof. By Theorem 8, it is straight forward that $\lim_{N^-\to1}\alpha^*_{\mathrm{ErL}}(N^-,N^+)=\alpha^*_{\mathrm{L}}(N^+)$ for all $N^+\in[N^-,+\infty)$, where $\alpha^*_{\mathrm{L}}(N^+)$ is the optimal competitive ratio for OTP-L, i.e., the special case of OTP-ErL with $N^-=1$. As a result, ErLHA is asymptotically optimal for OTP-ErL. □

N.3 Proof of Lemma 2

(Impact of Latest Horizon) Given L, U, and N^-, we have

$$\alpha^*_{\mathrm{ErL}}(N^-,N^+)\le\alpha^*_{\mathrm{ErL}}(N^-,N^++1). \tag{485}$$

Proof. This proof follows Lemma 26 (the optimization problem $\mathtt{P}_5$), Lemma 27, and Corollary 7 (the property of q_0^* that $q_0^*<U$). Observe that the problem $\mathtt{P}_5$ is equivalent to

$$\mathtt{P}_5\quad\max_{q_0\in[L,U)}\quad\frac{L}{Lf_M(q_0)+q_0} \tag{486}$$

$$\text{s.t.}\frac{LU}{(Lf_M(q_0)+q_0)}\le L+\inf_{q_{N^-}\in[q_0,U]}X_M(q_0,q_{N^-},N^-), \tag{487}$$

We set $t:=\frac{L}{Lf_M(q_0)+q_0}$, then the optimal value and optimal solution to $\mathtt{P}_5$ can be obtained by solving the following problem $\mathtt{P}_5'$:

$$\mathtt{P}_5'\quad\max_{q_0\in[L,U),t}\quad t \tag{488}$$

$$\text{s.t.}\quad Ut\le L+\inf_{q_{N^-}\in[q_0,U]}X_M(q_0,q_{N^-},N^-), \tag{489}$$

which is further equivalent to

$$\mathtt{P}_5'\quad\max_{q_0\in[L,U),t}\quad t \tag{490}$$

$$\text{s.t.} \quad Ut \le L + \frac{(U-q_0)q_{N^-}}{G(q_0,q_{N^-},N^-)(Lf_M(q_{N^-})+q_{N^-})+f_M(q_{N^-})(q_{N^-}-q_0)}, \tag{491}$$
$$\forall q_{N^-} \in [q_0, U].$$

by the definition of the function X_M. Suppose that q_0^* and t^* are the optimal solutions to $\mathtt{P}_5'$ when the earliest deadline equals $N^- = N_1$ and the latest deadline equals $N^+ = N_2$, along with $M = N_2 - N_1$

Now we show that, when the latest deadline N^+ decreases to $N_2 - 1$, q_0^* and t^* are still feasible solutions to $\mathtt{P}_5'$. When $N^+ = N_2 - 1$, let $M' = N_2 - 1 - N_1 < M$. Since for $f_M(q_{N^-})$ we have

$$f_0(q_{N^-}) = 0 \le 1 - \frac{q_{N^-}}{U} = f_1(q_{N^-}) \tag{492}$$

and

$$\begin{aligned}
\frac{df_M(q_{N^-})}{dM} =& \frac{d\{M[1-(\frac{q_{N^-}}{U})^{\frac{1}{M}}]\}}{dM} && (493)\\
=& \frac{d\{M[1-e^{\ln(\frac{q_{N^-}}{U})/M}]\}}{dM} && (494)\\
=& 1-[e^{\ln(\frac{q_{N^-}}{U})/M} - Me^{\ln(\frac{q_{N^-}}{U})/M}\ln(\frac{q_{N^-}}{U})M^{-2}] && (495)\\
=& 1-[e^{\ln(\frac{q_{N^-}}{U})/M} - M^{-1}\ln(\frac{q_{N^-}}{U})e^{\ln(\frac{q_{N^-}}{U})/M}] && (496)\\
=& 1-(1-\frac{\ln(\frac{q_{N^-}}{U})}{M})e^{\ln(\frac{q_{N^-}}{U})/M} && (497)\\
\ge& 1-(1-\frac{\ln(\frac{q_{N^-}}{U})}{M})\times\frac{1}{1-\frac{\ln(\frac{q_{N^-}}{U})}{M}} && (498)\\
& \text{(since } e^x \le \frac{1}{1-x} \text{ when } x \le 0 \text{ and } \frac{\ln(\frac{q_{N^-}}{U})}{M} \le 0) && (499)\\
=& 0, && (500)
\end{aligned}$$

so that $f_M(q_{N^-})$ is non-decreasing in M and $f_{M'}(q_{N^-}) \le f_M(q_{N^-})$. As a result, we have $X_{M'}(q_0^*, q_{N^-}, N^-) \ge X_M(q_0^*, q_{N^-}, N^-)$, such that it still holds true that $Ut^* \le L + X_{M'}(q_0^*, q_{N^-}, N^-)$ for all $q_{N^-} \in [q_0^*, U]$. Notice that the value of $1/t^*$ is the competitive ratio achieved by ErLHA, so the above result leads to

$$\alpha^*_{\text{ErL}}(N_1, N_2 - 1) \le \alpha^*_{\text{ErL}}(N_1, N_2), \tag{501}$$

which concludes this proof. □

O Proof of Lemma 3

(ErLHA behavior in OTP-Ex) When $N^+ = N^- = N$, the optimal parameter choices for ErLHA are $Y^*(q_{N^-}) = 1$ and $\beta_2^*(q_{N^-}) = 0$ for all q_{N^-}, $\beta_1^* = \frac{L}{q_0^*}$, and

$$q_0^* = L(1 + \Phi(N, \omega)), \tag{502}$$

with $\Phi(N,\omega) := \frac{(N-1)(\phi(N,\omega))^N}{\omega}$, $\omega := (N-1)\left[N\left(\frac{L}{U-L}\right)^{\frac{1}{N}}\right]^{\frac{-N}{N-1}}$, and

$$\phi(N,\omega) := \omega \cdot {}_{N-1}F_{N-2}\left(\begin{matrix}\frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N} \\ \frac{2}{N-1}, \cdots, \frac{N-2}{N-1}, \frac{N}{N-1}\end{matrix}\middle| -\frac{N^N}{(N-1)^{N-1}}\omega^{N-1}\right), \tag{503}$$

where ${}_{N-1}F_{N-2}$ is the hypergeometric function (Ritelli et al., 2021).

Proof. Please note that, by the definition of β_1^*, $\beta_2^*(q_{N^-})$, and $Y^*(q_{N^-})$, when the earliest and latest horizons are the same, i.e., $N^- = N^+$, we naturally have

$$\begin{cases} Y^*(q_{N^-}) = 1, & \forall q_{N^-} \in [q_0^*, U], \\ \beta_2^* = 0, & \forall q_{N^-} \in [q_0^*, U], \\ \beta_1^* = \frac{L}{q_0^*}, & \end{cases} \tag{504}$$

and q_0^* is the unique solution to $\frac{L}{q} G(q, U, N) = 1$. Recall that

$$G(q, U, N) = N[1 - (\frac{q - L}{U - L})^{\frac{1}{N}}] \tag{505}$$

$$= N[1 - (\frac{L}{U - L}(\frac{q}{L} - 1))^{\frac{1}{N}}], \tag{506}$$

so that the equation

$$\frac{q}{L} = N[1 - (\frac{L}{U - L}(\frac{q}{L} - 1))^{\frac{1}{N}}] \tag{507}$$

can be converted into a trinomial equation, for which we provide the following lemma to derive the expression of its solution.

Lemma 28. *Define $f(x) \triangleq n(1 - [a(x-1)]^{\frac{1}{n}})$, where $n > 1$ and $a > 0$, a unique solution x^* to $x = f(x)$ exists between $x \in (1, 1 + \frac{1}{a})$, such that $x^* = 1 + \Phi(n, (n-1)(na^{\frac{1}{n}})^{\frac{-n}{n-1}})$.*

Proof. We prove this lemma by converting the equation $x = f(x)$ into a trinomial equation. Set $t = (x-1)^{\frac{1}{n}}$, $b = na^{\frac{1}{n}}$, and $m = n - 1$, we have $x = t^n + 1$, and $x = f(x)$ is equivalent to

$$t^n + bt - m = 0. \tag{508}$$

We further set $v = tb^{\frac{-1}{n-1}}$ and $\omega = mb^{-\frac{n}{n-1}}$, then we have $t = vb^{\frac{1}{n-1}}$, $m = \omega b^{\frac{n}{n-1}}$, and $t^n + bt - m = 0$ is equivalent to

$$v^n b^{\frac{n}{n-1}} + vb^{\frac{n}{n-1}} - \omega b^{\frac{n}{n-1}} = 0 \tag{509}$$

$$\Leftrightarrow v^n + v - \omega = 0, \tag{510}$$

and $x = v^n b^{\frac{n}{n-1}} + 1$.

We note that Eq. (510) is the same as Eq. (9) in Ritelli et al. (2021) and the unique real value solution v^* to Eq. (510) is the $\phi(n, \omega)$ defined in Eq. (503), which is given by Eq. (32) in Ritelli et al. (2021) as a general solution.

Finally, let's put everything together.

$$x^* = 1 + (v^*)^n b^{\frac{n}{n-1}} \tag{511}$$

$$= 1 + \phi(n, \omega)^n b^{\frac{n}{n-1}} \tag{512}$$

$$= 1 + \frac{m\phi(n, \omega)^n}{\omega} \tag{513}$$

$$= 1 + \frac{(n-1)\phi(n, \omega)^n}{\omega} \tag{514}$$

$$= 1 + \Phi(n, \omega), \tag{515}$$

where Eq. (513) comes from the definition of ω, Eq. (514) comes from the definition of m, and Eq. (515) comes from the definition of function Φ. Lemma 28 is proved. □

By Lemma 28 and Eq. (507), Lemma 3 is proved. □

P PROOF OF THEOREM 10

(ExHA Competitive Ratio) The ExHA algorithm provides feasible solutions for OTP-Ex and achieves a competitive ratio of $\alpha^*_{\text{Ex}}(N) = 1 + \Phi(N, \omega)$ in the OTP-Ex problem.

Proof. First, since ExHA is a special case of ErLHA, and its parameter choices q_0^*, β_1^*, and Y^* are inherited directly from the feasible and optimal parameters of ErLHA in the general setting, it follows immediately that ExHA generates feasible solutions to the OTP-Ex problem under these parameter choices.

Then, by Theorem 5, Lemma 3, the fact that OTP-Ex is a special case of OTP-ErL, and the fact that ExHA is a special form of ErLHA, the competitive ratio of ExHA in this theorem holds true. □

Q PROOF OF THEOREM 11

(ExHA Optimality) Given L, U, and N, no online algorithm can achieve a competitive ratio smaller than $\alpha^*_{\text{Ex}}(N)$ in the OTP-Ex problem.

Proof. By Lemma 3 and Theorem 10, we have $\alpha^*_{\text{Ex}}(N) = \frac{q_0^*}{L}$. By the definition of q_0^* (Eq. (5)), when $N^- = N^+$, we have $q_0^*/L = G(q_0^*, U, N)$, which implies $\alpha^*_{\text{Ex}}(N) = G(L\alpha^*_{\text{Ex}}(N), U, N)$, which recovers the optimal result in El-Yaniv et al. (2001), such that

$$\alpha^*_{\text{Ex}}(N) = N\left[1 - [(L\alpha^*_{\text{Ex}}(N) - L)/(U - L)]^{\frac{1}{N}}\right]$$

. □

R PROOF OF LEMMA 4

(Infinite Exact Horizon) Given L and U, when N becomes arbitrarily large, we have

$$\lim_{N\to\infty} \alpha^*_{\text{Ex}}(N) = 1 + W(\frac{U-L}{eL}) < 1 + \ln(\frac{U}{L}), \tag{516}$$

where $W(\cdot)$ is the Lambert-W function.

Proof. By the definition of $\alpha^*_{\text{Ex}}(N)$, we have

$$\alpha^*_{\text{Ex}}(N) = N\left[1 - \left(\frac{\alpha^*_{\text{Ex}}(N)L - L}{U - L}\right)^{\frac{1}{N}}\right]. \tag{517}$$

Set $S_N := \alpha^*_{\text{Ex}}(N) - 1$ and $C := \frac{U-L}{L}$, we have

$$S_N + 1 = N\left[1 - \left(\frac{S_N}{C}\right)^{\frac{1}{N}}\right] \tag{518}$$

$$\Rightarrow \quad S_N = C\left(1 - \frac{S_N + 1}{N}\right)^N. \tag{519}$$

Since the LHS of Eq. (517) is non-decreasing in $\alpha^*_{\text{Ex}}(N)$ and the RHS of Eq. (517) is non-increasing $\alpha^*_{\text{Ex}}(N)$, and also since

$$1 \le N\left[1 - \left(\frac{1L - L}{U - L}\right)^{\frac{1}{N}}\right], \forall N \ge 1, \tag{520}$$

and

$$\frac{U}{L} > N\left[1 - \left(\frac{\frac{U}{L} \times L - L}{U - L}\right)^{\frac{1}{N}}\right], \forall N \ge 1, \tag{521}$$

the unique solution $\alpha^*_{\text{Ex}}(N)$ to Eq. (517) is within $[1, \frac{U}{L}]$. Therefore, we have $S_N \in [0, \frac{U}{L} - 1]$, and thereby

$$\frac{S_N + 1}{N} > 0, \forall N \geq 1, \tag{522}$$

$$\Rightarrow \quad 1 - \frac{S_N + 1}{N} \leq e^{-(S_N+1)/N}, \forall N \geq 1, \tag{523}$$

$$\Rightarrow \quad S_N \leq Ce^{-(S_N+1)} \leq Ce^{-1}, \forall N \geq 1. \tag{524}$$

By Eq. (519), we have

$$\ln\left(\frac{S_N}{C}\right) = N \ln\left(1 - \frac{S_N + 1}{N}\right) \tag{525}$$

$$= N\left[-\int_0^{\frac{S_N+1}{N}} \frac{1}{1-t} dt\right] \tag{526}$$

$$= -N \int_0^{\frac{S_N+1}{N}} 1 + \frac{t}{1-t} dt \tag{527}$$

$$= -(S_N + 1) - N \int_0^{\frac{S_N+1}{N}} \frac{t}{1-t} dt. \tag{528}$$

When N approaches infinity, we have $t \leq \frac{S_N+1}{N} \leq \frac{1}{2}$ and $\frac{1}{1-t} \leq 2$, which leads to

$$N \int_0^{\frac{S_N+1}{N}} \frac{t}{1-t} dt \leq 2N \int_0^{\frac{S_N+1}{N}} t dt \tag{529}$$

$$= \frac{(S_N + 1)^2}{N} \tag{530}$$

$$\leq \frac{(\frac{C}{e} + 1)^2}{N}. \tag{531}$$

Since $\lim_{N\to\infty}(C/e + 1)^2/N = 0$, by Eq. (528), we have

$$\lim_{N\to\infty} \ln\left(\frac{S_N}{C}\right) = \lim_{N\to\infty} -(S_N + 1) + o(1). \tag{532}$$

Define $S_\infty := \lim_{N\to\infty} S_N$, we have

$$\ln(\frac{S_\infty}{C}) = -S_\infty - 1 \tag{533}$$

$$\Rightarrow \quad \ln S_\infty + S_\infty = \ln C - \ln e \tag{534}$$

$$\Rightarrow \quad S_\infty e^{S_\infty} = \frac{C}{e} \tag{535}$$

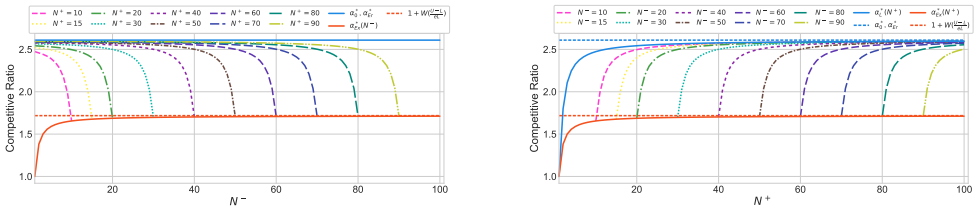
$$\Rightarrow \quad S_\infty = W(C/e) \tag{536}$$

$$\Rightarrow \quad \lim_{N\to\infty} \alpha^*_{\text{Ex}}(N) = 1 + W(\frac{U - L}{eL}). \tag{537}$$

What's left is to show that $W(\frac{U-L}{eL}) < \ln(U/L)$. Since W denotes the principal branch of the Lambert-W function, it suffices to show that $\frac{U-L}{eL} < \ln(U/L)e^{\ln(U/L)}$, which is equivalent to $\frac{(U/L)-1}{e} < (U/L)\ln(U/L)$. Since $e > 1$, it suffices to show $(U/L) - 1 < (U/L)\ln(U/L)$, which holds true by the standard inequality

$$\ln x > 1 - \frac{1}{x}, \forall x > 1. \tag{538}$$

This lemma is proved. □



(a) ErLHA's competitive ratio as N^- increases. (b) ErLHA's competitive ratio as N^+ increases.

Figure 2: The Competitive ratio of ErLHA approaches $\alpha^*_{\text{Ex}}(N)$. $U/L = 5$.

S Proof of Corollary 3

Given L and U, let N_1, N_2, and N_3 be three integers that satisfy $1 \le N_1 = N_2 \le N_3$, we have $\alpha^*_{\text{Ex}}(N_1) \le \alpha^*_{\text{ErL}}(N_2, N_3)$.

Proof. This corollary holds by $\alpha^*_{\text{Ex}}(N_1) = \alpha^*_{\text{ErL}}(N_1, N_1) = \alpha^*_{\text{ErL}}(N_2, N_2) \le \alpha^*_{\text{ErL}}(N_2, N_3)$, where the last inequality comes from Lemma 2. □

T Numerical Results for OTP-ErL: Unified Optimality

In this section, we present numerical results that illustrate the underlying connection among different forms of horizon information and provide an intuitive visualization of how ErLHA leverages different horizon information as a unified framework. In Figure 2, we plot ErLHA's competitive ratio $\alpha^*_{\text{ErL}}(N^-, N^+)$ (Theorem 5), together with the best achievable competitive ratios α^*_0 for OTP-NH, α^*_{Er} for OTP-Er (Theorem 1), $\alpha^*_{\text{L}}(N^-)$ for OTP-L (Theorem 4), and $\alpha^*_{\text{Ex}}(N)$ for OTP-Ex (Theorem 11), respectively. We plot the value and trend of these competitive ratios under different earliest and latest horizons, in order to visualize the bounded and convergent performance of ErLHA, with respect to the optimal solution to other settings (Theorem 8, Lemma 2, Corollaries 1, 2, and 3), and to reveal additional insights into the consistent connection among different forms of horizon information.

In Figure 2a, we plot $\alpha^*_{\text{ErL}}(N^-, N^+)$ under different fixed values of the latest horizon N^+, illustrating their trends and bounds as the earliest horizon N^- increases. The dashed curves are the curves of $\alpha^*_{\text{ErL}}(N^-, N^+)$ with the latest horizon N^+ fixed from 10 to 90. We also plot α^*_0, α^*_{Er}, where $\alpha^*_0 = \alpha^*_{\text{Er}} = 1 + \ln(U/L)$, so both are represented by the horizontal solid blue line. In addition, the solid red curve shows $\alpha^*_{\text{Ex}}(N^-)$, which is the best achievable competitive ratio for OTP-Ex when N^- is the exact horizon. Its limiting value $1 + W(\frac{U-L}{eL})$ as the exact horizon approaches infinity is shown by the horizontal red dashed line.

As shown in Figure 2a, the competitive ratios of ErLHA with different earliest and latest horizons stay above $\alpha^*_{\text{Ex}}(N^-)$, while converging to $\alpha^*_{\text{Ex}}(N^-)$ as the earliest horizon N^- approaches the latest horizon N^+ and finally the two horizons coincide. During this process, the best achievable results in OTP-Ex (solid orange curve) serve as a lower envelope for the competitive ratios of ErLHA, corroborating Corollary 3. Meanwhile, when the earliest horizon is small, the competitive ratios of ErLHA are close to, but remain well bounded by $1 + \ln(U/L)$ (solid blue line), and thereby $1 + \ln(U/L)$ serves as an upper envelope, which corroborates Theorem 8 and Corollaries 1–2. Every α^*_{ErL} curve stays above those with a smaller latest horizon N^+, corroborating Lemma 2. In addition, the decreasing trend of α^*_{ErL} demonstrates that an increasing earliest horizon reduces the uncertainty and the hardness of the OTP-ErL problem, until the earliest and latest horizons coincide into an exact horizon.

In Figure 2b, we plot $\alpha^*_{\text{ErL}}(N^-, N^+)$ under different fixed values of the earliest horizon N^-, illustrating their trends and bounds as the latest horizon N^+ increases. The dashed curves are the curves of $\alpha^*_{\text{ErL}}(N^-, N^+)$ with the earliest horizon N^- fixed from 10 to 90. We also plot $\alpha^*_{\text{L}}(N^+)$ as the solid blue curve, which represents the best achievable competitive ratio for OTP-L when the earliest horizon equals the value of N^+. The asymptote of $\alpha^*_{\text{L}}(N^+)$ is plotted in the horizontal dashed blue line, representing $\alpha^*_0 = \alpha^*_{\text{Er}} = 1 + \ln(U/L)$. In addition, the solid red curve shows $\alpha^*_{\text{Ex}}(N^+)$, which

is the best achievable competitive ratio for OTP-Ex when N^+ is the exact horizon. Its limiting value $1 + W(\frac{U-L}{eL})$ as the exact horizon approaches infinity is shown by the horizontal red dashed line.

As shown in Figure 2b, ErLHA's competitive ratio $\alpha^*_{\text{ErL}}(N^-, N^+)$ increases as the latest horizon N^+ increases, corroborating Lemma 2. As the latest horizon N^+ increases, every curve of $\alpha^*_{\text{ErL}}(N^-, N^+)$ "rises" from and stays above $\alpha^*_{\text{Ex}}(N^+)$, so that $\alpha^*_{\text{Ex}}(N^+)$ (solid red curve) serves as a lower envelope for the competitive ratio of ErLHA under all combinations of N^- and N^+, corroborating Corollary 3. Besides, $\alpha^*_{\text{ErL}}(N^-, N^+)$ stays below $\alpha^*_{\text{L}}(N^+)$, which converges to $\alpha^*_{\text{L}}(N^+)$ as N^+ grows, corroborating Theorem 8. In addition, $\alpha^*_{\text{L}}(N^+)$ (solid blue curve) serves as an upper envelope for the competitive ratio of ErLHA under all combinations of N^- and N^+, and every α^*_{ErL} curve stays above those with a larger earliest horizon N^-, demonstrating that an increasing earliest horizon reduces the uncertainty and the hardness of the OTP-ErL problem, until the earliest and latest horizons coincide into an exact horizon.

Figure 2, together with our optimality analysis in Theorems 1, 4, 9, and 11, reveals the underlying connection among different forms of horizon information by discovering the consistent pattern in how they interact with the uncertainty and hardness of OTP. When the latest horizon N^+ is extremely distant, it dominates the hardness of the problem, so that knowing only the earliest horizon brings no improvement. As N^+ decreases, the remaining uncertainty becomes manageable by the knowledge of an earliest horizon N^-, which further reduces the uncertainty. As the two horizons move closer, the uncertainty in OTP continues to shrink. Finally, when the two horizons coincide, the uncertainty and the hardness of OTP attain their minimum.

In Appendix U, we provide additional numerical results for OTP-Er, OTP-L, and OTP-Ex to illustrate how their optimal competitive ratios converge to their limiting constants, $1 + \ln(U/L)$ and $1 + W(\frac{U-L}{eL})$, in the corresponding settings.

U Numerical Results for OTP-Er, OTP-L, and OTP-Ex

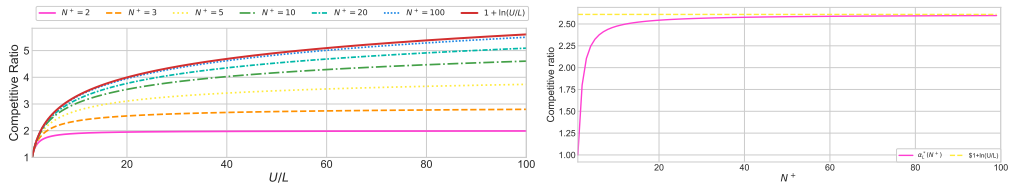
In addition to Appendix T, we provide additional experiments in this section to show the numerical results for OTP-Er, OTP-L, and OTP-Ex. We first present numerical results to illustrate that $\alpha^*_{\text{L}}(N^+)$ approaches and remains bounded below by α^*_{Er} (and hence α^*_0) in Section U.1, and we then show that $\alpha^*_{\text{Ex}}(N)$ approaches and remains bounded below by $1 + W(\frac{U-L}{eL})$ in Section U.2.

U.1 OTP-Er and OTP-L

To demonstrate the connection between the OTP-L and OTP-Er (and hence OTP-NH), we plot $\alpha^*_{\text{L}}(N^+)$ along with $\alpha^*_{\text{Er}} = \alpha^*_0 = 1 + \ln(U/L)$ in Figure 3, where $\alpha^*_{\text{L}}(N^+)$ approaches and remains bounded by α^*_{Er} (and hence α^*_0).

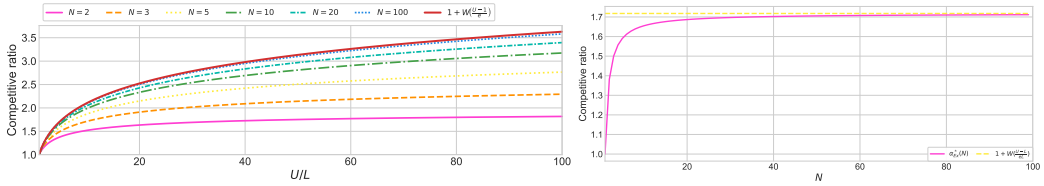
In Figure 3a, we show $\alpha^*_{\text{L}}(N^+)$ and $1 + \ln(U/L)$ across a range of U/L ratio and different N^+. In this figure, the red curve is $1 + \ln(U/L)$ at different U/L; other curves corresponds to $\alpha^*_{\text{L}}(N^+)$ increasing as the ratio U/L, with N^+ fixed at 2, 3, 5, 10, 20, and 100. The results show that $\alpha^*_{\text{L}}(N^+)$ stays strictly below $1 + \ln(U/L)$ for all tested combinations of U/L and N^+, consistent with Corollary 1. Moreover, as the latest horizon N^+ increases, $\alpha^*_{\text{L}}(N^+)$ approaches $1 + \ln(U/L)$. We further visualize the convergence of $\alpha^*_{\text{L}}(N^+)$ to $1 + \ln(U/L)$ in Figure 3b, where the horizontal line in yellow is $1 + \ln(U/L)$, and the increasing curve in pink plots $\alpha^*_{\text{L}}(N^+)$ as N^+ grows. In Figure 3b, U/L is fixed to 5. Figure 3b shows that, as N^+ grows, $\alpha^*_{\text{L}}(N^+)$ smoothly converges to but remains under $1 + \ln(U/L)$.

Figure 3 plots the best achievable competitive ratios for OTP-Er (and hence OTP-NH) and OTP-L, as established in Theorems 1 and 4. This comparison further illustrates the convergent yet bounded hardness of OTP-L relative to OTP-NH, as characterized in Corollaries 1 and 2. It numerically confirms that knowing the latest horizon improves online decision-making, while the gain diminishes as the latest horizon becomes increasingly distant. Since $\alpha^*_{\text{L}}(N^+)$ is precisely the competitive ratio attained by LHA in OTP-L (Theorem 2), Figure 3 also indicates that LHA remains optimal when OTP-L approaches OTP-NH. Thus, LHA provides optimal solutions for both settings.



(a) LHA's Competitive Ratio as U/L increases. (b) LHA's Competitive ratio as N^+ increases.

Figure 3: The Competitive ratio of LHA approaches $1 + \ln(U/L)$.



(a) ExHA's Competitive Ratio as U/L increases. (b) ExHA's Competitive ratio as N increases.

Figure 4: The Competitive ratio of ExHA approaches $1 + W(\frac{U-L}{eL})$.

U.2 OTP-Ex

To demonstrate the limit of $\alpha^*_{\text{Ex}}(N)$, we plot $\alpha^*_{\text{Ex}}(N)$ along with $1 + W(\frac{U-L}{eL})$ in Figure 4, where $\alpha^*_{\text{Ex}}(N)$ approaches and remains bounded by $1 + W(\frac{U-L}{eL})$.

In Figure 4a, we plot $\alpha^*_{\text{Ex}}(N)$ and $1+W(\frac{U-L}{eL})$ across a range of U/L ratios and different N. In this figure, the red curve is $1+W(\frac{U-L}{eL})$ at different U/L; other curves correspond to $\alpha^*_{\text{Ex}}(N)$ increasing as the ratio U/L increases, with N^+ fixed at 2, 3, 5, 10, 20, and 100. The results show that $\alpha^*_{\text{Ex}}(N)$ stays strictly below $1 + W(\frac{U-L}{eL})$ for all tested combinations of U/L and N. Moreover, as the exact horizon N increases, $\alpha^*_{\text{Ex}}(N)$ approaches $1+W(\frac{U-L}{eL})$. We further visualize the convergence of $\alpha^*_{\text{Ex}}(N)$ to $1 + W(\frac{U-L}{eL})$ in Figure 4b, where we use the horizontal curve in yellow to plot the constant $1 + W(\frac{U-L}{eL})$, and the increasing curve in pink plots $\alpha^*_{\text{Ex}}(N)$ for different values of N. In Figure 4b, U/L is fixed to 5. As N increases, $\alpha^*_{\text{Ex}}(N)$ approaches the limit $1 + W(\frac{U-L}{eL})$ from below,

Figure 4 plots the competitive ratio $\alpha^*_{\text{Ex}}(N)$ achieved by ExHA (Theorem 10), which is also the best achievable competitive ratio for OTP-Ex (Theorem 11). Figure 4b illustrates the relation between $\alpha^*_{\text{Ex}}(N)$ and its limit $1 + W(\frac{U-L}{eL})$, further validating the converging behavior of the hardness of OTP-Ex as stated in Lemma 4.