BEYOND ENDPOINTS: BÉZIER PATHS FOR VISUAL PROMPT TUNING

Anonymous authors
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ABSTRACT

Visual prompt tuning (VPT) adapts a frozen vision foundation model by learning a few tokens shared across all images. When prompts are updated as data shift or tasks change, the distribution of image features moves from one endpoint to another. Prior theory asks which endpoints prompts can reach; we ask whether they can travel between them along the shortest path. Because shared prompts compete with each image's own content for attention, they often cannot. We quantify this restriction as the extra transport cost that shared attention imposes over the shortest path. In an exactly solvable attention model, the reachable features form a cone; unrolling it into a plane gives the **exact minimum action** for every pair of endpoints and explicit optimal prompts on short arcs. The analysis reveals a **sharp threshold in content normalization**, which measures how much attention each image's own tokens absorb: with at most two distinct levels, prompts can follow the shortest path, whereas three or more force a strictly positive extra cost unless the endpoints differ only by rescaling. This gap persists with more prompts, small content perturbations, and aligned multihead outputs. On ViT-B/16 with VTAB-1k, a quadratic Bézier prompt path that minimizes feature action reduces held-out transport action relative to linear and cross-entropy-trained paths between the same endpoints, while cross-entropy-trained paths retain higher minimum accuracy. *Reachable endpoints do not guarantee a reachable shortest path*: how prompts move matters, not only where they end.

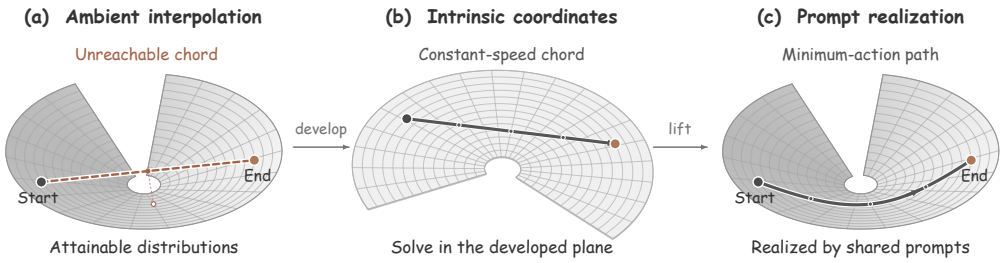

Figure 1: **Reachable endpoints do not guarantee a reachable shortest path.** Schematic; the surface is the family of feature distributions attainable by shared prompts. (a) Shared attention excludes the ambient shortcut. (b) Unrolling its cone geometry gives a straight path, (c) lifted to optimal prompts; its extra cost over (a) is the intrinsic gap. See Theorem 4.3 and Corollary 4.4.

1 INTRODUCTION

Vision foundation models are widely used across diverse tasks and domains, driving demand for adaptation that leaves the pretrained transformer untouched. Visual prompt tuning (VPT) meets this demand by appending a few learned tokens, called prompts, to the patch tokens of every image (Jia et al., 2022). These prompts are often updated as inputs shift or new tasks arrive, while the model remains in use (Gao et al., 2022; Lu et al., 2024). Since the same prompts serve all images, each update moves their feature distribution. A short transport path is a natural reference for gradual adaptation (Wang et al., 2022; He et al., 2024). Yet frozen attention makes each image respond differently to the

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same prompt update, so their features may be unable to take the shortest path together. This raises a question: *what extra transport cost do shared prompts impose between reachable endpoints?*

Existing work studies either distributional paths or what prompts can express, but not their intersection. Geodesic interpolation and stochastic interpolants construct paths without requiring shared-prompt realizability (He et al., 2024; Albergo et al., 2025; Ma et al., 2025). Parameterized Wasserstein flows constrain paths to model families but do not account for frozen attention (Jin et al., 2025). Prompt theory studies attainable outputs and capacity (Petrov et al., 2024; Hu et al., 2025), while mode connectivity studies low-loss bridges (Garipov et al., 2018b). The missing quantity is the extra transport action required to stay within the feature family of shared prompts.

To study the gap, we use a frozen head with different content normalizers across images; each normalizer measures how much attention an image's own tokens absorb. Prompt count leaves its attainable family unchanged: only aggregate prompt mass and weighted value matter. With zero content value, its positive branch is a **flat cone**. Unrolling the cone solves every endpoint sign and angle; short positive arcs have explicit optimal prompts (Figure 1; Appendix A.3). For positive endpoints that are not rescalings, two normalizers give no gap, whereas three or more give a gap at every angle (Corollary 4.4). Intuitively, with two normalizers the response direction stays in a plane and follows a great circle; a third bends it out of that plane.

The explicit head shows that adding prompt tokens cannot close this gap, and a small content-value perturbation preserves a positive gap (Theorem 4.5). We prove exact extensions for wider heads and independently controlled, aligned, unmixed multihead outputs, and give a projection test for more general mixing; independent head sums can remove the gap (Section 4.7). This cost has practical consequences: with a fixed classifier, excess action bounds the fraction of images whose prediction can change along a transition. Beyond the intrinsic gap, action separates into the cost of a longer route and the cost of uneven speed. Retiming removes the latter, while the developed chord, the straight path in the unrolled cone, removes both. We compare feature-action and loss-based paths between fixed endpoints on VTAB-1k (Zhai et al., 2019), tracking feature motion and intermediate accuracy (Section 5). A Bézier path trained for feature action lowers held-out transport action from 4.5 to 3.2 on VPT-Shallow endpoints relative to its cross-entropy-trained counterpart, whereas the latter retains higher minimum accuracy.

Our contributions can be summarized in three parts. ❶ We derive the **exact minimum action** for a realizable shared-prompt head across all endpoint signs and angles, and construct optimal prompts on short arcs. ❷ We prove a **sharp threshold at three normalizers** and show when the gap persists with more prompts, small content perturbations, and vector or multihead outputs. ❸ We separate **route and timing costs** and evaluate feature-action Bézier paths between fixed prompts in frozen ViTs. Together, these results extend prompt-tuning theory beyond endpoint reachability to the geometry and cost of feasible feature paths.

2 Related Work

Visual prompt tuning and its theoretical limits. Prompt and prefix tuning adapt frozen models through learned context (Li & Liang, 2021; Lester et al., 2021); VPT brings this idea to vision transformers (Jia et al., 2022). Later work improves initialization (Wang et al., 2024; Park & Chung, 2025), token statistics (Zeng et al., 2024; Wang et al., 2025), semantic guidance (Ren et al., 2025; Xiao et al., 2026), and block placement (Yoo et al., 2023; Shang et al., 2025). Other variants change cross-attention (Huang et al., 2025), input dependence (Xiao et al., 2025; Le et al., 2026), cross-layer dynamics (Wang et al., 2026), or information allocation (Li et al., 2026). Theory studies attention restrictions, capacity, memorization, and non-interfering updates (Petrov et al., 2024; Hu et al., 2025; Wang & Lin, 2025; Lu et al., 2024). We find minimum-action paths under shared attention.

Stochastic interpolants and constrained transport. Stochastic interpolants and flow matching learn fields for prescribed paths (Albergo & Vanden-Eijnden, 2023; Albergo et al., 2025; Lipman et al., 2023). Couplings use minibatch transport, conditional dependence, or model-aligned pairs (Tong et al., 2024; Ma et al., 2025; Lin et al., 2026); model-aware schedules adjust timing (Jia et al., 2026). Riemannian flow matching uses state-space geometry (Chen & Lipman, 2024), parameterized Wasserstein flows use model-family metrics (Jin et al., 2025), and PLOT aligns visual and

textual features (Chen et al., 2022a). Our constraint comes from shared attention: we derive its cone metric, minimum-action prompts, and sharp threshold for excess transport cost.

3 PRELIMINARIES

With softmax attention (Vaswani et al., 2017), a frozen scalar head with a common query isolates how content normalizers shape each sample's response to shared prompts.

3.1 SHARED ATTENTION AND THE ATTAINABLE FAMILY

Let $N \geq 1$ and $m \geq 1$ count samples and prompt tokens. Indices i, j, ℓ, b denote samples, prompts, content tokens, and endpoints, with $b \in \{0, 1\}$. Sample i has L_i fixed content tokens $s_{i\ell} \in \mathbb{R}^2$, $\ell = 0, \ldots, L_i - 1$, with readout token $\ell = 0$. The shared prompt tokens $p_j \in \mathbb{R}^2$ are unconstrained embeddings. Frozen linear maps $H, K, V : \mathbb{R}^2 \to \mathbb{R}$ produce queries, keys, and values. Orange marks prompt embeddings and coordinates; blue marks frozen model maps and parameters. Assume $H(s_{i0}) = q$ for a common scalar query q, with the attention scaling absorbed into q. Define the content and shared-prompt aggregates:

$$\begin{aligned} Z_i &= \sum_{\ell=0}^{L_i-1} e^{qK(s_{i\ell})}, \quad A_i = \sum_{\ell=0}^{L_i-1} e^{qK(s_{i\ell})} V(s_{i\ell}), \\ \kappa &= \sum_{j=1}^{m} e^{qK(p_j)}, \qquad w = \sum_{j=1}^{m} e^{qK(p_j)} V(p_j). \end{aligned} \tag{1}$$

Both Z_i and κ are positive. With $A = (A_i)_{i=1}^N$, the scalar output and its empirical law are

$$z_i^A(\kappa, w) = \frac{A_i + w}{Z_i + \kappa}, \qquad \mu_{\kappa,w}^A = \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i^A(\kappa,w)}, \qquad \mathcal{M}_A = \{\mu_{\kappa,w}^A : \kappa > 0,\ w \in \mathbb{R}\}. \tag{2}$$

Here δ_x is unit point mass at $x \in \mathbb{R}$ and $z^A = (z_i^A)_{i=1}^N$ the output vector. Order samples so $Z_1 \geq \cdots \geq Z_N$, with d_Z distinct values. For $A = 0$, omit the superscript and write $z, \mu_{\kappa,w}, \mathcal{M}$. Proposition 4.1 realizes every independent aggregate pair and smooth path with a fixed head. We solve the $A = 0$ core, then treat nonzero A in Section 4.5. Interpolation time is $t \in [0, 1]$.

3.2 DISTRIBUTION PATHS AND KINETIC ACTION

For laws with finite second moments, W_2 is the quadratic Wasserstein distance: $W_2^2(\mu, \nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^2} |x - y|^2 \, d\gamma(x, y)$, where $\Pi(\mu, \nu)$ contains their couplings. Let $\mathcal{P}_2(\mathbb{R})$ denote these laws and $\mathcal{P}_{2,N}(\mathbb{R})$ the equally weighted N-atom laws, including repeated atoms. The map Q lists an empirical law's atoms in increasing order. For $a, \widetilde{a} \in \mathbb{R}^N$, set $\langle a, \widetilde{a} \rangle_N = N^{-1} \sum_i a_i \widetilde{a}_i$ and $\|a\|_N = \sqrt{\langle a, a \rangle_N}$. Sorted matching gives $W_2(\mu, \nu) = \|Q\mu - Q\nu\|_N$ for empirical laws (Gabriel & Marco, 2019); the proof is in Appendix A.2.

Write $AC^2([0, 1]; \mathcal{P}_2(\mathbb{R}))$ for paths absolutely continuous in W_2 with square-integrable metric speed $|\dot{\mu}_t|_{W_2}$. Equivalently, some nonnegative $g \in L^2(0, 1)$ satisfies $W_2(\mu_s, \mu_t) \leq \int_s^t g(r) \, dr$ for $s \leq t$; the metric speed is the almost-everywhere limit of distance quotients (Ambrosio et al., 2008). For endpoint aggregates (κ_b, w_b), put $\mu_b^A = \mu_{\kappa_b,w_b}^A$ and $D_A = W_2(\mu_0^A, \mu_1^A)$. We minimize kinetic action over paths that remain attainable at every time:

$$\begin{aligned} \mathcal{E}[\mu_\cdot] &= \int_0^1 |\dot{\mu}_t|_{W_2}^2 \, dt, \\ \mathcal{E}_*(A) &= \inf\{\mathcal{E}[\mu_\cdot] : \mu_\cdot \in AC^2([0, 1]; \mathcal{P}_2(\mathbb{R})), \quad \mu_t \in \mathcal{M}_A \ (0 \leq t \leq 1),\ \mu_b = \mu_b^A\}. \end{aligned} \tag{3}$$

The action uses no factor $1/2$ (Benamou & Brenier, 2000). Without the constraint, the minimum is D_A^2. Thus $\mathcal{G}_A = \mathcal{E}_*(A) - D_A^2$ measures the cost of shared attention; write $D = D_0$ and $\mathcal{G} = \mathcal{G}_0$. We also use path length $\mathrm{Len}[\mu_\cdot] = \int_0^1 |\dot{\mu}_t|_{W_2} \, dt$. An empirical stochastic interpolant (Albergo et al., 2025; Ma et al., 2025) arises by drawing I uniformly from $\{1, \ldots, N\}$ and setting $X_t = z_I(\kappa_t, w_t)$, whose law is μ_{κ_t,w_t}.

3.3 Coordinates for the core result

On the positive branch $w > 0$ of $\mathcal{M}$, set $u_\kappa = ((Z_i + \kappa)^{-1})_{i=1}^N$, $v_\kappa = u_\kappa / \|u_\kappa\|_N$, and $R = w\|u_\kappa\|_N$. Then $Q\mu_{\kappa,w} = Rv_\kappa$. For a fixed $\kappa_{\mathrm{ref}} > 0$, define

$$\theta(\kappa) = \int_{\kappa_{\mathrm{ref}}}^{\kappa} \|v_s'\|_N \, ds, \qquad J = \theta((0,\infty)). \tag{4}$$

Primes here denote derivatives in κ; dots denote time derivatives. For $d_Z \geq 2$, θ is strictly increasing and has inverse $\theta^{-1} : J \to (0,\infty)$ (Appendix A.2). For either endpoint sign define $R_b = |w_b|\|u_{\kappa_b}\|_N$, $\theta_b = \theta(\kappa_b)$, $\Delta = |\theta_1 - \theta_0|$, and $\alpha = \arccos\langle v_{\kappa_0}, v_{\kappa_1}\rangle_N$. Δ measures travel along the direction curve; α is the direct angle between its endpoints. The subscript E denotes the Euclidean norm in planar or vector feature spaces.

For perturbation bounds, put

$$\eta = \left(\frac{1}{N}\sum_i \frac{A_i^2}{Z_i^2}\right)^{1/2}, \qquad h_A = \frac{Q\mu_0^A + Q\mu_1^A}{2}, \qquad \rho = \mathrm{dist}_N(h_0, Q(\mathcal{M})). \tag{5}$$

Here $\mathrm{dist}_N(a,S) = \inf_{s\in S}\|a-s\|_N$ and $[x]_+ = \max\{x,0\}$. Remaining symbols are defined where they are used.

4 Method: Optimal Interpolation within Shared Attention

Given normalizers $(Z_i)_{i=1}^N$, prompt count m, and endpoint aggregates (κ_b, w_b), we develop the $A = 0$ geometry into the plane and lift its optimal path to shared prompts. We then bound the effect of content perturbations. Figure 2 links the construction to frozen-ViT feature-action optimization.

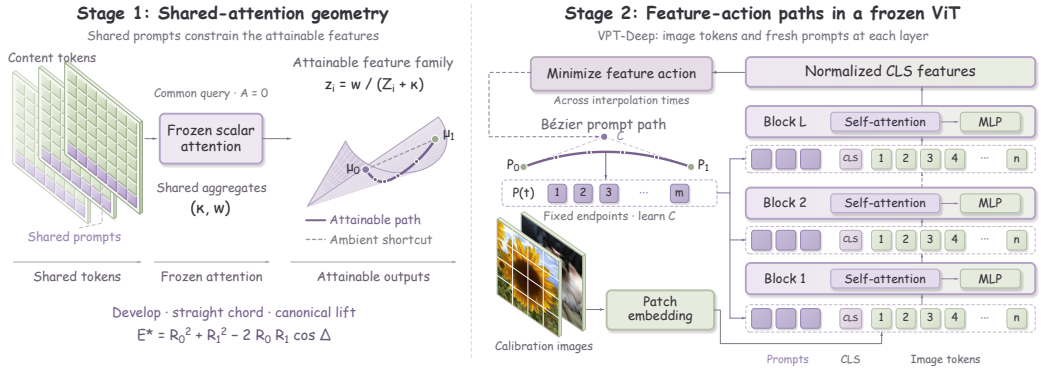

Figure 2: **Geometry-guided prompt interpolation.** *Left:* in the scalar core, we construct a minimum-action path within the attainable feature family. *Right:* in VPT-Deep, a Bézier path supplies prompts to each layer of a frozen ViT. We minimize feature action by optimizing its control tensor with both endpoints fixed.

4.1 An exactly realizable attention family

A frozen head independently controls κ, w in (1)–(2), realizing each path with shared tokens.

Proposition 4.1 (Exact realization for every prompt length) *For any positive $(Z_i)_{i=1}^N$ and fixed $A \in \mathbb{R}^N$, there is one frozen scalar attention head on two-dimensional tokens, without LayerNorm, whose attainable family equals $\mathcal{M}_A$ for every integer $m \geq 1$. Every smooth aggregate path with $\kappa_t > 0$ is realized by*

$$p_j(t) = \big(\log(\kappa_t/m),\ w_t/\kappa_t\big), \qquad j = 1,\dots,m. \tag{6}$$

Choose $\beta \neq 0$ with $e^\beta < \min_i Z_i$ and frozen maps $K(p) = p^{(1)}$, $V(p) = p^{(2)}$, and $H(p) = p^{(1)}/\beta$ for $p \in \mathbb{R}^2$. Fixed content tokens supply Z_i, A_i and common query one. Prompt count changes token representations, not attainable outputs. Please see Appendix A.1 for more details.

4.2 THE ATTAINABLE FAMILY HAS A CONE METRIC

Set $A = 0$. The scalar w scales outputs, while κ changes relative sample responses. These radial and directional motions give a planar polar-coordinate metric.

Proposition 4.2 (Exact cone coordinates) *If $d_Z \geq 2$, the positive branch of $\mathcal{M}$ is a smooth embedded surface in its sorted-vector representation. Its coordinates $(R, \theta) \in (0, \infty) \times J$ induce the Wasserstein line element*

$$ds^2 = dR^2 + R^2 d\theta^2. \tag{7}$$

Every admissible path remaining in this branch has $\mathcal{E}[\mu_\cdot] = \int_0^1 (\dot{R}_t^2 + R_t^2 \dot{\theta}_t^2)\, dt$.

Because v_κ has unit norm, its change is perpendicular to itself. Thus radial and angular motion contribute the two terms in (7). Mapping (R, θ) to $R(\cos\theta, \sin\theta)$ preserves their action. Please see Appendix A.2 for more details.

4.3 EXACT MINIMUM ACTION AND THE NORMALIZATION THRESHOLD

Reachable endpoints need not admit a reachable straight path. Comparing travel along the cone with the ambient chord in sorted output space gives the exact constraint cost.

Theorem 4.3 (Global minimum over all attainable paths) *Assume $d_Z \geq 2$, $w_0, w_1 > 0$, $\kappa_0 \neq \kappa_1$, and $\Delta < \pi$. The infimum in (3) for $A = 0$ is attained, and*

$$\begin{aligned} \mathcal{E}_*(0) &= R_0^2 + R_1^2 - 2R_0R_1\cos\Delta, \\ D^2 &= R_0^2 + R_1^2 - 2R_0R_1\cos\alpha, \\ \mathcal{G} &= 2R_0R_1(\cos\alpha - \cos\Delta). \end{aligned} \tag{8}$$

The infimum includes sign changes, zero crossings, and changes in atom matching.

The short-arc chord gives a smooth prompt optimum. For $d_Z \geq 2$, the global action is

$$\mathcal{E}_*(0) = \begin{cases} R_0^2 + R_1^2 - 2R_0R_1\cos(\min\{\Delta, \pi\}), & w_0w_1 > 0, \\ (R_0 + R_1)^2, & w_0w_1 \leq 0. \end{cases} \tag{9}$$

For $\Delta \geq \pi$ or opposite signs, a path through δ_0 attains the infimum among distribution paths; smooth prompt paths approach it. Please see Appendix A.3 for the proof and endpoint distances.

The number of distinct normalizers determines whether the direction curve exceeds the shortest spherical arc between its endpoints.

Corollary 4.4 (Two versus three distinct normalizers) *For $w_0, w_1 > 0$, if $d_Z \leq 2$, then $\mathcal{G} = 0$ for every pair $\kappa_0, \kappa_1 > 0$. If $d_Z \geq 3$ and $\kappa_0 \neq \kappa_1$, then $\mathcal{G} > 0$ at every angular span. For the head in Proposition 4.1, these statements hold for every $m \geq 1$.*

With two normalizers, the direction curve follows its shortest spherical arc, so $\Delta = \alpha$. A third distinct normalizer bends that curve away from the arc, giving $\Delta > \alpha$; the global formula covers long arcs through the apex as well. Please see Appendix A.4 for more details.

4.4 CONSTRUCTING THE OPTIMAL PROMPT INTERPOLANT

Write $e(\vartheta) = (\cos\vartheta, \sin\vartheta)$ and $c(t) = (1-t)R_0e(\theta_0) + tR_1e(\theta_1)$. Under Theorem 4.3, this developed chord never vanishes. Let $\vartheta(t)$ be its continuous angle between the endpoint angles. Recover the prompt aggregates by

$$R_t = \|c(t)\|_{\mathrm{E}}, \qquad \kappa_t = \theta^{-1}(\vartheta(t)), \qquad w_t = \frac{R_t}{\|u_{\kappa_t}\|_N}. \tag{10}$$

Equation (6) then gives every prompt. Computing the path requires one-dimensional quadrature for θ and monotone inversion between κ_0 and κ_1.

The lifted chord also gives the empirical interpolant X_t from the preliminaries. It has a bounded velocity field that is uniformly Lipschitz in space and satisfies the weak continuity equation. When $\kappa_0 = \kappa_1$ or $d_Z = 1$, simple linear paths suffice. The formal statement, construction, and degenerate cases are in Proposition A.2 and Appendix A.5.

4.5 STABILITY BEYOND ZERO CONTENT NUMERATORS

At fixed endpoint aggregates, small content perturbations preserve the core's midpoint separation from the attainable family. This separation bounds every perturbed path's action.

Theorem 4.5 (A neighborhood with positive excess action) *Suppose $d_Z \geq 3$, $w_0, w_1 > 0$, and $\kappa_0 \neq \kappa_1$. The midpoint distance ρ in (5) is strictly positive. For every $A \in \mathbb{R}^N$,*

$$\mathcal{E}_*(A) \geq D_A^2 + 4[\rho - 2\eta]_+^2. \tag{11}$$

In particular, $\eta < \rho/2$ implies $\mathcal{G}_A > 0$. This statement requires neither $\Delta < \pi$ nor preservation of sample ordering under perturbation.

The bound tracks how much content perturbations can reduce the midpoint's separation from the attainable family. Please see Appendix B for its proof and a one-dimensional evaluation of ρ.

4.6 FEATURE-ACTION PATHS IN A FROZEN VIT

The exact head motivates minimizing feature motion along a feasible prompt path. For a frozen ViT, let P_0, P_1 be trained endpoint prompt tensors. A quadratic Bézier path $P(t) = (1-t)^2 P_0 + 2t(1-t)C + t^2 P_1$ changes only its control tensor C. For the same N images, let $f_i(t) \in \mathbb{R}^{d_{\text{out}}}$ be their normalized readout features and $\nu_t = N^{-1} \sum_i \delta_{f_i(t)}$ their empirical law. At time nodes $t_k = k/S$, $k = 0, \ldots, S$, the practical objective is

$$\widehat{\mathcal{A}}_S(C) = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=0}^{S-1} \frac{\|f_i(t_{k+1}) - f_i(t_k)\|_{\mathrm{E}}^2}{t_{k+1} - t_k}. \tag{12}$$

The same-image cost avoids permutation ambiguity on a finite grid and, for each fixed path with square-integrable feature derivatives, converges to continuous feature action as the grid is refined. We optimize C on calibration images while holding both endpoints, the backbone, and the classifier fixed. The cross-entropy control uses the same Bézier family. Held-out transport action uses W_2 with squared Euclidean cost in $\mathbb{R}^{d_{\text{out}}}$; accuracy and loss test whether shorter paths preserve predictions. Please see Appendix D.3 and Appendix G for more details.

4.7 WHAT CHANGES BEYOND A SCALAR HEAD

These extensions locate the single-head obstruction. Increasing output width leaves the minimum action and gap unchanged when endpoints share one direction; a constructed pre-attention LayerNorm head also preserves the optimum for a fixed prompt count and compact aggregate range. For independently controlled, aligned, unmixed heads with positive endpoints, minimum actions and gaps add exactly (Proposition F.3). More generally, a scalar projection certifies a vector or multihead gap when its image stays close to the cone and extra endpoint separation is small (Proposition F.5). By contrast, summing independently controlled heads can trace the straight feature path and close the gap, while normalizing the measured output can erase the response direction. The certificate needs neither scalar outputs nor a common query. Please see Appendix F for more details.

5 EXPERIMENTAL ANALYSIS

We separate intrinsic gaps from avoidable path cost and test prediction quality. The scalar study tests ten paths on 24 cases with $d_Z \in \{3, 8\}$, positive endpoints, and $\Delta < \pi$. On ViT-B/16 (Dosovitskiy et al., 2021), we compare 15 trainers across 19 VTAB-1k tasks (Zhai et al., 2019) and six paths per VPT-Shallow or VPT-Deep endpoint pair. Please see Appendix D for more details.

The attainable optimum gives every path a common reference. Table 1 compares direct, retimed, and optimized paths. By Theorem 4.3, the developed chord has unit relative action. Cost above $\mathcal{E}_*(0)$ is avoidable, whereas $\mathcal{E}_*(0) - D^2$ is intrinsic to shared attention. Thus unit relative action need not mean zero excess over the ambient chord.

Table 1: **Fixed-endpoint comparison against direct, retimed, and optimized paths.** Relative action $\mathcal{E}/\mathcal{E}_*(0)$ is split by normalizer count; excess $(\mathcal{E} - D^2)/D^2$ and other diagnostics pool all 24 cases. Bézier and polygonal controls adapt the curve families of Garipov et al. (2018a) to action minimization. The ambient chord is the unconstrained lower bound.

	Relative action ↓		Pooled diagnostics			
Path	$d_Z = 3$	$d_Z = 8$	Excess ↓	End err. ↓	Build ms ↓	Valid ↑
Token-linear	5.3622	6.0701	4.7986	10^{-12}	1	24/24
Aggregate-linear	1.4300	1.3963	0.4316	10^{-12}	2	24/24
Cone-coordinate-linear	1.0153	1.0148	0.0254	10^{-12}	25	24/24
Token-linear + arclength	4.4514	4.9699	3.7815	10^{-12}	8	24/24
Aggregate-linear + arclength	1.0183	1.0137	0.0265	10^{-12}	10	24/24
Cone-linear + arclength	1.0146	1.0141	0.0247	10^{-12}	60	24/24
Quadratic Bézier (action)	1.0080	1.0070	0.0178	10^{-12}	2500	24/24
One-bend polygonal (action)	1.0160	1.0140	0.0253	10^{-12}	1600	24/24
16-interval action minimization	1.0002	1.0003	0.0104	10^{-12}	8000	24/24
Developed chord	1.0000	1.0000	0.0102	10^{-12}	40	24/24
Ambient chord	$D^2/\mathcal{E}_*(0)$	$D^2/\mathcal{E}_*(0)$	0	0	–	–

Route and clock account for different costs. Table 3 decomposes action into intrinsic, route, and timing costs: $\mathcal{E} - D^2 = \mathcal{G} + (\mathrm{Len}^2 - \mathcal{E}_*(0)) + (\mathcal{E} - \mathrm{Len}^2)$. Retiming preserves endpoints and route length, so only the last term can fall. The optimal chord pays $\mathcal{E}_*(0)/3$ more under $t \mapsto t^2$, entirely from changing speed along the same route. Please see Appendix D.1 for more details.

The threshold and the avoidable costs are directly visible. Figure 7(a,b) evaluates the exact optimum while varying both endpoint aggregate parameters κ_0, κ_1. With two distinct content normalizers, every pair has zero intrinsic gap. With three, non-rescaling pairs have positive excess; the displayed grid reaches 12.02% above unconstrained transport. Panel (c) separates route from timing on one shared endpoint pair. Retiming removes aggregate-linear's 72.73% timing excess but leaves 6.73% route excess. Token-linear retains 13.80% route excess. The developed chord sits at the origin of this plot; the clock t^2 adds 33.33% timing excess. Retiming cannot remove a curved route or the intrinsic gap. Please see Appendix C.1 for more details.

The intrinsic gap begins at three distinct normalizers. Table 4 varies normalizers, prompt count, and content values. With at most two distinct normalizers, the attention direction follows a shortest spherical arc and the intrinsic gap vanishes. A third bends it away: for positive, non-rescaling endpoints, the gap is then positive at any angular span (Corollary 4.4). Adding prompts at fixed aggregates cannot straighten this curve; the content-perturbation certificate shows where the gap persists beyond $A = 0$. Please see Appendix D.2 for more details.

Extra token coordinates can be exactly redundant. Figure 8 shows prompt-count invariance at fixed aggregates. Increasing m from 1 to 32 leaves two active output directions and adds $2m - 2$ null directions. At the equal-token lift, both raw singular values scale as $m^{-1/2}$; multiplying the Jacobian by $\sqrt{m}$ removes this replication factor. The active spectrum then agrees across counts, without a numerical rank threshold. Please see Appendix C.2 for more details.

Figure 9 shows the gap and its lower certificate under content perturbations; the certificate stays positive near zero content. Please see Appendix C.3 for more details.

Feature action probes what endpoint accuracy misses. Table 2 compares adaptation choices, with per-task results in Tables 6–8. Table 5 fixes endpoints and classifier, so all paths agree on endpoint predictions. Differences in minimum accuracy or loss barrier therefore reveal behavior inside the transition; the matched Bézier rows test how the CE and feature-action objectives shape that behavior. Please see Appendix D.3 for more details.

Table 2: **Accuracy and resource cost for 15 methods on ViT-B/16 and VTAB-1k.** All use matched task splits and validation selection; All (19) is the equally weighted mean of the 19 tasks. Trainable parameters, peak memory, and latency include the classifier.

Endpoint trainer	Top-1 accuracy (%) ↑				Resource cost ↓		
	Natural	Specialized	Structured	**All (19)**	Trainable	Peak mem.	Latency
	(7)	(4)	(8)		M	GiB	ms/img
Reference adaptation							
Linear probe	69.0	80.6	33.2	56.4	0.06	1.5	0.65
Full fine-tuning	76.2	85.0	53.4	68.5	86.0	10.0	0.65
Parameter-efficient updates							
BitFit (2022)	73.6	82.6	45.7	63.8	0.16	6.0	0.65
LoRA (2021)	79.4	85.2	58.4	71.8	0.35	6.2	0.65
AdaptFormer (2022b)	79.9	85.4	58.8	72.2	0.25	6.5	0.72
SSF (2022)	78.6	86.0	58.5	71.7	0.28	6.1	0.65
FacT-TT (2023)	79.5	86.3	59.7	72.6	0.14	6.4	0.65
RepAdapter (2023)	78.6	86.2	59.3	72.1	0.25	6.2	0.65
NOAH (2024)	79.8	86.8	59.7	72.8	0.48	6.8	0.79
Prompt adaptation							
VPT-Shallow (2022)	77.7	83.3	49.5	67.0	0.1	6.8	0.81
VPT-Deep (2022)	78.6	85.7	57.0	71.0	0.6	7.4	0.94
E^2VPT (2023)	79.6	86.2	59.6	72.6	0.4	7.2	0.91
CVPT (2025)	80.0	86.2	60.1	72.9	0.25	6.9	0.84
VAPT (2026)	80.7	86.4	60.7	73.5	0.8	7.8	1.02
VPT-Deep + PAE (2026)	79.8	86.4	60.5	73.0	0.65	7.6	0.98

Same endpoints, different motion. Figure 3 shows two flows with shared endpoints and contraction rate. The curved path costs 7, or about 6.756 after retiming. The constant-speed straight path costs 4, the squared endpoint distance. In the attention model, the developed chord lifts to shared prompts without changing action (Proposition A.2); its intrinsic gap can still be positive.

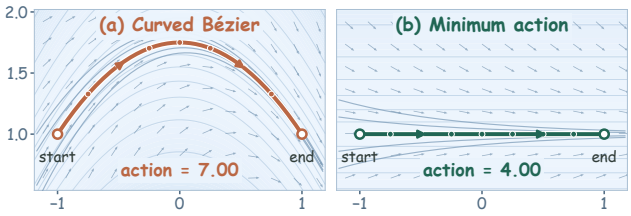

Figure 3: **Route geometry changes action.** Analytic illustration, not model outputs. Both flows share endpoints and contraction rate; dots mark equal times.

5.1 Dynamics, route, and timing

Most excess in the field pair is geometric. For the curved centerline $c(t) = (2t-1, 1+3t(1-t))$ in Figure 3, let $L_c = \int_0^1 \sqrt{4+9(1-2t)^2}\,dt$ be its length. Constant-speed retiming removes only $7 - L_c^2 \simeq 0.244$ of its three-unit excess, while $L_c^2 - 4 \simeq 2.756$ remains because the route is longer. Thus about 92% of the avoidable cost in this example requires changing the route itself.

Speed profiles explain excess action. The formula evaluations in Figure 4 show why the same endpoints can have different action: token-linear accelerates near the endpoint, while aggregate-linear spends more motion near the start. Constant-speed retiming removes speed variation but leaves each route unchanged. The developed chord removes both avoidable costs. Linearity in prompt or aggregate coordinates need not give a straight feature route. Timing excess is exactly the variance of metric speed; please see Appendix C.1 for more details.

5.2 Qualitative comparisons

Figures 5 and 6 pair CE and Feature-action Bézier maps on matched photographs at the midpoint. These snapshots illustrate spatial diagnostics; Table 5 tests action and prediction quality across the full path, which a single midpoint view cannot establish. Please see Appendix E for more cases.

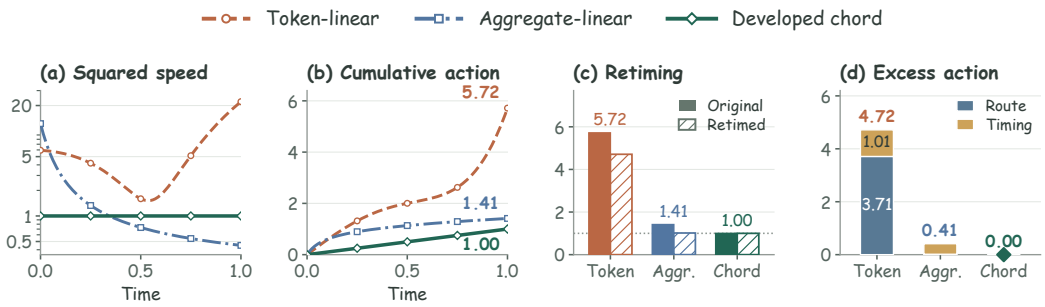

Figure 4: **Route and timing diagnostics.** Formula evaluations on 24 scalar-model cases, normalized by their attainable optimum. Panel (a) uses a log scale; (c) labels original action. Panels (c)–(d) separate timing cost removed by arclength retiming from route cost that remains. Comparing retimed action with the attainable optimum therefore tests the route, not just the clock. Panels (a)–(b) show that the baselines spend motion at different times despite equal endpoints. The cumulative-action curves expose this difference, whereas a midpoint snapshot cannot.



Figure 5: **Saliency on six Oxford-IIIT Pet photographs.** CE Bézier (top) and Feature-action Bézier (bottom).

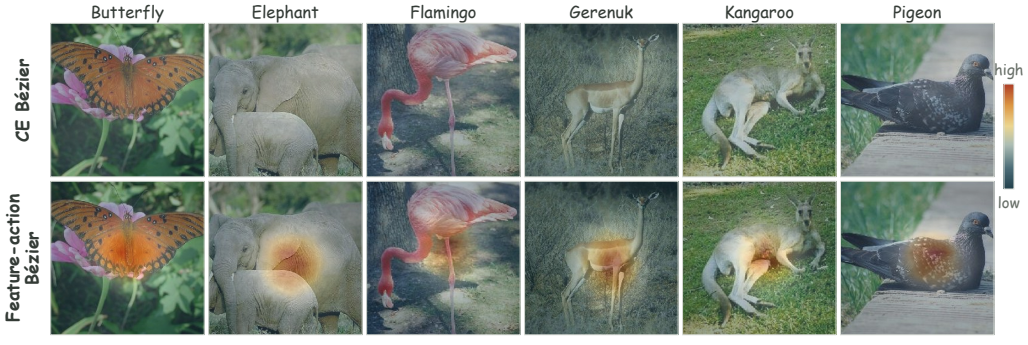

Figure 6: **Attention on six Caltech-101 animal photographs.** CE Bézier (top) and Feature-action Bézier (bottom).

6 Conclusion

In this work, we characterized the feature paths of shared visual prompts in an exactly realizable frozen attention head. Its cone geometry gives the minimum action across endpoint regimes, explicit short-arc prompts, and a sharp normalization threshold. Projection bounds show when the gap persists in wider and multihead features. These results motivate feature-action training for frozen ViTs: endpoint reachability alone cannot determine path cost.

AI USE STATEMENT

OpenAI Codex assisted with coding development, literature screening, mathematical formulation and proof review, manuscript writing, figure preparation, and numerical sanity checks. The authors take responsibility for the full content of this work.

REPRODUCIBILITY STATEMENT

Complete proofs accompany the analytic results. Please see Appendix A for more details. The evaluation uses matched baselines and reporting rules. Please see Appendix D for more details.

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Technical Appendices

Beyond Endpoints: Bézier Paths for Visual Prompt Tuning

Proofs, Experimental Protocols, and Additional Visualizations

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A PROOFS OF THE ATTENTION GEOMETRY AND OPTIMAL INTERPOLATION RESULTS

Throughout this appendix, all finite-dimensional norms and inner products on $\mathbb{R}^N$ are normalized by N, as defined in the preliminaries. Planar Euclidean norms are written $\|\cdot\|_{\mathrm{E}}$. An admissible distribution path takes values in the stated family at *every* time, including its endpoints.

A.1 EXACT REALIZATION BY A FIXED ATTENTION HEAD

Proof of Proposition 4.1. Choose a fixed $\beta \neq 0$ such that $e^\beta < \min_i Z_i$. Use two-dimensional token embeddings and scalar query, key, and value maps $H(p) = p^{(1)}/\beta$, $K(p) = p^{(1)}$, and $V(p) = p^{(2)}$, where $p = (p^{(1)}, p^{(2)}) \in \mathbb{R}^2$. For sample i, take the classification token to be $(\beta, 0)$ and the other content token to be $\big(\log(Z_i - e^\beta), A_i/(Z_i - e^\beta)\big)$. The classification query is identically one. The two fixed content tokens contribute partition $e^\beta + (Z_i - e^\beta) = Z_i$ and weighted value $0 + (Z_i - e^\beta)A_i/(Z_i - e^\beta) = A_i$. For any fixed prompt count $m \geq 1$, choose the shared tokens

$$p_j(\kappa, w) = \Big(\log(\kappa/m), \frac{w}{\kappa}\Big), \qquad j = 1, \ldots, m. \tag{13}$$

Their total partition is $m(\kappa/m) = \kappa$, and their total weighted value is $m(\kappa/m)(w/\kappa) = w$. The scalar classification head output is therefore exactly $z_i^A(\kappa, w) = (A_i + w)/(Z_i + \kappa)$. Conversely, arbitrary finite shared prompts in this fixed head contribute $\kappa = \sum_{j=1}^m e^{K(p_j)} > 0$ and $w = \sum_{j=1}^m e^{K(p_j)} V(p_j)$, so their outputs belong to $\mathcal{M}_A$. Thus every fixed $m \geq 1$ realizes exactly the same family, and (13) gives a smooth lift of every smooth aggregate path with $\kappa > 0$. The maps and content tokens remain fixed throughout the path. The construction realizes the scalar attention head output with unit scalar-key scaling, before LayerNorm or subsequent transformer blocks.

A.2 SORTED DISTRIBUTIONS AND THE CONE METRIC

The sorted-atom isometry. Let $\mu = N^{-1}\sum_i \delta_{a_i}$ and $\nu = N^{-1}\sum_i \delta_{b_i}$, with both vectors increasingly sorted. Any coupling of their atom locations can be lifted to a matrix $\Gamma = (\Gamma_{ir})$ on the labeled copies with row and column sums $1/N$, without changing its cost. Indeed, if a location occurs n_a times on the first side and n_b times on the second, distribute the mass assigned to this pair of locations equally among the $n_a n_b$ pairs of copies. For ordered real numbers,

$$(a_i - b_s)^2 + (a_r - b_j)^2 - (a_i - b_j)^2 - (a_r - b_s)^2 = 2(a_r - a_i)(b_s - b_j) \geq 0$$

whenever $i < r$ and $j < s$. To uncross a fractional plan, suppose its $(1,1)$ entry is below $1/N$. There are positive entries $(1, s)$ and $(r, 1)$ with $r, s > 1$. Move the smaller of these masses to $(1,1)$ and (r, s). This preserves every marginal and, by the displayed inequality, does not increase cost. It also removes at least one off-diagonal entry in the first row or column without adding any there. Finitely many such moves fill $(1,1)$ to $1/N$. Remove that row and column and repeat. The resulting diagonal plan has cost no greater than the original plan, including when locations repeat. Hence

$$W_2(\mu, \nu)^2 = \frac{1}{N}\sum_{i=1}^N (a_i - b_i)^2 = \|Q\mu - Q\nu\|_N^2. \tag{14}$$

In particular, a distribution path in the equal-weight empirical class is in AC^2 if and only if its sorted-vector path is in AC^2, and their metric speeds agree almost everywhere. This follows by applying the isometry to absolute-continuity bounds and metric-speed difference quotients.

For any path in $\mathcal{P}_2(\mathbb{R})$, the endpoint distance is at most its length. Cauchy–Schwarz on $[0,1]$ therefore bounds its action below by the squared endpoint distance. For the empirical endpoints here, the path whose sorted vector is $(1-t)Q\mu_0^A + tQ\mu_1^A$ remains sorted and has constant speed. It attains that lower bound. Thus the unconstrained action minimum is D_A^2 over all paths in $\mathcal{P}_2(\mathbb{R})$.

Proof of Proposition 4.2. Suppose $d_Z \geq 2$. On the positive branch, the prescribed nonincreasing order of the Z_i makes the output vector increasingly sorted, so $Q\mu_{\kappa,w} = wu_\kappa = Rv_\kappa$. The normalized direction is a smooth regular curve: if $v_\kappa' = 0$, differentiating $u_\kappa = \|u_\kappa\|_N v_\kappa$ would make

u'_κ proportional to u_κ. Coordinatewise, this would require all $(Z_i + \kappa)^{-1}$ to coincide, contrary to $d_Z \geq 2$. Thus $\theta'(\kappa) = \|v'_\kappa\|_N > 0$, and $\theta : (0, \infty) \to J$ is a smooth bijection with smooth inverse.

The parameters are identifiable on the positive branch. Choose two indices r, s with $Z_r \neq Z_s$. For a positive-branch vector x, define $\xi = x_r/x_s$. Then

$$\xi = \frac{Z_s + \kappa}{Z_r + \kappa}, \qquad \kappa = \frac{Z_s - \xi Z_r}{\xi - 1}, \qquad w = x_s(Z_s + \kappa), \qquad R = \|x\|_N. \tag{15}$$

Here $x_s > 0$ and $\xi \neq 1$. These formulas extend smoothly to an ambient neighborhood of each positive-branch point and give the continuous inverse of the parametrization on that branch.

Differentiating $\langle v_\kappa, v_\kappa \rangle_N = 1$ gives $\langle v_\kappa, v'_\kappa \rangle_N = 0$. Consequently

$$\|d(Rv_\kappa)\|_N^2 = dR^2 + R^2 \|v'_\kappa\|_N^2 \, d\kappa^2 = dR^2 + R^2 d\theta^2. \tag{16}$$

The map $Rv_\kappa \mapsto Re(\theta(\kappa))$ therefore preserves the speed of every differentiable curve. This developing map gives a local isometry, with no need for global planar injectivity.

To cover every AC^2 path in the positive branch, we use the inverse chart to recover parameter regularity. Let $x(t) = Q\mu_t$. Its image is compact. The smooth inverse in (15) shows that its parameters are continuous, and their image is a compact subset of $\{\kappa > 0, w > 0\}$. The smooth inverse and its derivatives are bounded on a neighborhood of the compact path image. Composition of an absolutely continuous finite-dimensional curve with these smooth maps, followed by the usual almost-everywhere chain rule, gives $R, \kappa, \theta(\kappa) \in AC^2$. Thus (16) holds almost everywhere. Each admissible positive path also avoids nonzero points on the limiting rays at $\kappa = 0$ and $\kappa = \infty$.

A.3 The minimum over all admissible paths

Proof of Theorem 4.3. Let $d_Z \geq 2$, $w_0, w_1 > 0$, and $\Delta < \pi$. For a path staying in the positive branch, apply the preceding chain rule to its developed curve $y(t) = R(t)e(\theta(\kappa(t)))$. The metric identity and Cauchy–Schwarz yield

$$\mathcal{E}[\mu] = \int_0^1 \|\dot{y}(t)\|_{\mathrm{E}}^2 \, dt \geq \|y(1) - y(0)\|_{\mathrm{E}}^2 \tag{17}$$

$$= R_0^2 + R_1^2 - 2R_0R_1 \cos\Delta. \tag{18}$$

The bound covers excursions past the endpoint parameters and self-overlapping developed curves.

Next consider paths that leave the positive branch. The mean is continuous in W_2: for any coupling, Cauchy–Schwarz bounds the difference of means by the root mean squared displacement. In $\mathcal{M}$, the mean has the sign of w and is zero exactly at δ_0. Any path from positive endpoints that visits the negative branch must therefore visit δ_0. If a path reaches δ_0 at some time, its total length is at least

$$W_2(\mu_0, \delta_0) + W_2(\delta_0, \mu_1) = R_0 + R_1.$$

Its action is at least $(R_0 + R_1)^2$. This is strictly larger than the right-hand side of (18), because $R_0R_1 > 0$ and $\cos\Delta > -1$. These two cases exhaust all admissible paths, including paths that change aggregate parameters at the apex. The lower bound also holds for paths of infinite action.

To attain the bound, take the planar chord $c(t) = (1-t)R_0e(\theta_0) + tR_1e(\theta_1)$. Since $|\theta_1 - \theta_0| < \pi$, the cone between the two endpoint rays has angle less than π. The chord never reaches the origin, and it has a unique continuous angular lift $\vartheta(t)$ in the closed interval with endpoints θ_0, θ_1. For distinct endpoint angles, the lift is monotone because

$$\dot{\vartheta}(t) = \frac{\det(c(t), \dot{c}(t))}{\|c(t)\|_{\mathrm{E}}^2} = \frac{R_0R_1 \sin(\theta_1 - \theta_0)}{\|c(t)\|_{\mathrm{E}}^2},$$

whose sign is constant. For equal angles the lift is constant. Since the angular interval is contained in J, define

$$\kappa(t) = \theta^{-1}(\vartheta(t)), \qquad w(t) = \frac{\|c(t)\|_{\mathrm{E}}}{\|u_{\kappa(t)}\|_N}. \tag{19}$$

These parameters are smooth, remain positive, and stay in a compact subset of their domain. Their developed curve is exactly c. Its speed is constant, so the action of the lifted path is $\|c(1) - c(0)\|_{\mathrm{E}}^2$, attaining (18). The fixed head realizes this path for any $m \geq 1$ by (13).

Finally, the sorted isometry gives

$$D_0^2 = \|R_1 v_{\kappa_1} - R_0 v_{\kappa_0}\|_N^2 = R_0^2 + R_1^2 - 2R_0 R_1 \cos\alpha.$$

Subtracting gives the formula for $\mathcal{G}_0$. The explicit chord construction attains the bound directly and uses the developing map only along the endpoint angular interval.

Proposition A.1 (Global action across signs and angles) *Let $A = 0$, $d_Z \geq 2$, $\kappa_b > 0$, and $w_b \in \mathbb{R}$. Set $R_b = |w_b| \|u_{\kappa_b}\|_N$, $\Delta = |\theta(\kappa_1) - \theta(\kappa_0)|$, and $\alpha = \arccos\langle v_{\kappa_0}, v_{\kappa_1}\rangle_N$. If $w_0 w_1 > 0$, put $\overline{\Delta} = \min\{\Delta, \pi\}$. Then*

$$\begin{aligned} \mathcal{E}_*(0) &= R_0^2 + R_1^2 - 2R_0 R_1 \cos\overline{\Delta}, \\ D^2 &= R_0^2 + R_1^2 - 2R_0 R_1 \cos\alpha, \\ \mathcal{G} &= 2R_0 R_1(\cos\alpha - \cos\overline{\Delta}). \end{aligned} \tag{20}$$

If $w_0 w_1 < 0$, let κ_+ and κ_- denote the parameters of the positive and negative endpoints, respectively. Define $\mathrm{rev}(a_1, \ldots, a_N) = (a_N, \ldots, a_1)$ and $c_{\mathrm{rev}} = \langle v_{\kappa_+}, \mathrm{rev}(v_{\kappa_-})\rangle_N$. The following formulas hold, with $0 < c_{\mathrm{rev}} < 1$:

$$\mathcal{E}_*(0) = (R_0 + R_1)^2, \qquad D^2 = R_0^2 + R_1^2 + 2R_0 R_1 c_{\mathrm{rev}}, \qquad \mathcal{G} = 2R_0 R_1(1 - c_{\mathrm{rev}}) > 0. \tag{21}$$

If either $w_b = 0$, then $\mathcal{E}_*(0) = D^2 = (R_0 + R_1)^2$. These infima are attained by distribution paths. For $\Delta \geq \pi$ with same-sign endpoints, or for opposite-sign endpoints, an optimal distribution path consists of radial legs meeting at δ_0. The infimum over smooth shared-prompt paths with the prescribed endpoint aggregates has the same value. An apex-crossing optimum with both endpoint weights nonzero has a smooth prompt lift if and only if $\kappa_0 = \kappa_1$; thus a same-sign optimum with $\Delta \geq \pi$ is never attained by continuous prompts. If exactly one endpoint weight vanishes, smooth-prompt attainment holds exactly when $\kappa_0 = \kappa_1$. If both weights vanish, a smooth zero-action prompt path exists for any endpoint parameters.

For $d_Z = 1$, every attainable law is a Dirac mass and $\mathcal{E}_*(0) = D^2$ for all endpoints.

Proof of Proposition A.1. When $w > 0$, the sorted vector is $R v_\kappa$; when $w < 0$, it is $-R\,\mathrm{rev}(v_\kappa)$. The reversal is an isometry, so either fixed-sign branch has line element $dR^2 + R^2 d\theta^2$. For same-sign endpoints with $\Delta < \pi$, the proof of Theorem 4.3 applies on the positive branch and, after reversal and a sign change, on the negative branch. Any competing path that changes sign visits δ_0, costs at least $(R_0 + R_1)^2$, and cannot improve the developed-chord action.

Suppose next that the endpoints have the same sign and $\Delta \geq \pi$. A path staying in that sign branch has continuous $R > 0$ and θ. In the direction from θ_0 to θ_1, it must first reach $\theta_0 \pm \pi$ at some time s; write $r = R(s) > 0$. Developing the first subpath into the plane puts its endpoints on opposite rays, hence its length is at least $R_0 + r$. Radial variation on the remaining subpath has length at least $|R_1 - r|$. Therefore its total length is at least $R_0 + r + |R_1 - r| \geq R_0 + R_1$. The inequality is strict for a path avoiding the apex: equality in the first bound would force a planar segment between opposite rays to pass through the origin, contradicting $R > 0$. Any path that reaches δ_0, whether or not it changes sign, has length at least $R_0 + R_1$ by the triangle inequality. Since action is at least squared length, every admissible path costs at least $(R_0 + R_1)^2$. This bound is attained as a law path: move radially from the first endpoint to δ_0 along its fixed-κ_0 ray, then radially from δ_0 to the second endpoint along its fixed-κ_1 ray, choosing the meeting time $R_0/(R_0 + R_1)$. Both legs then have metric speed $R_0 + R_1$. This proves the action formula in (20) also at and beyond a half-turn. The sorted-atom isometry gives its stated endpoint distance.

For opposite signs, the mean of an attainable law has the sign of w; continuity of the mean in W_2 forces every admissible path through δ_0. The same radial construction attains the lower bound $(R_0 + R_1)^2$. Its endpoint sorted vectors are $R_+ v_{\kappa_+}$ and $-R_-\,\mathrm{rev}(v_{\kappa_-})$, giving (21) by the sorted-atom isometry. Both direction vectors in c_{rev} have positive coordinates. The first is nonconstant and nondecreasing, whereas the reversed second is nonconstant and nonincreasing; they cannot coincide when $d_Z \geq 2$. Both have unit norm, so Cauchy–Schwarz gives $0 < c_{\mathrm{rev}} < 1$. If one endpoint is δ_0, its distance to the other is the latter's radius, and a single radial segment attains that distance. If both are δ_0, the constant law has zero action.

It remains to distinguish law-path attainment from prompt-path attainment. Equality in the apex lower bound requires constant metric speed, no positive-time stay at δ_0, and radial motion on each

side: equality in the Euclidean triangle inequality between each sorted endpoint vector and zero forces its leg to follow that ray. For $d_Z \geq 2$, the inverse chart (15) makes κ unique at every nonzero point on a branch. A continuous prompt lift must consequently keep $\kappa = \kappa_0$ up to the meeting time and $\kappa = \kappa_1$ afterward; this is possible exactly when those parameters agree. In that case, keep κ fixed and interpolate w linearly to obtain a smooth constant-speed lift. Same-sign endpoints with $\Delta \geq \pi$ necessarily have distinct parameters. With exactly one zero endpoint, equality likewise forces the nonzero ray's parameter immediately after or before that endpoint; matching parameters also suffice by linear interpolation of w. When both endpoints are zero, set $w \equiv 0$ and vary κ smoothly between them. For $R_0 + R_1 > 0$, to see that the prompt-path *infimum* still equals the law-path optimum in every apex case, reserve an interval of width ε at $w = 0$, vary κ smoothly there, flat at both ends. Give the radial legs durations $T_b = (1-\varepsilon)R_b/(R_0 + R_1)$, omitting a leg when $R_b = 0$. Choose a smooth increasing $\phi_\varepsilon : [0,1] \to [0,1]$ that is flat at its endpoints and satisfies $\int_0^1 |\phi_\varepsilon'|^2 \, dt \leq 1 + \varepsilon$; normalizing a smooth approximation to the constant speed 1 gives such a function. Use $R_0(1 - \phi_\varepsilon)$ on the first leg and $R_1\phi_\varepsilon$ on the last. The action is at most $(1+\varepsilon)(R_0 + R_1)^2/(1-\varepsilon)$ and tends to the optimum. On leg b, set $\kappa = \kappa_b$ and $w = \operatorname{sgn}(w_b)R/\|u_{\kappa_b}\|_N$; during the zero interval, set $w = 0$ and change κ smoothly. The lift (13) is smooth and has the specified endpoint aggregates. Finally, for $d_Z = 1$ all Z_i coincide, so $\mu_{\kappa,w}$ is the Dirac mass at $w/(Z_1 + \kappa)$; linear motion of its location attains D^2.

The long-arc regime occurs. Take six groups indexed by $j = 0, \ldots, 5$, with normalizer $Z_j = M^j$ and multiplicity $n_j = M^j$ for an integer $M > 1$. In orthonormal group coordinates, u_κ has components proportional to $\sqrt{n_j}/(Z_j + \kappa)$. At $\widehat{\kappa}_j = M^{j+1/2}$, $j = 0, \ldots, 4$, the normalized direction tends as $M \to \infty$ to $(e_j + e_{j+1})/\sqrt{2}$. Consecutive limits have spherical distance $\pi/3$. Hence the developed arc from $\widehat{\kappa}_0$ to $\widehat{\kappa}_4$ has length at least $4\pi/3 + o(1) > \pi$ for sufficiently large finite M.

A.4 Why three distinct normalizers change the answer

Proof of Corollary 4.4. First assume $d_Z \geq 3$ and $\kappa_0 \neq \kappa_1$. At any fixed $\kappa > 0$, the vectors $u_\kappa, u_\kappa', u_\kappa''$ are linearly independent. Indeed, if coefficients $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ satisfy $\lambda_0 u_\kappa + \lambda_1 u_\kappa' + \lambda_2 u_\kappa'' = 0$, then multiplying coordinate i by $(Z_i + \kappa)^3$ gives $\lambda_0(Z_i+\kappa)^2 - \lambda_1(Z_i+\kappa) + 2\lambda_2 = 0$. A quadratic polynomial with at least three distinct roots is identically zero, so all three coefficients vanish. Multiplication of a curve by a positive smooth scalar preserves the span of its first three derivative vectors: from $u_\kappa = \|u_\kappa\|_N v_\kappa$, the relation between (u, u', u'') and (v, v', v'') is triangular with nonzero diagonal. Hence $v_\kappa, v_\kappa', v_\kappa''$ are also independent.

If $\Delta \geq \pi$, positivity of the direction coordinates gives $\alpha < \pi/2$, and Proposition A.1 yields $\mathcal{G} = 2R_0R_1(1 + \cos\alpha) > 0$. It remains to treat $\Delta < \pi$. We prove strictness of the spherical length inequality by examining its equality case. Parametrize the direction curve between the endpoints by arc length $s \in [0, \Delta]$, writing it as $a(s)$ with $a(0) = v_{\kappa_0}$ and $\|a'(s)\|_N = 1$. Put $f(s) = \arccos\langle a(0), a(s)\rangle_N$. The positive coordinates give $f(s) < \pi/2$; injectivity of the direction curve gives $f(s) > 0$ for $s > 0$. Since $a'(s)$ is orthogonal to $a(s)$,

$$|f'(s)| = \frac{|\langle a(0) - \cos(f(s))a(s), a'(s)\rangle_N|}{\sin(f(s))} \leq 1.$$

Integrating from a positive lower limit and then letting that limit tend to zero proves $\alpha = f(\Delta) \leq \Delta$. Equality would require $f'(s) = 1$ throughout $(0, \Delta)$, and equality in Cauchy–Schwarz would give

$$a'(s) = \frac{\cos s\, a(s) - a(0)}{\sin s}.$$

It follows by differentiation that $(a(s) - \cos s\, a(0))/\sin s$ is a constant vector. Thus the entire direction curve would lie in a fixed two-dimensional linear plane, contradicting the independence just proved. Therefore $\alpha < \Delta$. Strict decrease of cosine on $[0, \pi]$ now gives $\mathcal{G}_0 > 0$.

If $d_Z = 2$, write the two normalizers as $\zeta_1 < \zeta_2$, with multiplicities n_1, n_2. The group-indicator vectors normalized to have unit $\|\cdot\|_N$ norm form an orthonormal basis of their two-dimensional subspace. In this basis, u_κ has coordinates

$$\left(\frac{\sqrt{n_1/N}}{\zeta_1 + \kappa}, \frac{\sqrt{n_2/N}}{\zeta_2 + \kappa} \right).$$

Its polar angle is

$$\psi(\kappa) = \arctan\left(\sqrt{\frac{n_2}{n_1}}\frac{\zeta_1+\kappa}{\zeta_2+\kappa}\right).$$

The ratio inside the arctangent has strictly positive derivative. Thus the unit direction traces a monotone great-circle arc in the positive quadrant, and $\Delta = |\psi(\kappa_1) - \psi(\kappa_0)| = \alpha < \pi/2$. Theorem 4.3 gives $\mathcal{G}_0 = 0$. If $d_Z = 1$, every distribution in the family is a Dirac mass and every Dirac mass is attainable; linearly interpolating its location attains action D_0^2. This directly solves the degenerate case. If the endpoint masses κ_0, κ_1 coincide, the two endpoints lie on the same ray and the gap is likewise zero, regardless of d_Z.

A.5 A REGULAR, ATTAINABLE STOCHASTIC INTERPOLANT

Proposition A.2 (Attainment and an empirical stochastic interpolant) *Under Theorem 4.3, (10) is a smooth positive aggregate path with the prescribed endpoints and action $\mathcal{E}_*(0)$, and (6) is a smooth token realization. For the uniform sample index I, the process $X_t = z_I(\kappa_t, w_t)$ admits a bounded velocity field B_t that is globally Lipschitz in space, uniformly in t. Its law $\mu_t = \mu_{\kappa_t, w_t}$ satisfies*

$$\frac{d}{dt}\int \varphi(x)\, d\mu_t(x) = \int \varphi'(x) B_t(x)\, d\mu_t(x), \qquad \varphi \in C_c^1(\mathbb{R}), \tag{22}$$

Here $C_c^1(\mathbb{R})$ denotes compactly supported C^1 functions; this is the weak continuity equation.

Proof of Proposition A.2. Use the smooth optimal parameters in (19). Let I be uniform on $\{1, \dots, N\}$, and set $X_t = z_I(\kappa(t), w(t))$. Then $\mathrm{Law}(X_t) = \mu_t$ and X_t is a smooth path for every value of I. Samples sharing the same Z_i have identical locations and identical velocities. Samples with different Z_i never collide, since $w(t) > 0$. There are finitely many distinct groups, and their locations are continuous on a compact time interval; their pairwise separations therefore have a strictly positive minimum. Their velocities are continuous and uniformly bounded.

Assign the value $\dot{z}_i(t)$ to $B_t(z_i(t))$. Coincident atoms have the same velocity, so this assignment is well-defined. Extend B_t affinely between successive distinct locations and constantly outside the two extreme locations. If there is only one distinct location, take the constant extension everywhere. The ratios of adjacent velocity differences to adjacent spatial gaps are uniformly bounded over all adjacent pairs and times. Their supremum is a uniform spatial Lipschitz constant. This gives a bounded, continuous field $(t, x) \mapsto B_t(x)$, globally Lipschitz in x uniformly over $t \in [0, 1]$, satisfying

$$\dot{X}_t = B_t(X_t), \qquad B_t(X_t) = \mathbb{E}[\dot{X}_t \mid X_t].$$

For every compactly supported C^1 test function φ, direct differentiation of the finite sum gives

$$\frac{d}{dt}\int \varphi(x)\, d\mu_t(x) = \frac{1}{N}\sum_i \varphi'(z_i(t))\dot{z}_i(t) = \int \varphi'(x) B_t(x)\, d\mu_t(x).$$

This is the weak continuity equation. The same identity for compactly supported C^1 space–time test functions follows by the product rule and integration in time. Finally, the fixed ordering and the sorted isometry imply

$$|\dot{\mu}_t|_{W_2}^2 = \frac{1}{N}\sum_i |\dot{z}_i(t)|^2 = \mathbb{E}|\dot{X}_t|^2 = \int |B_t(x)|^2\, d\mu_t(x).$$

Integrating identifies its kinetic action with the exact minimum from Theorem 4.3. The sample index supplies the randomness, and the optimal path explicitly determines the velocity field.

B STABILITY UNDER NONZERO CONTENT NUMERATORS

Starting from the exact cone formula at $A = 0$, we prove that the strict action obstruction persists for sufficiently small scalar content numerators $A = (A_i)_{i=1}^N$ in every direction. The argument works with distributions and allows the perturbed atoms to change order. Throughout this section, the fixed endpoint parameters satisfy $\kappa_0 \neq \kappa_1$ and $w_0, w_1 > 0$, and $d_Z \geq 3$. The bound also covers endpoint pairs outside $\Delta < \pi$. Recall the sorted midpoint $h_A = (Q\mu_0^A + Q\mu_1^A)/2$ and the unperturbed separation $\rho = \mathrm{dist}_N\big(h_0, Q(\mathcal{M}_0)\big)$.

Proposition B.1 (Strict midpoint separation) *The unperturbed midpoint satisfies $\rho > 0$, with separation given by the one-dimensional compact minimization*

$$\rho^2 = \min_{\kappa\in[0,\infty]} \left\{ \|h_0\|_N^2 - \langle h_0, v_\kappa\rangle_N^2 \right\}. \tag{23}$$

Here v_κ is extended continuously to the compactified parameter interval: at $\kappa = 0$ it is the normalization of $(Z_i^{-1})_{i=1}^N$, and at $\kappa = \infty$ it is $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$.

Proof. We first characterize the closed positive cone, then show that the endpoint midpoint lies outside it. The fixed order $Z_1 \geq \cdots \geq Z_N$ makes u_κ increasingly sorted. Since both endpoint numerators are positive, the sorted midpoint has coordinates

$$(h_0)_i = \frac{1}{2}\left(\frac{w_0}{Z_i + \kappa_0} + \frac{w_1}{Z_i + \kappa_1}\right) > 0. \tag{24}$$

First consider the positive part of the unperturbed sorted family. Writing a member as Rv_κ, with $R > 0$, shows that its closure is

$$\{Rv_\kappa : R \geq 0,\ \kappa \in [0,\infty]\}. \tag{25}$$

Indeed, v_κ extends continuously to the stated endpoints, so its image is compact. If a sequence Rv_κ converges in the normalized Hilbert norm, then R, which is exactly its norm, converges as well. After taking a subsequence, its directions converge in this compact image, establishing closedness of the displayed set. Conversely, each of its boundary rays is a limit of rays with $\kappa > 0$; scaling with $w = R/\|u_\kappa\|_N$ realizes every such approximation.

We next show that h_0 lies outside this closed positive cone. Suppose first that, for some finite $\kappa \geq 0$ and $w > 0$, $(h_0)_i = w/(Z_i + \kappa)$ for all i. The polynomial

$$\begin{aligned} P(s) = {} & [w_0(s + \kappa_1) + w_1(s + \kappa_0)]\,(s + \kappa) \\ & - 2w(s + \kappa_0)(s + \kappa_1) \end{aligned}$$

has degree at most two and vanishes at all distinct values of Z_i. Since $d_Z \geq 3$, it is identically zero. Evaluating it at $s = -\kappa_0$ yields $w_0(\kappa_1 - \kappa_0)(\kappa - \kappa_0) = 0$, hence $\kappa = \kappa_0$. Evaluating it at $s = -\kappa_1$ similarly forces $\kappa = \kappa_1$, a contradiction. The remaining boundary ray, at $\kappa = \infty$, contains only constant vectors. The function

$$s \longmapsto \frac{1}{2}\left(\frac{w_0}{s + \kappa_0} + \frac{w_1}{s + \kappa_1}\right)$$

is strictly decreasing for $s > 0$, so h_0 is not constant. It is also nonzero. Thus h_0 is outside the closed set in (25) and has strictly positive distance from it.

For each direction, $\langle h_0, v_\kappa\rangle_N > 0$ and $\|v_\kappa\|_N = 1$. Orthogonal projection onto its closed positive ray therefore gives

$$\min_{R\geq 0} \|h_0 - Rv_\kappa\|_N^2 = \|h_0\|_N^2 - \langle h_0, v_\kappa\rangle_N^2.$$

Every sorted vector generated by $w \leq 0$ has nonpositive coordinates, so its distance from h_0 is at least $\|h_0\|_N$. Positive-ray projection has strictly smaller distance. The infimum over the full family therefore equals the infimum over positive rays. Compact minimization proves both (23) and $\rho > 0$. ■

Proof of Theorem 4.5. The proof controls the displacement of the attainable family and then turns the remaining midpoint separation into a lower bound on action. Pairing atoms with the same index provides a valid transport plan, regardless of their sorted order. Consequently, uniformly for all $\kappa > 0$ and $w \in \mathbb{R}$,

$$\begin{aligned} W_2^2(\mu_{\kappa,w}^A, \mu_{\kappa,w}^0) &\leq \frac{1}{N}\sum_{i=1}^N \left|\frac{A_i}{Z_i + \kappa}\right|^2 \\ &\leq \eta^2. \end{aligned} \tag{26}$$

The sorted Wasserstein isometry and the triangle inequality then give $\|h_A - h_0\|_N \le \eta$. For an arbitrary candidate $Q\mu^A_{\kappa,w} \in Q(\mathcal{M}_A)$, the same estimates imply

$$\begin{aligned}\|h_A - Q\mu^A_{\kappa,w}\|_N &\ge \|h_0 - Q\mu^0_{\kappa,w}\|_N - \|h_A - h_0\|_N \\ &\quad - \|Q\mu^A_{\kappa,w} - Q\mu^0_{\kappa,w}\|_N \\ &\ge \rho - 2\eta.\end{aligned}$$

Taking the infimum over candidates shows

$$\operatorname{dist}_N\big(h_A, Q(\mathcal{M}_A)\big) \ge [\rho - 2\eta]_+. \tag{27}$$

We convert this midpoint separation into a lower bound on every admissible path's action. Suppose first that $D_A > 0$, and consider an arbitrary admissible path of finite action. Its sorted image $\omega_t = Q\mu_t$ is an absolutely continuous Hilbert-space path, with the same metric speed. The continuous function

$$t \longmapsto \langle \omega_t - h_A, Q\mu^A_1 - Q\mu^A_0 \rangle_N$$

is $-D_A^2/2$ at $t = 0$ and $D_A^2/2$ at $t = 1$. It vanishes at some $t_\circ \in (0,1)$. Thus $\omega_{t_\circ} - h_A$ is perpendicular to the endpoint chord and, by (27), has norm at least $[\rho - 2\eta]_+$. Splitting the path at $t_\circ$, using the triangle inequality for each piece, and applying Pythagoras gives

$$\begin{aligned}\int_0^1 \|\dot\omega_t\|_N \, dt &\ge \|\omega_{t_\circ} - Q\mu^A_0\|_N + \|Q\mu^A_1 - \omega_{t_\circ}\|_N \\ &= 2\sqrt{D_A^2/4 + \|\omega_{t_\circ} - h_A\|_N^2} \\ &\ge \sqrt{D_A^2 + 4[\rho - 2\eta]_+^2}.\end{aligned}$$

Cauchy–Schwarz on the unit time interval yields

$$\int_0^1 \|\dot\omega_t\|_N^2 \, dt \ge \left(\int_0^1 \|\dot\omega_t\|_N \, dt\right)^2,$$

which proves the asserted lower bound for every finite-action path. Infinite-action paths satisfy it automatically, so taking the infimum establishes (11).

Finally, if $D_A = 0$, then h_A equals the common sorted endpoint and belongs to $Q(\mathcal{M}_A)$. Equation (27) forces $[\rho - 2\eta]_+ = 0$. The claimed bound is therefore simply $\mathcal{E}_*(A) \ge 0$, which also holds. ∎

The weighted numerator norm η quantifies the stability neighborhood in Theorem 4.5 for perturbations of the scalar content numerators. The compact formula (23) reduces the separation constant to one-dimensional optimization. A certified lower estimate then bounds ρ rigorously.

C COMPUTATIONAL ILLUSTRATIONS OF THE THEOREMS

The figures in this section evaluate the scalar attention model directly. The phase diagram and path costs use the exact action formulas; the spectrum follows from an explicit Jacobian; and the stability example combines an exactly solvable perturbation with a rational certificate.

C.1 PHASE DIAGRAM AND ROUTE–TIMING DECOMPOSITION

Figure 7 separates the structural gap from the costs of a chosen route and clock. Panels (a,b) use $N = 24$ equally weighted samples. Two distinct normalizers, $Z_i \in \{16, 1\}$, each occur twelve times; three normalizers, $Z_i \in \{16, 4, 1\}$, each occur eight times. We fix $R_0 = 1$ and $R_1 = 3/2$ and vary both endpoint parameters on a logarithmic grid. The corresponding numerator is $w_b = R_b/\|u_{\kappa_b}\|_N$. Unequal radii keep $D^2 \geq 1/4$, so the relative gap remains defined throughout the grid.

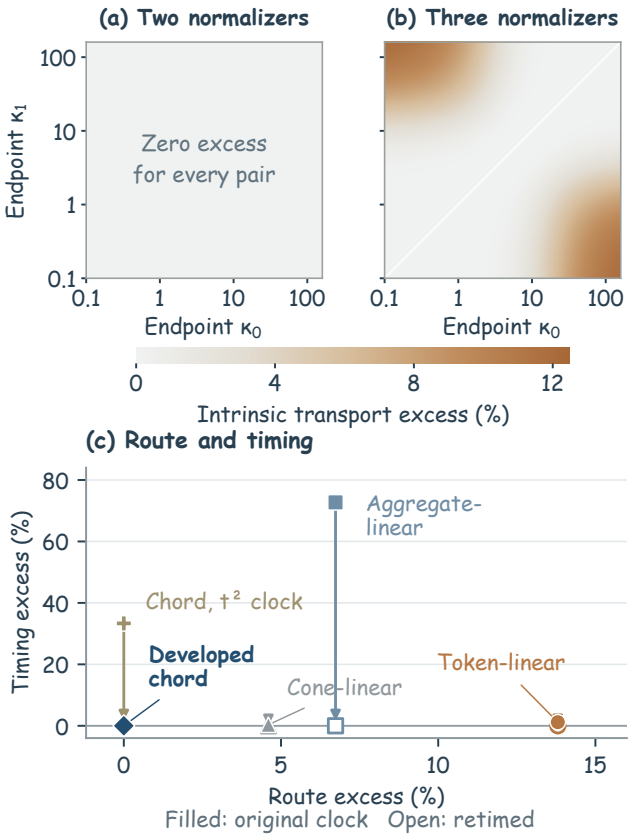

Figure 7: **When the gap appears, and where paths waste action.** (a,b) Exact intrinsic excess $100\mathcal{G}/D^2$ across endpoint pairs. (c) Route and timing excess, relative to the attainable optimum. Arrows retime each route to constant speed.

The diagonal consists of pure rescalings and has zero gap in both panels. Away from it, Corollary 4.4 gives the two-versus-three contrast: two normalizers permit an ambient chord, whereas three force a detour. The largest displayed excess is 12.02%. The two-normalizer panel uses its exact theoretical value zero; direct floating-point evaluation agrees to numerical precision.

Panel (c) fixes three normalizers and the endpoint pairs $(\kappa_0, R_0) = (1, 1)$ and $(\kappa_1, R_1) = (16, 2)$. Token-linear interpolates the two coordinates in (6); aggregate-linear interpolates (κ, w); and cone-linear interpolates (R, θ). The developed chord is evaluated with both a constant-speed clock and a quadratic clock. Each arrow preserves the feature route while removing speed variation. Its vertical displacement measures a cost that retiming can eliminate; its horizontal coordinate measures a cost that retiming must leave unchanged. Thus a low timing cost can coexist with a long route, and an optimal route can still use an inefficient clock. The developed chord with its constant-speed clock simultaneously removes both avoidable costs.

Phase computation and angular validity. Both endpoint parameters use a 181-point grid from 0.1 to 160. We evaluate the gap in the stable form

$$\mathcal{G} = 4R_0R_1 \sin\left(\frac{\Delta+\alpha}{2}\right)\sin\left(\frac{\Delta-\alpha}{2}\right), \qquad \alpha = 2\arcsin\left(\frac{\|v_{\kappa_1} - v_{\kappa_0}\|_N}{2}\right).$$

Angular integrals use Gauss–Legendre quadrature in $\log\kappa$. The angular assumption also has an analytic check. The squared speed $\|v'_\kappa\|_N^2$ is the variance of $(Z_i+\kappa)^{-1}$ under weights $u_{\kappa,i}^2/\sum_j u_{\kappa,j}^2$.

Variance is no larger than the mean squared deviation from the midpoint of the support, hence

$$\|v'_\kappa\|_N \le \frac{1}{2}\left(\frac{1}{1+\kappa} - \frac{1}{16+\kappa}\right).$$

Integrating over all $\kappa > 0$ bounds the entire angular span by $\frac{1}{2}\log 16 = \log 4 < \pi$. The largest displayed Δ is approximately 0.760. Using 32 versus 64 nodes on every logarithmic interval changes the heatmap by less than 3×10^{-14} percentage points.

The route–timing identity. For an absolutely continuous path with finite action, put $v(t) = |\dot{\mu}_t|_{W_2}$ and $\overline{v} = \int_0^1 v(t)\,dt = \mathrm{Len}[\mu_\cdot]$. Expanding the square gives the exact identity

$$\mathcal{E} - \mathrm{Len}^2 = \int_0^1 (v(t) - \overline{v})^2\,dt.$$

Thus timing excess is the variance of speed on the unit time interval. Constant-speed reparameterization leaves the route and length unchanged and attains action Len^2. It sends each plotted point vertically to the horizontal axis. Route excess remains nonnegative because the retimed path is admissible and $\mathrm{Len}^2 \ge \mathcal{E}_*(0)$. For a developed chord with the quadratic clock, speed is $2t\sqrt{\mathcal{E}_*(0)}$, so its action is $4\mathcal{E}_*(0)/3$ and its timing excess is exactly $\mathcal{E}_*(0)/3$. This example changes timing alone. The other actions and lengths use 256-node quadrature; doubling the order changes every reported quantity by less than 1.5×10^{-14}.

C.2 THE EXACT PROMPT-COUNT SPECTRUM

At the equal-token lift with $A = 0$, define the sample-weighted output Jacobian $\Xi_m = N^{-1/2}\partial z/\partial(p_1^{(1)}, p_1^{(2)}, \ldots, p_m^{(1)}, p_m^{(2)})$. Its parameter-direction spectrum consists of the square roots of all $2m$ eigenvalues of $\Xi_m^\top \Xi_m$, including zeros. Let $\sigma_1 \geq \sigma_2 > 0$ denote the two positive singular values of Ξ_1 when $d_Z \geq 2$ and $w \neq 0$.

Figure 8 shows that more prompt coordinates can preserve the same two active output directions. Fix $w \neq 0$ and $d_Z \geq 2$ in the head of Proposition 4.1. The canonical lift repeats $p_j = (\log(\kappa/m), w/\kappa)$, so the weighted output Jacobian Ξ_m has m identical two-column blocks.

The upper panel counts two active directions and $2m - 2$ null directions. Null perturbations redistribute token coordinates while leaving both first-order aggregate changes zero. The full parameter Gram matrix retains these zeros even when $2m > N$.

The lower panel isolates coordinate scaling: the two raw positive singular values decrease as $m^{-1/2}$. Multiplication by $\sqrt{m}$ removes this replication factor. The corrected spectrum is identical across counts at the specified lift, without a numerical rank threshold.

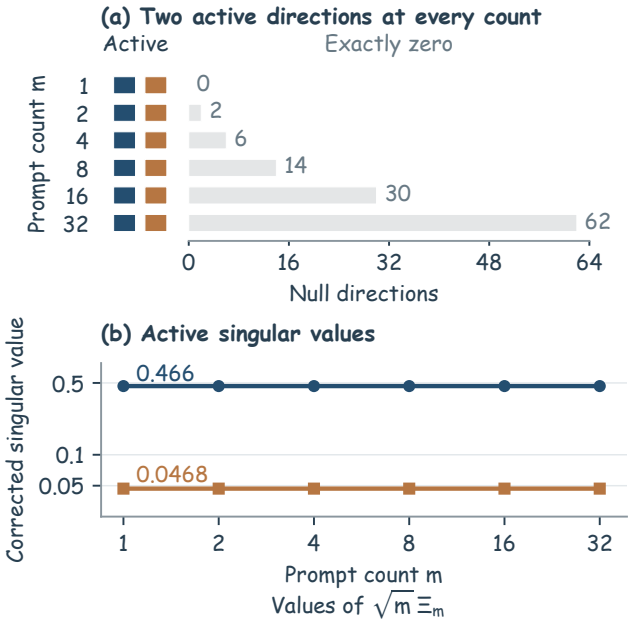

Figure 8: **More coordinates, the same two active directions.** At the equal-token lift, increasing m adds $2m - 2$ exact null directions. The two positive singular values coincide across counts after the replication correction $\sqrt{m}$.

We use $N = 24$, normalizers $(16, 4, 1)$ repeated eight times, $(\kappa, w) = (1, 2)$, and $m \in \{1, 2, 4, 8, 16, 32\}$. The positive singular values of Ξ_1 are approximately 0.4660274164 and 0.04681858105; the factorization below fixes all zeros exactly. Central finite differences on 48 random token realizations check the output derivative, with maximum relative error 8.9×10^{-11}.

Exact factorization and nullspace. Differentiating the aggregates and then the outputs gives

$$\frac{\partial z_i}{\partial p_j^{(1)}} = e^{p_j^{(1)}} \left(\frac{p_j^{(2)}}{Z_i + \kappa} - \frac{w}{(Z_i + \kappa)^2} \right), \qquad \frac{\partial z_i}{\partial p_j^{(2)}} = \frac{e^{p_j^{(1)}}}{Z_i + \kappa}.$$

At the canonical lift these are $wZ_i/[m(Z_i + \kappa)^2]$ and $\kappa/[m(Z_i + \kappa)]$, respectively. Consequently,

$$\Xi_m = \frac{1}{m} \underbrace{[\Xi_1 \ \cdots \ \Xi_1]}_{m \text{ copies}}, \qquad \Xi_m \Xi_m^\top = \frac{1}{m} \Xi_1 \Xi_1^\top.$$

The ratio of the first to second entry in row i of Ξ_1 is $(w/\kappa)Z_i/(Z_i + \kappa)$. This ratio takes distinct values at distinct normalizers, since $\kappa > 0$ and $w \neq 0$. The two columns are independent. Hence Ξ_m has rank two and exactly $2m - 2$ null parameter directions. The displayed Gram identity proves that the square roots of the $2m$ eigenvalues of $\Xi_m^\top \Xi_m$ are

$$\left(\frac{\sigma_1}{\sqrt{m}}, \frac{\sigma_2}{\sqrt{m}}, \underbrace{0, \ldots, 0}_{2m-2} \right).$$

In particular, the positive spectrum of $\sqrt{m}\,\Xi_m$ is independent of m. For a stacked token perturbation δp, the block formula gives $\Xi_m \delta p = m^{-1}\Xi_1(\sum_j \delta p_j^{(1)}, \sum_j \delta p_j^{(2)})^\top$. Since Ξ_1 has independent columns, this vanishes if and only if both coordinate sums vanish, establishing the stated nullspace. If $w = 0$ or $d_Z = 1$, the rank is instead one.

C.3 AN EXACTLY SOLVABLE PERTURBATION AND A RATIONAL CERTIFICATE

In this subsection, $\epsilon \geq 0$ controls the perturbation direction $A_i = \epsilon Z_i$, and ρ_{lb} denotes the certified lower estimate of ρ constructed below.

Figure 9 compares an exactly solvable perturbation with a uniform stability guarantee. It uses three equally weighted samples, $Z = (16, 4, 1)$, and endpoint aggregates $(\kappa_0, w_0) = (1/4, 2)$ and $(\kappa_1, w_1) = (16, 24)$. We perturb the content numerators along $A_i = \epsilon Z_i$, so the perturbation size is $\eta = \epsilon$ for $\epsilon \geq 0$.

This choice translates the entire attainable family after a change of aggregate numerator. Both sign branches and every admissible path are included. Translation preserves Wasserstein distances and action, so the core optimum gives the perturbed optimum exactly. The blue curve is therefore computed from a closed expression, with no path optimization error. It follows one specified perturbation direction.

The orange curve has a different role: it is a lower bound that holds for every content perturbation of the indicated norm. We obtain it from the midpoint-separation theorem and certify its constant with rational arithmetic. The shaded interval marks where this bound is strictly positive. This directly identifies a neighborhood in which the transport obstruction must persist.

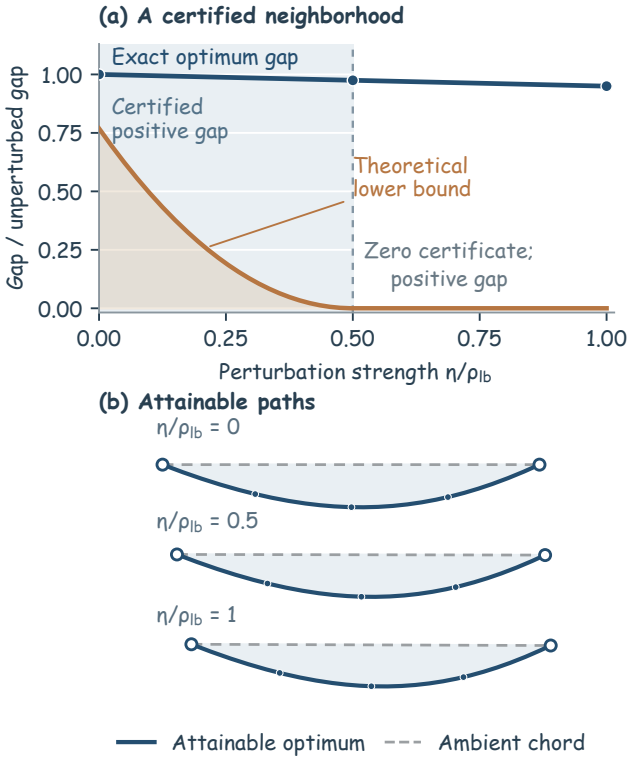

Figure 9: **A certified neighborhood of positive gap.** The exact gap along $A_i = \epsilon Z_i$ and a lower bound valid in every perturbation direction are normalized by the unperturbed gap. Shading marks the certified positive region. The path panels share one orthographic projection.

The lower bound begins at 76.9% of the unperturbed gap. At the right edge of the displayed interval, the exact gap along the chosen direction still retains 95.0% of its initial value. Thus a sufficient certificate can reach zero while the actual obstruction remains. The three path panels show how the attaining path and ambient chord change along the same perturbation, using a common projection and scale. All distances and actions are computed before projection, in the full three-dimensional output space.

The exact perturbation slice. The identity

$$z_i^A(\kappa, w) = \epsilon + \frac{w - \epsilon\kappa}{Z_i + \kappa}$$

maps the entire attainable family to a translation of the core. Indeed, $(\kappa, w) \mapsto (\kappa, w - \epsilon\kappa)$ is bijective on $(0, \infty) \times \mathbb{R}$, and translation is a Wasserstein isometry. It induces a bijection of admissible paths, including both sign branches, with endpoint numerators $w_b - \epsilon\kappa_b$ in the core. For the displayed interval $0 \leq \epsilon \leq \rho_{\mathrm{lb}}$, both transformed numerators are positive. Moreover, $\|v'_\kappa\|_N \leq \|u'_\kappa\|_N / \|u_\kappa\|_N \leq (1 + \kappa)^{-1}$ implies $\Delta \leq \log(68/5) < 3 < \pi$. Theorem 4.3 therefore yields the exact polynomial identity

$$\frac{\mathcal{G}_A}{\mathcal{G}} = \prod_{b=0}^{1} \left(1 - \frac{\epsilon\kappa_b}{w_b}\right) = (1 - \epsilon/8)(1 - 2\epsilon/3).$$

The blue curve follows this identity; the certificate is uniform at fixed η.

A rational separation certificate. To certify the midpoint separation, first compute $h_0 = (227/520, 71/85, 128/85)$. Define the following local auxiliary polynomials, using the same coordinate index $i = 1, 2, 3$:

$$d_i(\kappa) = \prod_{j \neq i} (\kappa + Z_j), \quad s(\kappa) = \sum_i (h_0)_i d_i(\kappa), \quad q_*(\kappa) = \sum_i d_i(\kappa)^2.$$

By Proposition B.1, ρ^2 is the minimum, including $\kappa = 0, \infty$, of $\|h_0\|_N^2 - s(\kappa)^2/[3q_*(\kappa)]$. Both polynomials are positive on $[0, \infty)$, and differentiation gives

$$-\frac{s(\kappa)[2s'(\kappa)q_*(\kappa) - s(\kappa)q_*'(\kappa)]}{3q_*(\kappa)^2}.$$

The bracket is a positive multiple of

$$9120064 - 316660\kappa - 310284\kappa^2 - 56312\kappa^3 - 7477\kappa^4.$$

This polynomial strictly decreases from a positive value to $-\infty$; it has exactly one positive root. The residual decreases before that root and increases after it, proving that the root is the global minimizer. A compact rational certificate locates this root between $\kappa_- = 3.6122954471836679$ and $\kappa_+ = 3.6122954471836680$: the displayed polynomial is positive at κ_- and negative at κ_+, as verified by integer arithmetic. To bound the residual, define

$$d_- = \sum_i \frac{(h_0)_i}{Z_i + \kappa_+}, \qquad d_+ = \sum_i \frac{(h_0)_i}{Z_i + \kappa_-},$$
$$t_- = \sum_i \frac{1}{(Z_i + \kappa_+)^2}, \quad t_+ = \sum_i \frac{1}{(Z_i + \kappa_-)^2}.$$

Bounding each positive sum separately gives the exact rational bounds

$$0.00407020925416 < \|h_0\|_N^2 - \frac{d_+^2}{3t_-} \leq \rho^2$$
$$\leq \|h_0\|_N^2 - \frac{d_-^2}{3t_+} < 0.00407020925417.$$

The squares of 0.063798191621 and 0.063798191622 lie below and above these bounds, respectively. Hence $0.063798191621 < \rho < 0.063798191622$. All decimal endpoints here denote exact rationals; the accompanying `compact_certificate.py` verifies each inequality without floating-point arithmetic. We use the lower endpoint, $\rho_{\mathrm{lb}} = 0.063798191621$. Theorem 4.5 then gives $\mathcal{G}_A \geq 4[\rho_{\mathrm{lb}} - 2\eta]_+^2$. The certificate begins at 76.9% of the unperturbed gap and remains strictly positive until $\eta = \rho_{\mathrm{lb}}/2$. At $\eta = \rho_{\mathrm{lb}}$, the exact gap in the displayed direction still retains 95.0% of its initial value. Thus a sufficient certificate can vanish while the obstruction persists.

All rational coefficients and enclosures are saved in `certificate.json`. Absolute gap values use angular quadrature; 64 versus 256 nodes changes them by less than 2.9×10^{-15}. Paths use bisection to invert the angle; doubling the quadrature from 128 to 256 nodes changes features by less than 3.7×10^{-14}. The insets use one fixed orthographic projection after scaling the three outputs by $1/\sqrt{3}$. Action and distance are computed in the full output space. The refinement checks concern curve evaluation; the separation certificate uses exact rational arithmetic.

D EXPERIMENTAL DESIGN AND REPORTING PROTOCOL

The experiments first test the exact construction, then identify what creates the transport gap, and finally examine paths in a frozen ViT. The protocol fixes the comparisons, numerical checks, and reporting rules. A dash in a table denotes a quantity that does not apply.

D.1 MATCHED PATHS AND EVALUATION IN THE EXACT CORE

Cases and controls. The main comparison uses $N = 24$, $m = 4$, and $d_Z \in \{3, 8\}$, with distinct normalizers geometrically spaced from 1 to 16 and equal multiplicities. Cross four endpoint pairs, $(\kappa_0, \kappa_1) \in \{(0.25, 1), (0.25, 4), (1, 16), (0.99, 1.01)\}$, with $R_0 = 1$ and $R_1 \in \{0.5, 1, 2\}$. This gives 24 prespecified cases, including close endpoints. Set w_b to $R_b/\|u_{\kappa_b}\|_N$ and verify $\Delta < \pi$ so the developed chord is a smooth attaining prompt baseline; Proposition A.1 also gives the optimum outside this range. All methods share endpoints and the explicit frozen head from Proposition 4.1, with canonical endpoint tokens $p_j^b = (\log(\kappa_b/m), w_b/\kappa_b)$.

Path definitions. Token-linear interpolates the canonical endpoint tokens. Aggregate-linear interpolates (κ, w), and cone-coordinate-linear interpolates (R, θ) before applying θ^{-1} and the prompt lift. An arclength clock reparameterizes the same curve by accumulated distributional length, using the generalized inverse at stationary intervals. The cone-linear row abbreviates cone-coordinate-linear. Quadratic Bézier and one-bend polygonal paths use the curve families of Garipov et al. (2018a) in $(\log \kappa, w)$ coordinates. For the quadratic curve, combine the first endpoint, a trainable two-dimensional control, and the second endpoint with coefficients $(1 - t)^2$, $2t(1 - t)$, and t^2. For the polygonal curve, traverse the first endpoint to the control on $[0, 1/2]$ and the control to the second endpoint on $[1/2, 1]$, linearly on each interval. Exponentiate the first coordinate and apply the canonical prompt lift, ensuring feasibility at every time. These two rows adapt the published parameterizations to action minimization. The higher-capacity numerical control uses a continuous piecewise-linear path in $(\log \kappa, w)$ with 16 equal time intervals and fixed endpoints.

Bound all optimized log-coordinates between $\log \min_b \kappa_b - \log 4$ and $\log \max_b \kappa_b + \log 4$, and impose $|w| \le 4 \max_b |w_b|$. This search allows negative values and zero crossings. All three optimized families use seven starts and 2,000 objective/gradient evaluations per bounded quasi-Newton run. For the single-control curves, start from the coordinate midpoint, the aggregate coordinates of the token-linear midpoint, and five uniformly sampled controls in the rectangle. The 16-interval family starts from token-linear, aggregate-linear, and five random interior paths. Use random seeds 0–4 in each family. Optimize with 32-panel, eight-point Gauss–Legendre quadrature, splitting panels at path knots; select the lowest independently evaluated action and record all failures. The developed chord uses Theorem 4.3's construction.

Metrics and accounting. In Tables 1 and 3, $\mathcal{E}$ and Len mean the action and length of the row's path. Compute normalized costs per case, then report their mean and standard deviation across the common case set. The two relative-action columns each use the 12 cases at their stated d_Z; other columns pool all 24. Use the common resolved-case intersection for comparative summaries and retain method-specific failures in the valid/total counts. In the ambient row, average the displayed analytic ratio within each normalizer group. End err. is the maximum endpoint W_2 residual divided by $\max(R_0, R_1)$; report its maximum across cases. Build time includes path construction, required quadrature and inversion, and materialization at 513 common time points, excluding the common independent evaluation step. Report median and interquartile range on the same device and precision after warm-up, a valid/total count for each method.

Continuous action and the timing control. Evaluate the analytic output derivative along each continuous parameter path by independent adaptive quadrature, splitting at parameter knots. For scalar laws, sort atoms independently at every Wasserstein comparison, preserving repeated atoms and their weights. Almost-everywhere sorting only permutes derivatives, so the squared metric speed is the mean of the squared atom derivatives, including for nonzero A. Squared length never exceeds action: Cauchy–Schwarz gives $(\int_0^1 |\dot{\mu}_t|_{W_2}\, dt)^2 \le \mathcal{E}$. Constant speed gives equality, proving the route/timing decomposition used in Table 3. For the developed chord, $t \mapsto t^2$ changes its constant speed by the factor $2t$, so its action is $(4/3)\mathcal{E}_*(0)$ while its length is unchanged. For the

Table 3: **Route choice and timing contribute distinct costs.** Five paired controls keep each geometric curve fixed and change only its clock. The normalized costs obey Total $= 1+$Route$+$Timing; the constant-speed developed chord predicts zero cost in both components.

Geometric curve	Clock	Normalized action and its components ↓		
		Total $\boldsymbol{\mathcal{E}/\mathcal{E}_*(0)}$	Route $\frac{\mathrm{Len}^2-\mathcal{E}_*(0)}{\mathcal{E}_*(0)}$	Timing $\frac{\mathcal{E}-\mathrm{Len}^2}{\mathcal{E}_*(0)}$
Token-linear	Original	5.7161	3.7107	1.0055
	Arclength	4.7107	3.7107	0.0000
Aggregate-linear	Original	1.4131	0.0160	0.3972
	Arclength	1.0160	0.0160	0.0000
Cone-coordinate-linear	Original	1.0150	0.0143	0.0007
	Arclength	1.0143	0.0143	0.0000
Quadratic Bézier (action)	Original	1.0075	0.0060	0.0015
	Arclength	1.0060	0.0060	0.0000
Developed chord	Original	1.0000	0.0000	0.0000
	$t \mapsto t^2$	1.3333	0.0000	0.3333

Bézier pair, hold the optimized control fixed when retiming. Numerical clock inversion is evaluated independently, so its measured timing residual remains visible rather than being replaced by zero.

Numerical accuracy. The sum of adjacent squared Wasserstein distances divided by their time increments is a lower approximation to continuous action. Establish between-knot feasibility from the continuous parameterization; certify action upper bounds only with validated integration. Cross-check derivative quadrature against nested grids of 32, 64, 128, 256, and 512 intervals. Independently vary path intervals in $\{4, 8, 16, 32\}$ and quadrature/inverse tolerances in $\{10^{-6}, 10^{-8}, 10^{-10}\}$, starting at 10^{-8} absolute and relative tolerance. Use multiprecision for unresolved small gaps and compute the predicted gap as $4R_0R_1 \sin((\Delta+\alpha)/2)\sin((\Delta-\alpha)/2)$ to reduce cancellation. Flag negative regret beyond estimated numerical error, endpoint mismatch, and unresolved refinement trends as failures. Keep raw values distinct from certified bounds.

D.2 Structural ablations and boundary cases

Normalizer diversity creates the obstruction; prompt count and content numerators test how it persists. Table 4 isolates these factors in three panels. Panel A holds $N = 24$ and normalizer multiplicities fixed within each setting, with the same range $[1, 16]$ for $d_Z \geq 2$; for $d_Z = 1$, use $Z_i = 1$. Fix $(\kappa_0, \kappa_1) = (1, 4)$ and $(R_0, R_1) = (1, 2)$, except in the stated equal-direction control. The latter sets both $\kappa_b = 1$. Normalize gaps by $R_0^2 + R_1^2$, which stays positive even for coincident endpoint laws. The measured gap is the best independently evaluated feasible action minus D^2; it includes optimization and evaluation error. Formula error is the absolute difference between the independently evaluated developed-chord action and $\mathcal{E}_*(0)$, with the same normalization. For $d_Z = 1$ or equal endpoint directions, use the radial path and compare against D^2.

Panel B uses the developed-chord aggregate path at $d_Z = 3$ and changes $m \in \{1, 2, 4, 16, 64\}$. Rebuild each prompt as $(\log(\kappa/m), w/\kappa)$ at every time to preserve the aggregates. Compare laws on the common refined grid and compare continuous actions. This isolates prompt-count invariance for unconstrained embeddings in the explicit head.

Panel C uses $d_Z = 3$ and the Panel A endpoints. Fix the endpoint aggregates and scale four content-numerator directions to the prescribed η: constant-sign, alternating-sign, seeded random, and a candidate designed to test atom-order crossings. Record the direction vectors and observed crossings at each scale, then recompute the perturbed endpoint laws and D_A. Compare $4[\rho - 2\eta]_+^2$ with the smallest evaluated feasible action minus D_A^2. A grid minimum over κ overestimates the infimum defining ρ. To certify the lower bound, use validated one-dimensional global minimization, including limiting directions at $\kappa = 0$ and $\kappa = \infty$. Otherwise label the result an uncertified numerical evaluation of the theoretical expression. For $2\eta/\rho \geq 1$, the bound no longer certifies a positive gap.

Table 4: **Testing the source and persistence of the transport gap.** A: vary normalizer diversity. B: vary prompt count at fixed aggregates; the shaded row is the canonical reference. C: vary the content numerator. Predictions state the theory; measurements test each factor independently.

A. Distinct normalizers ($N = 24$; fixed endpoint radii)			
		Normalized diagnostics	
Setting	Prediction for $\mathcal{G}$	**Gap**[†]	Formula error[†]
$d_Z = 1$	0	0	1.00×10^{-9}
$d_Z = 2$	0	0	1.00×10^{-9}
$d_Z = 3$	> 0	1.23×10^{-4}	1.00×10^{-9}
$d_Z = 8$	> 0	2.24×10^{-4}	1.00×10^{-9}
$d_Z = 8$, $\kappa_0 = \kappa_1$	0	0	1.00×10^{-9}
B. Prompt count ($d_Z = 3$; same aggregate path)			
Tokens m	Prediction	Law discrepancy[‡]	Action discrepancy[‡]
1	Reference lift	–	–
2	Same law and action	1.00×10^{-12}	1.00×10^{-12}
4	Same law and action	1.00×10^{-12}	1.00×10^{-12}
16	Same law and action	1.00×10^{-12}	1.00×10^{-12}
64	Same law and action	1.00×10^{-12}	1.00×10^{-12}
C. Nonzero content numerator (four perturbation directions)			
$2\eta/\rho$	Prediction	Lower expr.[§]	Feasible excess[§]
0	Positive gap	8.81×10^{-5}	1.23×10^{-4}
0.25	Positive gap	4.96×10^{-5}	1.21×10^{-4}
0.50	Positive gap	2.20×10^{-5}	1.18×10^{-4}
0.75	Positive gap	5.51×10^{-6}	1.16×10^{-4}
1.00	Bound inconclusive	0	1.14×10^{-4}
1.50	Bound inconclusive	0	1.12×10^{-4}

[†]Gap and absolute formula error are divided by $R_0^2 + R_1^2$. [‡]Maximum sampled law discrepancy from $m = 1$, divided by $\max_b R_b$; absolute action discrepancy divided by $\mathcal{E}_*(0)$. [§]Both excess quantities are divided by $R_0^2 + R_1^2$. A numerical certificate requires validated evaluation of ρ (see Appendix D.2).

Additional stress tests merge the third distinct normalizer toward the second, vary the normalizer range and endpoint radii, and separate sample count from diversity by repeating every atom equally. Architecture tests introduce input-dependent queries, additional heads, LayerNorm, and post-attention mixing one at a time with matched inputs and endpoints. Report path action and numerical residuals for each architecture. Separate pre-attention normalization from normalization of the measured readout, and concatenated heads from independently summed heads: the constructions in Appendix F predict different outcomes.

D.3 Downstream context and exploratory frozen-ViT paths

The ViT study tests whether path choice also affects feature transport in a full transformer. It pairs endpoint-quality baselines with a separate comparison of paths between fixed prompts. These measurements extend the geometric question beyond the scalar core, where the cone formula is exact.

Adaptation baselines. Table 2 compares 15 endpoint trainers. Linear probing and full fine-tuning anchor the comparison. Parameter-efficient updates include bias-only ViT tuning following BitFit (Zaken et al., 2022), LoRA (Hu et al., 2021), AdaptFormer (Chen et al., 2022b), scaling and shifting features (SSF) (Lian et al., 2022), tensor-train factor tuning (FacT-TT) (Jie & Deng, 2023), RepAdapter (Luo et al., 2023), and the hybrid search method NOAH (Zhang et al., 2024). Prompt baselines comprise VPT-Shallow and VPT-Deep (Jia et al., 2022), E^2VPT (Han et al., 2023), cross visual prompt tuning (CVPT) (Huang et al., 2025), visual adaptive prompt tuning (VAPT) (Le et al., 2026), and prompt-agnostic evolution (PAE) applied to VPT-Deep (Wang et al., 2026). This endpoint comparison complements analyses of when VPT or full tuning is preferable (Han et al., 2024); Table 5 evaluates paths connecting fixed endpoints.

Table 5: **Comparing transport and prediction quality along frozen-ViT paths.** Each block fixes the endpoints, backbone, classifier, and evaluation images. Mode-connectivity (MC) controls adapt Garipov et al. (2018a) to prompts and optimize cross-entropy (CE); the matched Bézier control changes only the objective to feature action. All paths are evaluated using the same transport, accuracy, loss, and cost metrics.

Path	Transport	Prediction quality			Implementation	
	Action ↓	Min. acc. ↑	Mean acc. ↑	Loss barrier ↓	End err. ↓	Build s ↓
VPT-Shallow endpoints (Jia et al., 2022)	(same pair for every row)					
Token-linear	4.8	65.5	69.0	0.18	10^{-7}	0.2
Token-linear + arclength	4.1	65.5	69.4	0.18	10^{-7}	20
MC Bézier (CE)	4.5	68.8	70.0	0.02	10^{-7}	2400
MC polygonal (CE)	4.4	68.5	69.8	0.03	10^{-7}	2200
Bézier (feature action)	3.2	67.4	69.5	0.06	10^{-7}	4200
8-interval feature-action path	3.0	67.8	69.6	0.05	10^{-7}	5600
VPT-Deep endpoints (Jia et al., 2022)	(same pair for every row)					
Token-linear	7.0	67.0	72.0	0.24	10^{-7}	0.4
Token-linear + arclength	5.8	67.0	72.6	0.24	10^{-7}	30
MC Bézier (CE)	6.3	71.8	73.5	0.02	10^{-7}	3000
MC polygonal (CE)	6.1	71.5	73.3	0.03	10^{-7}	2800
Bézier (feature action)	4.2	70.5	73.0	0.07	10^{-7}	5200
8-interval feature-action path	3.9	70.8	73.1	0.06	10^{-7}	6800

Accuracy is top-1 (%). Loss barrier is the maximum nonnegative CE excess above the linear interpolation of endpoint losses. Feature-action rows optimize same-image motion; reported action and endpoint error use empirical W_2.

Endpoint comparisons use matched backbones, preprocessing, data splits, and optimization budgets. Select hyperparameters using validation data with equal search budgets, including prompt counts, LoRA rank and target projections, adapter width, and method-specific settings. Count all trained parameters, including the classifier and any prompt generator or evolution operator. Average accuracy within each group and across all 19 tasks; the overall score weights groups by $7/19$, $4/19$, and $8/19$. Report means and standard deviations across seeds, with per-task values in Tables 6–8. BitFit updates all backbone biases and the classifier; FacT uses the TT variant, and RepAdapter uses its full configuration with supported merging for deployment. Count NOAH supernet training/search, E^2VPT pruning and rewinding, and PAE evolution in total adaptation time; log these stages separately from final training. Report the selected model's mean trainable parameter count over tasks and the maximum final-training GPU memory over tasks, including optimizer state. Record search-stage peak memory separately. Measure inference latency with batch size 128 on the same GPU and precision, after 50 warm-up batches and over 200 timed batches; report median milliseconds per image over task/seed runs, including the classifier and supported weight merging.

Fixed-endpoint path comparison. Table 5 uses two separately trained prompt settings on the same task and backbone, in separate VPT-Shallow and VPT-Deep blocks. Fit one linear probe on training images, then freeze it for both endpoint-prompt training runs and all intermediate evaluations. Train these endpoints specifically for the fixed-classifier protocol, so only prompts vary along the path. Keep prompt shape, insertion positions, readout layer, and feature normalization fixed within a block. Use final CLS features after backbone LayerNorm, normalized to unit Euclidean norm for both the probe and transport measurements. Use disjoint calibration and evaluation images. On each of the 19 VTAB tasks, pair five endpoint seeds cyclically as $(0,1),(1,2),(2,3),(3,4),(4,0)$. Average within each task, then across tasks; report pair variation separately. Choose paths with calibration data only; report on the common evaluation set.

Matched curve families and objectives. The mode-connectivity (MC) rows adapt Garipov et al. (2018a) to prompt coordinates, keeping the backbone and classifier fixed. Use one prompt-shaped control tensor in the same quadratic Bézier or one-bend polygonal formulas defined above. These MC rows minimize time-mean cross-entropy (CE) on 128 labeled calibration images drawn from

the training split. Use the 32 uniform-time midpoint nodes for this objective. The Bézier feature-action row uses the same curve, initial controls, optimizer, and calibration images, but minimizes discrete matched feature action over 32 uniform time intervals by pairing each image with itself at consecutive times. This objective is differentiable without solving an assignment problem. The higher-capacity feature-action row optimizes interior prompt tensors of a continuous piecewise-linear path with eight intervals and exact endpoints.

Give every optimized row three starts and 2,000 gradient evaluations per start, using the same Adam implementation (Kingma & Ba, 2015) and learning-rate search over $\{10^{-3}, 10^{-2}, 10^{-1}\}$. The single-control starts are the endpoint midpoint and two perturbations with random seeds 0 and 1. Use Gaussian noise with standard deviation 0.05 times the root-mean-square endpoint difference within each insertion layer; a zero difference gives zero noise. Initialize the multi-interval path linearly and perturb its interior tensors at the same layerwise scale. Select CE candidates by independently evaluated calibration time-mean CE on 128 midpoint nodes, and action candidates by calibration action at 128 intervals. Count starts, rate search, and calibration passes in build time.

Construct token-linear arclength retiming on the calibration features using 128 intervals and apply the resulting monotone clock unchanged to evaluation images. The held-out curve need not have constant speed. All path choices use calibration data only.

Use equally weighted empirical laws of 128 held-out image features and solve the squared-Euclidean assignment problem (Gabriel & Marco, 2019) for each adjacent time pair. Report the common readout and preprocessing, solver tolerances, and refinement at 32, 64, and 128 time intervals. Record assignment ties and solver failures. This estimator remains a lower approximation to continuous action. Compute minimum sampled accuracy and trapezoidal time-mean accuracy on the same grid and classifier; check both under refinement. For the loss barrier, take at each time the CE minus the linear interpolation of the two endpoint CEs, clip below at zero, and report the maximum. Evaluate all CEs on the same held-out images; this metric remains meaningful even when endpoint losses differ. Report raw endpoint W_2 residuals (maximum over the two endpoints), build-time accounting as in Appendix D.1, and valid/total counts. Label any sliced-Wasserstein, fixed-image, or entropy-regularized alternative separately from exact empirical W_2.

Table 6: **Per-task endpoint accuracy on natural VTAB-1k tasks.** Cells report mean top-1 accuracy (%); the final column averages this task group.

	Natural (7)							
Endpoint trainer	CIFAR 100	Caltech 101	DTD	Flowers 102	Pets	SVHN	SUN 397	**Mean**
Linear probe	66.0	78.6	56.8	87.2	77.1	72.1	45.2	69.0
Full fine-tuning	71.6	86.4	63.7	95.0	84.7	80.0	52.1	76.2
BitFit	71.1	82.9	61.1	92.4	80.7	77.8	49.2	73.6
LoRA	76.0	89.6	68.2	96.3	88.1	82.1	55.4	79.4
AdaptFormer	75.8	90.6	68.6	98.6	87.9	82.8	55.1	79.9
SSF	74.3	89.1	66.1	96.3	87.6	81.5	55.6	78.6
FacT-TT	76.8	90.2	67.3	96.6	87.6	81.9	55.9	79.5
RepAdapter	76.0	89.2	66.5	95.9	86.6	82.8	53.4	78.6
NOAH	77.5	89.0	67.1	96.8	87.6	84.1	56.3	79.8
VPT-Shallow	75.0	86.6	66.7	97.3	84.7	82.0	51.7	77.7
VPT-Deep	76.2	88.0	66.3	96.2	86.7	83.2	53.4	78.6
E^2VPT	75.5	89.8	67.8	96.9	87.3	84.5	55.6	79.6
CVPT	77.2	90.2	67.4	97.6	89.6	82.4	55.9	80.0
VAPT	77.1	90.2	69.0	98.6	89.9	83.6	56.3	80.7
VPT-Deep + PAE	76.3	91.7	67.7	97.5	87.5	82.5	55.2	79.8

Table 7: **Per-task endpoint accuracy on specialized VTAB-1k tasks.** Cells report mean top-1 accuracy (%); the final column averages this task group.

	Specialized (4)				
Endpoint trainer	Patch Camelyon	EuroSAT	RESISC45	Retinopathy	**Mean**
Linear probe	79.6	89.7	82.1	71.1	80.6
Full fine-tuning	84.2	94.7	86.0	74.9	85.0
BitFit	82.8	92.6	81.9	73.2	82.6
LoRA	85.7	94.0	85.1	76.1	85.2
AdaptFormer	84.1	94.4	85.0	78.1	85.4
SSF	84.2	96.5	87.1	76.2	86.0
FacT-TT	86.5	94.5	87.4	76.7	86.3
RepAdapter	85.9	96.3	86.1	76.5	86.2
NOAH	86.7	96.3	86.9	77.4	86.8
VPT-Shallow	82.9	92.2	82.6	75.3	83.3
VPT-Deep	84.5	94.9	84.8	78.4	85.7
E^2VPT	86.6	95.8	85.6	76.7	86.2
CVPT	85.6	95.7	86.7	76.8	86.2
VAPT	87.2	95.2	86.3	76.9	86.4
VPT-Deep + PAE	84.6	96.8	86.2	77.8	86.4

Table 8: **Per-task endpoint accuracy on structured VTAB-1k tasks.** Cells report mean top-1 accuracy (%); the final column averages this task group.

	Structured (8)								
Endpoint trainer	CLEVR count	CLEVR distance	DMLab	KITTI distance	dSprites location	dSprites orient.	sNORB azimuth	sNORB elev.	**Mean**
Linear probe	52.9	37.3	22.8	48.2	52.2	26.7	8.0	17.8	33.2
Full fine-tuning	74.5	56.2	45.3	68.8	71.0	45.9	27.1	38.5	53.4
BitFit	66.9	49.9	36.3	59.6	62.6	40.3	20.2	29.8	45.7
LoRA	80.5	61.9	47.8	73.3	77.0	52.0	30.7	44.2	58.4
AdaptFormer	80.1	63.8	48.7	72.5	77.0	53.2	31.9	43.5	58.8
SSF	79.0	62.1	49.1	71.8	76.1	52.8	33.3	44.1	58.5
FacT-TT	82.2	63.9	51.4	74.4	77.0	51.7	33.1	43.7	59.7
RepAdapter	80.7	63.7	48.1	74.8	78.5	52.2	33.3	43.2	59.3
NOAH	81.8	63.5	49.5	74.2	78.9	52.4	32.2	45.4	59.7
VPT-Shallow	69.8	53.0	40.0	64.9	67.9	41.8	24.1	34.4	49.5
VPT-Deep	77.8	59.7	46.8	70.6	76.6	51.0	32.0	41.7	57.0
E^2VPT	79.7	62.2	49.5	74.9	77.8	53.8	34.2	44.5	59.6
CVPT	82.4	62.2	50.2	74.6	78.0	54.3	34.9	44.2	60.1
VAPT	83.4	63.9	52.2	73.8	79.3	53.7	34.3	44.9	60.7
VPT-Deep + PAE	81.1	64.5	51.1	75.9	77.7	53.9	34.7	45.1	60.5

E Additional Qualitative Layouts

Four additional 2×6 groups extend the layouts in Figures 5 and 6. Each column shares a crop and scale across CE Bézier (top) and Feature-action Bézier (bottom).



Figure 10: **Additional saliency: transport and everyday objects.**



Figure 11: **Additional saliency: flower structure and texture.**

E.1 FURTHER ATTENTION CASES

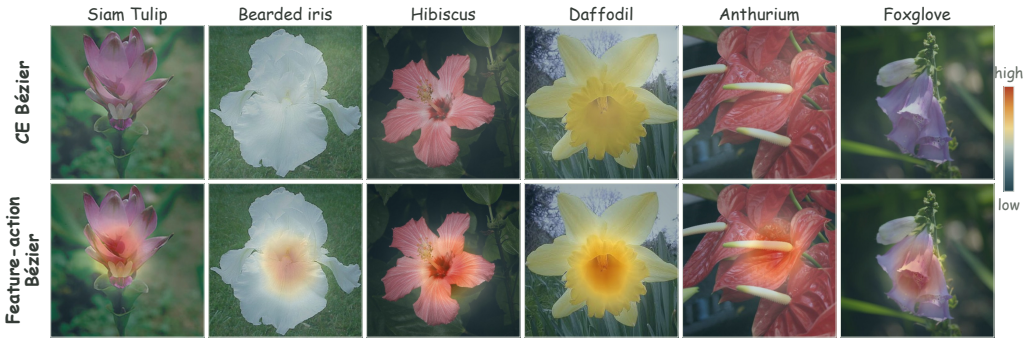

Figure 12: **Additional attention: six further flower categories.**



Figure 13: **Additional attention: pet poses and backgrounds.**

F FEATURE WIDTH, LAYERNORM, AND HEAD MIXING

These extensions separate three architectural operations: adding output coordinates, normalizing tokens before attention, and mixing independently controlled attention responses. All variational endpoints remain output laws, as in (3). For the explicit heads below, fix $\beta \neq 0$ with $e^{\beta} < \min_i Z_i$.

F.1 VECTOR OUTPUTS PRESERVE THE SCALAR OPTIMUM

Let $d_{\mathrm{out}} \geq 1$ be the output dimension and $\boldsymbol{w} \in \mathbb{R}^{d_{\mathrm{out}}}$ a variable shared numerator. Fix a unit output direction $\boldsymbol{a}$. The attainable laws and family are

$$\nu_{\kappa,\boldsymbol{w}} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\boldsymbol{w}/(Z_i+\kappa)}, \qquad \mathcal{M}^{\mathrm{vec}} = \{\nu_{\kappa,\boldsymbol{w}} : \kappa > 0,\ \boldsymbol{w} \in \mathbb{R}^{d_{\mathrm{out}}}\}.$$

Euclidean distance defines vector-valued W_2 and its AC^2 action. Write $\mathcal{E}_*^{\mathrm{vec}}$ for the constrained action infimum within $\mathcal{M}^{\mathrm{vec}}$ between the laws $\nu_{\kappa_b, w_b\boldsymbol{a}}$. Here $T_{\#}\nu$ is the pushforward of ν by T.

Proposition F.1 (Vector outputs preserve the optimum) *For every $d_{\mathrm{out}} \geq 1$ and unit $\boldsymbol{a}$, $\mathcal{E}_*^{\mathrm{vec}} = \mathcal{E}_*(0)$ and the vector endpoint distance equals D. Thus Theorem 4.3 and its strict-gap threshold remain exact even when paths can leave the fixed-direction slice $\boldsymbol{w} = w\boldsymbol{a}$.*

Proof of Proposition F.1. Let $T_{\boldsymbol{a}}(x) = \boldsymbol{a}^{\top}x$ and $\iota_{\boldsymbol{a}}(r) = r\boldsymbol{a}$, where $\|\boldsymbol{a}\|_{\mathrm{E}} = 1$. For any coupling γ between vector laws $\nu, \widetilde{\nu}$, $(T_{\boldsymbol{a}}, T_{\boldsymbol{a}})_{\#}\gamma$ couples their projected laws. Since $|T_{\boldsymbol{a}}(x) - T_{\boldsymbol{a}}(y)| \leq \|x - y\|_{\mathrm{E}}$, taking the infimum over couplings gives

$$W_2\big((T_{\boldsymbol{a}})_{\#}\nu, (T_{\boldsymbol{a}})_{\#}\widetilde{\nu}\big) \leq W_2(\nu, \widetilde{\nu}). \tag{28}$$

Moreover, $(T_{\boldsymbol{a}})_{\#}\nu_{\kappa,\boldsymbol{w}} = \mu_{\kappa,\boldsymbol{a}^{\top}\boldsymbol{w}} \in \mathcal{M}$. Every admissible vector path therefore projects to a scalar path with the prescribed endpoints. The contraction preserves the AC^2 inequality. At almost every time where both metric derivatives exist, applying (28) to difference quotients bounds the projected speed by the vector speed. Integrating their squares and taking the infimum gives $\mathcal{E}_*^{\mathrm{vec}} \geq \mathcal{E}_*(0)$. This permits atom matching, rotation of $\boldsymbol{w}$, zero outputs, and sign changes of its projection.

Conversely, lifting any scalar coupling by $(\iota_{\boldsymbol{a}}, \iota_{\boldsymbol{a}})$ preserves its cost. Together with (28), this proves

$$W_2\big((\iota_{\boldsymbol{a}})_{\#}\mu, (\iota_{\boldsymbol{a}})_{\#}\widetilde{\mu}\big) = W_2(\mu, \widetilde{\mu}). \tag{29}$$

Since $(\iota_{\boldsymbol{a}})_{\#}\mu_{\kappa,w} = \nu_{\kappa,w\boldsymbol{a}}$, every admissible scalar path lifts to an admissible vector path with identical speed and action. Taking infima proves the reverse inequality, without requiring scalar attainment. The same isometry proves that the endpoint distance is D. Under Theorem 4.3, lifting its attained optimum therefore preserves both the exact minimum and the excess action. ∎

A frozen vector head realizes the full family. Use token dimension $1 + d_{\mathrm{out}}$ and write a token as $p = (p^{(0)}, \boldsymbol{p})$. Set $K(p) = p^{(0)}$, $H(p) = p^{(0)}/\beta$, and $V(p) = \boldsymbol{p}$. The fixed readout and second content tokens are $(\beta, \mathbf{0})$ and $(\log(Z_i - e^{\beta}), \mathbf{0})$, respectively. Their readout query is one, their content partition is Z_i, and their vector content numerator is zero. For every $m \geq 1$, the shared tokens

$$p_j(\kappa, \boldsymbol{w}) = \Big(\log(\kappa/m), \frac{\boldsymbol{w}}{\kappa}\Big), \qquad j = 1, \ldots, m,$$

give aggregate mass κ and vector numerator $\boldsymbol{w}$. Conversely, arbitrary prompts produce a positive aggregate mass and a shared vector numerator, so the attainable family is exactly $\mathcal{M}^{\mathrm{vec}}$. The displayed lift is smooth for $\kappa > 0$, and the head remains fixed throughout each path.

Extrinsic bending and intrinsic flatness. The scalar positive branch has parametrization $F(\kappa, w) = wu_{\kappa}$ and tangent plane $\mathcal{T}_{\kappa,w} = \mathrm{span}\{u_{\kappa}, u_{\kappa}'\}$. For $d_Z \geq 3$, the independence in Appendix A.4 yields the nonzero normal component

$$\mathrm{Proj}_{\mathcal{T}_{\kappa,w}^{\perp}} \partial_{\kappa}^2 F = w\,\mathrm{Proj}_{\mathcal{T}_{\kappa,w}^{\perp}} u_{\kappa}'' \neq 0 \qquad (w > 0).$$

Here Proj denotes orthogonal projection in the normalized sample inner product. This nonzero normal component of a second derivative proves local extrinsic bending. In contrast, $\partial_w^2 F = 0$ and $\partial_w \partial_{\kappa} F = u_{\kappa}'$ is tangent. The cone metric $dR^2 + R^2 d\theta^2$ is intrinsically flat away from the apex. The fixed-direction vector slice is an isometric copy of this bent cone; the full vector family has additional directions, whose availability does not lower the action by Proposition F.1.

F.2 Exact realization with pre-attention LayerNorm

Let d_{tok} be the token dimension. For $x \in \mathbb{R}^{d_{\mathrm{tok}}}$ and $\varepsilon_{\mathrm{LN}} \geq 0$, identity-affine token LayerNorm (Ba et al., 2016) is

$$\mathrm{LN}_{\varepsilon_{\mathrm{LN}}}(x) = \frac{x - \bar{x}\mathbf{1}}{\sqrt{d_{\mathrm{tok}}^{-1}\|x - \bar{x}\mathbf{1}\|_{\mathrm{E}}^2 + \varepsilon_{\mathrm{LN}}}}, \qquad \bar{x} = d_{\mathrm{tok}}^{-1}\mathbf{1}^{\top}x.$$

Here $\mathbf{1}$ is the all-ones vector; when $\varepsilon_{\mathrm{LN}} = 0$, raw tokens must be nonconstant. Pre-attention LayerNorm applies this map before the frozen query, key, and value maps.

Proposition F.2 (Exact realization with pre-attention LayerNorm) *Fix m, (Z_i), A, and compact $\mathcal{K} \subset (0,\infty) \times \mathbb{R}$. For every $d_{\mathrm{tok}} \geq 4$ and $\varepsilon_{\mathrm{LN}} \geq 0$, one frozen pre-attention LayerNorm head has attainable family contained in $\mathcal{M}_A$ and smoothly realizes every aggregate pair in $\mathcal{K}$. For $A = 0$, choosing $\mathcal{K}$ to contain the optimal path in Theorem 4.3 preserves its exact minimum and gap.*

Proof of Proposition F.2. Fix m, $\mathcal{K}$, and $\varepsilon_{\mathrm{LN}}$ as in the proposition. Since $d_{\mathrm{tok}} \geq 4$, the centered subspace $\mathbf{1}^{\perp}$ contains three orthonormal vectors b_1, b_2, b_3. The desired key–value pairs lie in

$$\begin{aligned}\mathcal{B} = {} & \{(\log(\kappa/m), w/\kappa) : (\kappa, w) \in \mathcal{K}\} \cup \{(\beta, 0)\} \cup \\ & \left\{\left(\log(Z_i - e^{\beta}), \frac{A_i}{Z_i - e^{\beta}}\right) : 1 \leq i \leq N\right\}.\end{aligned} \tag{30}$$

Compactness follows because κ has a positive minimum on $\mathcal{K}$ and $Z_i - e^{\beta} > 0$ for every i. If $\varepsilon_{\mathrm{LN}} = 0$, set $r = \sqrt{d_{\mathrm{tok}}}$; otherwise choose $0 < r < \sqrt{d_{\mathrm{tok}}}$. Take $\Lambda > 0$ large enough that $k^2 + \xi^2 < \Lambda^2 r^2$ on a neighborhood of $\mathcal{B}$, where k, ξ are local scalar key and value coordinates. Define

$$Y(k,\xi) = \frac{k}{\Lambda}b_1 + \frac{\xi}{\Lambda}b_2 + \sqrt{r^2 - \frac{k^2+\xi^2}{\Lambda^2}}\, b_3. \tag{31}$$

This map is smooth on that neighborhood, is centered, and has constant Euclidean norm r. Its raw-token lift is

$$\Psi(k,\xi) = \begin{cases} Y(k,\xi), & \varepsilon_{\mathrm{LN}} = 0, \\ \sqrt{\dfrac{\varepsilon_{\mathrm{LN}}}{1 - r^2/d_{\mathrm{tok}}}}\, Y(k,\xi), & \varepsilon_{\mathrm{LN}} > 0. \end{cases} \tag{32}$$

For zero epsilon, Ψ has variance one, so its normalized value is Y. For positive epsilon, let a_{LN} denote the positive prefactor in the second line. The identity $a_{\mathrm{LN}}^2 r^2/d_{\mathrm{tok}} + \varepsilon_{\mathrm{LN}} = a_{\mathrm{LN}}^2$ yields

$$\mathrm{LN}_{\varepsilon_{\mathrm{LN}}}(\Psi(k,\xi)) = Y(k,\xi) \tag{33}$$

in both cases.

On normalized tokens y, choose the frozen maps $K(y) = \Lambda\langle b_1, y\rangle$, $V(y) = \Lambda\langle b_2, y\rangle$, and $H(y) = K(y)/\beta$. Equations (31)–(33) give key k and value ξ to raw token $\Psi(k,\xi)$. Choose the fixed raw readout and second content tokens

$$\Psi(\beta, 0), \qquad \Psi\left(\log(Z_i - e^{\beta}), \frac{A_i}{Z_i - e^{\beta}}\right).$$

Their normalized readout query is one. Their two key weights sum to $e^{\beta} + (Z_i - e^{\beta}) = Z_i$, and their weighted values sum to A_i. The shared raw prompts

$$p_j(\kappa, w) = \Psi(\log(\kappa/m), w/\kappa), \qquad j = 1, \ldots, m, \tag{34}$$

have total key weight κ and weighted value w. Thus they produce $(A_i + w)/(Z_i + \kappa)$, smoothly on a neighborhood of $\mathcal{K}$. Conversely, arbitrary valid raw prompts yield finite normalized keys and values, and hence some $\kappa > 0, w \in \mathbb{R}$. The fixed content aggregates are unchanged, so every resulting law belongs to $\mathcal{M}_A$.

For the action statement, set $A = 0$ and take the optimal aggregate path from Theorem 4.3. Its smooth image is compact in $\{\kappa > 0, w > 0\}$; choose $\mathcal{K}$ to contain it. Every admissible path of the constructed pre-LN head lies in $\mathcal{M}$, so its action is at least $\mathcal{E}_*(0)$. The optimal scalar path has the smooth lift (34) and attains that bound. Its endpoint laws and distance are unchanged, proving exact equality of the minimum and gap. Thus the three-normalizer gap survives LayerNorm. ∎

Attainable aggregates and the compact scope. For an arbitrary frozen pre-LN common-query head, the aggregate formula (2) still holds: normalization only changes which keys and values shared tokens can produce. In the construction, the frozen maps may depend on m and $\mathcal{K}$, but are fixed throughout the path. This scope is necessary. Normalized token norms are at most $\sqrt{d_{\mathrm{tok}}}$, so every fixed linear key map is bounded on them. If its scalar logits have absolute value at most C, then $me^{-C} \le \kappa \le me^{C}$. A fixed normalized head therefore cannot realize all positive aggregate masses. Any finite collection of prompt counts can share one construction by taking the union of their compact key–value sets in (30).

LayerNorm of the measured output can erase the obstruction. Let $d_{\mathrm{out}} \ge 2$, let $b \in \mathbb{R}^{d_{\mathrm{out}}}$ be a fixed nonconstant vector, and consider the output features $b + z_i(\kappa, w)\mathbf{1}$. Applying LayerNorm with the same definition in output dimension gives, for every $\varepsilon_{\mathrm{LN}} \ge 0$,

$$\mathrm{LN}_{\varepsilon_{\mathrm{LN}}}(b + z_i(\kappa, w)\mathbf{1}) = \mathrm{LN}_{\varepsilon_{\mathrm{LN}}}(b).$$

Indeed, subtracting the coordinate mean removes the added scalar multiple of $\mathbf{1}$, leaving the variance unchanged. Before normalization, this is an injective affine embedding of the scalar response, with distances multiplied by $\sqrt{d_{\mathrm{out}}}$. Afterwards the entire family is constant and its gap is zero. The example has nonzero variance even at zero epsilon. It distinguishes preservation by the pre-attention realization from a universal claim about normalized output features.

F.3 Concatenated heads and output mixing

For $h \in \{1, 2\}$, superscript (h) indexes head h's independently controlled prompt aggregates $\kappa^{(h)} > 0$ and $w^{(h)} \in \mathbb{R}$; y_i denotes the sum of their outputs.

Proposition F.3 (Exact action for aligned concatenated heads) *Consider $n_{\mathrm{h}} \ge 1$ independently controlled scalar heads with zero content numerators and positive normalizers $Z_i^{(h)}$, $h = 1, \dots, n_{\mathrm{h}}$, each with at least two distinct values. Suppose every head orders the samples in the same way, so $Z_1^{(h)} \ge \cdots \ge Z_N^{(h)}$ for all h. Measure their unmixed concatenated output law*

$$\nu_{\boldsymbol{\kappa},\boldsymbol{w}} = \frac{1}{N}\sum_{i=1}^{N} \delta_{(z_i^{(1)},\dots,z_i^{(n_{\mathrm{h}})})}, \qquad z_i^{(h)} = \frac{w^{(h)}}{Z_i^{(h)} + \kappa^{(h)}}.$$

For each head, fix endpoint aggregates with $w_0^{(h)}, w_1^{(h)} > 0$, $\kappa_0^{(h)} \ne \kappa_1^{(h)}$, and developed angle $\Delta_h < \pi$ as in Theorem 4.3. Write $\mathcal{E}_{*,h}$, D_h, and $\mathcal{G}_h = \mathcal{E}_{*,h} - D_h^2$ for its scalar quantities. For the concatenated family, with independent head controls,

$$\mathcal{E}_*^{\mathrm{cat}} = \sum_{h=1}^{n_{\mathrm{h}}} \mathcal{E}_{*,h}, \qquad D_{\mathrm{cat}}^2 = \sum_{h=1}^{n_{\mathrm{h}}} D_h^2, \qquad \mathcal{G}_{\mathrm{cat}} = \sum_{h=1}^{n_{\mathrm{h}}} \mathcal{G}_h. \tag{35}$$

The same identities hold after an isometric embedding of the concatenated output. In particular, one head with at least three distinct normalizers gives a strictly positive concatenated gap.

Proof of Proposition F.3. For any two vector laws $\nu, \widetilde{\nu}$, project an optimal W_2 coupling onto each coordinate. Each projection couples the corresponding scalar marginals, so summing their costs gives

$$W_2^2(\nu, \widetilde{\nu}) \ge \sum_{h=1}^{n_{\mathrm{h}}} W_2^2((\pi_h)_{\#}\nu, (\pi_h)_{\#}\widetilde{\nu}), \tag{36}$$

where π_h is coordinate projection. Every admissible concatenated path therefore projects to an admissible path for each head. Taking metric-derivative limits in (36) at almost every time and integrating gives $\mathcal{E}_*^{\mathrm{cat}} \ge \sum_h \mathcal{E}_{*,h}$.

For the reverse bound, run the attained scalar optima from Theorem 4.3 simultaneously. Independent controls realize this joint path, and all its numerators stay positive. The common normalizer order makes the same-index matching optimal in every coordinate between any two times. Its vector cost is the sum of these scalar optimal costs, so equality holds in (36) along this path. Consequently its action is $\sum_h \mathcal{E}_{*,h}$. The same matching argument at the positive endpoints gives $D_{\mathrm{cat}}^2 = \sum_h D_h^2$;

subtracting yields the gap identity. An isometric embedding preserves every W_2 distance and action. ■

Independent aggregate control is realized by a fixed shared-prompt block with two token coordinates per head. Choose $\beta_h \neq 0$ with $e^{\beta_h} < \min_i Z_i^{(h)}$ and repeat the frozen construction below in disjoint coordinate blocks; head h receives $(\log(\kappa^{(h)}/m), w^{(h)}/\kappa^{(h)})$ in every shared token, and its frozen maps select that coordinate pair. The decomposition does not apply to a non-isometric output projection that sums or otherwise mixes head responses.

Proposition F.4 (Two mixed heads can remove the gap) *Two independently controlled scalar heads with the same (Z_i), zero content numerators, and summed outputs connect any positive scalar endpoint laws μ_0, μ_1 with action D^2. An explicit frozen block with four-dimensional shared tokens realizes this connection for every $m \geq 1$.*

Proof of Proposition F.4. The summed output of two independently controlled heads is

$$y_i = \frac{w^{(1)}}{Z_i + \kappa^{(1)}} + \frac{w^{(2)}}{Z_i + \kappa^{(2)}}.$$

Keep $\kappa^{(1)} = \kappa_0$ and $\kappa^{(2)} = \kappa_1$ fixed, and set $w^{(1)}(t) = (1-t)w_0$, $w^{(2)}(t) = tw_1$. Then

$$y_i(t) = (1-t)\frac{w_0}{Z_i + \kappa_0} + t\frac{w_1}{Z_i + \kappa_1}. \tag{37}$$

The positive scalar endpoint vectors are increasingly ordered under the prescribed ordering of (Z_i). Their convex combinations remain ordered. The sorted-atom isometry therefore identifies the empirical law of (37) with the constant-speed chord between the sorted endpoints. Its action equals D^2, which bounds every path's action below. Thus this two-head family has zero excess action between the original endpoint laws, including when their single-head gap is positive.

To realize independent control by shared prompts, take $p = (p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}) \in \mathbb{R}^4$ and frozen maps

$$\begin{aligned} H_1(p) &= p^{(1)}/\beta, \quad K_1(p) = p^{(1)}, \quad V_1(p) = p^{(2)}, \\ H_2(p) &= p^{(3)}/\beta, \quad K_2(p) = p^{(3)}, \quad V_2(p) = p^{(4)}. \end{aligned}$$

For each sample, use readout token $(\beta, 0, \beta, 0)$ and second content token $(\log(Z_i - e^\beta), 0, \log(Z_i - e^\beta), 0)$. Both heads have readout query one, content partition Z_i, and zero content numerator. A frozen output projection sums their outputs. For any positive masses and real numerators, the shared prompts

$$p_j = \left(\log\frac{\kappa^{(1)}}{m}, \frac{w^{(1)}}{\kappa^{(1)}}, \log\frac{\kappa^{(2)}}{m}, \frac{w^{(2)}}{\kappa^{(2)}}\right), \qquad j = 1, \ldots, m,$$

realize the four aggregates independently. Conversely, arbitrary shared prompts produce such aggregates in their separate coordinate blocks. Hence this one frozen block realizes the full independent two-head family for every $m \geq 1$. The chord uses the explicit path

$$p_j(t) = \left(\log\frac{\kappa_0}{m}, \frac{(1-t)w_0}{\kappa_0}, \log\frac{\kappa_1}{m}, \frac{tw_1}{\kappa_1}\right). \tag{38}$$

This path is smooth and token-linear, with finite parameters and strictly positive partitions throughout. At each endpoint one numerator is zero and the other produces the prescribed law. ■

The construction specifies endpoint prompt arrays whose output laws are μ_0, μ_1. It does not prescribe an optimal path between arbitrary preassigned multihead prompt arrays. Its mechanism is the independent addition of two response families before feature motion is measured.

A protected concatenated channel retains the obstruction. For comparison, suppose the measured feature concatenates the vector head of Proposition F.1 with other head outputs. At both endpoints let the latter equal the same $r_i \in \mathbb{R}^{d_{\text{aux}}}$ for sample i, where d_{aux} is their total dimension. Assume the scalar optimum can be lifted in the first block while holding these other outputs fixed. Under Theorem 4.3, the minimum action of the concatenated family equals $\mathcal{E}_*(0)$, its endpoint distance equals D, and its gap equals $\mathcal{G}$.

To prove the lower bound, project onto $\boldsymbol{a}$ in the first block and ignore the remaining coordinates. This is a contraction that sends every admissible law into $\mathcal{M}$, with the original scalar endpoints. The action is therefore at least $\mathcal{E}_*(0)$. For the attaining path, use features $(z_i(t)\boldsymbol{a}, r_i)$, with $z_i(t)$ the positive scalar optimum. For any two times, matching the same sample indices has cost $N^{-1}\sum_i |z_i(t) - z_i(s)|^2$, which equals the scalar Wasserstein cost because the scalar vectors remain ordered. Projection gives the reverse bound, so these distances, metric speeds, and actions are exactly the scalar ones. The same argument at the endpoints proves the distance claim. If coupled prompt controls prevent this lift, the projection still gives the action lower bound whenever the stated projected family and endpoints are retained. Thus concatenation with a protected scalar channel and summation of independently normalized responses have different variational effects.

An approximate projected-channel certificate. The protected channel need not be exact. Let $\mathcal{V} \subset \mathcal{P}_{2,N}(\mathbb{R}^{d_{\mathrm{out}}})$ be any attainable vector-feature family, with endpoints $\nu_0, \nu_1 \in \mathcal{V}$. Write $\mathcal{E}_*^{\mathcal{V}}$ for the infimum of actions of AC^2 paths in $\mathcal{V}$ between these endpoints. For a unit vector $\boldsymbol{a} \in \mathbb{R}^{d_{\mathrm{out}}}$, let $\pi_{\boldsymbol{a}}(x) = \boldsymbol{a}^\top x$ and $(\pi_{\boldsymbol{a}})_\# \nu$ denote the projected law. Write $\mathrm{dist}_{W_2}(\lambda, \mathcal{M}) = \inf_{\mu \in \mathcal{M}} W_2(\lambda, \mu)$.

Proposition F.5 (Projected-channel gap certificate) *Suppose $d_Z \geq 3$, $w_0, w_1 > 0$, and $\kappa_0 \neq \kappa_1$. Assume $\varepsilon \geq 0$, $(\pi_{\boldsymbol{a}})_\# \nu_b = \mu_b$ for $b \in \{0,1\}$, and, for every $\nu \in \mathcal{V}$,*

$$\mathrm{dist}_{W_2}((\pi_{\boldsymbol{a}})_\# \nu, \mathcal{M}) \leq \varepsilon. \tag{39}$$

Set $\tau = W_2^2(\nu_0, \nu_1) - D^2 \geq 0$. Then

$$\mathcal{E}_*^{\mathcal{V}} - W_2^2(\nu_0, \nu_1) \geq 4[\rho - \varepsilon]_+^2 - \tau. \tag{40}$$

In particular, the vector-feature gap is positive when $2[\rho - \varepsilon]_+ > \sqrt{\tau}$.

Proof of Proposition F.5. Projection by $\pi_{\boldsymbol{a}}$ is 1-Lipschitz, so it contracts W_2 and metric action; in particular, it gives $\tau \geq 0$. For any admissible ν_t, put $\omega_t = Q((\pi_{\boldsymbol{a}})_\# \nu_t)$. The projected law path is AC^2, and the sorted-vector isometry makes ω_t an AC^2 path in the normalized sample Hilbert space with action no larger than that of ν_t. Cauchy–Schwarz and the parallelogram identity give

$$\begin{aligned} \mathcal{E}[\nu_\cdot] &\geq 2\|\omega_{1/2} - \omega_0\|_N^2 + 2\|\omega_1 - \omega_{1/2}\|_N^2 \\ &= D^2 + 4\|\omega_{1/2} - h_0\|_N^2. \end{aligned}$$

By (39) and the definition of ρ, $\|\omega_{1/2} - h_0\|_N \geq [\rho - \varepsilon]_+$. Taking the infimum and subtracting $W_2^2(\nu_0, \nu_1) = D^2 + \tau$ proves (40). ∎

For sample-aligned endpoints of the form $x_{i,b} = z_i(\kappa_b, w_b)\boldsymbol{a} + r_{i,b}$ with $\boldsymbol{a}^\top r_{i,b} = 0$, the same-index coupling gives the checkable estimate $\tau \leq N^{-1}\sum_i \|r_{i,1} - r_{i,0}\|_{\mathrm{E}}^2$. A uniform bound on projected auxiliary-head responses can likewise verify (39) by same-index coupling. Small mixing alone does not ensure a family-wide bound on the independently controlled head responses.

G ACTION BOUNDS FOR PROMPT TRANSITIONS AND ADAPTATION

This section translates feature action into bounds on representation and prediction changes. Throughout, the AC^2 trajectories $f_i : [0,1] \to \mathbb{R}^{d_{\text{out}}}$ track the same N images, as in Section 4.6. Define their straight endpoint bridges $\bar{f}_i(t) = (1-t)f_i(0) + tf_i(1)$ and

$$\mathcal{A}_{\text{pair}} = \frac{1}{N}\sum_i \int_0^1 \|\dot{f}_i(t)\|_{\text{E}}^2\, dt, \qquad \mathcal{B}_{\text{pair}} = \frac{1}{N}\sum_i \|f_i(1) - f_i(0)\|_{\text{E}}^2, \qquad \mathcal{R}_{\text{pair}} = \mathcal{A}_{\text{pair}} - \mathcal{B}_{\text{pair}}.$$

For $[s,t] \subseteq [0,1]$, let $\mathcal{A}_{\text{pair}}[s,t]$ restrict the integral to that interval. The bounds below use Euclidean energy estimates in any output dimension.

G.1 PROMPT TRANSITIONS WITH A FIXED CLASSIFIER

Continuous empirical action tracks individual motion. Define $\nu_t = N^{-1}\sum_i \delta_{f_i(t)}$. The identity coupling gives $W_2^2(\nu_s, \nu_t) \le N^{-1}\sum_i \|f_i(t) - f_i(s)\|_{\text{E}}^2$. In continuous time, the actions agree:

$$\int_s^t |\dot{\nu}_r|_{W_2}^2\, dr = \mathcal{A}_{\text{pair}}[s,t]. \tag{41}$$

Fix a time r and group atoms that coincide at r. If there are distinct groups, let $\text{sep}(r) > 0$ be their minimum separation. For small h, every $f_i(r+h)$ stays within $\text{sep}(r)/2$ of its group's position. An optimal coupling can be chosen as a permutation (Gabriel & Marco, 2019). A permutation crossing groups costs at least $\text{sep}(r)^2/(4N)$, while the identity cost tends to zero. Thus optimal permutations stay within groups for small h. Since each group has identical starting positions, every within-group permutation has the identity cost. If all atoms coincide at r, the conclusion holds directly. Therefore

$$W_2^2(\nu_{r+h}, \nu_r) = \frac{1}{N}\sum_i \|f_i(r+h) - f_i(r)\|_{\text{E}}^2$$

for sufficiently small h. Dividing by h^2 and differentiating the finitely many AC^2 trajectories proves the speed identity almost everywhere. The identity-coupling bound also makes $\nu_\cdot$ an AC^2 path, so integration proves (41), including paths with collisions.

Excess action measures departures from the endpoint bridge. The main bridge bound is

$$\frac{1}{N}\sum_i \sup_{t\in[0,1]} \|f_i(t) - \bar{f}_i(t)\|_{\text{E}}^2 \le \frac{1}{4}\mathcal{R}_{\text{pair}}. \tag{42}$$

The matched endpoint cost satisfies $\mathcal{B}_{\text{pair}} \ge W_2^2(\nu_0, \nu_1)$, because the identity coupling is one feasible coupling. Expanding the square and using $\int_0^1 \dot{f}_i(r)\, dr = f_i(1) - f_i(0)$ gives

$$\mathcal{R}_{\text{pair}} = \frac{1}{N}\sum_i \int_0^1 \big\|\dot{f}_i(r) - [f_i(1) - f_i(0)]\big\|_{\text{E}}^2\, dr \ge 0. \tag{43}$$

For each i, absolute continuity further gives

$$f_i(t) - \bar{f}_i(t) = \int_0^1 \big(\mathbf{1}_{\{r\le t\}} - t\big)\big(\dot{f}_i(r) - [f_i(1) - f_i(0)]\big)\, dr.$$

The squared integral norm of the scalar kernel is $t(1-t) \le 1/4$. Cauchy–Schwarz therefore yields

$$\|f_i(t) - \bar{f}_i(t)\|_{\text{E}}^2 \le \frac{1}{4}\int_0^1 \big\|\dot{f}_i(r) - [f_i(1) - f_i(0)]\big\|_{\text{E}}^2\, dr.$$

Taking a supremum in t, averaging, and applying (43) proves (42). The factor $1/4$ is sharp: for one scalar trajectory $f_1(t) = \min\{t, 1-t\}$, both endpoints are zero, the action is one, and the maximum squared departure is $1/4$. The endpoint bridges need not themselves be attainable by prompts for the bound to hold.

On the positive scalar branch, all samples maintain the order of their fixed normalizers. Identity matching is then optimal at the endpoints, so $\mathcal{B}_{\text{pair}} = D^2$ and $\mathcal{R}_{\text{pair}} = \mathcal{E}[\mu_\cdot] - D^2$. In particular, the exact minimum-action construction gives $\mathcal{R}_{\text{pair}} = \mathcal{G}$. For arbitrary vector trajectories, the matched endpoint cost must be retained; endpoint Wasserstein matching may permute images.

Corollary G.1 (Prediction changes along a prompt transition) *Let $\mathcal{C}$ be a finite class set and let $g_c(x) = \omega_c{}^\top x + \zeta_c$, $c \in \mathcal{C}$, be fixed affine logits, with $\omega_c \in \mathbb{R}^{d_{\text{out}}}$ and $\zeta_c \in \mathbb{R}$. Put $\Lambda_{\text{cls}} = \max_{c \neq c'} \|\omega_c - \omega_{c'}\|_{\text{E}}$, taking it as zero for a single class. For $\mathfrak{m} > 0$, let $\mathcal{I}_{\mathfrak{m}}$ contain the images with a common endpoint class $\chi_i \in \mathcal{C}$ such that*

$$g_{\chi_i}(f_i(b)) - g_c(f_i(b)) \geq \mathfrak{m} \quad \text{for } b \in \{0, 1\}, \quad c \neq \chi_i.$$

Let $\mathcal{F}_{\mathfrak{m}}$ consist of those $i \in \mathcal{I}_{\mathfrak{m}}$ for which some $t \in [0, 1]$ and $c \neq \chi_i$ satisfy $g_c(f_i(t)) \geq g_{\chi_i}(f_i(t))$. Thus ties count as failures. Then

$$\frac{|\mathcal{F}_{\mathfrak{m}}|}{N} \leq \min\left\{ \frac{|\mathcal{I}_{\mathfrak{m}}|}{N}, \frac{\Lambda_{\text{cls}}^2}{4\mathfrak{m}^2} \mathcal{R}_{\text{pair}} \right\}. \tag{44}$$

Proof of Corollary G.1. For $i \in \mathcal{I}_{\mathfrak{m}}$, every competing margin on $\bar{f}_i(t)$ is the affine interpolation of its endpoint margins and is at least $\mathfrak{m}$. At a failure time, Cauchy–Schwarz therefore implies $\mathfrak{m} \leq \Lambda_{\text{cls}} \|f_i(t) - \bar{f}_i(t)\|_{\text{E}}$. When $\Lambda_{\text{cls}} > 0$, bound each failure indicator by $\Lambda_{\text{cls}}^2 \mathfrak{m}^{-2} \sup_t \|f_i(t) - \bar{f}_i(t)\|_{\text{E}}^2$, sum over $\mathcal{I}_{\mathfrak{m}}$, and apply (42). If $\Lambda_{\text{cls}} = 0$, all logit differences are constant, so no such image can fail. The bound by $|\mathcal{I}_{\mathfrak{m}}|/N$ follows from set inclusion. ∎

For correct common endpoint classes, this also bounds classification failures. Samples whose desired class changes require a task-dependent criterion. Minimum action alone can pass through zero between opposite feature endpoints; the margin condition is what gives prediction control.

G.2 A DRIFT BUDGET FOR TEST-TIME ADAPTATION

Feature and score drift on any interval. For $0 \leq s < t \leq 1$, absolute continuity and Cauchy–Schwarz give

$$\frac{1}{N} \sum_i \|f_i(t) - f_i(s)\|_{\text{E}}^2 \leq (t - s) \mathcal{A}_{\text{pair}}[s, t]. \tag{45}$$

Indeed, apply $\| \int_s^t \dot{f}_i(r)\, dr \|_{\text{E}}^2 \leq (t-s) \int_s^t \|\dot{f}_i(r)\|_{\text{E}}^2\, dr$ to each image and average. If $\psi : \mathbb{R}^{d_{\text{out}}} \to \mathbb{R}$ is L_ψ-Lipschitz, then

$$\left(\frac{1}{N} \sum_i |\psi(f_i(t)) - \psi(f_i(s))|^2 \right)^{1/2} \leq L_\psi \sqrt{(t - s) \mathcal{A}_{\text{pair}}[s, t]}. \tag{46}$$

This follows by squaring the Lipschitz bound and using (45). For distribution statistics, the more general bound is

$$\left| \int \psi\, d\nu_t - \int \psi\, d\nu_s \right| \leq L_\psi W_2(\nu_s, \nu_t) \leq L_\psi \sqrt{(t - s) \int_s^t |\dot{\nu}_r|_{W_2}^2\, dr}. \tag{47}$$

For the first inequality, integrate the Lipschitz bound against an optimal coupling and use Cauchy–Schwarz. The second follows from metric absolute continuity and Cauchy–Schwarz. This last argument applies to any AC^2 path in $\mathcal{P}_2(\mathbb{R}^{d_{\text{out}}})$, not only empirical laws.

Using the bound during adaptation. An unlabeled adaptation objective can select an updated prompt while a constraint $\mathcal{A}_{\text{pair}}[s, t] \leq \mathsf{B}$ sets a feature-motion budget $\mathsf{B} \geq 0$. Equation (46) then limits the root-mean-square change of a fixed Lipschitz score to $L_\psi \sqrt{(t - s)\mathsf{B}}$. An action penalty gives a corresponding tradeoff between the adaptation objective and this motion cost. The same reference inputs must be evaluated throughout each update interval: comparing features of different incoming images mixes input variation with adaptation-induced motion. A new reference batch may be chosen for the next interval, with a separate budget. This is a mathematical design criterion for prompt adaptation; target accuracy must be evaluated with the adaptation objective.

Continuous certificates and sampled estimates. For any time grid $0 = t_0 < \cdots < t_{n_{\text{grid}}} = 1$, with n_{grid} the number of intervals, (45) gives

$$\begin{aligned} \sum_{k=0}^{n_{\text{grid}}-1} \frac{W_2^2(\nu_{t_k}, \nu_{t_{k+1}})}{t_{k+1} - t_k} &\leq \frac{1}{N} \sum_i \sum_{k=0}^{n_{\text{grid}}-1} \frac{\|f_i(t_{k+1}) - f_i(t_k)\|_{\text{E}}^2}{t_{k+1} - t_k} \\ &\leq \mathcal{A}_{\text{pair}}. \end{aligned} \tag{48}$$

Both sampled costs are lower approximations. Thus a rigorous drift certificate uses a validated upper bound on the continuous action, for example from derivative integration with a certified error bound. A coarse sampled action cannot be substituted as that upper bound. This is consistent with the refinement protocol in Appendix D.3.

For uniform grids, the paired middle term in (48) converges to $\mathcal{A}_{\mathrm{pair}}$ as the number of intervals grows. Indeed, on each interval the feature difference divided by interval length is the average of $\dot{f}_i$. The middle term is the squared L^2 norm of these piecewise-constant averages. They converge in $L^2(0,1)$ to $\dot{f}_i$ for every $f_i \in AC^2$, first for continuous derivatives and then by density. This proves consistency of (12) with the continuous action.