CLOSING GAPS IN ONLINE FAIR DIVISION

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ABSTRACT

We study the online fair division of indivisible items, where items arrive one at a time and must be allocated immediately and irrevocably. We address three central open questions in the literature.

First, we show that for every $n \geq 2$ agents, every fixed $k \geq 1$, and every $\alpha \in (0, 1]$, no online algorithm can guarantee α-PROPk against an adaptive adversary. This remains true even when the total number of goods is known in advance, all values lie in $[0, 1]$, and every good is positively valued by at most two agents. The impossibility extends to a broad range of standard envy-based, proportionality-based, and share-based fairness notions. We also establish an analogous impossibility for chores.

Second, with a prediction of each agent's maximum item value, a lightweight form of information about future goods, the best previous guarantee was $1/n$-PROP1, leaving open whether the dependence on n is necessary. We give a deterministic $19/30$-PROP1 algorithm against adaptive adversaries that does not require knowing the total number of goods. Given an additional upper bound $\kappa \in [2, n]$ on the number of agents who value any good positively, the guarantee improves to $\max\{19/30, n/(n + \kappa)\}$. The guarantee remains a positive constant under any fixed one-sided prediction error below one. When predictions are exact and the total number of goods $m \geq n \log n$ is known in advance, a deterministic algorithm achieves the same $19/30$-PROP1 factor together with $O(\sqrt{m \log n/n})$ maximum additive envy after normalizing each agent's values by their maximum item value.

Third, against a non-adaptive adversary, we determine the tight high-probability PROP1 guarantee of the classical LIKE rule, which assigns each good uniformly among the agents who value it positively. Unlike that of uniform random allocation, its guarantee improves when fewer agents value each good.

1 INTRODUCTION

Fair division of indivisible goods is a fundamental problem dealing with how to allocate discrete resources among agents with different preferences in a fair manner (Brams & Taylor, 1996; Moulin, 2003). Applications of this model are widespread in modern ML systems. Examples include assigning ad impressions or recommendation exposure, scheduling compute jobs in shared infrastructure, and distributing content moderation tasks or service requests as they arrive (Aleksandrov et al., 2015; Aleksandrov & Walsh, 2020; Neoh et al., 2026; Choo et al., 2026). In these applications, delaying every decision until the entire sequence is known is often impossible, and revising past decisions may be infeasible.

These applications can be studied through the *online fair division* model, where indivisible goods arrive sequentially and must be allocated immediately and irrevocably to an agent based on their reported values for the current good.[1] A key challenge is to provide ex post fairness guarantees on the final allocation, even though the algorithm made every decision online, without knowing the future. Key fairness properties studied in (both the offline and online) fair division literature include *envy-freeness up to one good* (EF1), *proportionality up to one good* (PROP1), and *maximin share fairness* (MMS). However, while these fairness notions are often possible to achieve exactly (or approximately) offline, they can become impossible to even approximate online.

[1] For our main impossibility result, we also establish a corresponding result for *chores*, modeled as items with *nonnegative costs*.

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For instance, for MMS, Zhou et al. (2023) showed that in the goods setting, even if each agent's total value for all goods is known in advance (*normalization* information), no algorithm can achieve any approximation to MMS for three or more agents, whereas a $1/2$-approximation is possible in the case of two agents. Subsequent work studies MMS under further assumptions. Kulkarni et al. (2025) consider sequentially arriving agents with known valuation types, whereas Wang & Wei (2026) retain sequentially arriving items and restrict valuations to binary and personalized two-value classes.

Another line of work asks whether information about future goods changes what can be achieved online. Neoh et al. (2026) study normalization information and frequency predictions, which give the multiset of each agent's values for the goods but not their arrival order. Without future information, they show that no deterministic online algorithm guarantees any positive approximation to EF1. With normalization information, they obtain PROP1 for every number of agents n and EF1 for $n = 2$, but no positive EF1 approximation is possible for $n \geq 3$. With frequency predictions, they match the best-known offline guarantees for a broad class of share-based fairness notions.

Benadè et al. (2018) and Neoh et al. (2026) show that exact PROP1 cannot be guaranteed online. PROP1 has a long history in the offline fair division literature as a weak benchmark that is implied by EF1 and MMS, so it remains a reasonable fairness target even when stronger notions are unattainable.

This leaves a basic and conceptually important question: are the widespread negative results in the online fair division literature caused mainly by the strength of notions such as EF1, EFX, or MMS, or does the standard online model already rule out every positive approximation to PROP1 (Neoh et al., 2026; Choo et al., 2026)? PROP1 asks only that each agent reach their proportional share after adding a single good outside their bundle. Thus, a positive approximation would identify a meaningful fairness guarantee under irrevocable online allocation. Conversely, an impossibility for PROP1 would show that the difficulty comes from the standard online model itself and would help explain why positive results change the information, valuation, or adversarial assumptions.

Choo et al. (2026) initiated a systematic study of this question. They showed that several natural greedy rules do not guarantee any positive multiplicative approximation to PROP1 against an adaptive adversary. Against a non-adaptive adversary, they proved that the uniform random allocation rule RAND, which assigns each good uniformly among all agents, has a PROP1 approximation factor of $\Theta(1/\log(n/\delta))$ with probability at least $1 - \delta$. With exact maximum item value (MIV) predictions, they obtained a deterministic $1/n$-PROP1 guarantee against an adaptive adversary.

The classical LIKE rule (Aleksandrov et al., 2015) is a natural alternative to RAND: it assigns each good uniformly among only the agents who value it positively. When few agents value a good, each such agent therefore receives it with greater probability. It was not known whether this improves the ex post PROP1 guarantee.

The current literature therefore leaves three central open questions:

1. Can any online algorithm guarantee a positive approximation to PROP1, or even to PROPk, against an adaptive adversary (Neoh et al., 2026; Choo et al., 2026)?
2. Can exact MIV predictions give a constant PROP1 approximation independent of n, improving on $1/n$ (Choo et al., 2026)?
3. Against a non-adaptive adversary, can the classical LIKE rule obtain a better PROP1 guarantee than RAND when few agents value the same good positively (Aleksandrov et al., 2015; Choo et al., 2026)?

Our contributions. We answer these questions through three results, summarized in Table 1.

1. **No positive approximation without future information.** For every $n \geq 2$, every fixed $k \geq 1$, and every $\alpha \in (0, 1]$, no online algorithm guarantees α-PROPk against an adaptive adversary. The adversary observes past assignments but not future random choices. The result applies to randomized algorithms and holds even if the number of goods is known in advance, every value lies in $[0, 1]$, and each good is valued positively by at most two agents (Theorem 3.1). It also implies impossibilities for EF1, MMS, and other standard notions (Appendix E). For chores, no factor below n is possible for PROPk (Appendix F).

Table 1: Summary of our main results. The last row gives guarantees with probability at least $1-\delta$, for $0 < \delta \leq 1/2$ and $2 \leq \kappa \leq n$. The prediction algorithms shown are deterministic and do not need the number of goods.

Setting and adversary	**Best known result**	**This paper**
Fully online; adaptive	For any $\alpha \in (0, 1]$, no α-EF1 for any algorithm (Neoh et al., 2026)	No approximation to PROPk for any algorithm
With maximum item value (MIV) predictions; adaptive	$1/n$-PROP1 (Choo et al., 2026)	$19/30$-PROP1; $\max\{19/30, n/(n+\kappa)\}$-PROP1 with known κ
Fully online; non-adaptive	$\Theta(1/\log(n/\delta))$-PROP1 for RAND (Choo et al., 2026)	$\Theta(\min\{1, n/(\kappa \log(n/\delta))\})$-PROP1 for LIKE

2. **A constant guarantee from one number per agent.** With exact MIV predictions, we give a deterministic $19/30$-PROP1 algorithm against adaptive adversaries that does not need the number of goods. If an upper bound κ on the number of agents who value any good positively is also given, we achieve $\max\{19/30, n/(n+\kappa)\}$-PROP1 (Theorem 4.1). The guarantee stays a positive constant under any fixed one-sided prediction error below one (Corollary 4.2). With exact predictions and a known number of goods $m \geq n \log n$, a different rule retains $19/30$-PROP1 and bounds normalized maximum additive envy by $O(\sqrt{m \log n/n})$ (Theorem 4.3).
3. **Tight bounds for allocating among agents who value a good.** Against a non-adaptive adversary, LIKE has a tight $\Theta(\min\{1, n/(\kappa \log(n/\delta))\})$ PROP1 guarantee with probability at least $1-\delta$, for $2 \leq \kappa \leq n$. In contrast, RAND retains its worst-case $\Theta(1/\log(n/\delta))$ factor even when each good is valued by just one agent (Corollary 5.2). Thus LIKE improves the guarantee by a factor of order n/κ, until the PROP1 guarantee reaches one. We also compare their additive envy bounds and give instances showing that better proportionality need not mean less envy (Theorem 5.3 and Proposition 5.4).

Relation to prior work. Our results concern fairness of the final allocation, measured directly against each agent's fairness requirement. They complement work that gives the algorithm other information about future goods, such as total values and value frequencies (Zhou et al., 2023; Neoh et al., 2026) or full valuation-vector predictions (Melissourgos & Protopapas, 2026). Positive results also arise from restricted valuations (Amanatidis et al., 2025; Wang & Wei, 2026), limited buffering (Amanatidis et al., 2026), or different repeated-assignment and additive-error requirements (Adams & Segal-Halevi, 2026; Kahana et al., 2026). Our impossibility theorem explains why such assumptions are needed: without them, no online algorithm guarantees any positive approximation to PROPk against an adaptive adversary. Our comparison of LIKE and RAND concerns two widely used rules; optimal envy rates for general algorithms are studied by Halpern et al. (2025). Appendix A gives further context, including the distinction from competitive analysis against an offline allocation (Chen & Tan, 2026).

2 Model and Fairness

All logarithms are natural. For a positive integer z, let $[z] := \{1, \ldots, z\}$. There are $n \geq 2$ *agents* $N = [n]$ and a finite sequence of $m \geq 1$ indivisible *goods* $G = \{g_1, \ldots, g_m\}$, indexed by arrival order. The algorithm need not know m (which we sometimes call the *horizon*) in advance unless stated otherwise. Each agent $i \in N$ has a nonnegative additive valuation $v_i : 2^G \to \mathbb{R}_{\geq 0}$, so $v_i(S) = \sum_{g \in S} v_i(g)$ for every $S \subseteq G$. We call an instance *binary* if $v_i(g) \in \{0, 1\}$ for every $i \in N$ and $g \in G$. A *(complete) allocation* is a tuple $A = (A_1, \ldots, A_n)$ that partitions G.

For a good g, let $N(g) := \{i \in N : v_i(g) > 0\}$ and $\kappa := \max_{g \in G} |N(g)|$. Thus, κ is the largest number of agents who value the same good positively. When used as an input to an algorithm, κ may instead be any given upper bound on $|N(g)|$. We use this parameter only in results that state it explicitly.

Definition 2.1 (α-PROPk)**.** Fix $\alpha \in [0,1]$ and an integer $k \geq 1$. An allocation A is *α-proportional up to k goods* (α-PROPk) if, for every agent i, there is a set $S_i \subseteq G \setminus A_i$ with $|S_i| \leq k$ such that $v_i(A_i \cup S_i) \geq \alpha \cdot \frac{v_i(G)}{n}$.

When $k = 1$, we write α-PROP1; when $\alpha = 1$, we omit α. In particular, PROP1 is the usual exact notion. Let $h_i := \max_{g \in G \setminus A_i} v_i(g)$ be agent i's largest value for a good outside her bundle, where, throughout, the maximum over an empty set is understood to be zero. For $k = 1$, the definition is equivalent to

$$v_i(A_i) + h_i \geq \alpha \cdot \frac{v_i(G)}{n} \quad \text{for every agent } i \in N. \tag{1}$$

For an allocation A, its *maximum additive envy* is $\mathrm{Env}(A) := \max_{i,j \in N}(v_i(A_j) - v_i(A_i))$. Appendix E defines the other fairness notions for goods mentioned in the paper and proves the implications used for our lower bounds. Appendix F gives the corresponding definitions and lower bounds for chores.

Online algorithms and adversaries. At time t, good g_t arrives and the vector $(v_1(g_t), \ldots, v_n(g_t))$ is revealed. The algorithm must assign g_t immediately to one agent. Every good is assigned exactly once, and earlier assignments cannot be changed. Fairness is evaluated on the final allocation. A randomized algorithm may use private random choices; a deterministic algorithm uses none. An *adaptive adversary* chooses each next valuation vector after seeing the earlier goods and their realized recipients; it does not see future private random choices. Unless the horizon is known in advance, it may also decide when the finite sequence ends. A *non-adaptive adversary* fixes the complete valuation sequence before the algorithm makes any random choice. Against an adaptive adversary, a randomized algorithm guarantees a property with probability at least $1 - \delta$ if the property holds with that probability against every allowed adversarial strategy. On a fixed instance, probability is only over the algorithm's random choices.

Maximum item value (MIV) predictions. For each agent i, let $p_i^{\max} := \max_{g \in G} v_i(g)$. An exact MIV prediction gives the algorithm $p_i = p_i^{\max}$ before any good arrives. It gives only one number per agent and reveals neither $v_i(G)$ nor the individual values or arrival order of future goods. Following Choo et al. (2026), predictions have one-sided error $\varepsilon \in [0,1)$ if $p_i^{\max} \in [(1-\varepsilon)p_i, p_i]$ for every agent i. Thus each prediction is an upper bound on the true maximum and overestimates it by at most the factor $1/(1-\varepsilon)$. The upper-bound direction ensures that $v_i(g)/p_i \leq 1$ for every agent with $p_i > 0$ and every good g. Choo et al. also observe that a multiplicative two-sided prediction can be converted to this form by scaling it upward.

3 Inapproximability of PROPk in the Fully Online Setting

We show that no positive multiplicative approximation to PROPk is possible in the standard fully online model.

Theorem 3.1. *Fix $n \geq 2$, $k \geq 1$, and $\alpha \in (0,1]$. For every online algorithm, there is an adaptive adversary under which the final allocation is not α-PROPk with probability one. The statement remains true even when (i) the final number of goods is known before the first good arrives; (ii) every value lies in $[0,1]$; and (iii) every good is positively valued by at most two agents.*

It suffices to rule out every positive PROP1 factor for two agents. An α-PROPk allocation is (α/k)-PROP1, since the most valuable added good accounts for at least $1/k$ of the added value. To obtain a two-agent algorithm, simulate an n-agent α-PROPk algorithm with $n-2$ zero-valued agents and redirect their assigned goods to agent 1. This does not decrease the left-hand side of either agent's PROP1 inequality, and the proportional share for two agents is $v_i(G)/2$ rather than $v_i(G)/n$, so the two-agent algorithm guarantees $(2\alpha/(kn))$-PROP1. Appendix B proves these reductions.

For the two-agent PROP1 problem, fix $\alpha \in (0,1]$ and set $\gamma = \alpha/2$ and $\beta = 1-\gamma$. Let G' contain the goods seen so far; for the current allocation of G', h_i is agent i's largest value for a good in the other agent's bundle. An initial sequence of goods worth one to both agents either violates α-PROP1 or makes both h_i positive. We can then define the normalized slack $r_i := (v_i(A_i) + h_i - \gamma v_i(G'))/h_i$. This is the amount by which agent i's current PROP1 inequality is satisfied, measured in units of the best good she could add. α-PROP1 requires $r_1, r_2 \geq 0$. If both slacks are below γ, a good worth h_i to each agent i makes the slack of the agent who does not receive it negative.

The difficulty is making both slacks small at the same time: giving a good to one agent normally improves that agent's position. The adversary uses goods whose values differ between the two agents. If the next good has value $u_i h_i$ to agent i, then

$$r_i^+ = \begin{cases} r_i + \beta u_i, & \text{if } i \text{ receives it,} \\ r_i - \gamma u_i, & \text{if } i \text{ does not receive it and } u_i \le 1, \\ \beta + (r_i - 1)/u_i, & \text{if } i \text{ does not receive it and } u_i > 1. \end{cases} \tag{2}$$

The last line is crucial: a sufficiently valuable good that agent i does not receive brings r_i close to β, regardless of how large r_i was. The adversary uses this to bring r_2 into a fixed range at only a small increase to r_1. A carefully chosen finite list of goods then either violates α-PROP1 or lowers r_1 by more than that increase. Repeating this construction eventually makes both slacks smaller than γ, unless a violation occurs earlier. One final good completes the argument.

The full proof defines one adaptive strategy that applies to every possible sequence of recipients, independently of the algorithm and of any horizon given to it in advance. Every such sequence reaches a violation after finitely many goods, and each good has only finitely many possible recipients. Thus there is a common finite upper bound on the number of goods needed. The adversary can give this bound as the horizon before the first good arrives and add zero-valued goods whenever a violation is reached earlier. This gives an instance with a known number of goods. Since only finitely many values occur, multiplying all values by one common positive constant makes them lie in $[0, 1]$ without changing the argument. Finally, the strategy leads to a violation for every possible sequence of realized recipients, so the result also applies to randomized algorithms.

Consequences for other fairness notions. Appendix E shows that positive multiplicative guarantees for the standard notions listed there would imply a positive PROPk approximation for some fixed k. Hence Theorem 3.1 also rules out those guarantees against an adaptive adversary; specifically, PROP, PROP1, PROPX, PROPm, PROPavg, Avg-EFX, EF, EFX, EF1, EEFX, MXS, MMS, PMMS, GMMS, and every fixed EFc notion (Amanatidis et al., 2018; 2023; Aziz et al., 2020; Baklanov et al., 2021a;b; Barman et al., 2018; Budish, 2011; Caragiannis et al., 2019; 2023; Conitzer et al., 2017; Kobayashi & Mahara, 2025; Lipton et al., 2004).[2]

For chores, approximation factors are at least one and smaller is better. Appendix F proves that no factor below n is possible for PROPk, whereas factor n holds for every allocation. It also derives consequences for standard chore fairness notions.

4 Constant PROP1 Approximation with MIV Predictions

Choo et al. (2026) show that exact MIV predictions give $1/n$-PROP1 and leave open the optimal dependence on n. They also show that no positive approximation to EF1, MMS, or PROPX is possible even with exact MIV predictions. Thus these predictions make approximate PROP1 attainable while leaving these stronger notions inapproximable. We improve the PROP1 factor to $19/30$, independently of n, and to $n/(n + \kappa)$ when an upper bound κ is known and this factor is larger. With a known number of goods, we also obtain the same constant PROP1 factor together with an additive envy bound.

Agents with $p_i = 0$ value every good at zero, so their fairness inequalities hold automatically. The definitions below therefore include only agents with $p_i > 0$, while n remains the total number of agents.

4.1 A Constant Factor PROP1 Approximation Independent of n

For an agent with $p_i > 0$, let t_i^* be the first time a good worth p_i arrives. Let G^t contain the first t goods, and let A_i^t be agent i's bundle at that time. Define

$$D_i^t := v_i(G^t) + p_i \mathbf{1}[t < t_i^*], \quad B_i^t := v_i(A_i^t \setminus \{g_{t_i^*}\}). \tag{3}$$

The quantity D_i^t includes the predicted maximum p_i before a good of that value arrives. At time t_i^*, the prediction is replaced by the actual value p_i, so D_i^t does not change. The quantity B_i^t counts the

[2] This is not an exhaustive list of the notions in the literature covered by the impossibility result.

value received except for this first maximum-valued good. Both quantities can be updated online: the prediction lets us recognize the first good worth p_i when it arrives.

At the end, $D_i^m = v_i(G)$ and $p_i + B_i^m \le v_i(A_i) + h_i$: if agent i received the first maximum-valued good, this sum equals their bundle value; otherwise, they can add that good. Excluding it from B_i^t therefore avoids counting it twice. For example, if $p_i = 10$ and the first good is worth 4 and assigned to agent i, then $(D_i^1, B_i^1) = (14, 4)$. If the next good is worth 10, both quantities stay unchanged, regardless of its recipient.

For a target coefficient $\rho \in (0, 1/n)$, define the normalized PROP1 slack

$$s_i^t := \frac{p_i + B_i^t - \rho D_i^t}{p_i}. \tag{4}$$

Initially, $s_i^0 = 1 - \rho$, and positive final slacks imply $n\rho$-PROP1. For the current good, let $z_i = v_i(g_t)/p_i \in [0, 1]$, except that $z_i = 0$ at agent i's first maximum-valued good and when $p_i = 0$. Assigning the good to j gives

$$s_i^t = s_i^{t-1} - \rho z_i + z_i \mathbf{1}[i = j]. \tag{5}$$

Thus an agent who does not receive the good loses ρz_i in slack, while its recipient gains $(1 - \rho)z_i$. The treatment of the first maximum-valued good follows Choo et al. (2026). We obtain the constant factor by comparing the total potential increase among nonrecipients with the decrease for the recipient. Let f be the fixed positive decreasing function defined in Appendix G.2. In particular,

$$f(x) > 1/x \quad \text{for } 0 < x \le 1, \quad 1 < f(1) < 30/19. \tag{6}$$

Set $\rho = 19/(30n)$ and define

$$\Phi^t := \sum_{i:p_i>0} f(s_i^t + \rho). \tag{7}$$

The shift by ρ makes every initial argument equal to one, so $\Phi^0 \le nf(1) < 1/\rho$. If all arguments are positive and $\Phi^t \le nf(1)$, then every slack is positive: if $s_i^t \le 0$, then $0 < s_i^t + \rho \le \rho$, and the agent's term alone exceeds $1/\rho$. Small slacks therefore contribute large potential values, so agents whose PROP1 inequality is close to being violated have more influence on the allocation decision.

Allocation rule. When g_t arrives, assign it to an agent maximizing

$$\Delta_i := f(s_i^{t-1} + \rho - \rho z_i) - f(s_i^{t-1} + \rho + (1 - \rho)z_i), \tag{8}$$

with $\Delta_i = 0$ for $p_i = 0$ and a fixed order for ties. This is the reduction in agent i's term from receiving the good rather than not receiving it. The rule therefore minimizes the next total potential.

Theorem 4.1. *With exact MIV predictions, the deterministic rule in (8) guarantees $19/30$-PROP1 against adaptive adversaries. If an upper bound $\kappa \in [2, n]$ on $|N(g)|$ is given in advance, a deterministic online algorithm guarantees $\max\{19/30, n/(n + \kappa)\}$-PROP1. Neither algorithm needs the number of goods in advance.*

Proof overview. Suppose the current slacks are positive and $\Phi^{t-1} \le nf(1)$. All candidate arguments are positive, since $s_i^{t-1} + \rho - \rho z_i \ge s_i^{t-1} > 0$. Let d_i be the increase in agent i's term if they do not receive the good, with $d_i = 0$ for $p_i = 0$. Assigning it to j changes the potential by

$$\Phi^t(j) - \Phi^{t-1} = \sum_i d_i - \Delta_j. \tag{9}$$

Appendix G.1 bounds the sum in two ways: by bounding each increase separately, and by comparing each increase with the agent's current potential term. The latter uses $\Phi^{t-1} \le nf(1)$, which limits the combined contribution of agents with small slacks. Together, the bounds give $\sum_i d_i \le n\rho f(1) \max_i \Delta_i < \max_i \Delta_i$ whenever some $z_i > 0$. If all $z_i = 0$, the potential is unchanged. The maximizing recipient therefore maintains the potential bound and positive slacks, and positive final slacks give $n\rho$-PROP1, that is, $19/30$-PROP1.

The function behaves as $1/x$ near zero and is logarithmic on an interval containing one. This keeps $f(1) < 30/19$, and hence $n\rho f(1) < 1$, while preserving both comparisons. Appendix C gives the complete induction; Appendix G.2 gives the formula and its verification. Assigning each good requires one score per agent.

For the $n/(n+\kappa)$ factor, instead set $\rho = 1/(n+\kappa)$ and minimize $\sum_{i:p_i>0} 1/s_i^t$. Only agents with $z_i > 0$ have changing terms, and there are at most κ of them. Appendix G.3 assigns these agents probabilities of receiving the good so that each of their reciprocal terms is unchanged in expectation. These probabilities sum to at most one; distributing the remainder can only decrease the expectation. Some recipient therefore keeps the total potential from increasing. When κ is given, choose the rule with the larger guarantee before the first good arrives. Both comparisons fix only the realized history and current good, so they apply against adaptive adversaries. □

Imperfect predictions. Following Choo et al. (2026), for each agent with $p_i > 0$, increase the value of the first good worth at least $(1-\varepsilon)p_i$ to p_i. The one-sided-error assumption ensures that this good appears and that all values are at most p_i. The modified instance thus has exact predictions, and running our rules on it gives the following result, proved in Appendix C.2.

Corollary 4.2. *Given predictions with one-sided error bound $\varepsilon \in [0,1)$, a deterministic online algorithm guarantees $\frac{19n(1-\varepsilon)}{30n-19\varepsilon}$-PROP1 against adaptive adversaries. With a known upper bound $\kappa \in [2,n]$, it guarantees the larger of this factor and $\frac{n(1-\varepsilon)}{n+\kappa-\varepsilon}$. Neither guarantee needs the number of goods in advance.*

The first factor is at least $\frac{19}{30}(1-\varepsilon)$, so it remains a positive constant for every fixed $\varepsilon < 1$, independently of n.

4.2 The Same PROP1 Factor with Additive Envy Bounds

Assume exact predictions and a known number of goods m. Define $\widehat{v}_i(g) = v_i(g)/p_i$ for $p_i > 0$, and zero otherwise. These values lie in $[0,1]$; the scaling preserves PROP1 and measures agent i's envy in units of their maximum item value.

Theorem 4.3. *With exact MIV predictions and known m, a deterministic online algorithm guarantees $19/30$-PROP1 against adaptive adversaries and $\max_{i,j}(\widehat{v}_i(A_j) - \widehat{v}_i(A_i)) = O(\log n + \sqrt{(m\log n)/n})$ with an absolute hidden constant. For $m \geq n\log n$, the bound is $O(\sqrt{m\log n/n})$.*

This result uses a different allocation rule. Set $\rho = 19/(30n)$ and let $E_{ij}^t = \widehat{v}_i(A_j^t) - \widehat{v}_i(A_i^t)$. Following the exponential-potential approach of Benadè et al. (2018), for $\eta > 0$ define $F_\eta^t := \sum_{i\neq j} e^{\eta E_{ij}^t}(1 + f(s_j^t + \rho))$, where the f term is zero when $p_j = 0$. The multiplier links the two fairness objectives: receiving a good may increase other agents' envy toward its recipient j, but it also decreases j's PROP1 term whenever $z_j > 0$. Using the *envied agent's* term lets this decrease compensate for the increase in envy. The envy terms use the normalized values $\widehat{v}_i$, which, unlike z_i, count each first maximum-valued good.

The rule minimizes a weighted sum of Φ^t and F_η^t. Either, in expectation, a uniformly random recipient does not increase Φ^t and increases F_η^t by at most a factor $1 + 2\eta^2/n$, or some single recipient decreases the weighted sum. The resulting bound keeps all slacks positive and bounds every exponential envy term. Appendix C.3 specifies the weights and full algorithm, and Appendix G.4 proves the comparison. The rule uses $O(n)$ evaluations of f and $O(n^2)$ additional operations per good. Appendix H gives a simpler alternative with explicit envy constant 11 and a smaller $\Theta(1/\log n)$ PROP1 factor.

5 Tight Guarantees for Like and Rand

Choo et al. (2026) showed that Rand, which assigns each good independently and uniformly among all n agents, has a tight $\Theta(1/\log(n/\delta))$ PROP1 guarantee with probability at least $1-\delta$ against a non-adaptive adversary. The classical Like rule of Aleksandrov et al. (2015), with truthful reports, instead assigns each good independently and uniformly among the agents who value it positively. Thus, when $i \in N(g)$, agent i receives g with probability $1/|N(g)|$ under Like rather than $1/n$ under Rand. We determine whether this larger probability improves PROP1 when few agents value the same good, and how the two rules compare for additive envy.

As in the result of Choo et al. for Rand, we assume a non-adaptive adversary. This is the setting needed for the concentration analysis below. Without future information, Theorem 3.1 rules out

every positive PROP1 approximation against an adaptive adversary, even for randomized algorithms, so the non-adaptive assumption is necessary for the guarantees studied here.

Let A^{LIKE} and A^{RAND} denote the final allocations. If $N(g) = \varnothing$, LIKE assigns g arbitrarily; this only makes the allocation complete and does not change any agent's value. Our analysis allows general nonnegative additive valuations.

Recall that $\kappa = \max_{g \in G} |N(g)|$. If $\kappa \leq 1$, every good with $N(g) \neq \varnothing$ is assigned by LIKE to the unique agent in $N(g)$, so LIKE is proportional. We therefore assume $2 \leq \kappa \leq n$.

We show that LIKE improves on the PROP1 factor of RAND by order n/κ, until the factor reaches one, and that the dependence on n, κ, and δ is tight up to constants. The envy comparison goes in the opposite direction because assigning a good among fewer agents increases pairwise variance.

5.1 Tight High-Probability PROP1 Guarantees

Theorem 5.1. *For every fixed instance with $2 \leq \kappa \leq n$ and every $\delta \in (0,1)$, LIKE returns an α-PROP1 allocation with probability at least $1-\delta$, where $\alpha = \min\{1, 3n/(32\kappa \log(n/\delta))\}$.*

The proof uses the one good that PROP1 allows an agent to add. Fix an agent i. If her largest good is worth at least the target $\alpha v_i(G)/n$, she satisfies α-PROP1 regardless of the allocation: she either receives that good or can add it. Otherwise, every good she values is smaller than the target. Since LIKE assigns each such good to her with probability at least $1/\kappa$, independently across goods, her received value concentrates around a value of at least $v_i(G)/\kappa$. More precisely, Lemma D.1 gives, for $r := \alpha\kappa/n < 1$,

$$\Pr[\text{agent } i \text{ does not satisfy } \alpha\text{-PROP1}] \leq \exp\left(-\frac{3(1-r)^2}{8r}\right).$$

The stated choice of α makes this at most δ/n, and a union bound proves the theorem. In particular, LIKE gives exact PROP1 with probability at least $1-\delta$ when $\kappa \leq 3n/(32\log(n/\delta))$.

Both the n/κ and the $\log(n/\delta)$ dependence are necessary up to constants. Form $\lfloor n/\kappa \rfloor$ disjoint groups of κ agents and give each group $\Theta(n/\alpha)$ goods, each worth one to the members of the group and zero to all other agents. Choose the number of goods so that the target is at least two. An agent who receives none then violates α-PROP1, since adding one good gives value at most one. The probability of receiving none is at least $\exp(-O(n/(\alpha\kappa)))$. A second-moment argument over the agents gives the matching upper limit on α (Theorem D.2). For RAND, a good valued by only one agent still reaches that agent with probability only $1/n$, so a similar instance shows that its $\Theta(1/\log(n/\delta))$ factor cannot be improved even when $\kappa = 1$ (Theorem D.3). Appendix D gives the complete proofs.

Corollary 5.2. *Fix $n \geq 2$, $2 \leq \kappa \leq n$, and $\delta \in (0, 1/2]$. Over fixed instances with at most κ agents valuing any good positively, the largest PROP1 factor guaranteed by LIKE with probability at least $1-\delta$ is $\Theta(\min\{1, \frac{n}{\kappa \log(n/\delta)}\})$. For RAND, the largest such factor over all fixed instances is $\Theta(1/\log(n/\delta))$, even if each good is positively valued by exactly one agent. The examples proving the upper limits on both guarantees use binary valuations.*

5.2 Comparing Additive Envy

The proportionality advantage of LIKE need not translate into less envy. For each agent, define

$$R_i^{\text{LIKE}} := \sum_{g : v_i(g) > 0} \frac{v_i(g)^2}{|N(g)|}, \quad R_i^{\text{RAND}} := \frac{1}{n} \sum_{g \in G} v_i(g)^2. \tag{10}$$

The pairwise concentration approach of Benadè et al. (2018) gives the RAND bound below. We obtain the corresponding bound for LIKE and compare the two quantities on the same instance.

Theorem 5.3. *Fix $\delta \in (0,1)$. For either rule $\mathcal{R} \in \{\text{LIKE}, \text{RAND}\}$ on a fixed instance, with probability at least $1-\delta$,*

$$\text{Env}(A^{\mathcal{R}}) \leq \max_i \left\{ 2\sqrt{R_i^{\mathcal{R}} \log \frac{n(n-1)}{\delta}} + \frac{p_i^{\max}}{3} \log \frac{n(n-1)}{\delta} \right\}.$$

Moreover, $R_i^{\text{RAND}} \le R_i^{\text{LIKE}}$ *for every agent, so the stated bound for* RAND *is no larger.*

To see where these quantities arise, fix agents $i \neq j$. Each good that i values positively contributes $v_i(g)$, $-v_i(g)$, or zero to $v_i(A_j) - v_i(A_i)$. Under RAND, its variance is $2v_i(g)^2/n$. Under LIKE, the variance is at most $2v_i(g)^2/|N(g)|$, and its expectation is nonpositive. Goods with $v_i(g) = 0$ contribute zero. Independence across goods and a union bound over pairs give the theorem; Appendix D.2 provides the details.

The comparison of upper bounds alone does not order actual envy. The next construction shows that larger envy under LIKE can occur. Let the same κ agents value every good at one, with all other values zero. Under LIKE, the difference between the bundle sizes of two of these agents has variance $2m/\kappa$. Bounding its fourth moment gives the following result.

Proposition 5.4. *For every* $2 \le \kappa \le n$ *and integer* $m \ge \kappa$*, there is a binary instance with* $\max_g |N(g)| = \kappa$ *such that* $\mathbb{E}[\text{Env}(A^{\text{LIKE}})] \ge \frac{9}{64\sqrt{2}}\sqrt{m/\kappa}$.

On these same instances, RAND has expected maximum envy $O(\sqrt{m \log n/n} + \log n)$ (Corollary D.5). For example, with $\kappa = 2$ and $m \ge n \log n$, this is asymptotically smaller than the $\Omega(\sqrt{m})$ lower bound for LIKE as n grows. Thus restricting recipients can improve proportionality while increasing envy.

6 CONCLUSION

This paper closes three gaps in online fair division. First, without information about future goods, no online algorithm, even a randomized one, guarantees any positive approximation to PROPk against an adaptive adversary; for chores, no factor below the trivial factor n is possible. Second, a prediction of each agent's maximum item value improves the previous $1/n$-PROP1 guarantee to $19/30$-PROP1, and to $\max\{19/30, n/(n+\kappa)\}$-PROP1 when an upper bound κ on the number of agents who value any good positively is known. The factor remains a positive constant under any fixed one-sided prediction error below one, and with exact predictions and a known horizon $m \ge n \log n$, the $19/30$ factor holds together with $O(\sqrt{m \log n/n})$ normalized maximum additive envy. Third, against a non-adaptive adversary, LIKE has a tight $\Theta(\min\{1, n/(\kappa \log(n/\delta))\})$ PROP1 guarantee, whereas RAND remains at $\Theta(1/\log(n/\delta))$ even when each good is valued by only one agent.

Returning to our motivating question, the negative results in online fair division are not merely a consequence of demanding fairness notions: PROP1 is implied by both EF1 and MMS, yet an adaptive adversary rules out every positive approximation to it. Positive results must therefore change the information, valuation, or adversarial assumptions, and we show that changing either the information or the adversary suffices. A maximum item value prediction reveals neither which good attains the maximum nor when it arrives, yet it is enough to account for the one good that PROP1 allows an agent to add; EF1, MMS, and PROPX remain inapproximable with the same predictions. Against a non-adaptive adversary, random allocation can instead rely on concentration. Randomization is then essential, since such an adversary can simulate any deterministic algorithm.

The parameter κ helps in both positive settings: $n/(n+\kappa)$ approaches one when κ is small relative to n, and LIKE is exactly PROP1 with probability at least $1 - \delta$ when $\kappa \le 3n/(32\log(n/\delta))$. By contrast, our impossibility for adaptive adversaries already holds when $\kappa = 2$. This advantage of LIKE has a cost: restricting each good to agents who value it raises their expected value but also the pairwise fluctuations behind additive envy, so proportionality and envy can favor different rules.

Several questions remain. What is the optimal PROP1 factor with exact MIV predictions, and can randomization improve it against an adaptive adversary? Our PROP1 guarantees need neither exact predictions nor the number of goods, whereas our simultaneous envy guarantee uses both; can it be retained without them? Against a non-adaptive adversary, can any randomized online algorithm improve on the $\Theta(1/\log(n/\delta))$ PROP1 factor of RAND on unrestricted instances, and can a single rule match both the PROP1 guarantee of LIKE and the envy bound of RAND? Finally, for chores, can a comparable one-number-per-agent prediction improve on the trivial factor n?

AI USE STATEMENT

In this work, we used generative AI tools to polish the exposition of the paper, including both the prose and the presentation of technical proofs, with the aim of improving clarity and readability. The key ideas, arguments, and results presented in an initial version of this work were developed independently by the authors. We subsequently used GPT-6 to improve our initial $1/2$-PROP1 guarantee to $19/30$-PROP1, and to extend the simultaneous PROP1 and additive-envy guarantee to the same factor. We verified its correctness and improved the argument substantially. Other uses (including generating synthetic datasets; designing or providing feedback on experiments; cleaning or reformatting datasets; supporting qualitative or thematic data analysis; and interpreting results) are not applicable to this work. All AI-assisted revisions and literature suggestions were reviewed and independently verified by the authors. In particular, the mathematical arguments and proofs were checked by the authors for correctness, and literature identified with the assistance of generative AI was verified against the original sources for relevance and accuracy. The authors take full responsibility for the final content of this work, including any text, claims, or other material produced with the assistance of generative AI.

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Appendix Guide

The appendices contain further related work (Appendix A), the online impossibility proof (Appendix B), the prediction algorithms and their complete proofs (Appendix C), and the random-allocation proofs (Appendix D). Appendices E and F give the other fairness notions and the chore results. Appendix G proves the technical inequalities for the prediction algorithms, and Appendix H gives the simpler simultaneous PROP1 and envy guarantee.

A Further Related Work

Online fair division. Aleksandrov et al. (2015) introduced the food bank model and studied the fairness and incentive properties of simple online rules, including LIKE. Under truthful reports, LIKE is envy-free ex ante, but its ex-post envy is unbounded even for binary valuations and two agents. Benadè et al. (2018) studied additive envy over time and designed an online allocation rule with vanishing envy per good. Aleksandrov & Walsh (2020) survey the earlier literature.

More recent work studies fairness of the final allocation. Zhou et al. (2023) study MMS under normalization information. Neoh et al. (2026) compare several kinds of information about future goods and give positive and negative results for EF, proportionality, and MMS. Choo et al. (2026) focus on approximate PROP1, including greedy lower bounds, uniform random allocation against a non-adaptive adversary, and maximum item value predictions. Melissourgos & Protopapas (2026) study online EFX with predictions of the agents' full valuation vectors and measure prediction error by total variation distance. Their prediction input and fairness objective differ from the MIV prediction input and PROP1 objective studied here.

Positive results are also known for binary and two-value valuation classes (Amanatidis et al., 2025; Wang & Wei, 2026). For chores, Seddighin & Seddighin (2025) and Song et al. (2025) give contemporaneous n-factor lower bounds for online MMS. Their constructions use recursion over the number of agents and rapidly increasing costs. Appendix F strengthens the inductive statement to control the cost left after removing any fixed number k of chores. This gives the PROPk lower bound, applies to randomized algorithms against adaptive adversaries, and gives consequences for several other notions.

Our first result differs from the earlier negative results in two ways. It applies to every online algorithm rather than selected greedy rules, and it rules out every positive approximation even when each agent may add any fixed number of goods. Our maximum item value result improves the previous $1/n$ PROP1 factor to $19/30$ without asking for more information.

Other online models. Some papers ask for fairness after every prefix, with an additive error that may grow with time (Kahana et al., 2026). Adams & Segal-Halevi (2026) study repeated assignments generated by sequences of permutations and show that certain balance conditions imply ordinal PROP1 or PROP2 after every day. Amanatidis et al. (2026) allow a bounded number of goods to be held temporarily and reordered before allocation. These models use different arrival or fairness requirements and are therefore complementary to our negative result. Cohen et al. (2026) study a different extension in which each assignment must satisfy a budget constraint and some goods may be left unallocated.

A related model is *temporal fair division*, which evaluates the cumulative allocation after every prefix. Elkind et al. (2025) assume that the complete arrival sequence is known in advance and study temporal EF1 for goods and chores. Choi et al. (2026) extend this model to mixed manna and identify restricted settings in which temporal fairness can be achieved by online rules. Goldberg et al. (2026) study computational questions related to minimizing the sum of envy in each round. These works require fairness throughout the sequence, whereas our guarantees concern the final allocation and, unless stated otherwise, use no information about future goods.

Chen & Tan (2026) give a broad competitive analysis for several fairness notions, goods and chores, and several valuation assumptions. Their ratio compares the fairness achieved online with the best fairness level attainable offline on the same instance. Our approximation factor instead compares each agent's final value directly with the PROPk requirement, as most other prior works in the area do. The two comparisons answer different questions.

For envy minimization, Halpern et al. (2025) determine the optimal order for general algorithms against a non-adaptive adversary through a connection with online multicolor discrepancy. Our Section 5 instead compares LIKE and RAND on the same fixed instance to determine the effect of assigning each good only among agents who value it positively. The comparison shows that this choice can improve proportionality while increasing additive envy, a question not answered by the optimal worst-case envy rate for general algorithms.

B PROOFS OF THE ONLINE IMPOSSIBILITY RESULT

We now show that no positive multiplicative approximation to PROPk is possible in the standard fully online model. We first prove the deterministic statement, then explain why the same construction applies to randomized algorithms with probability one.

Main-text result (Theorem 3.1). Fix $n \geq 2$, $k \geq 1$, and $\alpha \in (0, 1]$. For every online algorithm, deterministic or randomized, there is an adaptive adversary under which the final allocation is not α-PROPk with probability one. The statement remains true even when all of the following hold:

(i) the final number of goods is known before the first good arrives;

(ii) every value lies in $[0, 1]$; and

(iii) every good is positively valued by at most two agents.

The proof has three steps. First, allowing k added goods can improve the factor by at most k. Second, any guarantee for n agents gives a guarantee for two agents. We then construct the two-agent adversary.

Lemma B.1. *Under nonnegative additive valuations, every α-PROPk allocation is also (α/k)-PROP1.*

Proof. Fix an agent i. Since the allocation is α-PROPk, there is a set $S_i \subseteq G \setminus A_i$ with $|S_i| \leq k$ such that $v_i(A_i) + v_i(S_i) \geq \alpha \frac{v_i(G)}{n}$. Every good in S_i has value at most h_i, so $v_i(S_i) \leq kh_i$. Thus,

$$k(v_i(A_i) + h_i) \geq v_i(A_i) + kh_i \geq v_i(A_i) + v_i(S_i) \geq \alpha \frac{v_i(G)}{n},$$

where the first inequality uses $k \geq 1$ and $v_i(A_i) \geq 0$. Therefore,

$$v_i(A_i) + h_i \geq \frac{\alpha}{k} \frac{v_i(G)}{n}.$$

By (1), agent i satisfies (α/k)-PROP1. Since i was arbitrary, the result follows. □

Lemma B.2. *Fix $n \geq 3$ and $\alpha \in (0, 1]$. If a deterministic online algorithm guarantees α-PROP1 for n agents, then a deterministic two-agent online algorithm guarantees $(2\alpha/n)$-PROP1 against the same class of adversaries.*

Proof. Let $\mathcal{A}$ be the n-agent algorithm. We construct a two-agent algorithm $\mathcal{B}$. When a good g arrives with values $(v_1(g), v_2(g))$, algorithm $\mathcal{B}$ runs $\mathcal{A}$ on the value vector $(v_1(g), v_2(g), 0, \ldots, 0)$. If $\mathcal{A}$ assigns g to agent 2, then $\mathcal{B}$ also assigns it to agent 2; otherwise, $\mathcal{B}$ assigns it to agent 1.

The adversary class is preserved. A fixed two-agent sequence remains fixed after appending zeros. Given an adaptive two-agent adversary, the corresponding n-agent adversary appends zeros to each value vector and, after $\mathcal{A}$ assigns the good, maps recipient 2 to agent 2 and every other recipient to agent 1 before generating the next vector.

Let $(A_1, \ldots, A_n)$ be the allocation produced by $\mathcal{A}$, and let $D := A_3 \cup \cdots \cup A_n$. By construction, $\mathcal{B}$ produces the bundles $B_1 := A_1 \cup D$ and $B_2 := A_2$. For each $i \in \{1, 2\}$, the α-PROP1 guarantee of $\mathcal{A}$ gives

$$v_i(A_i) + \max_{g \in G \setminus A_i} v_i(g) \geq \frac{\alpha}{n} v_i(G). \tag{11}$$

For agent 1 under $\mathcal{B}$:

$$v_1(B_1) + \max_{g \in G \setminus B_1} v_1(g) = v_1(A_1) + v_1(D) + \max_{g \in A_2} v_1(g) \geq v_1(A_1) + \max_{g \in G \setminus A_1} v_1(g). \tag{12}$$

The inequality uses $G \setminus A_1 = A_2 \cup D$ and $v_1(D) \geq \max_{g \in D} v_1(g)$, which follows from nonnegativity and additivity. For agent 2 under $\mathcal{B}$:

$$v_2(B_2) + \max_{g \in G \setminus B_2} v_2(g) = v_2(A_2) + \max_{g \in G \setminus A_2} v_2(g). \tag{13}$$

Combining (12) and (13) with (11), each agent $i \in \{1, 2\}$ satisfies

$$v_i(B_i) + \max_{g \in G \setminus B_i} v_i(g) \geq \frac{\alpha}{n} v_i(G) = \frac{2\alpha}{n} \frac{v_i(G)}{2}.$$

Thus (B_1, B_2) is $(2\alpha/n)$-PROP1. □

Choo et al. (2026) use the current PROP1 ratio to analyze selected greedy rules. The construction below instead derives update formulas for the normalized slack in each agent's current PROP1 inequality and applies to every possible sequence of recipients. This is what allows the lower bound to cover every online algorithm.

It remains to prove the two-agent statement.

Lemma B.3. *Fix $\alpha \in (0, 1]$. There is an adaptive strategy for two agents such that every possible sequence of recipients reaches, after finitely many goods, a prefix that is not α-PROP1.*

Proof. Let G' be the goods seen so far, and let A_1, A_2 be the current bundles. Set $\gamma := \frac{\alpha}{2}$ and $\beta := 1 - \gamma$. Applying the notation of Section 2 to the current allocation of G', we have $h_i = \max_{g \in A_{-i}} v_i(g)$ for $i \in \{1, 2\}$, where A_{-i} is the other agent's bundle. The current allocation is α-PROP1 exactly when

$$v_i(A_i) + h_i \geq \gamma v_i(G') \quad \text{for } i = 1, 2. \tag{14}$$

We first make both h_i positive. Present goods with values $(1, 1)$. If one agent has received none after ℓ such goods, then that agent has $v_i(A_i) = 0$, $h_i = 1$, and $v_i(G') = \ell$. Inequality (14) is violated once $\ell > 1/\gamma$. Thus, after finitely many goods, either the allocation is not α-PROP1 or both agents have received a positively valued good. From then on, $h_1, h_2 > 0$.

For the rest of the proof, define the normalized slack in agent i's PROP1 inequality

$$r_i := \frac{v_i(A_i) + h_i - \gamma v_i(G')}{h_i}.$$

The current allocation satisfies α-PROP1 exactly when $r_1, r_2 \geq 0$. Suppose the next good has values $(u_1 h_1, u_2 h_2)$, where the h_i are measured immediately before the good arrives; we call (u_1, u_2) its *normalized values*. The update for agent i is as follows:

$$r_i^+ = \begin{cases} r_i + \beta u_i, & \text{if agent } i \text{ receives the good,} \\ r_i - \gamma u_i, & \text{if agent } i \text{ does not receive it and } u_i \leq 1, \\ \beta + \dfrac{r_i - 1}{u_i}, & \text{if agent } i \text{ does not receive it and } u_i > 1. \end{cases} \tag{15}$$

The first two lines follow because h_i does not change. In the third line, the new good becomes the best good in the other bundle, so the new denominator is $u_i h_i$.

In particular, if

$$r_1 < \gamma \quad \text{and} \quad r_2 < \gamma, \tag{16}$$

then one more good with normalized values $(1, 1)$ makes the resulting allocation not α-PROP1: whichever agent does not receive it has $r_i^+ = r_i - \gamma < 0$.

Let $q := \lfloor \beta/\gamma \rfloor$ and $\eta := \frac{\beta + (q+1)\gamma}{2}$. Since $\gamma \leq 1/2$, we have $q \geq 1$ and

$$q\gamma \leq \beta < \eta < (q+1)\gamma. \tag{17}$$

We next choose $t, s_1, \ldots, s_q \in (0, 1)$ so that

$$\gamma s_j > \beta \sum_{h<j} s_h + \beta t \quad \text{for all } j = 1, \ldots, q, \tag{18}$$

$$\gamma > \beta \sum_{h=1}^{q} s_h + \beta t. \tag{19}$$

The first inequality will be used when P_j is the first good in the list below assigned to agent 2, and the second when all q goods are assigned to agent 1. To see that such a choice exists, set $R := 1 + 4\beta/\gamma$. Choose $c > 0$ such that $cR^{q-1} < 1$ and $\beta c \sum_{h=0}^{q-1} R^h < \gamma/2$, and set $s_j := cR^{j-1}$ for all $j = 1, \dots, q$, and $t := \frac{\gamma c}{4\beta}$.

Then $t, s_1, \dots, s_q \in (0, 1)$. For every j,

$$\beta \sum_{h<j} s_h < \frac{\gamma}{4} s_j, \quad \beta t = \frac{\gamma c}{4} \le \frac{\gamma}{4} s_j,$$

which gives (18). Moreover, $\beta \sum_{h=1}^{q} s_h < \frac{\gamma}{2}$ and $\beta t < \frac{\gamma}{4}$, so (19) also holds.

Now, define $S_j := \sum_{h=1}^{j} s_h$ and $S_0 := 0$. Start from any allocation that still satisfies α-PROP1. Present a good with normalized values (t, U), where $U > 1$ is chosen from the current r_2. If $r_2 \le 1$, choose $U > \max\{1, \frac{1-r_2}{\beta}\}$; if $r_2 > 1$, choose $U > \max\{1, \frac{r_2-1}{\eta-\beta}\}$. If agent 1 receives this good, then the third line of (2) gives $0 < r_2^+ < \eta$, and r_1 increases by βt. If agent 2 receives it, then r_1 decreases by γt. Repeat with a newly chosen U until either the allocation is not α-PROP1 or agent 1 receives one of these goods. The repetition is finite: each consecutive assignment to agent 2 decreases r_1 by $\gamma t > 0$, so r_1 would eventually become negative. Thus, as long as the allocation remains α-PROP1, the adversary reaches an allocation with $r_2 \le \eta$. Over the whole repetition, r_1 increases by at most βt.

Suppose now that $r_2 \le \eta$. Present, in order, goods $P_1, \dots, P_q$ with normalized values $P_j = (s_j, 1)$, and stop the list when one is first assigned to agent 2. Suppose P_j is the first such good. The earlier goods $P_1, \dots, P_{j-1}$ went to agent 1, so the change in r_1 is $\beta S_{j-1} - \gamma s_j < -\beta t$ by (18). Bringing r_2 below η again can increase r_1 by at most βt. Hence r_1 decreases overall.

Suppose instead that all $P_1, \dots, P_q$ go to agent 1. As long as the allocation remains α-PROP1, agent 2 misses q goods with second normalized value one, and therefore $r_2 \le \eta - q\gamma < \gamma$ by definition of q, η, and (17). Present one more good with normalized values $(1, 1)$. If it goes to agent 1, the resulting allocation is not α-PROP1. Otherwise it goes to agent 2, and the total change in r_1 over this list and the additional good is $\beta S_q - \gamma < -\beta t$ by (19). Bringing r_2 below η again increases r_1 by at most βt.

Define

$$d := \min \left\{ \min_{1 \le j \le q} (\gamma s_j - \beta S_{j-1} - \beta t), \gamma - \beta S_q - \beta t \right\}. \tag{20}$$

Then, (18) and (19) give $d > 0$. We have shown that, whenever $r_2 \le \eta$, the adversary either makes the allocation not α-PROP1 or returns to an allocation with $r_2 \le \eta$ and lowers r_1 by at least d.

Now, first bring r_2 below η. Repeat the preceding decrease until either the allocation is not α-PROP1 or

$$r_1 < \gamma - \beta S_q. \tag{21}$$

This takes finitely many repetitions because every allocation that still satisfies α-PROP1 has $r_1 \ge 0$, while each completed repetition lowers r_1 by at least d.

Present $P_1, \dots, P_q$ again. If one of these goods goes to agent 2, bring r_2 below η again. Then r_1 decreases by at least d, and (21) remains true. This outcome cannot occur indefinitely without making r_1 negative. Therefore, if the allocation remains α-PROP1, at some repetition all $P_1, \dots, P_q$ go to agent 1. At that point, $r_1^+ < \gamma$ and $r_2^+ \le \eta - q\gamma < \gamma$. A final good with normalized values $(1, 1)$ makes the resulting allocation not α-PROP1, by (16). □

Proof of Theorem 3.1. Suppose, for contradiction, that a deterministic n-agent online algorithm guarantees α-PROPk. By Lemma B.1, it would guarantee (α/k)-PROP1. If $n = 2$, applying Lemma B.3 with factor α/k gives a contradiction. If $n \ge 3$, Lemma B.2 would give a two-agent algorithm guaranteeing $(2\alpha/(kn))$-PROP1, and applying Lemma B.3 with this factor again gives a contradiction. The reduction presents value vectors of the form $(v_1(g), v_2(g), 0, \dots, 0)$, so each good is positively valued by at most two agents.

It remains to justify the known horizon and the value bound. Fix a deterministic choice whenever the adversarial strategy permits several values, and apply the strategy to every possible recipient

history, stopping each branch when the allocation is no longer α-PROPk. This defines a rooted tree independently of the algorithm and of any horizon announced to it. Every nonterminal node has at most n children, one for each possible recipient of the next good, and the proof above shows that every branch is finite. Hence there is a common finite upper bound M on all branch lengths; otherwise, König's lemma would give an infinite branch.

The adversary sets the horizon to M and reveals it before the first good arrives. If a branch stops earlier, it fills the remaining positions with goods of value zero to every agent. These goods do not change any PROPk inequality, so the final allocation is still not α-PROPk. The resulting tree contains only finitely many values. Multiplying every value in the construction by one common positive constant makes all values lie in $[0, 1]$ without changing any allocation inequality or any normalized update along a given recipient history. The algorithm may choose different recipients after this multiplication or after learning M, but every such history is already included in the tree. This establishes the deterministic case.

Randomized algorithms. For every randomized online algorithm, there is an adaptive adversary under which the final allocation is not α-PROPk with probability one, and items (i)–(iii) of Theorem 3.1 continue to hold. The construction in Lemma B.2 is applied to the realized recipient of each good, while the strategy in Lemma B.3 uses only earlier realized recipients and reaches a violation for every possible sequence of recipients. Together with Lemma B.1, the violation occurs for every realization of the algorithm's private random choices, and hence with probability one, by definition. □

C Prediction Algorithms and Complete Proofs

Choo et al. (2026) show that exact MIV predictions give $1/n$-PROP1 and leave open the optimal dependence on n. We remove that dependence with a $19/30$ guarantee, and obtain $n/(n+\kappa)$ when an upper bound κ is known and this factor is larger. For a known horizon, we retain $19/30$-PROP1 together with normalized maximum additive envy $O(\log n + \sqrt{m \log n/n})$; when $m \geq n \log n$, this becomes $O(\sqrt{m \log n/n})$, with an absolute hidden constant. Agents with $p_i = 0$ value every good at zero, so their fairness inequalities hold automatically. Accordingly, the definitions below include only agents with $p_i > 0$, while n continues to denote the original number of agents.

C.1 A Constant Factor PROP1 Approximation Independent of n

For an agent with $p_i > 0$, let $t_i^* := \min\{t \in [m] : v_i(g_t) = p_i\}$ be the first time a maximum-valued good appears. For $t = 0, \ldots, m$, let $G^t := \{g_1, \ldots, g_t\}$ and let A_i^t be agent i's bundle after the first t goods. Define

$$D_i^t := v_i(G^t) + p_i \mathbf{1}[t < t_i^*], \quad B_i^t := v_i(A_i^t \setminus \{g_{t_i^*}\}). \tag{22}$$

The quantity D_i^t includes the predicted maximum p_i before a good of that value arrives. At time t_i^*, the prediction is replaced by the actual value p_i, so D_i^t does not change. The quantity B_i^t is the value received by agent i, except that it does not count this first maximum-valued good. Both quantities can be updated online.

At the end, $D_i^m = v_i(G)$ and $p_i + B_i^m \leq v_i(A_i) + h_i$. Indeed, if agent i did not receive $g_{t_i^*}$, they can add it and obtain $v_i(A_i) + p_i = B_i^m + p_i$. If they did receive it, then $B_i^m + p_i = v_i(A_i)$.

For a target coefficient $\rho \in (0, 1/n)$, define the normalized PROP1 slack

$$s_i^t := \frac{p_i + B_i^t - \rho D_i^t}{p_i}. \tag{23}$$

Initially, $s_i^0 = 1 - \rho$. By the preceding inequality and (1), positive final slacks imply $n\rho$-PROP1. For the current good g_t, write

$$z_i := \begin{cases} v_i(g_t)/p_i, & p_i > 0 \text{ and } t \neq t_i^*, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

The first maximum-valued good can be recognized using p_i and whether such a good has already appeared. For every agent with $p_i > 0$, assigning g_t to j gives us

$$s_i^t = s_i^{t-1} - \rho z_i + z_i \mathbf{1}[i = j]. \tag{25}$$

The treatment of the first maximum-valued good follows Choo et al. (2026).

From individual terms to the total potential. We obtain the $19/30$-PROP1 guarantee by showing that, for some recipient, the decrease in that agent's potential term offsets the total increase in the other agents' terms.

Let f be the fixed positive decreasing function defined in Appendix G.2. In particular,

$$f(x) > \frac{1}{x} \quad \text{for } 0 < x \le 1, \quad \text{and} \quad 1 < f(1) < \frac{30}{19}. \tag{26}$$

Set $\rho := 19/(30n)$ and define

$$\Phi^t := \sum_{i:p_i>0} f(s_i^t + \rho). \tag{27}$$

The shift by ρ makes every initial argument equal to one, so $\Phi^0 \le nf(1) < 1/\rho$. If all arguments of f are positive and $\Phi^t \le nf(1)$, then every slack is positive. Indeed, if $s_i^t \le 0$, then $0 < s_i^t + \rho \le \rho < 1$, and (6) gives us $f(s_i^t + \rho) > \frac{1}{s_i^t+\rho} \ge \frac{1}{\rho}$, contradicting $\Phi^t \le nf(1) < 1/\rho$. Thus it is enough to keep all arguments positive and maintain $\Phi^t \le nf(1)$. Our function behaves as $1/x$ near zero and is logarithmic on an interval containing one. This keeps $f(1) < 30/19$ while preserving the bounds used to compare the possible recipients.

Allocation rule. When g_t arrives, assign it to an agent maximizing

$$\Delta_i := f(s_i^{t-1} + \rho - \rho z_i) - f(s_i^{t-1} + \rho + (1-\rho)z_i), \tag{28}$$

with $\Delta_i := 0$ when $p_i = 0$ and a fixed order for breaking ties. This is exactly the recipient minimizing the next value of Φ^t.

We also use a reciprocal rule for instances in which few agents value the same good. With $\rho := 1/(n+\kappa)$, this rule minimizes

$$\Psi^t := \sum_{i:p_i>0} \frac{1}{s_i^t}. \tag{29}$$

The two rules are alternatives: when κ is given, choose the one with the larger guarantee before the first good arrives.

Then, our result is as follows.

Main-text result (Theorem 4.1). Suppose exact MIV predictions are available. Against adaptive adversaries:

(i) The rule in (8) returns a $19/30$-PROP1 allocation.

(ii) If the algorithm is given an upper bound $\kappa \in [2, n]$ on $|N(g)|$ for every good, the rule in (29) returns an $n/(n+\kappa)$-PROP1 allocation. Choosing between the two rules above gives a $\max\{19/30, n/(n+\kappa)\}$-PROP1 allocation.

Proof. For (i), take $\rho = 19/(30n)$. We maintain positive slacks and $\Phi^t \le nf(1)$. Both statements hold initially. Fix the realized history and current good, and suppose the statements hold at time $t-1$. Every candidate argument in (8) is positive, since $s_i^{t-1} + \rho - \rho z_i \ge s_i^{t-1} > 0$.

For every agent with $p_i > 0$, let $d_i := f(s_i^{t-1} + \rho - \rho z_i) - f(s_i^{t-1} + \rho)$, and set $d_i := 0$ when $p_i = 0$. Assigning the good to j changes the potential by exactly

$$\Phi^t(j) - \Phi^{t-1} = \sum_{i \in N} d_i - \Delta_j. \tag{30}$$

Thus it suffices to show that $\sum_i d_i \le \max_i \Delta_i$.

If all $z_i = 0$, the potential is unchanged. Otherwise, let $M := \max_i \Delta_i > 0$ and choose $S > \rho$ so that $f(S-\rho) - f(S+1-\rho) = M$. Such an S exists uniquely because f is strictly decreasing and

strictly convex, with limits $+\infty$ at zero and zero at infinity. The comparison proved in Appendix G.1 gives

$$\frac{\sum_i d_i}{M} \le n \frac{f(S-\rho)-f(S)}{f(S-\rho)-f(S+1-\rho)} \min\left\{1, \frac{f(1)}{f(S)}\right\} \le n\rho f(1) < 1. \tag{31}$$

The two bounds in the minimum have different roles. One bounds each increase d_i directly and then sums over at most n agents. The other bounds d_i relative to the agent's current potential term and uses $\Phi^{t-1} \le nf(1)$. The latter is stronger for $S \le 1$, and the former for $S \ge 1$. Together they show that the largest available decrease is enough; no distribution over recipients is required.

By (9), the maximizing recipient maintains $\Phi^t \le nf(1)$. All updated arguments are positive, and (6) now implies $s_i^t > 0$, completing the induction. At the final time, $D_i^m = v_i(G)$, so

$$p_i + B_i^m > \rho v_i(G) = \frac{19}{30}\frac{v_i(G)}{n}.$$

The interpretation following (3) proves (i).

For (ii), take $\rho = 1/(n+\kappa)$. Appendix G.3 proves that the reciprocal rule maintains $\Psi^t \le n/(1-\rho)$ and gives $n/(n+\kappa)$-PROP1. For the averaging calculation, each agent with $z_i > 0$ receives probability $\rho(1+(1-\rho)z_i/s_i^{t-1})$, which leaves her reciprocal term unchanged in expectation. These probabilities sum to at most $\rho\kappa + \rho(1-\rho)\Psi^{t-1} \le \rho(\kappa+n) = 1$. Choosing between that rule and (i) proves the stated maximum.

All comparisons use only the realized history and the current good, so the claims hold against adaptive adversaries without knowing the horizon. Each rule evaluates one score per agent. □

C.2 Proof of the Prediction-Error Guarantee

Combining Theorem 4.1 with the one-sided-error transformation of Choo et al. (2026) gives the following guarantee for imperfect MIV predictions.

Main-text result (Corollary 4.2). Suppose the algorithm is given $\kappa \in [2, n]$ such that $|N(g)| \le \kappa$ for every good, and a one-sided error bound $\varepsilon \in [0, 1)$ for the MIV predictions. Then a deterministic online algorithm guarantees

$$\max\left\{\frac{19n(1-\varepsilon)}{30n-19\varepsilon}, \frac{n(1-\varepsilon)}{n+\kappa-\varepsilon}\right\}\text{-PROP1}$$

against adaptive adversaries, without knowing the horizon. Without information about κ, the first factor applies.

Proof. We give the transformation explicitly. For each agent with $p_i > 0$, increase her value for the first good worth at least $(1-\varepsilon)p_i$ to p_i, leaving all other values unchanged. Run either rule from Theorem 4.1 on these modified values. The one-sided-error condition in Section 2 ensures that this good appears and that no value exceeds p_i. Thus the modified instance has exact MIV predictions. The modification is online and, since $\varepsilon < 1$, changes no set $N(g)$.

Fix an agent with $p_i > 0$. Let $d_i \in [0, \varepsilon p_i]$ be her single value increase. Suppose the chosen exact-prediction rule gives $n\rho$-PROP1. Returning to the original values reduces the value of a bundle with at most one added good by at most d_i, whereas it reduces the total value by d_i. Therefore

$$v_i(A_i) + h_i \ge \rho v_i(G) - (1-\rho)d_i.$$

Also, $v_i(A_i) + h_i \ge (1-\varepsilon)p_i$: the changed good is either in A_i or can be added to it. Hence

$$d_i \le \frac{\varepsilon}{1-\varepsilon}(v_i(A_i)+h_i), \quad v_i(A_i)+h_i \ge \frac{\rho(1-\varepsilon)}{1-\rho\varepsilon}\, v_i(G).$$

The latter inequality follows by substituting the former into the preceding display and rearranging. Agents with $p_i = 0$ satisfy PROP1 automatically. Taking $\rho = 19/(30n)$ or $\rho = 1/(n+\kappa)$ gives the displayed factors. Choose the rule with the larger factor before any good arrives. □

C.3 The Algorithm for PROP1 and Additive Envy

Assume that the horizon m is known. Define normalized values by

$$\widehat{v}_i(g) := \begin{cases} v_i(g)/p_i, & p_i > 0, \\ 0, & p_i = 0. \end{cases}$$

Then $\widehat{v}_i(g) \in [0, 1]$. This normalization does not change the PROP1 factor. The envy bounds below are stated in normalized values; multiplying the bound for agent i by p_i gives the corresponding bound in her original values. For agents $i, j \in N$, let $E_{ij}^t := \widehat{v}_i(A_j^t) - \widehat{v}_i(A_i^t)$, so $E_{ii}^t = 0$.

Retain $\rho = 19/(30n)$, the slacks s_i^t, and the potential Φ^t from (4) and (7). For $\eta > 0$, define

$$F_\eta^t := \sum_{i \neq j} e^{\eta E_{ij}^t}(1 + f(s_j^t + \rho)), \tag{32}$$

where a term written as $f(s_j^t + \rho)$ is defined to be zero when $p_j = 0$. The exponential terms follow the approach of Benadè et al. (2018). The multiplier links the two fairness objectives: receiving a good may increase other agents' envy toward its recipient j, but it also decreases j's PROP1 potential term whenever $z_j > 0$. Using the *envied agent's* term lets this decrease compensate for the increase in envy.

Allocation rule. Set $C := \frac{789}{500}$, $\eta := 10^{-6} \min\{1, \sqrt{(n \log n)/m}\}$, $A_\eta := 1 + \frac{2\eta^2}{n}$, $F_* := n(n-1)(1 + f(1))$, and $L := (m \log A_\eta + \log(2CF_*/(C - f(1))))/\eta$.

The bounds in Appendix G.2 give $f(1) < C < 30/19$. The first inequality leaves room for the envy potential in the initial bound $\Omega^0 < 1$ below, while the second ensures that $\Phi^t < nC$ still implies positive slacks. Define

$$\Omega^t := \frac{\Phi^t}{nC} + A_\eta^{m-t} e^{-\eta L} F_\eta^t. \tag{33}$$

When good g_t arrives, assign it to a recipient that minimizes the resulting value of Ω^t, breaking ties by a fixed order.

Main-text result (Theorem 4.3). Assume exact MIV predictions and a known horizon m. Against adaptive adversaries, the allocation rule above returns an allocation A that is $19/30$-PROP1 and satisfies

$$\max_{i,j \in N} (\widehat{v}_i(A_j) - \widehat{v}_i(A_i)) \leq \min\{m, L\}, \quad L = O\left(\log n + \sqrt{\frac{m \log n}{n}}\right), \tag{34}$$

where the hidden constant is absolute. In particular, if $m \geq n \log n$, the normalized maximum additive envy is $O(\sqrt{m \log n / n})$. For every known horizon m, the bound in (34) is also $O(\sqrt{m \log n})$. In the original values, (34) states that $v_i(A_j) - v_i(A_i) \leq p_i \min\{m, L\}$ for every $i, j \in N$.

Proof. Initially, $s_i^0 + \rho = 1$ for every i with $p_i > 0$, so $\Phi^0 \leq nf(1)$ and $F_\eta^0 \leq F_*$. By the definition of L,

$$\Omega^0 \leq \frac{f(1)}{C} + A_\eta^m e^{-\eta L} F_* = \frac{f(1)}{C} + \frac{C - f(1)}{2C} < 1.$$

We prove inductively that $s_i^t > 0$ for every i with $p_i > 0$ and that $\Omega^t < 1$. Suppose these inequalities hold at time $t - 1$. Every possible assignment has positive arguments in the potential, since $s_i^{t-1} + \rho - \rho z_i \geq s_i^{t-1} > 0$.

Appendix G.4 proves that some recipient satisfies $\Omega^t \leq \Omega^{t-1}$. The proof compares two possibilities. Either a uniformly random recipient does not increase any PROP1 term in expectation and increases F_η by a factor of at most A_η in expectation, or a single recipient decreases the combined potential. Thus the allocation rule maintains $\Omega^t < 1$. In particular, $\Phi^t < nC < 1/\rho$.

If $s_i^t \leq 0$, its positive argument $s_i^t + \rho$ would be at most ρ, and (6) would give $f(s_i^t + \rho) > 1/\rho$, a contradiction. Hence the selected assignment has positive slacks, completing the induction. The comparison in the appendix fixes only the realized history and current good, so it applies to adaptive adversaries.

At time m, positive slacks give us

$$p_i + B_i^m > \rho v_i(G) = \frac{19}{30}\frac{v_i(G)}{n}.$$

The interpretation following (3) proves $19/30$-PROP1. Agents with $p_i = 0$ satisfy this condition automatically. Also, (33) and $\Omega^m < 1$ imply

$$e^{\eta(E_{ij}^m - L)}(1 + f(s_j^m + \rho)) < 1 \quad (i \neq j).$$

The second factor is at least one, so $E_{ij}^m < L$. Normalized maximum additive envy is also at most m, since all normalized values lie in $[0, 1]$.

Finally, $F_* = O(n^2)$, and $C - f(1)$ is a positive absolute constant, so

$$L \leq \frac{2\eta m}{n} + \frac{\log(2CF_*/(C - f(1)))}{\eta} = O\left(\log n + \sqrt{\frac{m \log n}{n}}\right)$$

by the choice of η. If $m \geq n \log n$, the square-root term is at least $\log n$, giving the stated bound. For the $O(\sqrt{m \log n})$ bound, use the bound m when $m \leq \log n$; otherwise both terms in the last display are $O(\sqrt{m \log n})$. □

The rule uses $O(n)$ evaluations of f and $O(n^2)$ additional operations per good. Evaluate f at each agent's two possible updated arguments, and compute the needed row and column sums of the pairwise exponential terms once. For a fixed recipient, only her row and column of $(E_{ij}^t)_{i,j \in N}$ change, and only her column uses her multiplier after receiving the good. All candidate values can therefore be compared within the stated bound.

D Complete Proofs for Random Allocation

Choo et al. (2026) showed that RAND, which assigns each good independently and uniformly among all n agents, has a tight $\Theta(1/\log(n/\delta))$ PROP1 guarantee with probability at least $1 - \delta$ against a non-adaptive adversary. The classical LIKE rule of Aleksandrov et al. (2015), with truthful reports, instead assigns each good independently and uniformly among the agents who value it positively. Thus, when $i \in N(g)$, agent i receives g with probability $1/|N(g)|$ under LIKE rather than $1/n$ under RAND. We determine whether this larger probability improves PROP1 when few agents value the same good, and how the two rules compare for additive envy.

As in the result of Choo et al. for RAND, we assume a non-adaptive adversary. This is the setting needed for the concentration analysis below. Without future information, Theorem 3.1 rules out every positive PROP1 approximation against an adaptive adversary, even for randomized algorithms, so the non-adaptive assumption is necessary for the guarantees studied here.

Write A^{LIKE} and A^{RAND} for the final allocations. If $N(g) = \varnothing$, LIKE assigns g arbitrarily; this only makes the allocation complete and does not change any agent's value. Our analysis allows general nonnegative additive valuations.

Recall that $\kappa = \max_{g \in G} |N(g)|$. If $\kappa \leq 1$, every good with $N(g) \neq \varnothing$ is assigned by LIKE to the unique agent in $N(g)$, so LIKE is proportional. We therefore assume $2 \leq \kappa \leq n$.

The PROP1 factor improves by order n/κ until it reaches one, and the dependence on n, κ, and δ is tight up to constants. For additive envy, RAND has the smaller stated bound, and a binary construction shows that LIKE can have larger envy.

D.1 Tight High-Probability PROP1 Guarantees

For a fixed agent, the PROP1 condition is automatic if one good is already worth the target share. Otherwise, every good she values is smaller than that target, while LIKE assigns each such good to her with probability at least $1/\kappa$. This gives a lower-tail estimate for the value she receives. We state the one-agent estimate separately; the allocation guarantee then follows by a union bound over agents.

Lemma D.1. *Fix an agent $i \in N$ and $\alpha \in (0,1]$, and let $r := \frac{\alpha\kappa}{n}$. If $r < 1$, then under* LIKE*,*

$$\Pr[\textit{agent } i \textit{ does not satisfy the } \alpha\textit{-PROP1 condition}] \le \exp\left(-\frac{3(1-r)^2}{8r}\right).$$

Proof. If $v_i(G) = 0$, the claim is immediate. If $p_i^{\max} \ge \alpha\frac{v_i(G)}{n}$, then agent i satisfies the α-PROP1 condition for every realized allocation. Indeed, if a maximum-valued good lies outside A_i, add it to A_i; if it lies in A_i, take the set in the definition of PROP1 to be empty. Hence assume $p_i^{\max} < \alpha\frac{v_i(G)}{n}$.

Set $\mu := v_i(G)/\kappa$. Then $\alpha v_i(G)/n = r\mu$. For every good g with $v_i(g) > 0$, agent i belongs to $N(g)$, so

$$\Pr[g \in A_i] = \frac{1}{|N(g)|} \ge \frac{1}{\kappa}.$$

Realize the choices of LIKE for these goods with independent random variables $U_g \sim \mathrm{Unif}[0,1]$, one for each good with $v_i(g) > 0$, so that $g \in A_i$ exactly when $U_g \le \frac{1}{|N(g)|}$. Set $I_g := \mathbf{1}[U_g \le \frac{1}{|N(g)|}]$ and $B_g := \mathbf{1}[U_g \le \frac{1}{\kappa}]$. Then $I_g \ge B_g$ almost surely. Therefore, we get that

$$v_i(A_i) = \sum_{g:v_i(g)>0} v_i(g) I_g \ge Y := \sum_{g:v_i(g)>0} v_i(g) B_g.$$

The variables B_g are independent Bernoulli-$1/\kappa$ variables, and $\mathbb{E}[Y] = \mu$. Every summand of Y is at most $p_i^{\max} < r\mu$, and

$$\mathrm{Var}(Y) = \sum_{g:v_i(g)>0} v_i(g)^2 \frac{1}{\kappa}\left(1-\frac{1}{\kappa}\right) \le \frac{p_i^{\max}}{\kappa} \sum_{g:v_i(g)>0} v_i(g) < r\mu^2.$$

The centered variables $v_i(g)(B_g - 1/\kappa)$ are independent, have mean zero, and are bounded in absolute value by $r\mu$. If $Y \ge r\mu = \alpha v_i(G)/n$, then agent i satisfies α-PROP1 by taking the added set to be empty. Thus Bernstein's inequality gives

$$\begin{aligned}\Pr[\text{agent } i \text{ does not satisfy } \alpha\text{-PROP1}] &\le \Pr[Y - \mu < -(1-r)\mu] \\ &\le \exp\left(-\frac{(1-r)^2\mu^2}{2\mathrm{Var}(Y) + \frac{2}{3}(r\mu)(1-r)\mu}\right) \\ &\le \exp\left(-\frac{3(1-r)^2}{8r}\right),\end{aligned}$$

where the last inequality uses $\mathrm{Var}(Y) < r\mu^2$ and $1 - r \le 1$. □

Main-text result (Theorem 5.1). Fix $\delta \in (0,1)$. On every fixed instance with $2 \le \kappa \le n$, LIKE returns an α-PROP1 allocation with probability at least $1-\delta$, where $\alpha := \min\{1, \frac{3n}{32\kappa\log(n/\delta)}\}$.

Proof. Let $r := \alpha\kappa/n$. Then

$$r = \min\left\{\frac{\kappa}{n}, \frac{3}{32\log(n/\delta)}\right\} \le \frac{3}{32\log(n/\delta)} < \frac{1}{2}.$$

By Lemma D.1, each agent does not satisfy the α-PROP1 condition with probability at most

$$\exp\left(-\frac{3(1-r)^2}{8r}\right) \le \exp\left(-\frac{3}{32r}\right) \le \frac{\delta}{n}.$$

A union bound over the n agents proves the theorem. □

When every agent values every good, $\kappa = n$, and Theorem 5.1 gives the same $\Theta(1/\log(n/\delta))$ order as RAND. When fewer agents value each good, the factor improves by n/κ until it reaches one. In particular, if $\kappa \le 3n/(32\log(n/\delta))$, then LIKE returns a PROP1 allocation with probability at least $1-\delta$.

The next construction shows that both the n/κ term and the $\log(n/\delta)$ term are necessary, up to constants, whenever the displayed factor is below one.

Theorem D.2. *There is a universal constant $C > 0$ with the following property. Fix $n \geq 2$, $2 \leq \kappa \leq n$, $\delta \in (0, 1/2]$, and $\alpha \in (0, 1]$. If* LIKE *returns an α-PROP1 allocation with probability at least $1 - \delta$ on every fixed binary instance satisfying $\max_{g \in G} |N(g)| \leq \kappa$, then $\alpha \leq C \cdot \frac{n}{\kappa \log(n/\delta)}$.*

Proof. Let $\ell := \lfloor n/\kappa \rfloor$ and partition the first $\ell\kappa$ agents into ℓ disjoint sets $Q_1, \ldots, Q_\ell$, each of size κ. Set $\mu := \left\lceil \frac{2n}{\alpha\kappa} \right\rceil$. For each $h \in [\ell]$, create $\kappa\mu$ goods that have value 1 for the agents in Q_h and value 0 for every other agent. Every good is therefore positively valued by exactly κ agents.

Fix an agent i in one of these sets, and let X_i be the number of goods valued by i that LIKE allocates to i. Then, we have that $X_i \sim \text{Binomial}(\kappa\mu, 1/\kappa)$. Moreover, $v_i(G) = \kappa\mu$ and $\alpha \frac{v_i(G)}{n} = \frac{\alpha\kappa\mu}{n} \geq 2$. Thus, on the event $F_i := \{X_i = 0\}$, agent i does not satisfy α-PROP1: her own value is zero, and adding one good increases it by at most 1.

For every such agent,

$$\Pr[F_i] = \left(1 - \frac{1}{\kappa}\right)^{\kappa\mu} \geq e^{-2\mu} \geq \exp\left(-\frac{6n}{\alpha\kappa}\right).$$

Here we used $\log(1 - x) \geq -2x$ for $x \in [0, 1/2]$ and $\mu \leq 3n/(\alpha\kappa)$; the latter follows from $2n/(\alpha\kappa) \geq 2$.

Let $Z := \sum_{i=1}^{\ell\kappa} \mathbf{1}_{F_i}$. If two agents belong to different sets Q_h, their events are independent. If they belong to the same set, then

$$\Pr[F_i \cap F_j] = \left(1 - \frac{2}{\kappa}\right)^{\kappa\mu} \leq \left(1 - \frac{1}{\kappa}\right)^{2\kappa\mu} = \Pr[F_i] \Pr[F_j].$$

Thus the indicators in the sum defining Z have nonpositive pairwise covariances, and $\text{Var}(Z) \leq \mathbb{E}[Z]$. Since $\ell\kappa \geq n/2$ (because $\lfloor n/\kappa \rfloor \geq n/(2\kappa)$),

$$\mathbb{E}[Z] \geq \frac{n}{2} \exp\left(-\frac{6n}{\alpha\kappa}\right).$$

Since $Z = Z\mathbf{1}_{\{Z>0\}}$, Cauchy–Schwarz gives us $\mathbb{E}[Z]^2 \leq \mathbb{E}[Z^2] \Pr[Z > 0]$. Moreover, $\mathbb{E}[Z^2] = \text{Var}(Z) + \mathbb{E}[Z]^2 \leq \mathbb{E}[Z] + \mathbb{E}[Z]^2$. Therefore,

$$\Pr[Z > 0] \geq \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]} \geq \frac{\mathbb{E}[Z]}{1 + \mathbb{E}[Z]}.$$

The event $Z > 0$ implies that the allocation is not α-PROP1. By the assumed guarantee, $\Pr[Z > 0] \leq \delta$, and hence $\mathbb{E}[Z] \leq \frac{\delta}{1-\delta} \leq 2\delta$. Therefore, $\exp\left(-\frac{6n}{\alpha\kappa}\right) \leq \frac{4\delta}{n}$. If $\log(n/\delta) \geq 2 \log 4$, then

$$\frac{6n}{\alpha\kappa} \geq \log(n/\delta) - \log 4 \geq \frac{1}{2} \log(n/\delta),$$

and hence $\alpha \leq 12 \frac{n}{\kappa \log(n/\delta)}$. If $\log(n/\delta) < 2 \log 4$, then $\alpha \leq 1 \leq 2(\log 4) n/(\kappa \log(n/\delta))$. Increasing the constant if needed proves the theorem. □

A similar construction shows why RAND cannot benefit from small κ: its worst-case factor remains of order $1/\log(n/\delta)$ even when every good is valued positively by only one agent.

Theorem D.3. *There is a universal constant $C > 0$ with the following property. Fix $n \geq 2$, $\delta \in (0, 1/2]$, and $\alpha \in (0, 1]$. If* RAND *returns an α-PROP1 allocation with probability at least $1 - \delta$ on every fixed binary instance in which $|N(g)| = 1$ for every good g, then $\alpha \leq \frac{C}{\log(n/\delta)}$.*

Proof. Set $\mu := \left\lceil \frac{2}{\alpha} \right\rceil$. For each agent $i \in N$, create $n\mu$ goods that have value 1 for agent i and value 0 for every other agent. Let X_i be the number of these goods that RAND allocates to agent i. Then $X_i \sim \text{Binomial}(n\mu, 1/n)$, and the variables $X_1, \ldots, X_n$ are independent because they depend on disjoint sets of goods. Agent i has total value $n\mu$, so her proportional share is μ. On the

event $F_i := \{X_i = 0\}$, her own value is zero and adding one good increases it by at most 1, while $\alpha\mu \geq 2$. Thus F_i implies that agent i does not satisfy α-PROP1.

Moreover,

$$\Pr[F_i] = \left(1 - \frac{1}{n}\right)^{n\mu} \geq e^{-2\mu} \geq \exp\left(-\frac{6}{\alpha}\right),$$

where we used $\log(1-x) \geq -2x$ for $x \in [0, 1/2]$ and $\mu \leq 3/\alpha$. Independence gives

$$\Pr[\textsc{Rand} \text{ does not return an } \alpha\text{-PROP1 allocation}] \geq 1 - \left(1 - e^{-6/\alpha}\right)^n.$$

By the assumed guarantee, we have that $1 - \left(1 - e^{-6/\alpha}\right)^n \leq \delta$. Moreover, $(1 - e^{-6/\alpha})^n \leq \exp(-ne^{-6/\alpha})$, so $\exp(-ne^{-6/\alpha}) \geq 1 - \delta$ and therefore $ne^{-6/\alpha} \leq -\log(1-\delta) \leq 2\delta$. Consequently,

$$\frac{6}{\alpha} \geq \log\left(\frac{n}{2\delta}\right) \geq \frac{1}{2}\log\left(\frac{n}{\delta}\right),$$

where the last inequality uses $n/\delta \geq 4$. The result follows. □

The preceding results give the exact worst-case orders for the two rules. Combining Theorems D.2 and 5.1 gives the first statement below. The second follows from Theorem D.3 and the matching guarantee of Choo et al. (2026).

Main-text result (Corollary 5.2). Fix $n \geq 2$, $2 \leq \kappa \leq n$, and $\delta \in (0, 1/2]$.

(i) On the class of fixed instances satisfying $\max_g |N(g)| \leq \kappa$, the largest factor that LIKE guarantees with probability at least $1 - \delta$ is $\Theta\left(\min\left\{1, \frac{n}{\kappa \log(n/\delta)}\right\}\right)$.

(ii) On all fixed instances, the largest factor that RAND guarantees with probability at least $1 - \delta$ is $\Theta\left(\frac{1}{\log(n/\delta)}\right)$. The lower bound already holds when every good is positively valued by exactly one agent.

D.2 Comparing Additive Envy

Recall that $\mathrm{Env}(A)$ is the maximum additive envy and that $p_i^{\max}$ is agent i's maximum item value. The next theorem gives bounds for both rules on the same fixed instance. Define

$$R_i^{\textsc{Like}} := \sum_{g : v_i(g) > 0} \frac{v_i(g)^2}{|N(g)|}, \quad R_i^{\textsc{Rand}} := \frac{1}{n}\sum_{g \in G} v_i(g)^2. \tag{35}$$

For RAND, the proof follows the pairwise Bernstein approach of Benadè et al. (2018), while retaining the instance-specific quantity $R_i^{\textsc{Rand}}$. The novel parts are the corresponding calculation for LIKE, the direct comparison $R_i^{\textsc{Rand}} \leq R_i^{\textsc{Like}}$, and the lower bound showing that the larger quantity for LIKE can appear in the actual envy.

Main-text result (Theorem 5.3). Fix $\delta \in (0, 1)$. Under LIKE, with probability at least $1 - \delta$,

$$\mathrm{Env}(A^{\textsc{Like}}) \leq \max_{i \in N}\left(2\sqrt{R_i^{\textsc{Like}} \log\frac{n(n-1)}{\delta}} + \frac{p_i^{\max}\log\frac{n(n-1)}{\delta}}{3}\right).$$

Under RAND, with probability at least $1 - \delta$,

$$\mathrm{Env}(A^{\textsc{Rand}}) \leq \max_{i \in N}\left(2\sqrt{R_i^{\textsc{Rand}} \log\frac{n(n-1)}{\delta}} + \frac{p_i^{\max}\log\frac{n(n-1)}{\delta}}{3}\right).$$

Moreover, $R_i^{\textsc{Rand}} \leq R_i^{\textsc{Like}}$ for every agent $i \in N$.

Proof. Fix distinct agents i, j, and for each good g define $X_g := v_i(g)(\mathbf{1}[g \in A_j] - \mathbf{1}[g \in A_i])$. Then

$$v_i(A_j) - v_i(A_i) = \sum_{g \in G} X_g.$$

The variables X_g are independent across goods under both rules.

Consider LIKE first. If $v_i(g) = 0$, then $X_g = 0$. Suppose $v_i(g) > 0$. If $j \in N(g)$, then X_g equals $v_i(g)$ with probability $1/|N(g)|$, equals $-v_i(g)$ with probability $1/|N(g)|$, and is zero otherwise. If $j \notin N(g)$, then X_g equals $-v_i(g)$ with probability $1/|N(g)|$ and is zero otherwise. In either case,

$$\mathbb{E}[X_g] \le 0, \quad \mathrm{Var}(X_g) \le \frac{2v_i(g)^2}{|N(g)|}, \text{ and } \quad |X_g - \mathbb{E}[X_g]| \le v_i(g) \le p_i^{\max}.$$

Therefore, $\sum_{g \in G} \mathrm{Var}(X_g) \le 2R_i^{\mathrm{LIKE}}$. To state the concentration step precisely, let $S := \sum_g (X_g - \mathbb{E}[X_g])$ and $V := \sum_g \mathrm{Var}(X_g)$. The cases $V = 0$ or $p_i^{\max} = 0$ are immediate. Since the centered summands are independent and bounded in absolute value by $p_i^{\max}$, the Bernstein moment-generating-function bound gives

$$\log \mathbb{E}[e^{\lambda S}] \le \frac{V\lambda^2}{2(1 - p_i^{\max}\lambda/3)} \quad (0 < \lambda < 3/p_i^{\max}).$$

For any $x > 0$, applying Markov's inequality with $\lambda = \sqrt{2x/V}/(1 + (p_i^{\max}/3)\sqrt{2x/V})$ therefore yields

$$\Pr\left[S > \sqrt{2Vx} + \frac{p_i^{\max} x}{3}\right] \le e^{-x}.$$

Using $V \le 2R_i^{\mathrm{LIKE}}$ and $\sum_g \mathbb{E}[X_g] \le 0$, we obtain

$$\Pr\left[v_i(A_j) - v_i(A_i) > 2\sqrt{R_i^{\mathrm{LIKE}} x} + \frac{p_i^{\max} x}{3}\right] \le e^{-x}.$$

Under RAND, each good goes to i with probability $1/n$ and to j with probability $1/n$. Hence $\mathbb{E}[X_g] = 0$, $\mathrm{Var}(X_g) = \frac{2v_i(g)^2}{n}$, $|X_g| \le p_i^{\max}$, and the same inequality holds with R_i^{RAND} in place of R_i^{LIKE}.

There are $n(n-1)$ ordered pairs of distinct agents. Taking $x = \log(n(n-1)/\delta)$ and applying a union bound proves the corresponding inequality for each rule.

Finally, whenever $v_i(g) > 0$, we have $|N(g)| \le n$, and therefore $\frac{v_i(g)^2}{n} \le \frac{v_i(g)^2}{|N(g)|}$. Summing over the goods proves $R_i^{\mathrm{RAND}} \le R_i^{\mathrm{LIKE}}$. □

Corollary D.4. *If $v_i(g) \in [0,1]$ for every agent and good, then under* RAND*, with probability at least $1 - \delta$,*

$$\mathrm{Env}(A) \le 2\sqrt{\frac{m}{n} \log \frac{n(n-1)}{\delta}} + \frac{1}{3} \log \frac{n(n-1)}{\delta}.$$

Proof. For every agent i, $R_i^{\mathrm{RAND}} \le m/n$ and $p_i^{\max} \le 1$. Apply Theorem 5.3. □

Corollary D.5. *For each rule $\mathcal{R} \in \{\mathrm{LIKE}, \mathrm{RAND}\}$,*

$$\mathbb{E}[\mathrm{Env}(A^{\mathcal{R}})] = O\left(\max_i \left\{\sqrt{R_i^{\mathcal{R}} \log n} + p_i^{\max} \log n\right\}\right).$$

Proof. Fix one of the two rules, and set $R := \max_i R_i^{\mathcal{R}}$, $L := \log(n(n-1))$. For every $u > 0$, apply Theorem 5.3 with $\delta = e^{-u}$. Using $\sqrt{L+u} \le \sqrt{L} + \sqrt{u}$ gives

$$\Pr\left[\mathrm{Env}(A^{\mathcal{R}}) > 2\sqrt{RL} + \frac{L}{3}\max_i p_i^{\max} + 2\sqrt{Ru} + \frac{u}{3}\max_i p_i^{\max}\right] \le e^{-u}.$$

Let Z be an exponential random variable with mean one, so $\Pr[Z > u] = e^{-u}$. Since $u \mapsto 2\sqrt{Ru} + \frac{u}{3}\max_i p_i^{\max}$ is increasing, the preceding inequality and the tail-integration formula give

$$\mathbb{E}[\mathrm{Env}(A^{\mathcal{R}})] \le 2\sqrt{RL} + \frac{L}{3}\max_i p_i^{\max} + \mathbb{E}\left[2\sqrt{RZ} + \frac{Z}{3}\max_i p_i^{\max}\right].$$

Using $\mathbb{E}[\sqrt{Z}] = \sqrt{\pi}/2$ and $\mathbb{E}[Z] = 1$ gives us

$$\mathbb{E}[\mathrm{Env}(A^{\mathcal{R}})] \le 2\sqrt{RL} + \frac{L}{3}\max_i p_i^{\max} + \sqrt{\pi R} + \frac{1}{3}\max_i p_i^{\max}.$$

Since $L = \Theta(\log n)$ for $n \ge 2$, the stated bound follows. □

The quantities in (10) explain the difference between the two rules. Under RAND, a good contributes $v_i(g)^2/n$ to agent i's variance bound. Under LIKE, the contribution is $v_i(g)^2/|N(g)|$, which is larger when few agents value that good positively. The next result shows that this difference can appear in the actual envy, even for binary valuations.

Main-text result (Proposition 5.4). For every $2 \le \kappa \le n$ and every integer $m \ge \kappa$, there is a binary instance with $\max_{g\in G} |N(g)| = \kappa$ such that, under LIKE, $\mathbb{E}[\mathrm{Env}(A)] \ge \frac{9}{64\sqrt{2}} \cdot \sqrt{\frac{m}{\kappa}}$.

Proof. Let $Q := \{1, \dots, \kappa\}$. Every agent in Q values every good at 1, and every agent outside Q values every good at 0. Thus $N(g) = Q$ for every good, and LIKE assigns each good independently and uniformly among the agents in Q.

For $t = 1, \dots, m$, define

$$Z_t := \begin{cases} 1, & g_t \text{ is assigned to agent 2,} \\ -1, & g_t \text{ is assigned to agent 1,} \\ 0, & \text{otherwise,} \end{cases} \qquad D := \sum_{t=1}^{m} Z_t.$$

Then $v_1(A_2) - v_1(A_1) = D$, $v_2(A_1) - v_2(A_2) = -D$, so $\mathrm{Env}(A) \ge |D|$. The variables Z_t are independent and satisfy $\mathbb{E}[Z_t] = 0$, $\mathbb{E}[Z_t^2] = \mathbb{E}[Z_t^4] = \frac{2}{\kappa}$. Set $s := \sqrt{2m/\kappa}$. Then $\mathbb{E}[D^2] = s^2$. Since $m \ge \kappa$,

$$\mathbb{E}[D^4] = \sum_{t=1}^{m} \mathbb{E}[Z_t^4] + 6 \sum_{1\le r<t\le m} \mathbb{E}[Z_r^2]\mathbb{E}[Z_t^2] \le s^2 + 3s^4 \le 4s^4.$$

Applying the Paley–Zygmund inequality to D^2 gives us

$$\Pr\left[|D| \ge \frac{s}{2}\right] \ge \left(1 - \frac{1}{4}\right)^2 \frac{\mathbb{E}[D^2]^2}{\mathbb{E}[D^4]} \ge \frac{9}{64}.$$

Therefore

$$\mathbb{E}[\mathrm{Env}(A)] \ge \mathbb{E}[|D|] \ge \frac{9}{64}\frac{s}{2} = \frac{9}{64\sqrt{2}}\sqrt{\frac{m}{\kappa}}.$$ □

On the instance in Proposition 5.4, the variance quantities are $R_i^{\mathrm{LIKE}} = m/\kappa$ and $R_i^{\mathrm{RAND}} = m/n$ for the agents in Q. Thus the lower bound has the same square-root dependence on m/κ as the upper bound in Theorem 5.3, up to the logarithmic factor in the simultaneous high-probability guarantee over all pairs. Together with Theorems D.3 and 5.1, this gives a clear comparison between the two rules: LIKE can give much better proportionality when few agents value each good, while RAND has the smaller variance bound for additive envy.

E Definitions and Consequences for Other Fairness Notions

We use the notation of Section 2, in particular h_i and the form (1) of α-PROP1.

E.1 Definitions

We use the direct multiplicative versions of the following notions.

Proportionality-based axioms. An allocation is α-PROP if $v_i(A_i) \geq \alpha \cdot \frac{v_i(G)}{n}$ for every agent $i \in N$. It is α-PROPX if, for every i and every good $g \in G \setminus A_i$, $v_i(A_i) + v_i(g) \geq \alpha \frac{v_i(G)}{n}$; when $A_i = G$, the condition for i is automatic (Aziz et al., 2020).

For distinct agents i, j, let

$$\ell_{ij} := \begin{cases} \min_{g \in A_j} v_i(g), & A_j \neq \varnothing, \\ 0, & A_j = \varnothing. \end{cases}$$

An allocation is α-PROPm, α-PROPavg, or α-Avg-EFX if, respectively,

$$v_i(A_i) + \max_{j \neq i} \ell_{ij} \geq \alpha \frac{v_i(G)}{n},$$
$$v_i(A_i) + \frac{1}{n-1} \sum_{j \neq i} \ell_{ij} \geq \alpha \frac{v_i(G)}{n},$$
$$v_i(A_i) + \frac{1}{n} \sum_{j \neq i} \ell_{ij} \geq \alpha \frac{v_i(G)}{n}$$

for every agent i (Baklanov et al., 2021a;b; Kobayashi & Mahara, 2025). At $\alpha = 1$, these are the usual exact notions.

Envy-based and share-based axioms. An allocation is α-EF if $v_i(A_i) \geq \alpha v_i(A_j)$ for every i, j. It is α-EFX if, for every i, j and every $g \in A_j$ with $v_i(g) > 0$, $v_i(A_i) \geq \alpha v_i(A_j \setminus \{g\})$. It is α-EF1 if, for every i, j, there is a set $R_{ij} \subseteq A_j$ with $|R_{ij}| \leq 1$ such that $v_i(A_i) \geq \alpha v_i(A_j \setminus R_{ij})$ (Lipton et al., 2004; Amanatidis et al., 2018).

For a positive integer q and a set of goods S, let $\Pi_q(S)$ be the set of partitions of S into q bundles. Agent i's maximin share is $\mathrm{MMS}_i := \max_{(X_1,\ldots,X_n) \in \Pi_n(G)} \min_{j \in N} v_i(X_j)$. An allocation is α-MMS if $v_i(A_i) \geq \alpha \mathrm{MMS}_i$ for every agent i (Budish, 2011).

An allocation is α-EEFX if, for every agent i, there is an allocation $B^i = (B^i_1, \ldots, B^i_n)$ of G such that $B^i_i = A_i$ and agent i satisfies α-EFX in B^i. Let $\mathcal{E}_i$ be the set of allocations in which agent i satisfies EFX, and define $\mathrm{MXS}_i := \min_{B \in \mathcal{E}_i} v_i(B_i)$. The set $\mathcal{E}_i$ is nonempty: the allocation with $B_i = G$ and $B_j = \varnothing$ for every $j \neq i$ belongs to $\mathcal{E}_i$. Since G is finite, the minimum is attained. An allocation is α-MXS if $v_i(A_i) \geq \alpha\, \mathrm{MXS}_i$ for every agent i (Caragiannis et al., 2023).

For a nonempty set of agents $S \subseteq N$ and $i \in S$, let $\mu_i(S) := \max_{(X_1,\ldots,X_{|S|}) \in \Pi_{|S|}(\bigcup_{j \in S} A_j)} \min_{r \in [|S|]} v_i(X_r)$. An allocation is α-GMMS if $v_i(A_i) \geq \alpha \mu_i(S)$ for every nonempty $S \subseteq N$ and every $i \in S$ (Barman et al., 2018).

E.2 Implications

Lemma E.1. *Fix $\alpha \in (0, 1]$. Each of the following conditions implies α-PROP1:*

$$\alpha\text{-PROP},\ \alpha\text{-PROP1},\ \alpha\text{-PROPX},\ \alpha\text{-PROPm},\ \alpha\text{-PROPavg},\ \alpha\text{-Avg-EFX},$$
$$\alpha\text{-EF},\ \alpha\text{-EFX},\ \alpha\text{-EF1},\ \alpha\text{-EEFX},\ \alpha\text{-MXS},\ \alpha\text{-MMS},\ \alpha\text{-GMMS}.$$

Proof. The implications from α-PROP and α-PROP1 are immediate. For α-PROPX, if $A_i \neq G$, apply its defining inequality to a good attaining h_i; if $A_i = G$, the PROP1 condition is automatic. For PROPm, PROPavg, and Avg-EFX, every ℓ_{ij} is at most h_i, so each correction term in the definition is at most h_i. In every case, (1) follows.

Next, α-EF implies α-EFX, and α-EFX implies α-EF1. To prove the remaining implication, suppose that A is α-EF1 and fix an agent i. If $A_i = G$, the PROP1 condition is immediate. For each $j \neq i$, the α-EF1 condition allows the removal of at most one good from A_j. Since every good in A_j has value at most h_i to agent i, $v_i(A_j) \leq \frac{v_i(A_i)}{\alpha} + h_i$. Therefore

$$v_i(G) \leq v_i(A_i) + (n-1)\left(\frac{v_i(A_i)}{\alpha} + h_i\right).$$

Rearranging gives

$$v_i(A_i) + h_i - \alpha\frac{v_i(G)}{n} \geq \frac{1-\alpha}{n}v_i(A_i) + \frac{n-\alpha(n-1)}{n}h_i \geq 0.$$

Thus agent i satisfies α-PROP1.

The same calculation applies to α-EEFX. Fix i and choose an allocation B^i with $B^i_i = A_i$ in which agent i satisfies α-EFX. For $j \neq i$, if $v_i(B^i_j) = 0$, then $v_i(B^i_j) \leq v_i(A_i)/\alpha + h_i$ is immediate. Otherwise, choose a good $g_j \in B^i_j$ of maximum value to agent i. Then $v_i(g_j) > 0$, and $v_i(g_j) \leq h_i$ because $B^i_j \subseteq G \setminus A_i$. Moreover, α-EFX gives

$$v_i(A_i) \geq \alpha v_i(B^i_j \setminus \{g_j\}).$$

Hence $v_i(B^i_j) \leq v_i(A_i)/\alpha + h_i$. Summing over $j \neq i$ and applying the same rearrangement as above gives $v_i(A_i) + h_i \geq \alpha v_i(G)/n$.

Now suppose that A is α-MXS. Fix an agent i with $A_i \neq G$ and assume for contradiction that

$$v_i(A_i) + h_i < \alpha\frac{v_i(G)}{n}. \tag{36}$$

Choose $Y \in \mathcal{E}_i$ with $v_i(Y_i) = \mathrm{MXS}_i$. Since $v_i(A_i) \geq \alpha\,\mathrm{MXS}_i$, we must have $\mathrm{MXS}_i < v_i(G)/n$; otherwise $v_i(A_i) \geq \alpha v_i(G)/n$. The bundles in Y have total value $v_i(G)$ to agent i, so some $j \neq i$ satisfies $v_i(Y_j) > v_i(G)/n$. Since

$$v_i(A_i) < \alpha\frac{v_i(G)}{n} \leq \frac{v_i(G)}{n} < v_i(Y_j),$$

additivity and nonnegativity give

$$v_i(Y_j \setminus A_i) = v_i(Y_j) - v_i(Y_j \cap A_i) \geq v_i(Y_j) - v_i(A_i) > 0.$$

Hence there is a good $g \in Y_j \setminus A_i$ with $v_i(g) > 0$. Agent i satisfies EFX in Y, and hence

$$\mathrm{MXS}_i = v_i(Y_i) \geq v_i(Y_j \setminus \{g\}) > \frac{v_i(G)}{n} - v_i(g).$$

Using $v_i(g) \leq h_i$ and $\alpha \leq 1$,

$$v_i(A_i) + h_i \geq \alpha\,\mathrm{MXS}_i + h_i > \alpha\left(\frac{v_i(G)}{n} - v_i(g)\right) + h_i \geq \alpha\frac{v_i(G)}{n},$$

contradicting (36).

It remains to consider MMS and GMMS. The exact implication from MMS to PROP1 is known (Caragiannis et al., 2023); we give the multiplicative calculation. Fix an agent i with $A_i \neq G$ and assume for contradiction that $v_i(A_i) + h_i < \alpha\frac{v_i(G)}{n}$. Set $\tau := v_i(A_i)/\alpha$. Since A is α-MMS, $\mathrm{MMS}_i \leq \tau$. The assumed inequality gives

$$\tau < \frac{v_i(G)}{n}, \quad h_i < \alpha\left(\frac{v_i(G)}{n} - \tau\right) \leq \frac{v_i(G)}{n} - \tau.$$

We construct an n-partition in which every bundle has value more than τ to agent i. Start with A_i and add goods from $G \setminus A_i$ until its value first exceeds τ. Every added good has value at most h_i, so the completed bundle has value less than $v_i(G)/n$. Form each later bundle from the remaining goods, stopping when its value first exceeds τ. Again, each completed bundle has value less than $v_i(G)/n$. Before the rth bundle is formed, the first $r-1$ bundles have total value less than $(r-1)v_i(G)/n$, so the remaining goods have value more than

$$v_i(G) - \frac{(r-1)v_i(G)}{n} \geq \frac{v_i(G)}{n} > \tau.$$

Thus all n bundles can be formed. Add any goods left afterward to arbitrary bundles. The least valuable bundle is worth more than τ, contradicting $\mathrm{MMS}_i \leq \tau$.

Finally, α-GMMS implies α-MMS by applying the groupwise condition to the full set of agents. This completes the proof. □

For a positive integer c, say that A is α-EFc if, for every ordered pair i, j, there is a set $R_{ij} \subseteq A_j$ with $|R_{ij}| \leq c$ such that $v_i(A_i) \geq \alpha v_i(A_j \setminus R_{ij})$.

Lemma E.2. *Fix $\alpha \in (0, 1]$ and a positive integer c. Every α-EFc allocation is α-PROP$c(n-1)$.*

Proof. Fix an agent i. For each $j \neq i$, choose a set $R_{ij} \subseteq A_j$ with $|R_{ij}| \leq c$ and $v_i(A_i) \geq \alpha v_i(A_j \setminus R_{ij})$. Let $S_i := \bigcup_{j \neq i} R_{ij}$. Then $S_i \subseteq G \setminus A_i$ and $|S_i| \leq c(n-1)$. Also,

$$\begin{aligned} v_i(G) &= v_i(A_i) + \sum_{j \neq i} v_i(A_j) \\ &\leq v_i(A_i) + \sum_{j \neq i} \left(\frac{v_i(A_i)}{\alpha} + v_i(R_{ij}) \right) \\ &= v_i(A_i) + \frac{(n-1)v_i(A_i)}{\alpha} + v_i(S_i). \end{aligned}$$

Therefore

$$v_i(A_i \cup S_i) - \alpha \frac{v_i(G)}{n} \geq \frac{1-\alpha}{n} v_i(A_i) + \left(1 - \frac{\alpha}{n}\right) v_i(S_i) \geq 0.$$

Thus agent i satisfies α-PROP$c(n-1)$. □

For PMMS, let $\mu_i^2(S) := \max_{(B_1, B_2) \in \Pi_2(S)} \min\{v_i(B_1), v_i(B_2)\}$. An allocation is α-PMMS if $v_i(A_i) \geq \alpha \mu_i^2(A_i \cup A_j)$ for every ordered pair of distinct agents $i, j \in N$ (Caragiannis et al., 2019).

Lemma E.3. *Fix $\alpha \in (0, 1]$. Every α-PMMS allocation is $\frac{\alpha n}{\alpha + 2(n-1)}$-PROP1.*

Proof. Fix an agent i.

We first use a simple two-partition bound. If a set of goods has total value W and every good has value at most q, then its two-way maximin share is at least $(W - q)/2$. To see this, add goods to one bundle until its value first reaches $(W - q)/2$. The value of that bundle is at most $(W + q)/2$, so the other bundle also has value at least $(W - q)/2$.

Apply this bound to $A_i \cup A_j$. Its total value is $v_i(A_i) + v_i(A_j)$, and every good in it has value at most $v_i(A_i) + h_i$. Hence

$$\mu_i^2(A_i \cup A_j) \geq \frac{v_i(A_j) - h_i}{2}.$$

The α-PMMS condition gives us $v_i(A_i) \geq \frac{\alpha}{2}(v_i(A_j) - h_i)$, so we have that $v_i(A_j) \leq \frac{2v_i(A_i)}{\alpha} + h_i$. Summing over $j \neq i$, we get

$$\begin{aligned} v_i(G) &= v_i(A_i) + \sum_{j \neq i} v_i(A_j) \\ &\leq v_i(A_i) + (n-1)\left(\frac{2v_i(A_i)}{\alpha} + h_i \right) \\ &\leq \left(1 + \frac{2(n-1)}{\alpha}\right)(v_i(A_i) + h_i). \end{aligned}$$

Therefore,

$$v_i(A_i) + h_i \geq \frac{\alpha n}{\alpha + 2(n-1)} \frac{v_i(G)}{n}.$$

Equation (1) proves the claim. □

Corollary E.4. *Fix $n \geq 2$. For every notion covered by Lemmas E.1 to E.3, with c fixed for EFc, no randomized online algorithm can guarantee any positive multiplicative approximation against an adaptive adversary. The same conclusion holds for any simultaneous guarantee that includes one of these notions.*

Proof. A positive approximation to any listed notion would, by the preceding lemmas, give β-PROPk for some $\beta > 0$ and some fixed k. This contradicts Theorem 3.1. The same argument applies to a simultaneous guarantee containing one of the listed notions. □

F CHORES: INAPPROXIMABILITY AND CONSEQUENCES FOR OTHER FAIRNESS NOTIONS

The main text and Appendix E consider goods. In this appendix, we use the same sets of agents and items, the same arrival order, and the same online model as in Section 2, but each item is a *chore*. Agent i has a nonnegative additive cost function $c_i : 2^G \to \mathbb{R}_{\geq 0}$. Thus $c_i(S) = \sum_{g \in S} c_i(g)$ for every $S \subseteq G$.

For a nonnegative integer k and a set $X \subseteq G$, let

$$\widehat{c}_i^k(X) := \min_{\substack{S \subseteq X \\ |S| \leq k}} c_i(X \setminus S). \tag{37}$$

This is the least cost remaining after removing at most k chores from X. For chores, a smaller approximation factor is stronger, and we therefore take $\lambda \geq 1$.

Definition F.1 (λ-PROPk for chores)**.** Fix $\lambda \geq 1$ and an integer $k \geq 1$. An allocation A is λ-PROPk for chores if $\widehat{c}_i^k(A_i) \leq \lambda \frac{c_i(G)}{n}$ for every $i \in N$.

Every allocation is n-PROPk, since $\widehat{c}_i^k(A_i) \leq c_i(A_i) \leq c_i(G)$. The next theorem shows that no smaller universal factor is possible online, even for a fixed k.

F.1 THE PROPk LOWER BOUND

We will use the following monotonicity property of (37): if $X \subseteq Y$, then $\widehat{c}_i^k(X) \leq \widehat{c}_i^k(Y)$. Indeed, for every $S \subseteq Y$ with $|S| \leq k$, nonnegativity and additivity give

$$c_i(Y \setminus S) \geq c_i(X \setminus (S \cap X)) \geq \widehat{c}_i^k(X).$$

Taking the minimum over all such S gives the stated inequality.

Fix $k \geq 1$ for the rest of this subsection. The induction below uses the same broad recursive idea as the online MMS lower bounds of Seddighin & Seddighin (2025) and Song et al. (2025): an r-agent construction is repeated for the first r agents, while the costs of the additional agent increase geometrically. The new inductive statement controls the cost left after removing any k chores. It therefore gives a PROPk lower bound rather than only an MMS lower bound.

More precisely, the next lemma constructs an adaptive adversary such that, regardless of the recipients, at some prefix one agent bears almost all of her total cost for the chores seen so far even after removing up to k chores from her bundle. Each new copy of the r-agent construction is scaled so that earlier chores contribute little to the relevant agent's total cost.

Lemma F.2. *For every positive integer r and every $\varepsilon \in (0, 1)$, there is an adaptive adversary for r agents together with a positive integer $T(r, \varepsilon)$ such that, for every possible sequence of recipients, some prefix G' of at most $T(r, \varepsilon)$ chores and some agent $i \in [r]$ satisfy*

$$\widehat{c}_i^k(A_i) > (1 - \varepsilon) c_i(G'), \tag{38}$$

where A_i is agent i's bundle at that prefix. Thus, the cost remaining after removing up to k chores from A_i exceeds a $(1-\varepsilon)$ fraction of agent i's cost for all chores in G'. The adversary chooses each cost vector using only the recipients of earlier chores. Its first chore has cost one for every agent, and every chore it presents has positive cost for every agent.

Proof. We proceed by induction on r.

For $r = 1$, set $T(1, \varepsilon) := \lfloor k/\varepsilon \rfloor + 1$ and present $T(1, \varepsilon)$ chores, each of cost one. The only agent receives every chore. Since $\varepsilon T(1, \varepsilon) > k$,

$$\widehat{c}_1^k(A_1) = T(1, \varepsilon) - k > (1 - \varepsilon) T(1, \varepsilon) = (1 - \varepsilon) c_1(G').$$

Now assume the statement for r agents, and consider agents $1, \ldots, r + 1$. Set

$$\eta := \frac{\varepsilon}{4}, \quad T_0 := T(r, \eta), \quad q := 1 + \frac{1}{\eta}, \quad b_s := q^{s-1} \quad (s = 1, \ldots, T_0).$$

The strategy proceeds in blocks. At the beginning of a block, let P_i be agent i's total cost for all preceding chores, for each $i \in [r]$, and set $w_i := \max\{1, P_i/\eta\}$. Start a new copy of the r-agent strategy with parameter η. If its sth chore has costs $(d_1, \dots, d_r)$, present an actual chore with costs $(w_1 d_1, \dots, w_r d_r, b_s)$. If this chore is assigned to agent $r+1$, end the current block and start a new one. Otherwise, pass its recipient to the r-agent strategy and continue the current block.

Suppose first that the r-agent strategy reaches (38) within a block before agent $r+1$ receives a chore. Let $i \in [r]$ be the agent given by the induction hypothesis, let X_i be the chores allocated to i in the current block up to that point, and let L_i be agent i's total cost for all chores in that part of the block. Scaling agent i's costs by w_i gives

$$\widehat{c}_i^k(X_i) > (1-\eta)L_i.$$

The first chore of the r-agent strategy has cost one to every participating agent, so $L_i \geq w_i$. By the definition of w_i, $P_i \leq \eta w_i \leq \eta L_i$. Since $X_i \subseteq A_i$, the monotonicity shown above gives

$$\frac{\widehat{c}_i^k(A_i)}{c_i(G')} \geq \frac{\widehat{c}_i^k(X_i)}{P_i + L_i} > \frac{1-\eta}{1+\eta} > 1-\varepsilon.$$

The last inequality follows from $\eta = \varepsilon/4$. Thus (38) holds. Consequently, unless the desired inequality has already been reached, agent $r+1$ receives a chore within the first T_0 positions of every block.

It remains to consider the case in which agent $r+1$ repeatedly ends the blocks. If the first $s-1$ chores of a block are allocated to agents in $[r]$ and the sth chore is allocated to agent $r+1$, then the cost missed by agent $r+1$ in that block is

$$\sum_{h=1}^{s-1} b_h = \frac{q^{s-1}-1}{q-1} = \eta(b_s - 1) < \eta b_s. \tag{39}$$

Let $Q := \lfloor k b_{T_0}/\eta \rfloor + 1$. Complete Q blocks, unless (38) has already been reached for an agent in $[r]$. Let L be agent $r+1$'s total cost for the Q chores she receives, and let O be her total cost for the chores allocated to the other agents. Summing (39) over the blocks gives $O < \eta L$. Also, every received chore has cost at most b_{T_0} and at least one, so $L \geq Q > \frac{k b_{T_0}}{\eta}$. Removing at most k chores can remove cost at most $k b_{T_0}$. Therefore

$$\widehat{c}_{r+1}^k(A_{r+1}) \geq L - k b_{T_0} > (1-\eta)L.$$

It follows that

$$\frac{\widehat{c}_{r+1}^k(A_{r+1})}{c_{r+1}(G')} > \frac{1-\eta}{1+\eta} > 1-\varepsilon.$$

Thus the induction closes with $T(r+1, \varepsilon) := QT_0$. At the beginning of the first block, every P_i is zero and every w_i is one, so the first chore has cost one for all $r+1$ agents. All other costs in the construction are positive as well. □

Theorem F.3. *Fix $n \geq 2$, $k \geq 1$, and $\lambda \in [1, n)$. For every (deterministic or randomized) online algorithm, there is an adaptive adversary under which the final allocation is not λ-PROPk; for a randomized algorithm, this holds with probability one. The statement remains true even when both of the following hold:*

(i) the final number of chores is known before the first chore arrives; and

(ii) every cost lies in $[0, 1]$.

The adversary observes only the realized recipients of earlier chores.

Proof. Set $\varepsilon := \frac{n-\lambda}{2n}$. Then $1 - \varepsilon > \lambda/n$. Apply Lemma F.2 with $r = n$. Along every sequence of recipients, some prefix G' and some agent i satisfy

$$\widehat{c}_i^k(A_i) > (1-\varepsilon)c_i(G') > \lambda \frac{c_i(G')}{n}.$$

The adversary ends the instance at this prefix. The allocation is therefore not λ-PROPk. Since the argument applies to every sequence of recipients, it applies to every realization of a randomized algorithm's private choices.

The bound $T(n, \varepsilon)$ in Lemma F.2 depends only on n, k, and λ. The adversary may reveal this number before the first chore arrives and, after the displayed inequality is reached, add zero-cost chores until exactly $T(n, \varepsilon)$ chores have arrived. These added chores do not change the inequality.

Finally, consider the finite tree containing every recipient history of length at most $T(n, \varepsilon)$ under the strategy. Only finitely many cost vectors occur in this tree. Multiplying all of them by one common positive constant makes every cost lie in $[0, 1]$. Each inequality above is homogeneous, so the same conclusion holds after this scaling. □

F.2 Consequences for Proportionality and Envy

We next use direct multiplicative cost versions of standard chore fairness notions (Chen & Tan, 2026; Garg & Sharma, 2026). An allocation is λ-PROP if

$$c_i(A_i) \le \lambda \frac{c_i(G)}{n} \quad \text{for every } i.$$

It is λ-PROP1 when Definition F.1 holds with $k = 1$. It is λ-PROPX if, for every agent i and every chore $g \in A_i$ with $c_i(g) > 0$,

$$c_i(A_i \setminus \{g\}) \le \lambda \frac{c_i(G)}{n};$$

the condition is automatic when A_i contains no positive-cost chore. Under additive chore costs, the standard definitions of PROPm and PROPavg reduce to PROPX (Garg & Sharma, 2026); their direct multiplicative versions therefore coincide with λ-PROPX. Hence

$$\lambda\text{-PROP} \implies \lambda\text{-PROPX} \implies \lambda\text{-PROP1}. \tag{40}$$

The same conclusion holds for λ-PROPm and λ-PROPavg.

For a nonnegative integer c, an allocation is λ-EFc for chores if, for every ordered pair i, j, there is a set $S_{ij} \subseteq A_i$ with $|S_{ij}| \le c$ such that

$$c_i(A_i \setminus S_{ij}) \le \lambda c_i(A_j). \tag{41}$$

Thus λ-EF is λ-EF0, and λ-EF1 is the case $c = 1$. An allocation is λ-EFX if, for every ordered pair i, j and every $g \in A_i$ with $c_i(g) > 0$,

$$c_i(A_i \setminus \{g\}) \le \lambda c_i(A_j).$$

In particular, λ-EFX implies λ-EF1. An allocation is epistemic λ-EFc if, for every agent i, there is a complete allocation B^i with $B^i_i = A_i$ in which agent i satisfies λ-EFc. The cases $c = 0$ and $c = 1$ are denoted EEF and EEF1, respectively; epistemic EFX is denoted EEFX.

Define

$$\theta_n(\lambda) := \frac{n\lambda}{\lambda + n - 1}. \tag{42}$$

For every finite $\lambda \ge 1$, we have $\theta_n(\lambda) < n$.

Lemma F.4. *Fix a nonnegative integer c and a finite $\lambda \ge 1$. Every λ-EFc allocation and every epistemic λ-EFc allocation is $\theta_n(\lambda)$-PROPc, where PROP0 means PROP. In particular, every λ-EFX and every λ-EEFX allocation is $\theta_n(\lambda)$-PROP1.*

Proof. Fix an agent i and set $r_i := \widehat{c}^c_i(A_i)$. For ordinary EF$c$, take $B^i = A$. For epistemic EFc, take the allocation B^i from the definition. For each $j \ne i$, (41) and the definition of r_i give

$$r_i \le \lambda c_i(B^i_j),$$

so $c_i(B^i_j) \ge r_i/\lambda$. Also $c_i(A_i) \ge r_i$. Therefore

$$c_i(G) = c_i(A_i) + \sum_{j \ne i} c_i(B^i_j) \ge r_i + (n-1)\frac{r_i}{\lambda}.$$

Rearranging gives

$$r_i \le \frac{\lambda}{\lambda + n - 1} c_i(G) = \theta_n(\lambda) \frac{c_i(G)}{n}.$$

For EFX and EEFX, use the implication to EF1 and take $c = 1$. □

When $c = 0$, $\theta_n(\lambda)$-PROP is stronger than $\theta_n(\lambda)$-PROP1, so Theorem F.3 applies in this case as well. It follows from Theorem F.3 and Lemma F.4 that no randomized online algorithm guarantees any finite multiplicative approximation to EF, EF1, EFX, EEF, EEF1, EEFX, EFc, or epistemic EFc for any fixed c.

F.3 Consequences for Share-Based Notions

For a set of chores $H \subseteq G$ and a positive integer q, using the partition notation $\Pi_q(H)$ from Appendix E, define

$$\mu_i^{\mathrm{ch}}(H, q) := \min_{(X_1,\dots,X_q)\in\Pi_q(H)} \max_{t\in[q]} c_i(X_t). \tag{43}$$

For the AnyPrice share for chores (Babaioff et al., 2024; Garg & Sharma, 2026), define $\mathrm{APS}_i^{\mathrm{ch}}(\varnothing, q) := 0$, and, for $H \neq \varnothing$,

$$\mathrm{APS}_i^{\mathrm{ch}}(H, q) := \max_{\substack{x\in\mathbb{R}^H_{\ge 0} \\ \sum_{g\in H} x_g = 1}} \min_{Y\subseteq H} \; c_i(Y). \tag{44}$$
$$\sum_{g\in Y} x_g \ge 1/q$$

Agent i's chore maximin share and AnyPrice share are $\mathrm{MMS}_i^{\mathrm{ch}} := \mu_i^{\mathrm{ch}}(G, n)$ and $\mathrm{APS}_i^{\mathrm{ch}} := \mathrm{APS}_i^{\mathrm{ch}}(G, n)$. An allocation is λ-MMS or λ-APS if, respectively,

$$c_i(A_i) \le \lambda \mathrm{MMS}_i^{\mathrm{ch}} \quad \text{or} \quad c_i(A_i) \le \lambda \mathrm{APS}_i^{\mathrm{ch}} \quad \text{for every } i.$$

It is λ-PMMS or λ-PAPS if, respectively,

$$c_i(A_i) \le \lambda \mu_i^{\mathrm{ch}}(A_i \cup A_j, 2) \quad \text{or} \quad c_i(A_i) \le \lambda \mathrm{APS}_i^{\mathrm{ch}}(A_i \cup A_j, 2)$$

for every $i \neq j$. It is λ-GMMS (respectively, λ-GAPS) if, for every nonempty $S \subseteq N$ and every $i \in S$, the first (respectively, second) inequality below holds:

$$c_i(A_i) \le \lambda \mu_i^{\mathrm{ch}} \left(\bigcup_{j\in S} A_j, |S| \right), \quad c_i(A_i) \le \lambda \mathrm{APS}_i^{\mathrm{ch}} \left(\bigcup_{j\in S} A_j, |S| \right).$$

Let $\mathcal{E}_i^{\mathrm{ch}}$ and $\mathcal{E}_i^{1,\mathrm{ch}}$ be the sets of allocations in which agent i satisfies exact EFX and exact EF1 for chores, respectively. Both sets are nonempty because they contain every allocation that gives agent i the empty bundle. Define

$$\mathrm{MXS}_i^{\mathrm{ch}} := \max_{B\in\mathcal{E}_i^{\mathrm{ch}}} c_i(B_i), \quad \mathrm{M1S}_i^{\mathrm{ch}} := \max_{B\in\mathcal{E}_i^{1,\mathrm{ch}}} c_i(B_i). \tag{45}$$

These are the cost forms of the minimum EFX share and minimum EF1 share: utilities for chores are $-c_i$, so minimizing utility over the relevant allocations is equivalent to maximizing cost. An allocation is λ-MXS or λ-M1S if, respectively,

$$c_i(A_i) \le \lambda \mathrm{MXS}_i^{\mathrm{ch}} \quad \text{or} \quad c_i(A_i) \le \lambda \mathrm{M1S}_i^{\mathrm{ch}} \quad \text{for every } i.$$

For an agent i, let

$$U_i := \frac{c_i(G)}{n} + \left(1 - \frac{1}{n}\right) \max_{g\in G} c_i(g). \tag{46}$$

Lemma F.5. *For every agent i,*

$$\mathrm{MMS}_i^{\mathrm{ch}} \le U_i, \quad \mathrm{APS}_i^{\mathrm{ch}} \le U_i, \quad \mathrm{MXS}_i^{\mathrm{ch}} \le U_i, \quad \mathrm{M1S}_i^{\mathrm{ch}} \le U_i.$$

Proof. For MMS, assign the chores one at a time to a currently least costly one of n bundles, according to c_i. Consider a bundle of maximum final cost, and let g be the last chore assigned to it. Immediately before g was assigned, that bundle had cost at most the average current cost, and hence at most $(c_i(G) - c_i(g))/n$. Its final cost is therefore at most

$$\frac{c_i(G) - c_i(g)}{n} + c_i(g) \leq U_i.$$

This gives an n-partition whose maximum cost is at most U_i, and hence $\mathrm{MMS}_i^{\mathrm{ch}} \leq U_i$.

For APS, fix a nonnegative price vector x with total price one, and take an MMS-optimal partition $(X_1, \ldots, X_n)$. Some bundle X_j has price at least $1/n$, and every bundle has cost at most $\mathrm{MMS}_i^{\mathrm{ch}}$. Hence

$$\min_{\substack{Y \subseteq G \\ \sum_{g \in Y} x_g \geq 1/n}} c_i(Y) \leq c_i(X_j) \leq \mathrm{MMS}_i^{\mathrm{ch}}.$$

Taking the maximum over x gives $\mathrm{APS}_i^{\mathrm{ch}} \leq \mathrm{MMS}_i^{\mathrm{ch}} \leq U_i$.

For MXS and M1S, fix B in $\mathcal{E}_i^{\mathrm{ch}}$ or $\mathcal{E}_i^{1,\mathrm{ch}}$, and let $q_i := \max_{g \in B_i} c_i(g)$. In either case, agent i satisfies EF1, so

$$c_i(B_i) - q_i \leq c_i(B_j) \quad \text{for every } j \neq i.$$

Consequently,

$$c_i(G) = c_i(B_i) + \sum_{j \neq i} c_i(B_j) \geq n c_i(B_i) - (n-1) q_i.$$

Since $q_i \leq \max_{g \in G} c_i(g)$, rearranging gives $c_i(B_i) \leq U_i$. Taking the maximum over the appropriate set of allocations proves both bounds. □

Define $\psi_n(\lambda) := \frac{\lambda n^2}{n + \lambda(n-1)}$. If $1 \leq \lambda < n$, then $\psi_n(\lambda) < n$.

Lemma F.6. *Fix $1 \leq \lambda < n$. Every λ-MMS, λ-APS, λ-MXS, or λ-M1S allocation is $\psi_n(\lambda)$-PROP1. The same conclusion holds for λ-GMMS and λ-GAPS.*

Proof. Fix an agent i and set $r_i := \widehat{c}_i^{\,1}(A_i)$. By Lemma F.5, each of the λ-MMS, λ-APS, λ-MXS, and λ-M1S conditions gives

$$c_i(A_i) \leq \lambda U_i. \tag{47}$$

We also have

$$r_i \leq c_i(G) - \max_{g \in G} c_i(g). \tag{48}$$

Indeed, if a maximum-cost chore belongs to A_i, remove it; otherwise, $c_i(A_i) \leq c_i(G) - \max_{g \in G} c_i(g)$ even without removing a chore.

If $c_i(G) = 0$, the claim is immediate. Otherwise, set $x := \max_{g \in G} c_i(g)/c_i(G)$. From (47)–(48),

$$\frac{r_i}{c_i(G)} \leq \min\left\{ \frac{\lambda}{n}(1 + (n-1)x), 1 - x \right\}.$$

The first expression increases with x, and the second decreases. They are equal at $x = \frac{n-\lambda}{n+\lambda(n-1)}$, where their common value is $\frac{\lambda n}{n+\lambda(n-1)}$. Therefore

$$r_i \leq \psi_n(\lambda) \frac{c_i(G)}{n}.$$

Finally, λ-GMMS implies λ-MMS, and λ-GAPS implies λ-APS, by taking $S = N$. □

Unlike the goods setting, exact MMS need not imply exact PROP1 for chores (Garg & Sharma, 2026). The quantitative implication in Lemma F.6 is enough here because $\psi_n(\lambda) < n$ whenever $\lambda < n$. Thus no randomized online algorithm guarantees λ-MMS, λ-APS, λ-MXS, λ-M1S, λ-GMMS, or λ-GAPS for any $1 \leq \lambda < n$. The MMS conclusion agrees with the independent $(n - \varepsilon)$ online lower bound of Song et al. (2025).

PMMS has a different universal factor, so we treat it separately. We use the following elementary bound: if a set H has total cost W and maximum single-chore cost p, then

$$\mu_i^{\mathrm{ch}}(H,2) \le \frac{W+p}{2}. \tag{49}$$

To see this, assign each chore to a currently less costly one of two bundles. If g is the last chore assigned to a final maximum-cost bundle, its cost before receiving g is at most $(W - c_i(g))/2$.

Lemma F.7. *Fix $1 \le \lambda < 2$.*

(i) If $n = 2$, every λ-PMMS allocation is $\frac{4\lambda}{2+\lambda}$-PROP1.

(ii) If $n \ge 3$, every λ-PMMS allocation is $\frac{n\lambda}{2}$-PROP1.

In either case, the resulting PROP1 factor is smaller than n.

Proof. When $n = 2$, PMMS is MMS on the full set of chores, and the first statement is Lemma F.6 with $n = 2$.

Suppose $n \ge 3$. Fix an agent i, let $q_i := \max_{g \in A_i} c_i(g)$, and set $d := (2-\lambda)/\lambda$. For each $j \ne i$, the largest cost of a chore in $A_i \cup A_j$ is at most $q_i + c_i(A_j)$. By (49) and λ-PMMS, $c_i(A_i) \le \frac{\lambda}{2}(c_i(A_i) + q_i + 2c_i(A_j))$, and hence $c_i(A_j) \ge \frac{d\,c_i(A_i) - q_i}{2}$. Since every $c_i(A_j)$ is nonnegative and there are at least two agents other than i,

$$\sum_{j \ne i} c_i(A_j) \ge (d\,c_i(A_i) - q_i)_+. \tag{50}$$

If $q_i \ge d\,c_i(A_i)$, then

$$\widehat{c}_i^1(A_i) = c_i(A_i) - q_i \le (1-d)c_i(A_i) \le \frac{\lambda}{2}c_i(A_i) \le \frac{\lambda}{2}c_i(G),$$

where $1 - d \le \lambda/2$ is equivalent to $(\lambda - 2)^2 \ge 0$. If $q_i < d\,c_i(A_i)$, then (50) gives

$$c_i(G) \ge c_i(A_i) + d\,c_i(A_i) - q_i = \frac{2}{\lambda}c_i(A_i) - q_i.$$

Therefore

$$\frac{\lambda}{2}c_i(G) \ge c_i(A_i) - \frac{\lambda q_i}{2} \ge c_i(A_i) - q_i = \widehat{c}_i^1(A_i),$$

where the second inequality uses $\lambda \le 2$. Thus in all cases

$$\widehat{c}_i^1(A_i) \le \frac{\lambda}{2}c_i(G) = \frac{n\lambda}{2}\frac{c_i(G)}{n}.$$

□

Combining Theorem F.3 and Lemma F.7, no randomized online algorithm guarantees λ-PMMS for any $1 \le \lambda < 2$. The same conclusion holds for λ-PAPS because $\mathrm{APS}_i^{\mathrm{ch}}(H,2) \le \mu_i^{\mathrm{ch}}(H,2)$ for every set H: for every price vector, one bundle in a two-way min-max partition has price at least $1/2$ and cost at most $\mu_i^{\mathrm{ch}}(H,2)$. Thus λ-PAPS implies λ-PMMS.

An allocation is *λ-pairwise proportional* (λ-PPROP) if $c_i(A_i) \le \frac{\lambda}{2}c_i(A_i \cup A_j)$ for every $i \ne j$. Since $\mu_i^{\mathrm{ch}}(A_i \cup A_j, 2) \ge c_i(A_i \cup A_j)/2$, λ-PPROP also implies λ-PMMS. Hence no randomized online algorithm guarantees λ-PAPS or λ-PPROP for any $1 \le \lambda < 2$.

Corollary F.8. *Against an adaptive adversary, the following statements hold for randomized online algorithms.*

(i) No finite multiplicative approximation is guaranteed for EF, EF1, EFX, EEF, EEF1, EEFX, EFc, or epistemic EFc for any fixed c.

(ii) For every $1 \le \lambda < n$, no λ-approximation is guaranteed for PROP, PROP1, PROPk for any fixed k, PROPX, PROPm, PROPavg, MMS, APS, MXS, M1S, GMMS, or GAPS.

(iii) For every $1 \le \lambda < 2$, no λ-approximation is guaranteed for PMMS, PAPS, or PPROP.

The same statements hold for any simultaneous guarantee containing one of these notions. In each case, the adaptive adversary obtains the stated violation with probability one, and the known-horizon and bounded-cost conclusions of Theorem F.3 apply.

The factors n and 2 in this corollary are universal. Every allocation is n-PROP, and hence also n-PROPk, n-PROPX, n-PROPm, and n-PROPavg. Also, $\mathrm{MMS}_i^{\mathrm{ch}} \ge c_i(G)/n$, so every allocation is n-MMS. For additive chores, MMS implies MXS (Garg & Sharma, 2026), equivalently $\mathrm{MMS}_i^{\mathrm{ch}} \le \mathrm{MXS}_i^{\mathrm{ch}}$, and $\mathrm{MXS}_i^{\mathrm{ch}} \le \mathrm{M1S}_i^{\mathrm{ch}}$. Thus every allocation is also n-MXS and n-M1S. The same averaging argument within every group shows that every allocation is n-GMMS.

For APS, if $c_i(H) > 0$, set $x_g := c_i(g)/c_i(H)$ in (44); then every set of price at least $1/q$ has cost at least $c_i(H)/q$. The case $c_i(H) = 0$ is immediate. Therefore

$$\mathrm{APS}_i^{\mathrm{ch}}(H, q) \ge \frac{c_i(H)}{q}.$$

It follows that every allocation is n-APS and n-GAPS, and every allocation is 2-PAPS. Finally,

$$\mu_i^{\mathrm{ch}}(A_i \cup A_j, 2) \ge \frac{c_i(A_i \cup A_j)}{2} \ge \frac{c_i(A_i)}{2},$$

so every allocation is 2-PMMS; every allocation is also 2-PPROP directly from its definition.

Remark F.9. The restriction in Theorem 3.1 that every good is positively valued by at most two agents has no analogue here when $n \ge 3$. If every chore has positive cost for at most two agents, then every arriving chore has a zero-cost agent. Assigning each chore to such an agent gives every agent zero cost for her own bundle and satisfies exact proportionality and envy-freeness.

G Technical Inequalities for the Prediction Algorithms

G.1 Comparing the Potential Changes

We prove the comparison used in (31). The argument uses two bounds on each increase: an absolute bound and a bound relative to the current potential term. Together, they give the total-change inequality needed for $19/30$-PROP1.

In this subsection, let $f : (0, \infty) \to (0, \infty)$ be twice continuously differentiable, strictly decreasing and strictly convex, with $f(0+) = +\infty$ and $f(\infty) = 0$. Write $a := -f' > 0$. Assume that f and a are log-convex (that is, their logarithms are convex), that $1/a$ is convex, and that

$$a(x) \le f(x)(f(x) - f(x+1)) \quad (0 < x \le 1). \tag{51}$$

Appendix G.2 verifies these properties for the explicit function used by the algorithm. Condition (51) relates the rate of change of f to its decrease over an interval of length one, the largest normalized value of a good.

Lemma G.1. *Fix $\rho \in (0,1)$ and $M > 0$. Let $S > \rho$ satisfy*

$$f(S-\rho) - f(S+1-\rho) = M,$$

and set $d_* := f(S-\rho) - f(S)$. For $0 \le z \le 1$ and $x > \rho z$, if

$$f(x-\rho z) - f(x+(1-\rho)z) \le M,$$

then

$$f(x-\rho z) - f(x) \le d_*, \quad \frac{f(x-\rho z) - f(x)}{f(x)} \le \frac{d_*}{f(S)}. \tag{52}$$

Moreover,

$$\frac{d_*}{M} \le \begin{cases} \rho f(S), & S \le 1, \\ \rho f(1), & S \ge 1. \end{cases} \tag{53}$$

Finally, $f(x) > 1/x$ for $0 < x \le 1$.

Proof. The case $z = 0$ in (52) is immediate. Otherwise, write $\ell := x - \rho z$ and $u := \ell + z$, so $x = (1-\rho)\ell + \rho u$. At fixed ℓ, increasing z increases both $f(\ell) - f(x)$ and its ratio to $f(x)$.

First suppose $\ell \ge S - \rho$. Increase z to one. Convexity of f implies that $f(\ell) - f(\ell+\rho)$ is nonincreasing in ℓ. Log-convexity of f implies the same for $f(\ell)/f(\ell+\rho) - 1$. Evaluating at $\ell = S - \rho$ proves both bounds.

Now suppose $\ell < S - \rho$. Since $f(\ell) - f(\ell+1) > M$, increase z until $f(\ell) - f(u) = M$. Along the curve defined by this equality,

$$u'(\ell) = \frac{a(\ell)}{a(u)}, \quad x'(\ell) = (1-\rho) + \rho\frac{a(\ell)}{a(u)} > 0.$$

Convexity of $1/a$ gives

$$\frac{1}{a(x)} \le \frac{1-\rho}{a(\ell)} + \frac{\rho}{a(u)},$$

and therefore

$$\frac{d}{d\ell}(f(\ell) - f(x)) = -a(\ell) + a(x)\left((1-\rho) + \rho\frac{a(\ell)}{a(u)}\right) \ge 0.$$

Meanwhile $f(x)$ decreases, so the ratio $(f(\ell) - f(x))/f(x)$ also increases. Increase ℓ to $S - \rho$, where $u = S + 1 - \rho$ and $x = S$. This proves (52). The curve stays within $u - \ell \le 1$: this difference increases to one because a is decreasing.

For (53), first let $S \le 1$. By (51),

$$\begin{aligned} \frac{1}{f(S)} - \frac{1}{f(S-\rho)} &= \int_{S-\rho}^{S} \frac{a(t)}{f(t)^2}\,dt \\ &\le \int_{S-\rho}^{S} \left(1 - \frac{f(t+1)}{f(t)}\right) dt \\ &\le \rho\left(1 - \frac{f(S+1-\rho)}{f(S-\rho)}\right). \end{aligned}$$

The last inequality holds because log-convexity of f makes $f(t+1)/f(t)$ nondecreasing. Multiplying by $f(S)f(S-\rho)$ and dividing by M gives $d_*/M \le \rho f(S)$.

For every $S > \rho$, the ratio in question can be written as

$$\frac{d_*}{M} = \frac{\int_0^\rho a(S-\rho+t)\,dt}{\int_0^1 a(S-\rho+t)\,dt}.$$

This ratio is nonincreasing in S. Indeed, for $h_2 > h_1$, log-convexity of a makes $a(h_2+t)/a(h_1+t)$ nondecreasing in t. Hence, whenever $0 \le t \le \rho \le u \le 1$,

$$a(h_2+t)a(h_1+u) \le a(h_1+t)a(h_2+u).$$

Integration over $t \in [0, \rho]$ and $u \in [\rho, 1]$, followed by cross-multiplication, proves the assertion. When $S \geq 1$, the ratio is thus at most its value at $S = 1$, which is at most $\rho f(1)$ by the preceding case.

Finally, integrating (51) after division by $f(t)^2$ and letting the lower endpoint tend to zero gives

$$\frac{1}{f(x)} \leq \int_0^x \left(1 - \frac{f(t+1)}{f(t)}\right) dt < x \quad (0 < x \leq 1).$$

The last inequality is strict since the subtracted ratio is positive. This proves the last assertion. □

To obtain (31), apply (52) with $x = s_i^{t-1} + \rho$ and $z = z_i$. Summing the first bound gives $\sum_i d_i \leq n d_*$. Summing the second and using $\Phi^{t-1} \leq n f(1)$ gives $\sum_i d_i \leq n f(1) d_* / f(S)$. If $S \leq 1$, combine the latter with the first case of (53); if $S \geq 1$, combine the former with the second case. In either case, $\sum_i d_i / M \leq n\rho f(1)$, as required.

G.2 The Function Giving the 19/30 Factor

We give an explicit f satisfying the properties in Appendix G.1 and $f(1) < 30/19$. Set

$$b := \frac{1431}{1000}, \quad r := \frac{1517}{6775}, \quad K := \frac{1423249}{1300800},$$
$$c := K\left(1 + \log \frac{327121}{47720}\right) - \frac{1157}{480},$$
$$f(x) := \begin{cases} \dfrac{1}{x} + c + \dfrac{x}{16} - \dfrac{25x^2}{12} + \dfrac{875x^3}{256}, & 0 < x \leq 2/5, \\ K\left(1 + \log \dfrac{b-r}{x-r}\right), & 2/5 \leq x \leq b, \\ K \exp\left(-\dfrac{x-b}{b-r}\right), & x \geq b. \end{cases} \tag{54}$$

The constants make the pieces and their first two derivatives agree at both endpoints. The logarithmic part gives $1/(-f'(x)) = (x-r)/K$, an affine function, so the convexity used to compare potential changes holds there with equality. The reciprocal term makes small slacks contribute large potential values; the polynomial terms let (51) hold with leading term $1/x$ rather than $2/x$. The exponential part makes $f(x+1)$ small enough for the same inequality.

Monotonicity and convexity. With $a = -f'$ as above,

$$a(x) = \begin{cases} x^{-2} - \dfrac{1}{16} + \dfrac{25x}{6} - \dfrac{2625x^2}{256}, & 0 < x \leq 2/5, \\ \dfrac{K}{x-r}, & 2/5 \leq x \leq b, \\ \dfrac{K}{b-r} \exp\left(-\dfrac{x-b}{b-r}\right), & x \geq b. \end{cases}$$

At $x = 2/5$, the first two expressions give

$$a(2/5) = \frac{1193}{192} = \frac{K}{2/5 - r}, \quad a'(2/5) = -\frac{6775}{192} = -\frac{K}{(2/5-r)^2}.$$

The value of the first expression for $f(2/5)$ is $c + 1157/480$, which agrees with the second expression by the definition of c. Matching at b is immediate. Thus a is continuously differentiable and f is twice continuously differentiable.

For $x \in [0, 2/5]$, define

$$H(x) := 1 - \frac{x^2}{16} + \frac{25x^3}{6} - \frac{2625x^4}{256},$$
$$J(x) := 2 - \frac{25x^3}{6} + \frac{2625x^4}{128},$$
$$Q(x) := 6 - \frac{2625x^4}{128}.$$

Then $a(x) = H(x)/x^2$, $-a'(x) = J(x)/x^3$, and $a''(x) = Q(x)/x^4$ for $0 < x < 2/5$. We claim that

$$\frac{99}{100} \le H(x) \le \frac{11}{10}, \quad \frac{19}{10} \le J(x) \le \frac{271}{120}, \quad \frac{219}{40} \le Q(x) \le 6.$$

For the lower bound on H, use $25/6 - 2625x/256 \ge 0$ and $x^2/16 \le 1/100$. Differentiating the two polynomials on the left below gives

$$x^3\left(\frac{25}{6} - \frac{2625x}{256}\right) \le \frac{25}{24}\left(\frac{32}{105}\right)^3 < \frac{1}{10}, \quad x^3\left(\frac{25}{6} - \frac{2625x}{128}\right) \le \frac{25}{24}\left(\frac{16}{105}\right)^3 < \frac{1}{10}.$$

These prove the upper bound on H and the lower bound on J. Since $J'(x) = x^2(-25/2 + 2625x/32)$ changes sign only from negative to positive, the maximum of J is at an endpoint; its endpoint values are 2 and $271/120$. The bounds on Q follow directly from $0 \le x \le 2/5$. Consequently,

$$HQ - J^2 \ge \frac{99}{100}\frac{219}{40} - \left(\frac{271}{120}\right)^2 = \frac{23053}{72000} > 0,$$

$$2J^2 - HQ \ge 2\left(\frac{19}{10}\right)^2 - \frac{11}{10}\cdot 6 = \frac{31}{50} > 0.$$

Thus $a > 0$, $a' < 0$, $aa'' - (a')^2 > 0$, and $2(a')^2 - aa'' > 0$ on $(0, 2/5)$. These are the required monotonicity of a, log-convexity of a, and convexity of $1/a$. On $[2/5, b]$, $\log a = \log K - \log(x - r)$ is convex and $1/a = (x - r)/K$ is affine. On $[b, \infty)$, $\log a$ is affine and $1/a$ is exponential. The matching of a and a' makes both convexity properties hold on $(0, \infty)$.

Since $f(x) = \int_x^\infty a(t)\,dt$, the function f is positive, strictly decreasing and strictly convex, with the required limits. Log-convexity of a and Hölder's inequality also give log-convexity of f: for $x, y > 0$ and $\theta \in (0, 1)$,

$$\begin{aligned} f(\theta x + (1-\theta)y) &\le \int_0^\infty a(x+t)^\theta a(y+t)^{1-\theta}\,dt \\ &\le f(x)^\theta f(y)^{1-\theta}. \end{aligned}$$

Bounds on the constants. The value at one is

$$f(1) = K\left(1 + \log\frac{327121}{210320}\right).$$

For $q \ge 1$, put $z = (q-1)/(q+1)$. For any positive integer h,

$$2\sum_{j=0}^{h-1}\frac{z^{2j+1}}{2j+1} \le \log q \le 2\sum_{j=0}^{h-1}\frac{z^{2j+1}}{2j+1} + \frac{2z^{2h+1}}{(2h+1)(1-z^2)}. \tag{55}$$

This follows by summing the positive series for $\log((1+z)/(1-z))$ and bounding its remaining terms by a geometric series. Apply (55) with $h = 10$ to $\log(327121/210320)$ and to the two logarithms in

$$\log\frac{327121}{47720} = 3\log 2 - \log\frac{381760}{327121}.$$

Substitution in the definitions of $f(1)$ and c gives the rational bounds

$$\frac{157741237}{10^8} < f(1) < \frac{157741238}{10^8}, \quad \frac{78990175}{10^8} < c < \frac{78990176}{10^8}. \tag{56}$$

In particular, $1 < f(1) < 30/19$.

Proving (51) for $0 < x \le 2/5$. The terms of order $1/x^2$ cancel in $f(x)(f(x) - f(x+1)) - a(x)$. We bound the remaining expression by a rational polynomial and then by a positive cubic. Let $C_- := 7887/5000$ and $C_+ := 78871/50000$, so $C_- < f(1) < C_+$ by (56). Define

$$P(x) := \frac{7899}{10000} + \frac{x}{16} - \frac{25x^2}{12} + \frac{875x^3}{256},$$

$$U(x) := C_+ - K\sum_{j=1}^{6}\frac{(-1)^{j+1}}{j}\left(\frac{x}{1-r}\right)^j,$$

$$R_0(x) := 2P(x) - U(x) + x(P(x)^2 - P(x)U(x) + P'(x)).$$

Since $x+1 \le 7/5 < b$ and $0 \le x/(1-r) < 1$, the even partial sum for $\log(1+z)$ gives

$$f(x+1) = f(1) - K\log\left(1+\frac{x}{1-r}\right) \le U(x) \le C_+.$$

The last inequality follows by pairing consecutive terms in the sum defining U. Also, $P(x) \ge 7899/10000 - 1/3 > 0$. Write $f(x) = x^{-1} + \widetilde{P}(x)$, where $\widetilde{P} = P + c - 7899/10000 \ge P$. Direct expansion gives

$$\begin{aligned} x[f(x)(f(x)-f(x+1)) - a(x)] &= 2\widetilde{P} - f(x+1) \\ &\quad + x(\widetilde{P}^2 - \widetilde{P}f(x+1) + P'). \end{aligned}$$

This expression decreases with $f(x+1)$. After replacing $f(x+1)$ by U, it increases with $\widetilde{P} \ge P$, since its derivative with respect to $\widetilde{P}$ is at least $2 - xC_+ > 0$. Therefore

$$x[f(x)(f(x)-f(x+1)) - a(x)] \ge R_0(x). \tag{57}$$

Expanding R_0 and comparing the rational coefficients of the same powers of x gives, for $x \ge 0$,

$$\begin{aligned} R_0(x) \ge \frac{119}{50000} + \frac{39x}{40} - \frac{813x^2}{100} + \frac{86x^3}{5} - \frac{17x^4}{5} + \frac{117x^5}{10} \\ - \frac{193x^6}{10} + \frac{153x^7}{10} - \frac{17x^8}{4} - 3x^{10}. \end{aligned}$$

For $0 \le x \le 2/5$, the last three terms are at least $13x^7$, because $153/10 - (17/4)x - 3x^3 \ge 153/10 - (17/4)(2/5) - 3(2/5)^3 > 13$. Furthermore, the identity

$$\begin{aligned} &\frac{2}{5} - \frac{17x}{5} + \frac{117x^2}{10} - \frac{193x^3}{10} + 13x^4 \\ &\quad = \frac{3x}{125} + \left(\frac{2}{5}-x\right)^2\left[\frac{51}{50} - \frac{3}{2}\left(\frac{2}{5}-x\right) + 13\left(\frac{2}{5}-x\right)^2\right] \ge 0 \end{aligned}$$

holds because the bracket is at least $21/50$. Combining these two bounds yields

$$R_0(x) \ge \frac{119}{50000} + \frac{39x}{40} - \frac{813x^2}{100} + \frac{84x^3}{5} > 0.$$

To justify the strict inequality, for $0 \le x \le 1/5$ the quadratic $39/40 - 813x/100 + 84x^2/5$ is decreasing and is at least its value $21/1000$ at $1/5$. For $1/5 \le x \le 2/5$, the same cubic equals

$$\begin{aligned} &\left(\frac{84x}{5} + \frac{27}{100}\right)\left(x-\frac{1}{4}\right)^2 + \frac{3}{50}\left(x-\frac{1}{4}\right) + \frac{101}{200000} \\ &\quad \ge \frac{363}{100}\left(x-\frac{1}{4}\right)^2 + \frac{3}{50}\left(x-\frac{1}{4}\right) + \frac{101}{200000} \ge \frac{6221}{24200000} > 0, \end{aligned}$$

where the last bound follows by completing the square. This proves (51) on $(0, 2/5]$.

Proving (51) for $2/5 \le x \le 1$. Here $a(x) = K/(x-r)$. Define

$$R_1(x) := \frac{x-r}{K} f(x)(f(x) - f(x+1)) - 1. \tag{58}$$

We show that R_1 is concave on $[2/5, 1]$. Its minimum is then at an endpoint, so only two values need to be bounded. Note that $6/5 < b - r < 5/4$.

For $2/5 \le x \le b-1$, both function values use the logarithmic expression. Differentiating gives

$$R_1''(x) = \frac{K}{x-r}\left[\frac{2(x+1-r) - f(x)/K}{(x+1-r)^2} - \log\frac{x+1-r}{x-r}\right] < 0.$$

Indeed, $\log q \ge 1 - 1/q$ for $q \ge 1$ gives

$$\frac{f(x)}{K} \ge 1 + \frac{1}{b-r} > \frac{9}{5} > x + 1 - r, \quad \log\frac{x+1-r}{x-r} > \frac{1}{x+1-r},$$

which prove the sign of the displayed derivative.

For $b-1 \le x \le 1$, $f(x+1)$ uses the exponential expression. In the derivative calculation, write $y = (x-r)/(b-r)$, so $0 < y < 1$ and $y \ge 1 - 1/(b-r)$. Then

$$R_1''(x) = \frac{K}{x-r}\left[2\log y + e^{1-1/(b-r)-y}(1-y^2-(2y-y^2)\log y)\right].$$

The coefficient of the exponential is positive. Since $e^{-u} \le 1/(1+u)$ for $u \ge 0$, we obtain

$$R_1''(x) \le \frac{K}{x-r}\frac{1-y^2+(y^2+2/(b-r))\log y}{y+1/(b-r)} < 0.$$

For the final sign, the first term of (55) gives

$$-\log y \ge \frac{2(1-y)}{1+y} > \frac{1-y^2}{y^2+2/(b-r)}.$$

The strict inequality follows from $2(y^2+2/(b-r)) - (1+y)^2 = (1-y)^2 + 4/(b-r) - 2 > 0$. The pieces of f have matching derivatives, so R_1 is concave on the whole interval.

For the endpoint at $2/5$, (56) and $\log q \le q-1$ give

$$f(2/5) = c + \frac{1157}{480} > \frac{16}{5}, \quad f(7/5) \le K\frac{b-r}{7/5-r} < \frac{9}{8}.$$

For the endpoint at one, use $f(1) > C_-$ and the even Taylor polynomial for the exponential:

$$f(2) \le K\sum_{j=0}^{4}\frac{1}{j!}\left(\frac{b-2}{b-r}\right)^j < \frac{6831}{10000}.$$

Since $u(u-v)$ increases with u whenever $2u > v$, substitution gives

$$R_1(2/5) > \frac{2/5-r}{K}\frac{16}{5}\left(\frac{16}{5}-\frac{9}{8}\right) - 1 > \frac{1}{20},$$
$$R_1(1) > \frac{1-r}{K}C_-\left(C_- - \frac{6831}{10000}\right) - 1 > \frac{1}{2000}.$$

By concavity, $R_1(x) > 0$ throughout $[2/5, 1]$. This proves (51) on the remaining interval. All hypotheses of Lemma G.1 have now been established by algebra and calculus, completing the proof of the $19/30$-PROP1 guarantee.

G.3 The Guarantee When Few Agents Value Each Good

We prove the $n/(n+\kappa)$ guarantee used in Theorem 4.1. The reciprocal rule assigns each good to a recipient minimizing (29).

Proof. Set $\rho := 1/(n+\kappa)$ and use s_i^t and Ψ^t from (4) and (29). Initially, $s_i^0 = 1-\rho$ and $\Psi^0 \le n/(1-\rho)$.

Assume inductively that $s_i^{t-1} > 0$ for every agent with $p_i > 0$ and $\Psi^{t-1} \le \frac{n}{1-\rho}$. This holds at time zero. Since each reciprocal term is at most the sum,

$$s_i^{t-1} \ge \frac{1}{\Psi^{t-1}} \ge \frac{1-\rho}{n} \tag{59}$$

for every agent with $p_i > 0$.

For the current good, use z_i from (24) and let $Q_t := \{i : z_i > 0\}$. Only agents in Q_t have changing slacks, and $|Q_t| \le \kappa$. Their updates are given by (5).

Equation (59) and $z_i \le 1$ give us

$$s_i^{t-1} - \rho z_i \ge \frac{1-\rho}{n} - \rho = \frac{\kappa-1}{n(n+\kappa)} > 0.$$

Thus, every possible assignment keeps all s_i^t positive.

We now show that some recipient does not increase the potential. Consider the following probability distribution only for this calculation. For each $i \in Q_t$, set $\pi_i := \rho(1 + (1-\rho) \cdot \frac{z_i}{s_i^{t-1}})$. Algebraic manipulation gives us

$$\frac{1-\pi_i}{s_i^{t-1} - \rho z_i} + \frac{\pi_i}{s_i^{t-1} + (1-\rho)z_i} = \frac{1}{s_i^{t-1}}. \tag{60}$$

Moreover,

$$\sum_{i \in Q_t} \pi_i = \rho|Q_t| + \rho(1-\rho) \sum_{i \in Q_t} \frac{z_i}{s_i^{t-1}} \le \rho\kappa + \rho(1-\rho)\Psi^{t-1} \le \rho\kappa + \rho n = 1.$$

Assign probability π_i to each $i \in Q_t$ and distribute the remaining probability arbitrarily. For each $i \in Q_t$, the probability that i receives the good is then at least π_i. Since $s_i^{t-1} + (1-\rho)z_i > s_i^{t-1} - \rho z_i > 0$, the value of $1/s_i^t$ is smaller when i receives the good than when it does not. Increasing the probability that i receives the good above π_i can therefore only decrease its expectation. By (60), we get that

$$\mathbb{E}\left[\frac{1}{s_i^t}\right] \le \frac{1}{s_i^{t-1}} \quad \text{for every } i \in Q_t.$$

For i with $p_i > 0$ and $i \notin Q_t$, $s_i^t = s_i^{t-1}$. Therefore, $\mathbb{E}[\Psi^t] \le \Psi^{t-1}$ under this distribution over recipients. At least one recipient has $\Psi^t \le \Psi^{t-1}$, and the allocation rule chooses a minimizing recipient. By induction,

$$\Psi^t \le \Psi^0 \le \frac{n}{1-\rho} \quad \text{for all } t = 0, \dots, m. \tag{61}$$

In particular, $s_i^m > 0$ for every agent with $p_i > 0$. Since $D_i^m = v_i(G)$, we get that

$$p_i + B_i^m > \rho v_i(G) = \frac{n}{n+\kappa} \frac{v_i(G)}{n}.$$

By the interpretation following (3), every agent satisfies $n/(n+\kappa)$-PROP1.

The expectation bound above is obtained after fixing the realized history and the current good. It therefore remains valid when the adversary chooses later goods after seeing earlier recipients. □

G.4 The Potential Decrease for the Simultaneous Guarantee

We prove the assertion used in the proof of Theorem 4.3 in Appendix C.3: if the current slacks are positive and $\Omega^{t-1} < 1$, some recipient satisfies $\Omega^t \le \Omega^{t-1}$. Throughout, use f from Appendix G.2, $a = -f'$ as in Appendix G.1, and ρ, C, η from Section C.3. In particular, $0 < \rho < 1/3$ and $0 < \eta \le 10^{-6}$.

Two estimates for one agent. For $0 \le z \le 1$ and $x > \rho z$, write

$$\ell := x - \rho z, \quad u := x + (1-\rho)z, \quad d := f(\ell) - f(x), \quad \Delta := f(\ell) - f(u).$$

Thus d is the increase when the agent does not receive the good, and Δ is the decrease obtained by changing that decision. Define

$$Q := \Delta - 2\eta(1 + f(u)), \quad \tau := f(1-\rho) - f(2-\rho).$$

The value τ is Δ at $x = z = 1$. The subtraction in Q accounts for the increase in envy toward a recipient: $e^{\eta \widehat{v}_i(g_t)} - 1 \le 2\eta$, and $1 + f(u)$ is the recipient's multiplier after receiving the good. We will show

$$Q \le \tau \implies d \le \rho C \Delta, \tag{62}$$

$$M \ge \tau, \quad Q \le M \implies d \le \frac{2001}{2000} \rho f(x) M. \tag{63}$$

The first estimate will be used for a uniformly random recipient in Case 1 below. The second allows us to sum the increases with the weights contributed by the envy potential.

First, Lemma G.1 gives

$$\frac{d}{\rho f(x)} \le \max\{\Delta, \tau\}. \tag{64}$$

Indeed, apply (52) with $M = \max\{\Delta, \tau\}$; its corresponding S is at most one, so (53) proves the displayed bound. A second consequence is

$$\left.\begin{array}{c} \rho < S \le 1, \\ \Delta \le f(S-\rho) - f(S+1-\rho) \end{array}\right\} \quad \Longrightarrow \quad d \le \rho f(S)\Delta. \tag{65}$$

For $z > 0$, first apply (52) with $M = \Delta$. Its corresponding value of S is no smaller than the one in (65). The ratio

$$\frac{f(S-\rho) - f(S)}{f(S-\rho) - f(S+1-\rho)}$$

is nonincreasing on $S > \rho$, as proved in Appendix G.1. Combining this monotonicity with (53) proves (65). If $z = 0$, then $d = \Delta = 0$, and all the required estimates are immediate.

We use the following elementary bounds for the explicit function:

$$\begin{array}{c} f(2499/2500) < C, \quad 1 + f(1/100) < 102, \\ f(1) - f(2) > 4/5, \quad a(1) - a(2) > 4/5. \end{array} \tag{66}$$

For the first bound, monotonicity of a and the logarithmic piece of f give

$$f(2499/2500) \le f(1) + \frac{K}{2500(2499/2500 - r)} < \frac{789}{500},$$

using the upper bound on $f(1)$ in (56). The second follows from the first piece of f and $c < 4/5$. For the last two, the endpoint estimate in Appendix G.2 gives $f(2) < 7/10$. Now use $f(1) > 3/2$, $a(1) = K/(1-r) > 7/5$, and $a(2) = f(2)/(b-r)$ with $b - r > 6/5$. All constants are those in (54). Since a is decreasing and $1 - \rho > 2/5$, these bounds also imply

$$\frac{4}{5} < f(1) - f(2) \le \tau < f(2/5) < 4. \tag{67}$$

Finally, direct substitution into the first pieces of f and a gives

$$f(v) \le \frac{2}{v}, \quad \frac{99}{100v^2} \le a(v) \le \frac{101}{100v^2} \quad (0 < v \le 1/100). \tag{68}$$

For the derivative bounds, multiply by v^2 and bound the monomials of $1 - v^2/16 + 25v^3/6 - 2625v^4/256$ on $[0, 1/100]$; the bound on f uses $c < 4/5$.

Suppose first that $u \ge 1/100$, so $1 + f(u) < 102$. The positive log-convex function a is convex, and hence $a(v) - a(v+1)$ is nonincreasing. Consequently,

$$\begin{aligned} &f(2499/2500 - \rho) - f(2499/2500 + 1 - \rho) - \tau \\ &\quad = \int_{2499/2500}^{1} (a(v-\rho) - a(v+1-\rho))\, dv \\ &\quad \ge \frac{a(1) - a(2)}{2500} > \frac{4}{12500} > 204\eta. \end{aligned}$$

If $Q \le \tau$, then $\Delta \le \tau + 204\eta$, so (65) at $S = 2499/2500$, together with $f(2499/2500) < C$, proves (62). If $Q \le M$ and $M \ge \tau$, then (64) gives

$$\frac{d}{\rho f(x)} \le M + 204\eta \le \frac{2001}{2000} M,$$

where the last inequality uses $M > 4/5$ and $\eta \le 10^{-6}$. This proves (63).

It remains to consider $0 < u < 1/100$ and $z > 0$. Since $x = \ell + \rho(u - \ell)$, (68) gives

$$\frac{d}{\Delta} \le \frac{101}{99} \frac{\int_\ell^x v^{-2}\, dv}{\int_\ell^u v^{-2}\, dv} = \frac{101}{99} \frac{\rho u}{x}.$$

Using $f(x) > 1/x$ and $u(1 + f(u)) < 3$, if $Q \le M$ and $M \ge \tau$ we obtain

$$\frac{d}{\rho f(x)} \le 2u\Delta \le 2uM + 4\eta u(1 + f(u)) < \frac{M}{50} + 12\eta < M.$$

This is stronger than (63). If instead $Q \le \tau$, then (67) gives

$$\frac{99}{100}\left(\frac{u}{\ell} - 1\right) \le u\Delta \le 4u + 2\eta u(1 + f(u)) < \frac{4}{100} + 6\eta < \frac{7}{100}.$$

Thus $u/\ell < 11/10$. By monotonicity of a and (68),

$$\frac{d}{\rho\Delta} \le \frac{a(\ell)}{a(u)} \le \frac{101}{99}\left(\frac{u}{\ell}\right)^2 < \frac{101}{99}\left(\frac{11}{10}\right)^2 < \frac{3}{2} < C.$$

This proves (62) and completes both estimates.

Comparing recipients. Fix the realized history and current good, and suppose the current slacks are positive and $\Omega^{t-1} < 1$. For each agent with $p_i > 0$, apply the preceding estimates with $x = s_i^{t-1} + \rho$ and $z = z_i$, adding subscript i to ℓ, u, d, Δ, Q. Then Δ_i is exactly (8). For $p_i = 0$, set $f(\ell_i) = f(u_i) = d_i = \Delta_i = 0$ and $Q_i = -2\eta$, consistently with the convention in Section C.3. In the envy calculation, we always use the actual normalized value $\widehat{v}_i(g_t)$, not z_i. In particular, a first maximum-valued good has $z_i = 0$ but $\widehat{v}_i(g_t) = 1$.

Case 1: $Q_i \le \tau$ for every agent. Since $n\rho C = (19/30)C < 1$, (62) implies

$$d_i - \frac{\Delta_i}{n} \le \left(\rho C - \frac{1}{n}\right)\Delta_i \le 0.$$

Under the uniform distribution over all n recipients, it follows that $\mathbb{E}[\Phi^t] \le \Phi^{t-1}$. For distinct agents i, j, direct expansion gives

$$\begin{aligned}
&\frac{\mathbb{E}\left[e^{\eta E_{ij}^t}(1 + f(s_j^t + \rho))\right]}{e^{\eta E_{ij}^{t-1}}} \\
&\quad = 1 + f(s_j^{t-1} + \rho) + d_j - \frac{\Delta_j}{n} \\
&\qquad + \frac{e^{\eta\widehat{v}_i(g_t)} - 1}{n}(1 + f(u_j)) + \frac{e^{-\eta\widehat{v}_i(g_t)} - 1}{n}(1 + f(\ell_j)) \\
&\quad \le \left(1 + \frac{2\eta^2}{n}\right)(1 + f(s_j^{t-1} + \rho)).
\end{aligned}$$

For the last inequality, use $f(\ell_j) \ge f(u_j)$, $f(u_j) \le f(s_j^{t-1} + \rho)$, and $e^h + e^{-h} - 2 \le 2\eta^2$ for $0 \le h \le \eta \le 1$. The negative coefficient of $1 + f(\ell_j)$ lets us replace it by $1 + f(u_j)$ in an upper bound. These observations also apply when $p_j = 0$. Summing over ordered pairs gives $\mathbb{E}[F_\eta^t] \le A_\eta F_\eta^{t-1}$, and therefore

$$\mathbb{E}[\Omega^t] \le \frac{\Phi^{t-1}}{nC} + A_\eta^{m-t+1} e^{-\eta L} F_\eta^{t-1} = \Omega^{t-1}.$$

Some recipient consequently does not increase Ω.

Case 2: $Q_i > \tau$ for some agent. Use the coefficient of the envy potential at the *new* time t, and define

$$\lambda := nCA_\eta^{m-t}e^{-\eta L}, \quad w_j := 1 + \lambda \sum_{i \ne j} e^{\eta E_{ij}^{t-1}} \ge 1.$$

With the exponential factors held fixed, w_j is the coefficient of agent j's PROP1 term in $\Phi + \lambda F_\eta$. Moreover,

$$\sum_j w_j f(s_j^{t-1} + \rho) \le \Phi^{t-1} + \lambda F_\eta^{t-1} \le nC\Omega^{t-1} < nC. \tag{69}$$

The middle inequality uses the larger coefficient $A_\eta\lambda$ at time $t-1$. Choose a recipient j maximizing w_jQ_j and put $M := w_jQ_j$. Since some $Q_i > \tau$ and every $w_i \geq 1$, we have $M > \tau > 0$ and $Q_i \leq M$ for every agent. Applying (63) with this common M and then (69) gives

$$\sum_i w_i d_i \leq \frac{2001}{2000}\rho M \sum_i w_i f(s_i^{t-1} + \rho) < \frac{2001}{2000}\frac{19}{30} CM < M. \tag{70}$$

Here $(2001/2000)(19/30)C = 9998997/10000000 < 1$.

Assign the good to this j. Updating the PROP1 terms while holding the exponentials fixed changes $\Phi + \lambda F_\eta$ by exactly $\sum_i w_i d_i - w_j\Delta_j$. Updating the exponentials decreases the terms for agent j's envy toward the other agents. The only positive changes are in the terms for other agents' envy toward j, and their sum is at most

$$\lambda \sum_{i \neq j} e^{\eta E_{ij}^{t-1}} (e^{\eta \widehat{v}_i(g_t)} - 1)(1 + f(u_j)) \leq 2\eta(w_j - 1)(1 + f(u_j)).$$

Consequently,

$$\begin{aligned}
&(\Phi^t + \lambda F_\eta^t) - (\Phi^{t-1} + \lambda F_\eta^{t-1}) \\
&\quad \leq \sum_i w_i d_i - w_j\Delta_j + 2\eta(w_j - 1)(1 + f(u_j)) \\
&\quad \leq \sum_i w_i d_i - w_j Q_j < 0,
\end{aligned}$$

by (70). Dividing by nC and again using the larger envy coefficient at time $t-1$ yields $\Omega^t \leq \Omega^{t-1}$.

In both cases, some recipient does not increase Ω. The algorithm chooses a recipient minimizing Ω^t, so it has the same property. This proves the required assertion without requiring a single distribution to preserve every PROP1 term at every time.

H A Simpler Simultaneous Guarantee with MIV Predictions

The rule in Section C.3 uses envy terms that depend on the PROP1 slacks. Here we give a simpler rule with separate PROP1 and envy potentials, both of which satisfy the same uniform averaging inequality: assigning the good to a uniformly random agent does not increase the potential in expectation. Its PROP1 factor is smaller, but its normalized envy bound has the explicit constant 11. In this appendix, the symbols Φ^t, Ω^t, C, and L are redefined below. For envy, the form of Γ^t and its uniform averaging calculation follow the exponential potential method of Benadè et al. (2018). We choose its parameters so that the initial envy potential is below $1/2$, and then add it to the new PROP1 potential. Their common uniform averaging inequality gives both guarantees for the same allocation.

Set

$$q := \frac{2n^2 - 1}{n - 1}, \quad \zeta := \log q, \quad \psi := \log\left(\frac{(n-1)e^{\zeta/n} + e^{-(n-1)\zeta/n}}{n}\right), \tag{71}$$

and define

$$\alpha_n := 1 - \frac{n\psi}{\zeta} = \frac{n}{\log q}\log\left(\frac{q}{2n}\right). \tag{72}$$

For every agent with $p_i > 0$, let

$$\phi_i^t := \frac{1}{q}\exp\left(\zeta\left(\frac{D_i^t}{np_i} - \frac{B_i^t}{p_i}\right) - \psi\frac{D_i^t}{p_i}\right), \quad \Phi^t := \sum_{i:p_i>0}\phi_i^t. \tag{73}$$

For a possible recipient a, write $\Phi^t(a)$ for the value after assigning the current good to a.

Lemma H.1. *The potential in (73) has the following properties:*

(i) for every realized history and current good, $\frac{1}{n}\sum_{a\in N}\Phi^t(a) \leq \Phi^{t-1}$;

(ii) $\Phi^0 \leq 1/2$;

(iii) if $\phi_i^m \leq 1$, *then agent* i *satisfies* α_n*-PROP1; and*

(iv) $0 < \alpha_n < 1$, *and* $\frac{1}{2\log(4n)} \leq \alpha_n \leq \frac{2}{\log(2n)}$.

Proof. Fix a time t and an agent i with $p_i > 0$. If $t = t_i^*$, then D_i and B_i do not change, so $\phi_i^t(a) = \phi_i^{t-1}$ for every recipient a.

Suppose $t \neq t_i^*$, so that $z_i = v_i(g_t)/p_i \in [0,1]$ by (24). If i receives the good, then

$$\phi_i^t(i) = \phi_i^{t-1} \exp\left(-\frac{n-1}{n}\zeta z_i - \psi z_i\right).$$

If another agent receives it, then

$$\phi_i^t(a) = \phi_i^{t-1} \exp\left(\frac{\zeta z_i}{n} - \psi z_i\right).$$

Therefore

$$\frac{1}{n}\sum_{a\in N}\phi_i^t(a) = \phi_i^{t-1} e^{-\psi z_i} \frac{e^{-(n-1)\zeta z_i/n} + (n-1)e^{\zeta z_i/n}}{n}.$$

Let

$$H(z) := \log\left(\frac{e^{-(n-1)\zeta z/n} + (n-1)e^{\zeta z/n}}{n}\right).$$

The function H is convex, $H(0) = 0$, and $H(1) = \psi$. Hence $H(z) \leq z\psi$ for $z \in [0,1]$. The average candidate value of ϕ_i is therefore at most its current value. Summing over agents proves the first statement.

The identity $(n-1)q + 1 = 2n^2$ gives $e^{\psi} = 2nq^{1/n-1}$. At time zero, $D_i^0 = p_i$ and $B_i^0 = 0$, so $\phi_i^0 = \frac{1}{q}e^{\zeta/n-\psi} = \frac{1}{2n}$. There are at most n agents with positive p_i, and therefore $\Phi^0 \leq 1/2$.

At the final time, $D_i^m = v_i(G)$. The inequality $\phi_i^m \leq 1$ is equivalent to

$$p_i + B_i^m \geq \left(\frac{1}{n} - \frac{\psi}{\zeta}\right) v_i(G) = \frac{\alpha_n}{n} v_i(G).$$

The interpretation of $p_i + B_i^m$ following (3) proves α_n-PROP1.

For the bounds, note that $\frac{q}{2n} = 1 + z$ with $z := \frac{2n-1}{2n(n-1)}$. Then

$$\alpha_n = \frac{n\log(1+z)}{\log q}.$$

Moreover, $0 < \alpha_n < 1$. Positivity follows from $q > 2n$. Also,

$$(n-1)\log(1+z) < (n-1)z = \frac{2n-1}{2n} < 1 < \log(2n).$$

Thus $(n-1)\log(1+z) < \log(2n)$, which is equivalent to $\alpha_n < 1$. For $n \geq 2$, we have $z \leq 1$, $1 \leq nz \leq 2$, and $2n < q \leq 4n$. Using $\frac{z}{1+z} \leq \log(1+z) \leq z$ gives us the stated bounds. □

The lemma already implies that assigning each good to a recipient minimizing Φ^t gives α_n-PROP1. Its main use here is that the averaging inequality can be combined with the next potential.

Adding the envy potential. Assume that the horizon $m \geq n\log n$ is known. We use the normalized valuations $\widehat{v}_i$ and the quantities E_{ij}^t defined in Section C.3. Set $s := \sqrt{2\log(1 + \frac{n\log n}{m})}$, $C := 1 + \frac{e^s + e^{-s} - 2}{n}$, and $L := 10\sqrt{\frac{m\log n}{n}} + \frac{\log 2}{s}$. Also define

$$\xi_{ij}^t := C^{m-t}\exp(s(E_{ij}^t - L)), \quad \Gamma^t := \sum_{i\neq j}\xi_{ij}^t. \tag{74}$$

For a possible recipient a, write $\Gamma^t(a)$ for the value after assigning the current good to a.

Lemma H.2. *The potential in (74) has the following properties:*

(i) for every realized history and current good, $\frac{1}{n}\sum_{a\in N}\Gamma^t(a) \leq \Gamma^{t-1}$;

(ii) $\Gamma^0 < 1/2$;

(iii) if $\xi_{ij}^m \leq 1$, *then* $E_{ij}^m \leq L$; *and*

(iv) $L \leq 11\sqrt{\frac{m\log n}{n}}$.

Proof. Fix t, distinct agents i, j, and let $w := \widehat{v}_i(g_t) \in [0,1]$. Assigning g_t to j increases E_{ij} by w, assigning it to i decreases E_{ij} by w, and every other assignment leaves it unchanged. Hence

$$\frac{1}{n}\sum_{a\in N}\xi_{ij}^{t}(a) = C^{m-t}e^{s(E_{ij}^{t-1}-L)}\,\frac{e^{sw}+e^{-sw}+n-2}{n}$$
$$\leq C^{m-t}e^{s(E_{ij}^{t-1}-L)}\,\frac{e^{s}+e^{-s}+n-2}{n} = \xi_{ij}^{t-1}.$$

Summing over all ordered pairs (i,j) with $i \neq j$ proves the first statement.

At time zero,

$$\Gamma^0 = \frac{1}{2}n(n-1)C^m \exp\left(-10s\sqrt{\frac{m\log n}{n}}\right) < \frac{1}{2}\exp\left(2\log n + m\log C - 10s\sqrt{\frac{m\log n}{n}}\right).$$

Using $\log(1+x) \leq x$ and $\cosh x \leq e^{x^2/2}$, $m\log C \leq 2\log n$. Let $x := m/(n\log n) \geq 1$. Since $x\log(1+1/x) \geq \log 2$ for $x \geq 1$,

$$10s\sqrt{\frac{m\log n}{n}} = 10\sqrt{2x\log(1+1/x)}\,\log n \geq 10\sqrt{2\log 2}\,\log n > 4\log n.$$

Thus $\Gamma^0 < 1/2$.

At time m, $\xi_{ij}^m = \exp(s(E_{ij}^m - L))$, so $\xi_{ij}^m \leq 1$ implies $E_{ij}^m \leq L$.

Finally, $\log(1+u) \geq u/2$ for $u \in [0,1]$ gives us

$$s^2 = 2\log(1+1/x) \geq \frac{1}{x}, \quad \frac{1}{s} \leq \sqrt{\frac{m}{n\log n}}.$$

Since $\log 2 \leq \log n$, by the definition of L, this proves (iv). □

The recipients minimizing the two potentials separately need not be the same. Their common averaging inequality lets us avoid this issue: we minimize the sum $\Phi^t + \Gamma^t$, whose initial value is below one. Keeping this sum below one makes every term in both potentials less than one at the end, so both guarantees hold for the same allocation.

Joint allocation rule. When good g_t arrives, assign it to a recipient that minimizes the combined potential $\Omega^t := \Phi^t + \Gamma^t$.

Theorem H.3. *Assume exact MIV predictions and a known horizon $m \geq n\log n$. Against adaptive adversaries, the joint allocation rule returns an allocation A that simultaneously satisfies the following:*

(i) A is α_n-PROP1, where α_n is defined in (72) and $\frac{1}{2\log(4n)} \leq \alpha_n \leq \frac{2}{\log(2n)}$;

(ii) the maximum additive envy under the normalized valuations satisfies $\max_{i,j\in N}(\widehat{v}_i(A_j) - \widehat{v}_i(A_i)) \leq 11\sqrt{\frac{m\log n}{n}}$. Equivalently, for every $i, j \in N$, $v_i(A_j) - v_i(A_i) \leq 11p_i\sqrt{\frac{m\log n}{n}}$.

Proof. Both component potentials satisfy the same averaging inequality, so $\frac{1}{n}\sum_{a\in N}\Omega^t(a) \leq \Omega^{t-1}$. The joint allocation rule chooses a minimizing recipient, and therefore $\Omega^t \leq \Omega^{t-1}$ for every t. Initially, $\Omega^0 = \Phi^0 + \Gamma^0 < \frac{1}{2} + \frac{1}{2} = 1$. Thus, $\Omega^m < 1$. Every term in both potentials is nonnegative, so $\phi_i^m < 1$ and $\xi_{ij}^m < 1$ for all $i \neq j$. The conclusions follow from Lemmas H.1 and H.2. □