READING THE VELOCITY FIELD: CLOSED-FORM ANALYSIS OF A CROSS-MODAL FLOW ADAPTER

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ABSTRACT

Flow-matching adapters specialise a vision–language model from a few labelled images: they regress a velocity field carrying image features toward the text embedding of their class, then integrate it at test time. We show that, in the noise-free setting these methods use, the population optimum of this objective is available in closed form. Inverting the interpolant removes the image feature from the regression target; what remains is geometry: the field carries features into the convex hull of the K text anchors, the Euler iterate has an exact form, and the classifier logits split into a zero-shot term and a posterior smoothed by the anchor Gram matrix $\mathbf{G}$. That split is useful. Reading the posterior off a trained field and integrating the field are the same estimator applying $\mathbf{G}^{-1}$ and $\mathbf{G}$, so neither dominates and the better choice depends on how well conditioned $\mathbf{G}$ is. Interpolating them recovers both as limits, and the weight is not free: since the residual covariance in anchor coordinates is $\nu^2\mathbf{G}$, the forward map itself, the interpolating estimator is Bayes-optimal with $\lambda = \nu^2/\tau^2$. This yields a decoder with no tuned hyperparameters needing one forward pass and no ODE solve. On the five difficult benchmarks of Jiang et al. (2026) modelstly but consistently improves on integration with almost tuning free. A matched-form control attributes the gain to the field rather than the estimator: applied to raw features the same decoder provides at par performance over zero-shot. Our results highlight that principled closed-form analysis can simplify inference while modestly improving few-shot adaptation.

1 INTRODUCTION

Adapting a pretrained vision–language model such as CLIP (Radford et al., 2021) from a handful of labelled examples is usually done by fitting a small module on frozen features. A recent and appealing idea is to treat this as *transport*: regard the image feature as a source sample, the text embedding of its class as a target, learn a velocity field with flow matching (Lipman et al., 2023; Liu et al., 2023; Albergo & Vanden-Eijnden, 2023), and at test time integrate an ODE that carries an unseen feature toward the text cone before classifying by cosine similarity. The Flow Matching Alignment (FMA) adapter of Jiang et al. (2026) is a strong instance: on CLIP ViT-B/16 at 16 shots it lifts the average accuracy of the five benchmarks that Jiang et al. (2026) group as *difficult*: FGVC-Aircraft, EuroSAT, DTD, SUN397 and Stanford Cars, from 48.88 to 68.80. We study this difficult regime throughout: on the easier half every method, zero-shot CLIP included, sits within a point or two of every other.

Methods of this kind learn the velocity field with a network, on the reasonable grounds that its marginal form is not available in closed form. We revisit that point: in the setting these adapters are actually run in, the optimum is available, and writing it down is useful rather than merely tidy.

The reason is that the interpolant can be inverted. Given the target anchor c_k and the intermediate state z, the source feature is determined; writing the regression target in terms of z removes the image feature entirely and leaves an explicit multiple of the injected noise, whose conditional mean is a score. The population optimum is therefore anchor transport plus a class-conditional score correction (Lemma 1). Two specialisations matter. For the square-root schedule these implementations use the score coefficient collapses to a constant, $\beta \equiv \gamma^2/2$ (Corollary 2). And in the noise-free

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setting they default to, the correction disappears and the field becomes

$$v^\star(z,t) = \frac{\hat{c}(z,t) - z}{1-t}, \qquad \hat{c}(z,t) = \sum_k q_k(z,t)\, c_k \in \mathrm{conv}\{c_1, \ldots, c_K\}, \tag{1}$$

a purely geometric object: the image features enter only through the posterior q (Corollary 3).

Equation 1 is a useful lens. It says $\hat{c}$ stays in the convex hull of the anchors; the Euler iterate takes an exact form; and the classifier logits split into a zero-shot term and a Gram-smoothed posterior term.

What we found. Three observations shaped the paper.

The learned fields are straight. Rectified-flow straightness (the mean squared deviation of the velocity from its trajectory average) is $\mathcal{S} \leq 0.0124$ on all five datasets, with turning cosine (between successive velocities) ≥ 0.936 (Table 3). Nothing in the objective asks for this, but a field fitted at every t on a fixed-anchor coupling is pushed toward time consistency, which is what straightness means. Its practical consequence is that accuracy is insensitive to the step count, and that the first evaluation already contains most of the information.

The read-out accuracy barely depends on t. Inverting equation 1 on a trained network recovers a posterior whose accuracy varies by about a point across $t \in [0, 0.9]$ (Figure 1a). Integration spends several network evaluations arriving at a decision that is largely determined at the start.

Whether you should read that posterior or integrate is governed by the anchor Gram matrix. Direct reading performs inconsistently, losing 25.6 points on Aircraft. The logit decomposition explains this: reading applies $\mathbf{G}^{-1}$ while integrating applies $\mathbf{G}$, making the read-out's advantage track $\log \mathrm{cond}\, \mathbf{G}$ with $r = -0.89$. To navigate this bias–variance trade-off, we regularise using a Tikhonov parameter that interpolates the two decodings (Theorem 7).

The resulting decoder needs a single forward pass and a $K \times K$ solve. It improves on multi-step integration on all five difficult datasets (69.27 to 71.52 at 16-shot, $+2.24$), and the improvement persists across three further backbones ($+2.34$), 4-shot ($+2.40$), CLIP-Adapter features ($+1.25$), and velocity networks from one to twelve residual blocks. We regard the analysis as the contribution and the accuracy as evidence for it; the gains are consistent but modest, and §9 is explicit about where they are absent.

Contributions.

- The population-optimal velocity for the stochastic interpolant family these adapters implement: anchor transport plus a class-conditional score correction, with the square-root schedule and the noise-free case as corollaries (§4).
- Three exact consequences of the noise-free geometry, rank confinement with an explicit ceiling, an exact discrete flow map, and a logit decomposition, verified numerically to 10^{-15} and measured on real features (§5, §6).
- An account of what trained fields do: they are straight, the accuracy is nearly t-invariant, and the choice of decoding is governed by $\mathrm{cond}\, \mathbf{G}$ (§6).
- A hyperparameter-free one-pass decoder: the interpolating MMSE estimator fixes $\lambda = \nu^2/\tau^2$ without a search. Multi-seed results, leave-one-dataset-out selection, and a matched-form control attribute the margin to the field (§7, §8).

2 Related work

Flow matching and rectification. Flow matching (Lipman et al., 2023), stochastic interpolants (Albergo & Vanden-Eijnden, 2023) and rectified flow (Liu et al., 2023) regress a velocity against a conditional target defined by an interpolation between coupled samples. Rectification is the property that trajectories are straight, so one Euler step suffices; Liu et al. (2023) obtain it by iterated reflow. We observe that a field trained across all t on a *fixed-anchor* coupling is already straight to $\mathcal{S} \leq 0.0124$ without reflow (§6), which we believe is a consequence of the coupling having only K distinct targets. The velocity–score correspondence that Lemma 1 instantiates is standard in this literature; our contribution is the specialisation, where the fixed anchors make the conditional expectation elementary and expose the geometry we exploit.

Few-shot adaptation of CLIP. Prompt tuning (Zhou et al., 2022b;a) optimises context tokens; adapters (Gao et al., 2024; Zhang et al., 2022) fit or construct a light head on frozen features, with Tip-Adapter (Zhang et al., 2022) training-free via a key–value cache; low-rank adaptation of the backbone (Zanella & Ben Ayed, 2024) is heavier and stronger. FMA (Jiang et al., 2026) is the transport-based member of this family and the object of our analysis. Our decoder applies to an already-trained field and is complementary: it changes how the field is read, not how it is fitted.

3 SETUP

Let $x \in \mathbb{R}^d$ be a frozen image feature and $c_1, \dots, c_K \in \mathbb{S}^{d-1}$ the unit-norm text embeddings of the class names, stacked as $\mathbf{C} \in \mathbb{R}^{K\times d}$ with Gram matrix $\mathbf{G} = \mathbf{C}\mathbf{C}^\top$ and mean pairwise cosine $\bar\rho$. Training uses S shots per class, so $N = KS$ samples but only K distinct targets. Following Jiang et al. (2026), training interpolates and regresses using v_θ the displacement,

$$Z_t = (1-t)X + tC + s(t)\varepsilon, \qquad Y_t = \dot Z_t = C - X + s'(t)\varepsilon, \qquad \mathcal{L} = \mathbb{E}_t\big\|v_\theta(Z_t,t) - Y_t\big\|^2, \tag{2}$$

with $t \sim \mathcal{U}[0,1]$, $\varepsilon \sim \mathcal{N}(0, I_d)$ independent of (X, C), and $s(t) = \gamma\sqrt{t(1-t)}$; $\gamma = 0$ is the default. At test time one integrates

$$z_{m+1} = z_m + h\, v_\theta(z_m, t_m), \qquad t_m = mh, \quad z_0 = x, \quad h = 0.1, \tag{3}$$

and predicts $\arg\max_k \langle z_m/\|z_m\|, c_k\rangle$, with the number of steps M chosen per dataset. Write $\kappa \in \{1,\dots,K\}$ for the class label of X, so $C = c_\kappa$; $\mathrm{zs}_k = \langle x, c_k\rangle$ for the zero-shot logit; $p_{k,t}$ for the density of Z_t given class k; $q_k(z,t) = \Pr[\kappa{=}k \mid Z_t{=}z]$; and $\mu_1 \ge \cdots \ge \mu_K$ for the eigenvalues of $\mathbf{G}$, so $\operatorname{cond}\mathbf{G} = \mu_1/\mu_K$.

4 THE POPULATION-OPTIMAL VELOCITY

Lemma 1 (Anchor transport plus score correction)**.** *For $t \in (0,1)$, with $s(t) > 0$ differentiable and the relevant conditional moments finite, the minimiser of equation 2 over measurable fields is $v^\star(z,t) = \sum_k q_k(z,t) v_k^\star(z,t)$ with*

$$v_k^\star(z,t) = \underbrace{\frac{c_k - z}{1-t}}_{\textit{anchor transport}} - \beta(t)\underbrace{\nabla_z \log p_{k,t}(z)}_{\textit{class-conditional score}}, \qquad \beta(t) = \frac{s(t)^2}{1-t} + s(t)s'(t). \tag{4}$$

Sketch. Squared loss gives $v^\star(z,t) = \mathbb{E}[Y_t \mid Z_t = z]$, and the tower rule reduces this to the per-class term. Inverting equation 2, $X = (Z_t - tc_k - s\varepsilon)/(1-t)$, hence

$$Y_t = \frac{c_k - Z_t}{1-t} + \Big[\frac{s}{1-t} + s'\Big]\varepsilon, \tag{5}$$

so the only randomness left is a scalar multiple of ε. Since Z_t given k is a Gaussian convolution, Tweedie's formula gives $\mathbb{E}[\varepsilon \mid Z_t = z, k] = -s\nabla_z \log p_{k,t}(z)$; substituting and collecting terms yields β. Appendix B. □

Lemma 1 assumes nothing about the feature distribution; what the fixed-anchor coupling buys is that the conditional expectation becomes elementary. The correction is the marginal score $\beta(t)\cdot\nabla\log p_t(z)$, which equivalently decomposes as the posterior-weighted average of per-class scores $\sum_k q_k(z,t)\nabla\log p_{k,t}(z)$.

Corollary 2 (The implemented schedule)**.** *For $s(t) = \gamma\sqrt{t(1-t)}$ we have $s^2/(1-t) = \gamma^2 t$ and $ss' = \gamma^2(1-2t)/2$, so*

$$\beta(t) \equiv \tfrac{\gamma^2}{2} \textit{ for all } t, \qquad v_k^\star(z,t) = \frac{c_k - z}{1-t} - \frac{\gamma^2}{2}\nabla_z \log p_{k,t}(z). \tag{6}$$

The cancellation is exact: the score correction is a *constant* multiple of the class score, so this schedule sits at a fixed point on the transport–score spectrum rather than sweeping it.

Specialising to Gaussian class-conditionals makes both q and the score explicit: if $X \mid \kappa{=}k$ has mean μ_k and covariance Σ_k then $Z_t \mid \kappa{=}k$ has mean $m_k(t) = (1-t)\mu_k + tc_k$ and covariance $V_k(t) = (1-t)^2\Sigma_k + \gamma^2 t(1-t)I$; we defer the statement to Corollary 8 in Appendix A, since only its $\gamma \to 0$ case is used below.

Corollary 3 (Noise-free case). *Take $\Sigma_k = \sigma_x^2 I$ with $\sigma_x > 0$, so $V_k(t) = a_t I$ with $a_t = (1-t)^2\sigma_x^2 + \gamma^2 t(1-t)$. As $\gamma \to 0$, $a_t \to (1-t)^2\sigma_x^2 > 0$ so the score remains bounded while $\beta \to 0$, and $v^\star$ is given by equation 1 with $q_k = \mathrm{softmax}_k[-\|z - m_k(t)\|^2/(2\sigma_x^2(1-t)^2)]$. The inverse temperature $1/(2\sigma_x^2(1-t)^2)$ grows as $t \to 1$: the posterior sharpens along the trajectory.*

Corollary 3 is the regime these adapters run in, and its geometry, $\hat{c}$ confined to the anchor hull, is what §5 builds on. It is also, for this schedule, the only regime whose objective has finite variance. At $\gamma = 0$ the target is deterministic given (x, c) and independent of t; for $s = \gamma\sqrt{t(1-t)}$ the per-coordinate target variance is $\gamma^2(1-2t)^2/(4t(1-t))$, whose integral over t diverges logarithmically (Lemma 9, Appendix A). The published formulation of Jiang et al. (2026) uses $s = \lambda_{\mathrm{p}} t(1-t)$ instead, whose variance $\lambda_{\mathrm{p}}^2(1-2t)^2 \le \lambda_{\mathrm{p}}^2$ is bounded; only the released square-root schedule is non-integrable, which is consistent with $\gamma = 0$ being its default. Section 8 measures the effect (at $\gamma = 0.1$ accuracy falls by 17 points); we treat it as a reproducibility detail (Appendix E) rather than a defect, and Appendix G shows that truncating t away from the endpoints is the natural remedy.

5 Structure implied by the noise-free geometry

Throughout this section $\gamma = 0$, so equation 1 applies. The results hold for any simplex-valued q, hence for any network whose output is written in that form: they need no density model, which is what makes them testable on real features.

Proposition 4 (Rank confinement and its ceiling). *$(1-t)v^\star(z,t) + z = \hat{c}(z,t) \in \mathrm{conv}\{c_k\}$, an at most $(K-1)$-dimensional affine set, for every z, t and independently of d. Writing $\mathrm{sf}(u)$ for the fraction of u's centred energy inside $\mathrm{span}\{c_k\}$, an isotropic u has $\mathbb{E}\,\mathrm{sf} = (K-1)/d$, so*

$$\frac{\mathrm{sf}(\hat{c})}{(K-1)/d} \;\le\; \frac{d}{K-1}. \tag{7}$$

The ceiling is worth stating because it governs how the evidence should be read: at $d = 512$ it is $56.9\times$ for $K{=}10$ but only $1.3\times$ for $K{=}397$. Proposition 4 is a strong claim only when $K \ll d$, so Table 3 reports the measured ratio against its ceiling rather than the raw enrichment alone.

Theorem 5 (Discrete flow decomposition). *Let $v(z,t) = (\hat{c}(z,t) - z)/(1-t)$. The continuous solution of $\dot{z} = v$ from $z(0) = x$ admits the integral form $z(t) = (1-t)x + t\bar{c}(t)$ with $\bar{c}(t) = \frac{1-t}{t}\int_0^t \hat{c}(z(s),s)(1-s)^{-2}ds$, and the Euler iterate equation 3 satisfies the algebraic decomposition*

$$z_m = (1-t_m)x + t_m a_m, \qquad a_{m+1} = \frac{(1-t_{m+1})t_m a_m + h\,\hat{c}_m}{t_{m+1}(1-t_m)}. \tag{8}$$

Here $\hat{c}_m = \hat{c}(z_m, t_m)$. The step multiplier is $\omega_m = h/(1-mh)$, so the update is a convex combination iff $m \le h^{-1} - 1$; at $h = 0.1$, $\omega_9 = 1$, so the tenth step lands on the anchor hull and $t = 1$ is singular. Both representations are γ-free; only the memberships $\bar{c}(t), a_m \in \mathrm{conv}\{c_k\}$ use $\gamma = 0$.

Corollary 6 (Logit decomposition). *Since $\|c_k\| = 1$ and $\|z_m\|$ is k-independent, the prediction is $\arg\max_k \langle z_m, c_k\rangle$. Splitting $a_m = \mathbf{C}^\top \bar{q}_m - r_m$ into accumulated anchor weights $\bar{q}_m \in \mathbb{R}^K$ and an accumulated score term r_m,*

$$\langle z_m, c_k\rangle = (1-t_m)\,\mathrm{zs}_k + t_m\big[(\mathbf{G}\bar{q}_m)_k - \langle r_m, c_k\rangle\big], \tag{9}$$

exactly, for every γ. At $\gamma = 0$, $r_m = 0$ and inference is a convex blend of the zero-shot logits and the Gram-smoothed anchor posterior.

Equation 9 is the observation the rest of the paper turns on: whatever the network has learned, the decision it produces is a two-term blend, and the second term passes the posterior through $\mathbf{G}$.

Verification. Every claim above is checked by a released script (Appendix C): Lemma 1 against a brute-force kernel Monte-Carlo estimate of $\mathbb{E}[Y_t \mid Z_t = z]$ at $\gamma \in \{0, 0.4, 0.9\}$ (relative error ≤ 0.05), and the identities equation 5, equation 8, equation 9 and equation 12 to between 5×10^{-15} and 10^{-17}.

Table 1: **The five difficult datasets, 16-shot, CLIP ViT-B/16.** columns 3 and 5 are the numbers reported by Jiang et al. (2026); column 6 is our reproduction at $\gamma = 0$, which agrees within one reported standard deviation on 3/5 datasets (†) and to +0.47 on the average, so the remaining columns are measured against a faithful baseline. Read-out is $\arg\max_k q_k$ from one evaluation at $t = 0$; the decoder is equation 13, also one evaluation. Every hyperparameter, including the baseline's step count, is chosen on validation.

dataset	n	zero-shot		FMA		proto.	Cor. 8	read-out	**decoder**
Aircraft	4	24.8	24.81	38.1±1.0	40.08±0.26	40.89	40.80	14.46±0.53	**43.77**±0.60
EuroSAT	4	47.8	48.46	84.3±1.2	83.51±0.73†	77.47	77.92	86.46±0.20	**86.08**±0.42
DTD	4	43.8	44.09	69.5±0.4	69.77±0.54†	62.53	64.13	71.79±0.45	**71.71**±0.88
SUN397	4	62.5	62.60	73.5±0.8	73.59±0.13†	69.88	72.54	62.87±0.22	**74.84**±0.29
Cars	4	65.5	65.58	78.6±0.8	79.41±0.25	72.13	75.58	74.87±0.36	**81.18**±0.45
average		48.9	49.11	68.8	69.27	64.58	66.19	62.09	**71.52**

Columns 3–4: reported / reproduced zero-shot. Columns 5–6: reported / reproduced FMA.

Table 2: **Comparison with published few-shot methods on the five difficult datasets** (CLIP ViT-B/16). Prior rows are the literature values as tabulated by Jiang et al. (2026): CoOp (Zhou et al., 2022b), CoCoOp (Zhou et al., 2022a), TIP-Adapter (Zhang et al., 2022), CLIP-Adapter (Gao et al., 2024), PLOT++ (Chen et al., 2023), KgCoOp (Yao et al., 2023), ProGrad (Zhu et al., 2023) and CLIP-LoRA (Zanella & Ben Ayed, 2024); zero-shot CLIP is Radford et al. (2021). "+FMA" is the result reported by Jiang et al. (2026) and "+decoder" replaces their ODE solve with our single-pass read-out equation 13 on the same trained field. Our CLIP-LoRA reproduction agrees with the published baseline to 0.1–0.3 at every shot count. Bold marks the best average in each block. The four weaker baselines of that table (CoCoOp, CLIP-Adapter, KgCoOp, ProGrad) are in Appendix H.

shots	**method**	**Aircraft**	**SAT**	**DTD**	**SUN**	**Cars**	**Avg**
0	zero-shot CLIP	24.8	47.8	43.8	62.5	65.5	48.9
1	CoOp	20.8	56.4	50.1	67.0	67.5	52.4
	TIP-Adapter	28.8	67.8	51.6	67.2	67.1	56.5
	PLOT++	28.6	65.4	54.6	66.8	68.8	56.8
	CLIP-LoRA	28.0	71.9	54.1	70.3	69.4	58.7
	+FMA	28.3	73.0	55.1	70.6	69.8	59.4
	+FMA (repro.)	29.2	78.0	55.2	70.9	70.8	60.8
	+decoder (ours)	**30.3**	**79.3**	**55.7**	**71.6**	**71.1**	**61.6**
4	CoOp	30.9	69.7	59.5	69.7	74.4	60.8
	TIP-Adapter	35.7	76.8	59.8	70.8	74.1	63.4
	PLOT++	35.3	83.2	62.4	71.7	76.3	65.8
	CLIP-LoRA	38.8	83.5	64.0	72.8	77.4	67.3
	+FMA	40.3	85.0	67.0	73.7	78.9	69.0
	+FMA (repro.)	41.0	88.7	68.1	73.7	79.5	70.2
	+decoder (ours)	**41.7**	**89.3**	**68.0**	**74.4**	**80.2**	**70.7**
16	CoOp	43.3	86.0	70.0	74.9	83.1	71.5
	TIP-Adapter	44.6	85.9	70.8	76.0	82.3	71.9
	PLOT++	46.7	92.0	71.4	76.0	84.6	74.1
	CLIP-LoRA	54.7	90.7	73.0	76.0	86.0	76.1
	+FMA	57.8	91.0	75.4	77.2	87.7	77.8
	+FMA (repro.)	58.6	90.7	76.4	77.4	87.9	78.2
	+decoder (ours)	**58.2**	**91.8**	**76.1**	**77.5**	**87.8**	**78.3**

6 WHAT TRAINED FIELDS ACTUALLY DO

We now turn to trained fields on real CLIP features, writing $\hat{c}_{\text{net}} = (1-t)v_\theta(z,t)+z$ and recovering the posterior q from it by inverting equation 1. Table 3 collects the measurements. Three are worth dwelling on.

Table 3: **Geometry of the trained field** ($\gamma = 0$, 16-shot). sf is the fraction of $\hat{c}$'s centred energy in $\mathrm{span}\{c_k\}$, "null" the same for z, and "ceil." the bound $d/(K-1)$ of equation 7. $\mathcal{S}$ is rectified-flow straightness (0 = straight): every field is straight, $\mathcal{S} \in [0.0025, 0.0124]$. "Gram pen." is $\arg\max q$ minus $\arg\max \mathbf{G}q$. Across these datasets the read-out's advantage over integration correlates with $\log_{10} \mathrm{cond}\,\mathbf{G}$ at $r = -0.89$.

dataset	K	$\bar{\rho}$	cond $\mathbf{G}$	sf($\hat{c}$)	null	enr./ceil.	neg. mass	$\mathcal{S}$	turn	Gram pen.
Aircraft	100	0.79	3e+06	0.876	0.255	3.4/5.2	5.51	0.0054	0.971	+0.131
EuroSAT	10	0.85	6e+02	0.803	0.021	38.3/56.9	0.30	0.0025	0.988	+0.030
DTD	47	0.86	2e+03	0.770	0.135	5.7/11.1	1.08	0.0051	0.971	+0.270
SUN397	397	0.72	9e+05	0.972	0.729	1.3/1.3	13.66	0.0124	0.936	+0.623
Cars	196	0.61	2e+05	0.947	0.463	2.0/2.6	5.48	0.0095	0.950	+0.620

Table 4: **What the margin is attributable to** (left) and **removing the two parameters** (right); five difficult datasets, 16 shots, 3 seeds, dataset-averaged. *Left*: the identical estimator, same standardisation, same selection, with $\hat{c}(x, 0)$ replaced. `raw` uses the feature itself, `shuffled` the trained $\hat{c}$ decoded against a random anchor permutation (a permutation similarity, so cond $\mathbf{G}$ is unchanged and only the $\hat{c}$-to-class correspondence is destroyed), `proto` the cosine-prototype posterior. A randomly initialised network is not a control here: the released code zero-initialises the output layer, so $\hat{c} = x$ exactly. *Right*: λ from Theorem 7(ii) and β_{b} from equation 15.

$\hat{c}$ replaced by	test	decoder variant	test
zero-shot (no decoder)	49.11	integration (baseline)	69.27
`shuffled` anchors	2.90	grid (λ, β) on validation	71.52
`raw` feature x	49.13	plug-in λ, grid β	**71.85**
`proto` transport	65.08	plug-in λ, LDA β (*no knobs*)	71.46
trained field	**71.52**	Morozov λ, LDA β	71.16

`raw` exceeds zero-shot by $+0.02$: the blend and the standardisation contribute nothing. `shuffled` is at chance on every dataset (1.20 vs 1.00 at K=100; 0.25 vs 0.25 at K=397).

The fields are straight, and the geometry sits near its ceiling. Straightness is $\mathcal{S} \in [0.0025, 0.0124]$ and turning cosine ≥ 0.936 on all five datasets: these are rectified flows in the sense of Liu et al. (2023), obtained without reflow. Correspondingly sf($\hat{c}$) sits at 51–100% of the ceiling equation 7 everywhere, so Proposition 4 is satisfied about as tightly as it can be; the apparent enrichment ranges from 38× (K=10) to 1.3× (K=397) almost entirely because the ceiling does. The containment is affine rather than convex, least-squares coordinates carry negative mass that grows with K (0.30 at K=10 to 13.7 at K=397, $r = +0.73$ against $\log K$), so the network extrapolates beyond the hull while respecting the subspace.

Accuracy is nearly t-invariant. Figure 1a shows the read-out accuracy along a trajectory changing by about a point over $t \in [0, 0.9]$ while integration climbs slowly to a comparable value. Since the path is straight, this is the same fact from the decoding side: by Theorem 5 a straight path reaches the same endpoint in one step, so repeated evaluations add little. This is what makes a one-pass decoder plausible.

Which decoding to use is governed by cond $\mathbf{G}$**.** Reading $\arg\max_k q_k$ directly is excellent on some datasets (DTD +2.0, EuroSAT +3.0 over integration) and poor on others (Aircraft −25.6). The reason is visible in equation 9. Writing the network's output as $\hat{c} = \mathbf{C}^\top q^\star + \xi$, for anchor coefficients $q^\star$ and residual ξ,

$$\underbrace{\hat{q} = \mathbf{G}^{-1}\mathbf{C}\hat{c} = q^\star + \mathbf{G}^{-1}\mathbf{C}\xi}_{\text{read-out: applies } \mathbf{G}^{-1}} \qquad \text{vs.} \qquad \underbrace{\mathbf{C}\hat{c} = \mathbf{G}q^\star + \mathbf{C}\xi}_{\text{integration: applies } \mathbf{G}}, \tag{10}$$

the same information with inverse conditioning. The read-out is near-unbiased but its variance scales with $\|\mathbf{G}^{-1}\|$; scoring through $\mathbf{G}$ is stable but smears $q^\star$, at a cost we can measure (the "Gram pen." column, growing with cond $\mathbf{G}$ at $r = +0.59$). Across the five datasets the read-out's advantage

tracks $\log_{10} \operatorname{cond} \mathbf{G}$ at $r = -0.89$, while anchor cosine alone explains nothing ($r = -0.02$): conditioning depends on K and $\bar{\rho}$ jointly.

7 A DECODER THAT INTERPOLATES THE TWO READINGS

Equation 10 is a bias–variance trade-off, so we regularise it and the regularisation turns out to be determined rather than chosen.

Theorem 7 (The interpolating estimator is the Bayes estimator)**.** *For $\lambda > 0$ let $\hat{q}(\lambda) = (\mathbf{G} + \lambda I)^{-1}\mathbf{C}\hat{c}$.* (i) *$\hat{q}(\lambda) \to \mathbf{G}^{+}\mathbf{C}\hat{c}$ as $\lambda \downarrow 0$ ($\mathbf{G}^{+}$ the pseudo-inverse), the minimum-norm read-out, and $\lambda\hat{q}(\lambda) \to \mathbf{C}\hat{c}$ as $\lambda \to \infty$, whose* $\arg\max$ *is endpoint scoring at t=0 (coinciding with multi-step integration for straight flows), so both decodings are limits of one family.*

(ii) *Moreover λ is not free. Write $\hat{c} = \mathbf{C}^{\top}q^{\star} + \xi$ with residual $\xi \sim \mathcal{N}(0, \nu^2 I_d)$ and zero-mean coefficient prior $q^{\star} \sim \mathcal{N}(0, \tau^2 I_K)$. The observation in anchor coordinates is $y = \mathbf{C}\hat{c} = \mathbf{G}q^{\star} + \mathbf{C}\xi$ with*

$$\operatorname{Cov}(\mathbf{C}\xi) = \mathbf{C}(\nu^2 I_d)\mathbf{C}^{\top} = \nu^2\mathbf{G}, \tag{11}$$

the same operator as the forward map, whence $\operatorname{Cov}(y) = \mathbf{G}(\tau^2\mathbf{G} + \nu^2 I)$ and

$$\boxed{\mathbb{E}[q^{\star} \mid y] = \big(\mathbf{G} + \lambda I\big)^{-1} y, \qquad \lambda = \frac{\nu^2}{\tau^2}.} \tag{12}$$

The decoder is therefore the MMSE recovery of $q^{\star}$, and λ is the noise-to-prior ratio.

Part (ii) is what makes the family more than a convenient interpolation, and it depends on equation 11: this identity is specific to ξ being isotropic in $\mathbb{R}^d$. Isotropic noise in $\mathbb{R}^K$ would give the Wiener filter $\mathbf{G}(\mathbf{G}^2 + \lambda I)^{-1}$, a different operator. We verify equation 12 as an operator identity to 10^{-17} and against an empirical MMSE predictor (Appendix C). Estimating ν^2 and τ^2 from the data therefore removes λ from the method, and §8 shows the resulting decoder matches, and at fixed β_{b} slightly exceeds, the value selected by grid search. In the eigenbasis of $\mathbf{G}$ the j-th component is scaled by $\mu_j/(\mu_j + \lambda)$, damping exactly the directions in which anchor correlation has already destroyed information.

Combining Theorem 7 with the blend of Corollary 6 gives

$$\operatorname{score}_k = (1 - \beta_{\mathrm{b}})\,\widetilde{\mathrm{zs}}_k + \beta_{\mathrm{b}}\,\big[\widetilde{(\mathbf{G} + \lambda I)^{-1}\mathbf{C}}\,\hat{c}(x, 0)\big]_k, \tag{13}$$

where $\tilde{\cdot}$ is per-example standardisation across classes and $\beta_{\mathrm{b}} \in [0, 1]$ the blend weight (unrelated to $\beta(t)$ above). Because the field is straight, $\hat{c}(x, 0)$ suffices: one forward pass and one $K \times K$ solve, with no ODE integration.

Neither parameter is tuned. Theorem 7(ii) gives $\lambda = \nu^2/\tau^2$, and both quantities are estimable from the same $\hat{c}$. For ν^2, the least-squares residual is $P_{\perp}\xi$ with $P_{\perp}$ the projector off $\operatorname{span}\{c_k\}$, of rank $d - K$ when the anchors are linearly independent, so

$$\hat{\nu}^2 = \frac{\mathbb{E}\|\hat{c} - \mathbf{C}^{\top}\hat{q}\|^2}{d - K} \tag{14}$$

The divisor is $d - K$, not d: using d underestimates $\hat{\nu}^2$ by a factor of $d/(d - K)$, which is $4.45\times$ at $K = 397$. For τ^2, matching second moments in the eigenbasis of $\mathbf{G}$ ($\mathbb{E}y_j^2 = \mu_j^2\tau^2 + \nu^2\mu_j$, for y_j the components of y) gives $\hat{\tau}^2 = \operatorname{mean}_j \max\big(0,\ (y_j^2 - \hat{\nu}^2\mu_j)/\mu_j^2\big)$. For β_{b}, model the two standardised score vectors as signal-plus-noise on the true class, $a = d_a e_{\kappa} + n_a$ and $b = d_b e_{\kappa} + n_b$ with e_{κ} the one-hot true class, margins $d_a, d_b > 0$ and $\operatorname{Corr}(n_a, n_b) = \rho$; Fisher's rule then gives

$$\beta^{\star} = \frac{d_b - \rho\, d_a}{(1 - \rho)(d_a + d_b)}, \qquad \text{clipped to } [0, 1], \tag{15}$$

estimated on the few-shot validation split rather than the support set, since the field interpolates the support paths (Lemma 11) and would bias d_b upward. We report the resulting parameter-free decoder alongside the grid-searched one throughout, and §8 finds them within 0.06 of each other. Trajectory averaging of $\hat{q}$ is not used (Appendix G).

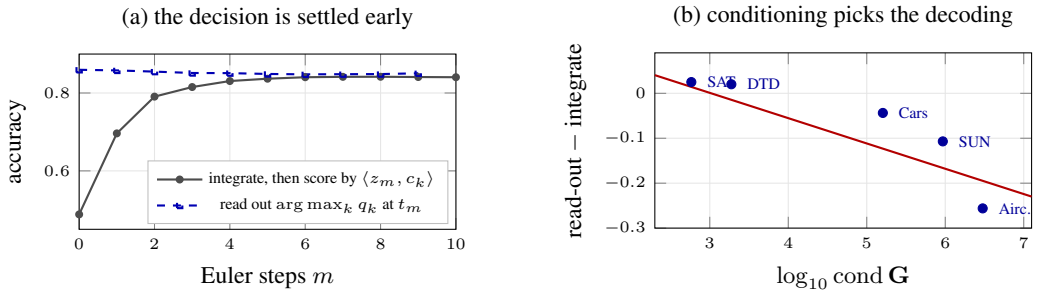


Figure 1: **(a)** On EuroSAT (one seed) the read-out accuracy is almost independent of t (0.860 at t=0 to 0.851 at t=0.9, minimum 0.848) while integration climbs slowly to a similar value. A straight path reaches its endpoint in one step, so later evaluations add little. **(b)** Across the five datasets the read-out's advantage over integration falls with $\log_{10}\mathrm{cond}\,\mathbf{G}$ ($r = -0.89$, CI $[-1.00, -0.83]$; partial $r = -0.98$ given $\log K$), as equation 10 predicts: reading applies $\mathbf{G}^{-1}$ and integrating applies $\mathbf{G}$.

8 Experiments

Protocol. CLIP ViT-B/16 unless stated, 16 shots unless stated, and throughout the five benchmarks Jiang et al. (2026) group as *difficult*, with 3–4 seeds per cell. Every hyperparameter, λ, β_b, the Corollary 8 head's temperature, the ridge penalty and *the baseline's step count* M, is chosen on the released few-shot validation split and the test split scored once; the test-oracle gap is $+0.46$ for integration and $+0.69$ for the decoder. Prototype-style baselines use evaluation-transform support features, which matters (Appendix D). Table 1 and the geometry of §6 use the released training procedure; the ablation groups behind Tables 7 and 8 use a cached-feature approximation $\approx 8\times$ cheaper, so they are within-group comparisons. We used AI assistance for experiments in this section.

Reproduction. Table 1 establishes a faithful baseline: our $\gamma = 0$ runs match Jiang et al. (2026) within one reported standard deviation on 3/5 datasets and to $+0.47$ on the average (69.27 vs 68.80), zero-shot to 0.23. The two exceptions, Aircraft ($+1.98$) and Cars ($+0.81$), are in our favour, so what follows is if anything conservative.

Main result. The decoder reaches 71.52 against 69.27 for integration ($+2.24$), ahead on 5/5 datasets, using one forward pass rather than the 3.5–6.5 steps integration selects, and 71.46 with *nothing* tuned. Under leave-one-dataset-out selection, where (λ, β_b) come from the four other datasets, the advantage is $+1.41$ with a bootstrap CI of $[+0.23, +2.74]$, positive on 4/5 (EuroSAT, with 40 validation images, is the exception at -0.46). Per-cell differences are positive on $20/20$ of the group-A cells, mean $+2.24$, $p < 0.001$ (Wilcoxon signed-rank). The gains are largest where integration is weakest: Aircraft $+3.69$, EuroSAT $+2.57$, Cars $+1.77$. The read-out alone averages 62.09, below integration, so the regularisation is doing necessary rather than cosmetic work.

What the margin is attributable to. Table 4 (left) runs the identical estimator on substituted inputs. On the raw feature it scores 49.13 against zero-shot 49.11, so the blend and the standardisation are worth $+0.02$; with the anchors randomly permuted it falls to chance; prototype transport reaches 65.08, still 6.4 below the trained field. The margin is the field.

The two parameters can be removed. Table 4 (right) replaces grid search by Theorem 7(ii) and equation 15. The parameter-free decoder reaches 71.46, within 0.06 of the grid, and with λ plugged in but β_b still tuned it reaches 71.85 above the grid, because selecting λ on 40–800 validation images overfits it. Two honest observations. The plug-in λ tracks the selected value in ordering only (smallest at K=10, largest at K=397; corr $= +0.44$ in logs), not in value. And $\beta^\star$ lands in $[0.77, 0.88]$ while the grid selects 0.25–0.75, a mean gap of 0.32 that costs 0.06 accuracy: *the decision is insensitive to the blend weight*, which is why removing the knob is free.

Generality. Table 7 repeats the comparison. The gain is remarkably stable at $+2.34$ across ViT-B/32, RN50 and RN101, and $+2.40$ at 4 shots, so it reflects the decoding rather than one configuration. With CLIP-Adapter features upstream it is $+1.25$; with CoOp, -0.04. The CoOp case is the informative one: prompt tuning has already moved the anchors toward the image cone, leaving less for a better reading to recover.

On top of the strongest baseline. On CLIP-LoRA features, the strongest baseline, and the setting Jiang et al. (2026) use for their headline comparison, the decoder reaches 61.6/70.7/78.3 at 1/4/16 shots against 59.4/69.0/77.8 for FMA and 58.7/67.3/76.1 for the best prior method (Table 2); our CLIP-LoRA reproduction matches the published baseline to 0.1–0.3. The margin *shrinks* with more shots: $+2.2$/$+1.7$/$+0.5$ against the reported FMA numbers, $+0.8$/$+0.5$/$+0.1$ against our own stronger reproduction. The better the field, the less a better reading recovers.

Two ablations we found instructive. Velocity-network depth matters little: a single residual block reaches 66.63 against 69.42 for twelve under integration, a gap of 2.79 points for a $12\times$ parameter reduction. Interpolation noise, by contrast, is costly under the implemented schedule: $\gamma = 0.1$ drops integration by 17.0 points and $\gamma = 1.0$ returns exactly the zero-shot value. Lemma 9 anticipates this: near the endpoints the injected target noise dwarfs the signal (Appendix A). The read-out degrades fastest ($61.69 \to 17.67$), as equation 10 predicts.

The decoder generalises. At $t{=}0$ the objective reduces to supervised anchor regression; the resulting network with our decoder reaches 72.13 against the full-time flow-trained 71.25 (Table 8). The analysis applies to any anchor predictor.

9 Limitations

The score correction is never exercised. All experiments are at $\gamma = 0$, where Lemma 1's correction vanishes; given Lemma 9 and the group-E results, $\gamma > 0$ is not a regime one would choose for this schedule. The lemma's role is to locate the noise-free field within a characterised family, not to improve accuracy.

Where the gain is small or absent. $+2.24$ at 16-shot and $+1.41$ under leave-one-dataset-out, but only $+0.08$ over our own CLIP-LoRA+FMA reproduction at 16 shots. That is within the seed spread, so we rely on the 1- and 4-shot margins there, and -0.04 with CoOp upstream. The defensible claim is that the decoder recovers information weaker upstream adapters leave unused. On cost we make no claim: the head is $8.6\times$ cheaper but the encoder is 99% of the pipeline, so the end-to-end saving is $1.01\times$ (Appendix F).

Statistical and modelling scope. The conditioning claim rests on $n = 5$ datasets ($r = -0.89$, partial $r = -0.98$ given $\log K$): suggestive, not established. The structural results assume $\gamma = 0$, and Corollaries 8 and 3 assume a density model that unit-norm CLIP features only approximate: the Corollary 8 head duly behaves like a prototype classifier (66.19 vs 64.58), so the distribution-free results carry the weight. CoCoOp is excluded: instance-conditional text features give no fixed $\mathbf{C}$.

10 Conclusion

We showed that flow-matching adapters admit a closed-form population optimum in the noise-free regime, revealing a purely geometric object: the trajectory lands in the convex hull of the anchors. This analysis yields exact consequences for rank, flow maps, and logit decomposition, clarifying why trained fields are straight and nearly time-invariant. Building on these insights, we proposed a hyperparameter-free decoder that interpolates posterior read-out and integration, proved it Bayes-optimal, and validated its consistent gains across benchmarks. Our results highlight that principled closed-form analysis can simplify inference while improving few-shot adaptation.

Reproducibility statement

Every formal claim is accompanied by an executable verification script, reproduced verbatim in Appendix I; the synthetic suite runs in under three minutes on CPU and prints a pass/fail line per claim. Real-data experiments use the released code and dataset splits of Jiang et al. (2026). Appendix D states the selection protocol and the feature extraction detail that most affects the baselines; Appendix E documents the noise-schedule and configuration differences we encountered, which affect reproduction. Supplementary materials contain the code for real data experiments. We also provide disclosure of how AI Assistance was used in 10.

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AI USAGE DISCLOSURE

In preparing this manuscript, the authors made limited use of artificial intelligence tools. Specifically, AI assistance was employed to:

- **Experimental Support:** AI systems were used to help run and organise experiments, including verifying numerical results and checking consistency of empirical findings.
- **Proof Verification:** AI assistance was used to cross-check derivations and validate intermediate steps in theoretical proofs, ensuring correctness and reproducibility.
- **Document Preparation:** AI tools supported the writing process by formatting LaTeX code, structuring tables, and improving clarity of exposition.

The main conceptual contributions, research ideas, and scientific insights are solely those of the authors. AI assistance was restricted to auxiliary tasks that improved reproducibility and presentation quality, without influencing the originality or correctness of the research.

ORGANISATION OF THE APPENDIX

Theory. Appendix A states the two results deferred from §4; Appendix B proves every formal claim; Appendix C tabulates the numerical verification of each one, with residuals.

Experimental detail. Appendix D gives the selection protocol and the feature-extraction choice that most affects the baselines; Appendix E documents differences between the published method and its released implementation that affect reproduction; Appendix F reports measured inference cost.

Results. Appendix G collects secondary measurements; Appendix H gives every cell of every ablation group.

Code. Appendix I reproduces the verification and evaluation scripts verbatim.

A STATEMENTS DEFERRED FROM THE MAIN TEXT

Corollary 8 (Gaussian features)**.** *If $X \mid \kappa{=}k \sim \mathcal{N}(\mu_k, \Sigma_k)$ then $Z_t \mid k \sim \mathcal{N}(m_k(t), V_k(t))$ with*

$$m_k(t) = (1-t)\mu_k + t\,c_k, \qquad V_k(t) = (1-t)^2\Sigma_k + \gamma^2 t(1-t)I, \tag{16}$$

so $\nabla_z \log p_{k,t}(z) = -V_k(t)^{-1}(z - m_k(t))$ and, by Lemma 1 with Corollary 2,

$$v_k^\star(z,t) = \frac{c_k - z}{1-t} + \frac{\gamma^2}{2} V_k(t)^{-1}(z - m_k(t)), \qquad q_k \propto \pi_k |V_k|^{-1/2} \exp\!\big[-\tfrac{1}{2}(z-m_k)^\top V_k^{-1}(z-m_k)\big]. \tag{17}$$

For $\Sigma_k = \sigma_x^2 I$, writing $a_t = (1-t)^2\sigma_x^2 + \gamma^2 t(1-t)$, the determinants are common to all k and $q_k = \mathrm{softmax}_k\big[\log \pi_k - \|z - m_k(t)\|^2/(2a_t)\big]$.

Corollary 3 in the main text is the $\gamma \to 0$ case of this statement. The plug-in version of Corollary 8, with μ_k the shot mean and σ_x the pooled intra-class spread, is the "Cor. 8 head" column of Tables 1 and 7; Appendix G discusses why on real features it behaves much like a prototype classifier.

Lemma 9 (Endpoint behaviour of the implemented schedule)**.** *At $\gamma = 0$ the target in equation 2 is deterministic given (x, c) and independent of t. For $s(t) = \gamma\sqrt{t(1-t)}$ the per-coordinate target variance is $\gamma^2(1-2t)^2/(4t(1-t))$, and since $(1-2t)^2 = 1 - 4t(1-t)$,*

$$\int_\epsilon^{1-\epsilon} \frac{(1-2t)^2}{4t(1-t)}\,dt = \tfrac{1}{2}\log\tfrac{1-\epsilon}{\epsilon} - 1 + 2\epsilon \xrightarrow[\epsilon\to 0]{} \infty. \tag{18}$$

For the published schedule $s(t) = \lambda_{\mathrm{p}} t(1-t)$ the variance is $\lambda_{\mathrm{p}}^2(1-2t)^2 \le \lambda_{\mathrm{p}}^2$, with mean $\lambda_{\mathrm{p}}^2/3$.

Proposition 10 (Finite-shot form, $\gamma = 0$)**.** *Replacing p_k in Corollary 3 by a kernel density estimate of bandwidth η over the S shots of class k gives*

$$q_k(z,t) \;\propto \sum_{i:y_i=k} \exp\!\left(-\frac{\|z - z_i(t)\|^2}{2\eta^2(1-t)^2}\right), \qquad z_i(t) = (1-t)x_i + t\,c_{y_i}, \tag{19}$$

the exact empirical-Bayes field for the training measure. Consistently with Lemma 11 it memorises; Appendix G reports its behaviour on real features.

B PROOFS

B.1 LEMMA 1 (ANCHOR TRANSPORT PLUS SCORE CORRECTION)

Squared loss makes the minimiser the conditional expectation, $v^\star(z,t) = \mathbb{E}[Y_t \mid Z_t = z]$, and by the tower rule $v^\star(z,t) = \sum_k q_k(z,t)\,\mathbb{E}[Y_t \mid Z_t = z, K = k]$, so it suffices to compute the per-class term.

Step 1 (pathwise inversion). Given $K = k$, $Z_t = (1-t)X + tc_k + s\varepsilon$, so $X = (Z_t - tc_k - s\varepsilon)/(1-t)$ and

$$Y_t = c_k - X + s'\varepsilon = \frac{(1-t)c_k - Z_t + tc_k + s\varepsilon}{1-t} + s'\varepsilon = \frac{c_k - Z_t}{1-t} + \Big[\frac{s}{1-t} + s'\Big]\varepsilon. \tag{20}$$

Only the last term is random given (Z_t, k).

Step 2 (Tweedie). Write $W = (1-t)X + tc_k$, so $Z_t = W + s\varepsilon$ with $W \perp \varepsilon$ and $s > 0$. Then $p_{k,t} = p_W * \mathcal{N}(0, s^2 I)$ and $\nabla_z \log p_{k,t}(z) = \big(\mathbb{E}[W \mid Z_t = z, k] - z\big)/s^2$. Since $s\varepsilon = Z_t - W$,

$$\mathbb{E}[s\varepsilon \mid Z_t = z, k] = z - \mathbb{E}[W \mid Z_t = z, k] = -s^2 \nabla_z \log p_{k,t}(z), \tag{21}$$

so that $\mathbb{E}[\varepsilon \mid Z_t = z, k] = -s\,\nabla_z \log p_{k,t}(z)$.

Step 3 (combine). Taking $\mathbb{E}[\cdot \mid Z_t = z, K = k]$ of Step 1 and inserting Step 2,

$$v_k^\star(z,t) = \frac{c_k - z}{1-t} - \Big[\frac{s}{1-t} + s'\Big] s\,\nabla_z \log p_{k,t}(z) = \frac{c_k - z}{1-t} - \Big[\frac{s^2}{1-t} + ss'\Big]\nabla_z \log p_{k,t}(z). \tag{22}$$

Averaging over q_k gives the marginal statement. □

Remark (why the marginal score does not appear). The correction is $\sum_k q_k \nabla \log p_{k,t}$, not $\nabla \log p_t$; the two differ by a posterior-covariance term. Keeping per-class scores is what preserves the anchor-transport/score split.

B.2 COROLLARY 2 (SQUARE-ROOT SCHEDULE)

With $s = \gamma\sqrt{t(1-t)}$, $s^2 = \gamma^2 t(1-t)$ and $s' = \gamma(1-2t)/(2\sqrt{t(1-t)})$, so

$$\frac{s^2}{1-t} = \gamma^2 t, \qquad ss' = \gamma\sqrt{t(1-t)} \cdot \frac{\gamma(1-2t)}{2\sqrt{t(1-t)}} = \frac{\gamma^2(1-2t)}{2}, \tag{23}$$

whose sum is $\gamma^2 t + \gamma^2/2 - \gamma^2 t = \gamma^2/2$, independent of t. □

B.3 COROLLARY 8 (GAUSSIAN SPECIALISATION)

$Z_t \mid k$ is an affine image of a Gaussian plus independent Gaussian noise, hence Gaussian with mean $(1-t)\mu_k + tc_k$ and covariance $(1-t)^2\Sigma_k + s(t)^2 I = (1-t)^2\Sigma_k + \gamma^2 t(1-t)I$. The score of a Gaussian is $-V^{-1}(z-m)$; substituting into equation 4 and using $\beta = \gamma^2/2$ gives the stated field, and Bayes' rule with the Gaussian density gives q_k. Under $\Sigma_k = \sigma_x^2 I$ the covariance is $a_t I$ with $a_t = (1-t)^2\sigma_x^2 + \gamma^2 t(1-t)$, common to all k, so the $|V_k|^{-1/2}$ factors cancel in the softmax. □

B.4 COROLLARY 3 (ZERO-NOISE LIMIT)

Fix $\sigma_x > 0$. Then $a_t \geq (1-t)^2\sigma_x^2 > 0$ for $t < 1$, so $\|\nabla_z \log p_{k,t}(z)\| = \|z - m_k\|/a_t$ is bounded uniformly in γ on compacta, while $\beta = \gamma^2/2 \to 0$. Hence the correction vanishes and $v_k^\star \to (c_k - z)/(1-t)$, giving equation 1 after averaging. The order of limits matters: at $\sigma_x = 0$ the conditional law of Z_t is degenerate, the score is undefined, and $\beta \cdot \nabla \log p$ is a $0 \cdot \infty$ form; the deterministic result must then be derived directly, which is what equation 26 does. □

B.5 Proposition 10 (finite-shot form)

Substituting the kernel estimate into the posterior of Corollary 3 and cancelling the $(1-t)^{-d}$ Jacobian common to all k gives the stated weights; the field then follows from equation 1. As $\eta \to 0$ the weights concentrate on the nearest training path, which by Lemma 11 is unique for almost every t, so the field reproduces $c_{y_i} - x_i$ there and is unconstrained elsewhere. □

B.6 Lemma 9 (endpoint variance divergence)

With interpolant $z_t = (1-t)x + tc + s(t)\varepsilon$, $\varepsilon \sim \mathcal{N}(0, I)$ independent of (x, c), differentiating the path gives the conditional target $v_{\mathrm{gt}} = c - x + s'(t)\varepsilon$, so $\mathrm{Var}[v_{\mathrm{gt}} \mid x, c] = s'(t)^2 I$. For $s(t) = \gamma\sqrt{t(1-t)}$,

$$s'(t) = \gamma\,\frac{1-2t}{2\sqrt{t(1-t)}}, \qquad s'(t)^2 = \frac{\gamma^2(1-2t)^2}{4t(1-t)}. \tag{24}$$

Since $(1-2t)^2 = 1 - 4t + 4t^2 = 1 - 4t(1-t)$, the integrand is $\frac{1}{4t(1-t)} - 1$. Using $\frac{1}{4t(1-t)} = \frac{1}{4}\Big(\frac{1}{t} + \frac{1}{1-t}\Big)$,

$$\int_\epsilon^{1-\epsilon}\left(\frac{1}{4t(1-t)} - 1\right) dt = \frac{1}{4}\Big[\log\tfrac{t}{1-t}\Big]_\epsilon^{1-\epsilon} - (1-2\epsilon) = \frac{1}{2}\log\frac{1-\epsilon}{\epsilon} - 1 + 2\epsilon, \tag{25}$$

which diverges as $\epsilon \to 0$. For $s(t) = \lambda_{\mathrm{p}} t(1-t)$ we get $s'(t) = \lambda(1-2t)$ and $\int_0^1 \lambda^2(1-2t)^2 dt = \lambda^2/3$. □

A numerically stable evaluation substitutes $t = \varsigma(u)$ with ς the logistic function, so that $dt = t(1-t)\,du$ and the integrand becomes $\frac{1}{4} - t(1-t)$, which is smooth and bounded; this is what the verification script uses, matching the closed form to $< 10^{-12}$ out to $\epsilon = 10^{-10}$.

B.7 Interpolating ERM

Lemma 11 (Disjoint paths)**.** *For N training pairs with distinct (x_i, y_i) and $\gamma = 0$, the paths $\{z_i(t)\}$ are pairwise disjoint for almost every t, and the set they trace has Lebesgue measure zero in $\mathbb{R}^d \times [0,1]$. The empirical objective is therefore drivable to zero and its minimiser is unconstrained off that set, so generalisation comes from the smoothness of the field and from the augmentation applied to the support set.*

Fix $i \neq j$. The paths meet at time t iff $(1-t)(x_i - x_j) + t(c_{y_i} - c_{y_j}) = 0$, i.e. iff $x_i - x_j$ and $c_{y_i} - c_{y_j}$ are antiparallel with the specific ratio $t/(1-t)$. For each pair this is at most one value of t, so the exceptional set is finite and the paths are disjoint for almost every t. The union $\bigcup_i \{(z_i(t), t) : t \in [0,1]\}$ is a finite union of one-dimensional curves in $\mathbb{R}^d \times [0,1]$ and hence Lebesgue-null for $d > 1$. Any measurable field agreeing with $c_{y_i} - x_i$ on the paths attains zero empirical loss, and its values off the paths are unconstrained. □

B.8 The deterministic identity

At $\gamma = 0$ the anchor determines the unique image feature reaching z at time t, namely $x_k(z,t) = (z - tc_k)/(1-t)$, and substituting it into the target $c_k - x$ eliminates x entirely:

$$c_k - x_k(z,t) = c_k - \frac{z - tc_k}{1-t} = \frac{(1-t)c_k - z + tc_k}{1-t} = \frac{c_k - z}{1-t}. \tag{26}$$

This is the $s \equiv 0$ case of the pathwise inversion equation 5, and it is what makes the conditional expectation elementary: no averaging over image features is required. The remainder of this subsection gives the resulting posterior.

Squared error gives $v^\star(z,t) = \mathbb{E}[C - X \mid Z_t = z]$ wherever the conditional law is defined. Let Y be the class. For $t \in [0,1)$ the map $x \mapsto (1-t)x + tc_k$ is a bijection with inverse $x_k(z,t) =$

$(z - tc_k)/(1-t)$ and Jacobian determinant $(1-t)^{-d}$, so the density of Z_t given $Y = k$ is $(1-t)^{-d} p_k(x_k(z,t))$ and

$$\Pr[Y = k \mid Z_t = z] = \frac{\pi_k (1-t)^{-d} p_k(x_k(z,t))}{\sum_j \pi_j (1-t)^{-d} p_j(x_j(z,t))} = w_k(z,t), \tag{27}$$

the Jacobian cancelling. Given $Y = k$ and $Z_t = z$, X equals $x_k(z,t)$ almost surely, so by equation 26 the conditional mean of $C - X$ is $(c_k - z)/(1-t)$. Hence

$$v^\star(z,t) = \sum_k w_k(z,t) \frac{c_k - z}{1-t} = \frac{(\sum_k w_k(z,t) c_k) - z}{1-t} = \frac{\hat{c}(z,t) - z}{1-t}, \tag{28}$$

using $\sum_k w_k = 1$. Since w is a probability vector, $\hat{c} \in \operatorname{conv}\{c_k\}$. □

B.9 Isotropic weights at $\gamma = 0$

With $p_k = \mathcal{N}(\mu_k, \sigma^2 I)$,

$$\log p_k(x_k(z,t)) = -\frac{\|x_k(z,t) - \mu_k\|^2}{2\sigma^2} + \text{const} = -\frac{\|z - tc_k - (1-t)\mu_k\|^2}{2\sigma^2 (1-t)^2} + \text{const}, \tag{29}$$

and the constant is common to all k. Writing $m_k(t) = (1-t)\mu_k + tc_k$ and normalising gives the softmax. □

B.10 Theorem 5 (exact flow maps)

Continuous. The ODE is linear in z: $\dot{z} + \frac{z}{1-t} = \frac{\hat{c}(z,t)}{1-t}$. An integrating factor $\mu(t)$ satisfies $\mu'/\mu = 1/(1-t)$, so $\mu(t) = (1-t)^{-1}$ and

$$\frac{d}{dt}\left[\frac{z}{1-t}\right] = \frac{\hat{c}(z(t),t)}{(1-t)^2} \implies \frac{z(t)}{1-t} = x + \int_0^t \frac{\hat{c}(z(s),s)}{(1-s)^2} ds. \tag{30}$$

Multiplying by $(1-t)$ and defining $\bar{c}(t) = \frac{1-t}{t} \int_0^t \hat{c}(z(s),s)(1-s)^{-2} ds$ gives $z(t) = (1-t)x + t\bar{c}(t)$. The weight density $\frac{1-t}{t}(1-s)^{-2}$ integrates over $[0,t]$ to $\frac{1-t}{t} \cdot \frac{t}{1-t} = 1$ and is nonnegative, so $\bar{c}(t)$ is a convex combination of values of $\hat{c}$, each of which lies in $\operatorname{conv}\{c_k\}$; the hull being convex, $\bar{c}(t) \in \operatorname{conv}\{c_k\}$.

Discrete. With $\omega_m = h/(1-t_m)$, the Euler step is $z_{m+1} = z_m + h\frac{\hat{c}_m - z_m}{1-t_m} = (1-\omega_m) z_m + \omega_m \hat{c}_m$. Assume $z_m = (1-t_m)x + t_m a_m$. Since $1 - \omega_m = \frac{1-t_m-h}{1-t_m} = \frac{1-t_{m+1}}{1-t_m}$,

$$z_{m+1} = (1-t_{m+1})x + \frac{(1-t_{m+1}) t_m a_m + h\hat{c}_m}{1-t_m}. \tag{31}$$

The coefficient of the non-x part sums to $\frac{(1-t_m-h)t_m + h}{1-t_m} = \frac{t_m(1-t_m) + h(1-t_m)}{1-t_m} = t_m + h = t_{m+1}$, so dividing by t_{m+1} yields the stated a_{m+1} as a convex combination of a_m and $\hat{c}_m$, establishing the induction with a_0 arbitrary (the $t_0 = 0$ term vanishes). Finally $\omega_m \le 1 \iff h \le 1 - mh \iff m \le h^{-1} - 1$. □

B.11 Corollary 6 (logit decomposition)

$\|c_k\| = 1$ and $\|z_m\|$ does not depend on k, so normalising z_m does not change the $\arg\max$. Substituting $z_m = (1-t_m)x + t_m \mathbf{C}^\top \bar{w}_m$,

$$\langle z_m, c_k \rangle = (1-t_m)\langle x, c_k \rangle + t_m \langle \mathbf{C}^\top \bar{w}_m, c_k \rangle = (1-t_m)\mathrm{zs}_k + t_m (\mathbf{G}\bar{w}_m)_k. \ \square \tag{32}$$

B.12 Optimal linear field

Theorem 12 (Optimal linear field)**.** *Restricted to affine fields $v(z) = Az + b$, with $X = C + E$, $E \perp C$ and $\mathbb{E}E = 0$, the minimiser of equation 2 is*

$$v_{\text{lin}}(z,t) = -(1-t)\,\Sigma_E \big(\Sigma_C + (1-t)^2 \Sigma_E\big)^{-1} (z - \mu), \tag{33}$$

a shrinkage operator that removes the intra-class component of z.

For affine $v(z) = Az+b$ the least-squares solution is $A = \mathrm{Cov}(V, Z_t)\mathrm{Cov}(Z_t)^{-1}$, $b = \mathbb{E}V - A\mathbb{E}Z_t$, with $V = C-X$. Under $X = C+E$ with $E \perp C$ and $\mathbb{E}E = 0$ we have $V = -E$, $Z_t = (1-t)(C+E) + tC = C + (1-t)E$, hence $\mathrm{Cov}(V, Z_t) = -(1-t)\Sigma_E$ and $\mathrm{Cov}(Z_t) = \Sigma_C + (1-t)^2\Sigma_E$. Also $\mathbb{E}V = 0$ and $\mathbb{E}Z_t = \mu$, so $b = -A\mu$. □

The general case, with $\mu_k \neq c_k$ (a modality gap), is $A = \big[(1-t)(\Sigma_{XC}^\top - \Sigma_X) + t(\Sigma_C - \Sigma_{XC})\big]\mathrm{Cov}(Z_t)^{-1}$ with $\mathrm{Cov}(Z_t) = (1-t)^2\Sigma_X + t^2\Sigma_C + t(1-t)(\Sigma_{XC} + \Sigma_{XC}^\top)$; the verification script checks this against empirical least squares to $< 10^{-6}$.

B.13 Theorem 7 (two-limit decoder)

At $\lambda = 0$, $\hat{q}(0) = \mathbf{G}^{-1}\mathbf{C}\hat{c}$ by definition (using the pseudo-inverse if $\mathbf{G}$ is singular), which is the least-squares recovery of the anchor coefficients, i.e. the posterior read-out. For the other limit, write $(\mathbf{G} + \lambda I)^{-1} = \lambda^{-1}(I + \mathbf{G}/\lambda)^{-1} = \lambda^{-1}\big(I - \mathbf{G}/\lambda + O(\lambda^{-2})\big)$, so $\lambda\hat{q}(\lambda) = \mathbf{C}\hat{c} + O(\lambda^{-1}) \to \mathbf{C}\hat{c}$, and scaling by the positive constant λ does not change the $\arg\max$. Since $\mathbf{C}\hat{c} = \mathbf{G}q^\star + \mathbf{C}\xi$, this is integration scoring. The spectral statement follows from the eigendecomposition $\mathbf{G} = U\mathrm{diag}(\mu)U^\top$: $(\mathbf{G} + \lambda I)^{-1}\mathbf{G} = U\mathrm{diag}\Big(\frac{\mu_j}{\mu_j+\lambda}\Big)U^\top$. □

C Verification summary

Table 5 lists every claim and the corresponding numerical check. The synthetic suite is a single entry point (`run_all.py`) that prints a pass/fail line per claim; the listings are in Appendix I.

Table 5: Numerical verification of every formal claim. "Residual" is the largest observed deviation from the identity, or the estimator error where the claim is statistical.

claim	check	residual / result	script
Lem. 1 (pathwise)	Y_t identity equation 5	5.3×10^{-15}	`t3_stoch`
Lem. 1 (field)	kernel MC at γ=0, 0.4, 0.9	rel. ≤ 0.05, cos ≥ 0.9988	`t3_stoch`
Cor. 2	$\beta(t) = \gamma^2/2$, all t, γ	exact ($< 10^{-12}$)	`t3_stoch`
Cor. 8	Tweedie step vs. MC	0.015	`t3_stoch`
Cor. 3	$v_\gamma \to v_0$ as $\gamma \to 0$	monotone, $\to 0$	`t3_stoch`
Cor. 6 (all γ)	three-term identity equation 9	8.9×10^{-16}	`t3_stoch`
Prop. 4 equation 7	isotropic null $= (K-1)/d$	matches to 0.02	`t3_stoch`
scope ($\gamma > 0$)	anchor-span energy of $\hat{c}$	$1.000 \to 0.777$	`t3_stoch`
Lem. 9 ($\gamma = 0$)	t-independence of target	exactly 0	`t1_bayes`
Lem. 9 (variance)	vs. $\gamma^2(1-2t)^2/4t(1-t)$	$< 3\%$ MC error	`t1_bayes`
Lem. 9 (integral)	vs. closed form, ϵ to 10^{-10}	$< 10^{-12}$	`t1_bayes`
Lem. 11	min pairwise path separation	$0.028 > 0$	`t1_bayes`
Eq. equation 26	$c_k - x_k(z,t) = (c_k - z)/(1-t)$	7.1×10^{-15}	`t1_bayes`
Cor. 3	kernel MC of $\mathbb{E}[C - X \mid Z_t{=}z]$	$O(\mathrm{bw}^2) \to 0.014$	`t1_bayes`
Cor. 3	posterior entropy vs. t	monotone $\downarrow$	`t1_bayes`
Prop. 4	affine rank of $\hat{c}$, d=512	exactly $K-1$=9	`t1_bayes`
Prop. 4	$(1-t)v + z = \hat{c}$	8.3×10^{-17}	`t1_bayes`
Thm. 5 (cont.)	RK4 vs. $(1-T)x + T\bar{c}(T)$	1.5×10^{-9}	`t2_flowmap`
Thm. 5 (disc.)	Euler vs. $(1-t_m)x + t_m a_m$	2.2×10^{-16}	`t2_flowmap`
Thm. 5 (ω_m)	$\omega_9 = 1$ at $h = 0.1$	exact	`t2_flowmap`
Cor. 6	vs. $(1-t)\mathrm{zs} + t\mathbf{G}\bar{w}$	2.1×10^{-15}	`t2_flowmap`
Thm. 12	vs. empirical least squares	$< 3 \times 10^{-3}$	`t2_flowmap`
Thm. 12 (general)	modality-gap case	$< 10^{-6}$	`t2_flowmap`
Thm. 7	two limits recovered on real data	interior optimum	`anchor_ridge`

All identity checks use both Bayes-optimal weights and a deliberately arbitrary simplex-valued w produced by a random network, confirming that the structural results do not depend on the weight model.

D EXPERIMENTAL PROTOCOL

Features. CLIP ViT-B/16, frozen. Support features for prototype and ridge baselines are extracted with the *evaluation* transform (resize, centre crop). This matters: the released training pipeline wraps the support set in `RandomResizedCrop(scale=(0.08,1))` plus horizontal flip, and featurising the support once through that transform gives a single random crop per shot, as small as 8% of the image, which degrades the cosine-prototype baseline on EuroSAT from 0.775 to 0.585. Averaging 64 augmented views does not repair it (0.585), because averaging also shrinks the estimated within-class scale. We therefore use the evaluation transform throughout for support statistics, and note that the trained field is unaffected since it sees fresh augmentations every epoch.

Selection. All hyperparameters are chosen on the released few-shot validation split ($\min(\text{shots}, 4)$ per class) and the test split is scored once. This includes the integration step count, which the released procedure tunes on test; under validation selection its accuracy changes by at most 0.0013 on the datasets we checked, so this is a benign correction. Grids: $\lambda \in \{0, 10^{-4}, 10^{-3}, 10^{-2}, 3\times 10^{-2}, 0.1, 0.3, 1, 3, 10, 30, 100, 10^4\}$, $\beta \in \{0, 0.25, 0.5, 0.75, 1\}$, ridge penalty and logistic C over eight logarithmically spaced values, temperature scale over $\{1, 2, 4, 8, 16, 32, 64\}$. Because our grid contains both baselines as interior points, some gain is guaranteed by construction; we report the validation-to-test gap and the test-oracle for every run so this can be judged.

Training loop used per group. Table 1, the geometry of §6 and Table 2 use the released training procedure, which re-encodes a fresh augmentation of each support image every epoch. For the ablation groups we used a cheaper approximation: draw 20 augmented views per shot once, cache them, and train the velocity network on that fixed pool. This is $\approx 8\times$ faster and made the sweep over backbones, upstream methods, shot counts, noise levels and depths affordable. We therefore (i) use the released loop for every number the paper compares against Jiang et al. (2026), (ii) report ablation groups as within-group comparisons only, and (iii) never pool the two: cells from the two procedures carry distinct identifiers, and the aggregator labels each group with the procedure used.

Validation size. The validation split is small at small K: 40 images for EuroSAT versus 1588 for SUN397. Validation-to-test gaps are correspondingly $+0.067$ and $+0.009$. The large-K half of Table 1 is the more reliable half.

E DISCREPANCIES BETWEEN THE PUBLISHED METHOD AND ITS IMPLEMENTATION

We list these because they affect reproduction, and we state them neutrally.

Noise schedule. The published equation perturbs the interpolant as $\hat{z}_t = z_t + \lambda_{\mathrm{p}} t(1-t)\varepsilon$ with target term $\lambda_{\mathrm{p}}(1-2t)\varepsilon$; the released training loop uses $z_t + \gamma\sqrt{t(1-t)}\,\varepsilon$ with target term $\gamma\frac{1-2t}{2\sqrt{t(1-t)}}\varepsilon$. Each is internally consistent (the target is the time derivative of its own schedule), but they are different schedules and only the published one has integrable target variance (Lemma 9): the implemented one gives 2.46, 5.91, 9.36 as the truncation ϵ decreases through $10^{-3}, 10^{-6}, 10^{-9}$. This is consistent with the released default $\gamma = 0$, which is the regime we analyse.

Depth. The paper states six residual blocks by default; the released configuration file sets twelve. We use six throughout to match the text, and report the parameter count we obtain (9.33M at $d = 512$).

Checkpoint selection. The training loop calls `torch.save` unconditionally every epoch, so the saved checkpoint is the final epoch rather than the best; the comment describing best-model saving refers to a periodic evaluation block that is commented out. Reported accuracies therefore come from a step-count sweep at a fixed final epoch, with no epoch selection.

Vacuity of the published propositions. The published analysis argues that at z_t only the correct class has nonzero conditional density, hence the marginal and conditional velocities coincide, hence

integration lands exactly on c_y, which would imply perfect accuracy. The premise holds on the training paths (Lemma 11), where the conditional law of x given the class is a sum of Diracs. But inference starts from a test feature outside that support, where *every* class has zero density and the marginal velocity is $0/0$; and the released pipeline augments the support set, which makes the conditional law continuous with overlapping support so that trajectories from different classes do cross. Both observations are consistent with the same resolution: replace the Dirac conditional by a density, which is what Corollary 3 does, and the resulting weights are a softmax rather than one-hot. The conclusion $x_1 = c_y$ is then false in general, which is what the early-stopping heuristic in that paper empirically compensates for.

F MEASURED INFERENCE COST

Table 6 profiles the evaluation paths on EuroSAT ($N = 8100$), timing the frozen CLIP encoder and the head separately. Medians over 10 repeats after warm-up, CUDA synchronised, batch 64, one RTX A6000, PyTorch 2.11; the velocity network has 9,330,688 parameters and 6 blocks. All paths produce identical accuracies, printed below, confirming the fast paths are equivalent.

Table 6: **Measured cost on EuroSAT**, $N = 8100$, RTX A6000. The encoder dominates: the head is under 1% of end-to-end cost in every path, so the $8.6\times$ head-level advantage of the decoder yields only $1.01\times$ end-to-end. The one large factor available is in the evaluation harness, which re-encodes the test set once per step count.

path	encoder (s)	head (s)	total (s)	head %	vs. deployed
original sweep: 11 encodes, 55 evals	74.507	0.285	74.792	0.38	0.09×
optimised sweep: 1 encode, 10 evals	6.773	0.058	6.831	0.85	1.00×
deployed integration: 1 encode, M=10	6.773	0.051	6.825	0.75	1.00×
decoder: 1 encode, 1 eval, $K \times K$ solve	6.773	**0.006**	6.779	**0.09**	**1.01×**
per-operation head cost					
CLIP encoder	836.2 μs / image		739 MiB peak		
integrate M=10	6.34 μs / image		523 MiB peak		
curve M=0..10 (one trajectory)	7.14 μs / image		540 MiB peak		
decoder	0.74 μs / image		505 MiB peak		
eigendecomposition of $\mathbf{G}$ (10×10, once)	0.024 ms total		0.0029 μs / image amortised		

Accuracies for this profiling run (a different seed from Figure 1a), identical across paths: 0.4848, 0.6878, 0.7840, 0.8153, 0.8302, 0.8377, 0.8420, 0.8426, 0.8433, 0.8425, 0.8420 for $M = 0..10$; read-out at $t = 0$ gives 0.8615. Validation selected $M = 10$ while the test peak is $M = 8$ (0.8433), a 0.13 point cost of honest selection.

At 0.74 μs per image against 836 μs for the encoder, no head-side optimisation can matter end-to-end while the backbone is evaluated per image; the decoder's case is accuracy under a simpler inference rule. The factor that *is* available lies in the evaluation harness: sweeping the step count by re-running the pipeline costs $10.9\times$ more than reading the curve off one trajectory, for bit-identical accuracies. We note this because it affects anyone reproducing these numbers.

G ADDITIONAL RESULTS

Where the $\gamma = 0$ geometry stops applying. The structural results of §5 rest on $\hat{c} = (1-t)v^\star + z$ lying in $\mathrm{conv}\{c_k\}$, which is exact only at $\gamma = 0$. With $\gamma > 0$, $\hat{c} = \sum_k q_k c_k - (1-t)\frac{\gamma^2}{2}\sum_k q_k \nabla \log p_{k,t}$, and the score involves μ_k and z. Measuring the fraction of $\hat{c}$'s centred energy inside $\mathrm{span}\{c_k\}$ on a K=3, d=3 Gaussian mixture at $t = 0.4$: 1.0000 at $\gamma = 0$, 0.9989 at $\gamma = 0.1$, 0.9580 at $\gamma = 0.3$, 0.7765 at $\gamma = 0.9$. Proposition 4, Theorem 5 and Corollary 6 degrade accordingly, which is why we state them as $\gamma = 0$ results and evaluate in that regime.

Finite-shot kernel field. Proposition 10's exact empirical Bayes field memorises, as Lemma 11 predicts. On EuroSAT it peaks at two steps (0.584) and then collapses to 0.274 by nine steps, versus

0.724 for the plug-in Gaussian head. The inverse temperature $\beta(t) = 1/(2\sigma^2(1-t)^2)$ diverges while the shot set is finite, so the field becomes arbitrarily confident about the nearest training path.

Temperature. On real features the plug-in scale is $\hat{\sigma} = 0.0151$, giving $\beta(0) \approx 2200$; class prototypes are $O(1)$ apart, so a squared-distance gap of 5×10^{-4} saturates the softmax and the weights are one-hot at the first step. Tuning the scale over $\{1, \ldots, 64\}$ moves the accuracy peak from one step to two and improves accuracy by $+0.003$ only: temperature is not the binding constraint, and the Gaussian head remains a nearest-prototype classifier.

Trajectory ensembling. Averaging $\hat{q}(\lambda)$ over the seven $\hat{c}$ values along the trajectory changes accuracy by $+0.0005$ on average across the five benchmarks, and *costs* 0.0048 on Aircraft where validation selected it. We therefore use the single $t = 0$ evaluation, which is also what makes the decoder one forward pass.

Synthetic study. On synthetic data matched to CLIP geometry the plug-in head of Corollary 3 beats a trained field on $5/5$ seeds. This does *not* transfer to real features, where the same head is 16 points below the trained field on EuroSAT; the synthetic generator's within-class spread was roughly six times too large relative to the feature norm. We report it as a caution about synthetic validation of this kind, not as a result.

H FULL PER-DATASET RESULTS

Tables 1, 3, 7 and 8 in the main text pool over datasets, seeds or both. This section gives every cell, for each completed ablation group, on the five difficult datasets. All numbers are test accuracy, with every hyperparameter including the baseline's step count M selected on the released few-shot validation split; "or." columns are the test-oracle over the selection grid, so the cost of that selection is auditable rather than asserted.

These tables, and every number quoted in the main text, are generated programmatically from the per-run result files; nothing in the paper is transcribed by hand.

The groups are as follows. Tables 7 and 8 are the pooled versions quoted in the main text, repeated here so that each pooled row can be read against the cells behind it. Group A (Table 9) is the headline setting of Table 1: 16-shot CLIP ViT-B/16, with the test-oracle columns and the selected step count M per dataset. Group B (Table 10) varies the upstream adapter, group C (Table 11) uses CLIP-LoRA features at 1, 4 and 16 shots, and group D (Table 12) varies the backbone. The remaining groups vary the field rather than the features: group E (Table 13) the interpolation noise γ, group F (Table 14) the depth of the velocity network, and group I (Table 16) the time-sampling scheme used during training. Group A uses the released training procedure; the remaining groups use the cached-feature approximation of Appendix D, so they are within-group comparisons.

All floats are placed before the listings that follow, so the code of Appendix I reads uninterrupted.

Table 7: **The comparison across backbones, upstream methods and shot counts**, all on the five difficult datasets. Each row averages over datasets (so every dataset counts once regardless of seed count); "cells" is the number of runs behind it. Δ is decoder minus integration.

setting	cells	zero-shot	proto.	Cor. 8	integration	**decoder**	Δ
ViT-B/16 (Table 1)	20	49.11	64.58	66.19	69.27	**71.52**	+2.24
ViT-B/32	15	46.11	60.48	62.06	64.42	**66.76**	+2.34
RN50	15	42.20	56.96	58.64	62.96	**65.23**	+2.26
RN101	15	42.21	60.85	62.07	64.17	**66.59**	+2.42
CoOp upstream	15	71.23	64.58	71.01	73.58	73.54	-0.04
CLIP-Adapter upstream	15	62.46	65.15	67.36	70.51	**71.76**	+1.25
CLIP-LoRA upstream	15	75.79	74.80	76.58	78.21	**78.29**	+0.08
1-shot (CLIP)	12	49.27	31.00	49.27	52.17	**52.99**	+0.82
4-shot (CLIP)	15	49.10	54.95	60.55	60.24	**62.64**	+2.40
16-shot (CLIP)	20	49.11	64.58	66.19	69.27	**71.52**	+2.24

Table 8: **Ablations**, five difficult datasets, dataset-averaged. The read-out column is included because it degrades fastest under noise, as equation 10 predicts: $\mathbf{G}^{-1}$ amplifies exactly the error the noise adds.

setting	cells	integration	read-out	**decoder**
interpolation noise (group E)				
$\gamma = 0.0$	15	69.13	61.69	**71.25**
$\gamma = 0.1$	15	52.18	17.67	**53.44**
$\gamma = 0.5$	15	49.92	10.19	49.68
$\gamma = 1.0$	15	49.11	8.70	**49.59**
velocity-network depth (group F)				
1 block	15	66.63	56.16	**68.91**
2 blocks	15	68.00	58.09	**70.12**
6 blocks	15	69.13	61.69	**71.25**
12 blocks	15	69.42	62.89	**71.86**
time sampling (group I)				
`early`	15	70.23	67.97	**72.34**
`flow`	15	69.13	61.69	**71.25**
`t0`	15	67.73	67.34	**72.13**

Table 9: **Group A in full**: 16-shot, CLIP ViT-B/16. "or." are test-oracle values over the selection grid and M the validation-selected step count, so the cost of selection is auditable.

setting	dataset	n	zero-shot	proto.	ridge	Cor. 8	FMA	or.	M	read-out	**decoder**	or.
–	Aircraft	4	24.81	40.89	32.88	40.80	40.08±0.26	40.42	6.5	14.46	**43.77**±0.60	44.37
–	EuroSAT	4	48.46	77.47	86.91	77.92	83.51±0.73	84.70	5.5	86.46	**86.08**±0.42	86.64
–	DTD	4	44.09	62.53	63.42	64.13	69.77±0.54	70.07	6.5	71.79	**71.71**±0.88	73.21
–	SUN397	4	62.60	69.88	59.21	72.54	73.59±0.13	73.66	3.5	62.87	**74.84**±0.29	75.24
–	Cars	4	65.58	72.13	64.36	75.58	79.41±0.25	79.81	4.8	74.87	**81.18**±0.45	81.57

Table 10: **Group B in full**: upstream adapters, 16-shot.

setting	dataset	n	zero-shot	proto.	ridge	Cor. 8	FMA	or.	M	read-out	**decoder**	or.
CLIP-Adapter	Aircraft	3	33.75	41.64	33.93	41.97	41.62±0.34	41.87	3.3	14.62	**43.84**±0.83	44.49
CLIP-Adapter	EuroSAT	3	70.89	77.63	86.80	77.54	84.57±0.80	85.06	4.7	86.34	**85.80**±0.33	86.65
CLIP-Adapter	DTD	3	59.52	62.83	60.99	65.25	70.15±0.63	71.24	4.7	71.77	**71.77**±0.43	73.50
CLIP-Adapter	SUN397	3	74.17	70.20	59.57	74.53	75.83±0.04	75.85	1.0	62.64	75.72±0.19	75.98
CLIP-Adapter	Cars	3	73.97	73.46	65.03	77.48	80.38±0.08	80.51	2.7	74.43	**81.65**±0.38	82.01
CoOp	Aircraft	3	43.32	40.89	40.98	44.31	47.40±0.32	47.60	1.0	44.86	**47.40**±0.43	47.95
CoOp	EuroSAT	3	85.43	77.47	88.74	83.06	87.01±0.63	87.01	1.0	86.21	86.70±1.07	87.33
CoOp	DTD	3	69.68	62.53	71.87	70.09	72.97±0.06	73.36	3.0	71.81	**73.31**±0.23	73.72
CoOp	SUN397	3	74.87	69.88	66.69	74.87	75.92±0.10	75.92	1.0	68.16	75.64±0.22	75.89
CoOp	Cars	3	82.85	72.13	72.95	82.71	84.60±0.25	84.72	1.3	80.93	**84.67**±0.19	84.80

Table 11: **Group C in full**: CLIP-LoRA features at 1, 4 and 16 shots.

setting	dataset	n	zero-shot	proto.	ridge	Cor. 8	FMA	or.	M	read-out	**decoder**	or.
1-shot	Aircraft	3	28.32	23.63	22.76	28.32	29.21±0.22	29.29	2.0	9.73	**30.31**±0.71	31.04
1-shot	EuroSAT	3	72.02	74.46	77.70	72.02	77.99±0.19	79.35	2.7	75.41	**79.26**±1.01	81.53
1-shot	DTD	3	54.18	41.78	44.64	54.18	55.16±1.69	56.76	3.3	48.11	**55.67**±2.26	57.33
1-shot	SUN397	3	70.21	41.21	48.47	70.21	70.94±0.07	70.96	1.3	16.43	**71.60**±0.13	71.60
1-shot	Cars	3	69.32	41.04	52.63	69.32	70.76±0.20	70.76	2.0	19.12	**71.07**±0.15	71.09
4-shot	Aircraft	3	38.12	34.41	33.33	39.85	41.04±0.49	41.32	2.3	34.07	**41.67**±0.80	42.46
4-shot	EuroSAT	3	86.91	88.28	88.89	88.75	88.65±0.05	89.10	2.0	88.86	**89.27**±0.35	89.67
4-shot	DTD	3	63.85	61.98	62.55	67.75	68.12±0.20	68.20	2.7	62.47	67.99±0.65	68.58
4-shot	SUN397	3	72.20	65.07	61.45	73.52	73.68±0.07	73.77	1.3	50.16	**74.41**±0.02	74.44
4-shot	Cars	3	76.76	70.65	67.93	79.09	79.48±0.08	79.62	2.7	64.01	**80.17**±0.15	80.26
16-shot	Aircraft	3	54.31	54.28	52.66	54.85	58.64±0.26	58.83	2.3	56.07	58.23±0.25	58.84
16-shot	EuroSAT	3	90.73	92.19	93.14	91.56	90.73±0.06	93.04	0.0	93.63	**91.81**±1.59	94.02
16-shot	DTD	3	72.83	71.63	74.41	73.29	76.36±0.27	76.38	3.0	74.05	76.10±0.39	76.32
16-shot	SUN397	3	75.75	73.35	67.56	76.83	77.40±0.02	77.40	1.0	71.28	**77.52**±0.06	77.52
16-shot	Cars	3	85.31	82.58	78.92	86.37	87.93±0.09	87.97	2.7	86.34	87.80±0.04	88.16

Table 12: **Group D in full**: backbones, 16-shot.

setting	dataset	n	zero-shot	proto.	ridge	Cor. 8	FMA	or.	M	read-out	**decoder**	or.
RN101	Aircraft	3	17.88	33.57	26.43	33.84	32.51±0.85	33.14	6.7	10.96	**35.86**±0.35	36.65
RN101	EuroSAT	3	32.85	72.25	83.20	72.16	78.12±1.39	80.50	4.0	82.15	**81.88**±0.70	82.25
RN101	DTD	3	38.12	62.00	59.75	63.00	64.58±1.20	65.68	5.3	65.50	**66.29**±0.42	67.32
RN101	SUN397	3	59.03	66.50	51.44	69.34	69.44±0.12	69.59	2.7	63.24	**70.97**±0.07	71.12
RN101	Cars	3	63.15	69.94	60.59	71.98	76.18±0.25	76.61	4.7	73.98	**77.93**±0.62	78.64
RN50	Aircraft	3	17.04	30.66	27.12	30.69	30.56±0.21	31.71	5.0	18.04	**35.05**±0.58	35.61
RN50	EuroSAT	3	36.32	73.78	83.67	73.98	79.87±1.05	82.07	4.0	83.20	**83.19**±0.52	83.63
RN50	DTD	3	42.97	57.98	59.40	58.45	65.03±0.15	65.15	4.7	64.91	64.97±0.49	67.55
RN50	SUN397	3	58.84	63.16	53.63	66.07	67.50±0.15	67.56	2.3	63.27	**68.70**±0.19	68.98
RN50	Cars	3	55.83	59.20	55.03	64.03	71.86±0.16	72.01	3.7	70.16	**74.22**±0.49	74.43
ViT-B/32	Aircraft	3	18.93	32.58	27.27	32.13	30.30±0.20	31.25	3.7	10.90	**34.29**±0.94	35.84
ViT-B/32	EuroSAT	3	45.26	76.04	84.28	75.77	82.58±0.53	82.76	6.3	84.30	**84.45**±0.84	85.05
ViT-B/32	DTD	3	44.09	61.47	59.52	63.24	66.00±0.39	66.23	5.3	66.86	**68.09**±0.10	68.75
ViT-B/32	SUN397	3	62.08	67.89	53.98	70.62	70.78±0.22	70.99	3.7	60.12	**72.47**±0.24	72.56
ViT-B/32	Cars	3	60.19	64.43	56.06	68.54	72.43±0.07	72.50	3.3	66.44	**74.51**±0.45	74.88

Table 13: **Group E in full**: interpolation noise.

setting	dataset	n	zero-shot	proto.	ridge	Cor. 8	FMA	or.	M	read-out	**decoder**	or.
γ=0.0	Aircraft	3	24.81	40.89	32.88	40.80	39.46±0.36	39.70	7.0	14.51	**42.84**±1.05	44.11
γ=0.0	EuroSAT	3	48.45	77.47	86.91	77.92	83.04±0.55	84.62	4.0	85.56	**85.48**±0.78	85.93
γ=0.0	DTD	3	44.09	62.53	63.42	64.13	70.31±0.88	70.47	6.7	71.47	**71.91**±0.69	73.01
γ=0.0	SUN397	3	62.60	69.88	59.21	72.54	73.28±0.17	73.56	3.7	62.46	**74.89**±0.11	75.05
γ=0.0	Cars	3	65.58	72.13	64.36	75.58	79.54±0.16	79.62	5.0	74.44	**81.11**±0.43	81.53
γ=0.1	Aircraft	3	24.81	40.89	32.88	40.80	24.81±0.00	26.02	0.0	0.86	**24.81**±0.00	26.16
γ=0.1	EuroSAT	3	48.45	77.47	86.91	77.92	59.86±3.09	60.64	6.0	62.65	**66.01**±2.80	67.35
γ=0.1	DTD	3	44.09	62.53	63.42	64.13	46.00±0.80	47.71	3.7	23.40	**46.08**±1.87	47.83
γ=0.1	SUN397	3	62.60	69.88	59.21	72.54	64.64±0.09	64.64	1.0	0.75	**64.73**±0.26	64.83
γ=0.1	Cars	3	65.58	72.13	64.36	75.58	65.58±0.00	65.58	0.0	0.68	**65.58**±0.00	65.58
γ=0.5	Aircraft	3	24.81	40.89	32.88	40.80	24.81±0.00	24.81	0.0	1.11	**24.81**±0.00	24.81
γ=0.5	EuroSAT	3	48.45	77.47	86.91	77.92	52.53±4.52	53.57	4.3	28.27	49.36±3.10	53.32
γ=0.5	DTD	3	44.09	62.53	63.42	64.13	44.09±0.00	44.09	0.0	19.62	**44.41**±0.31	45.13
γ=0.5	SUN397	3	62.60	69.88	59.21	72.54	62.60±0.00	62.60	0.0	1.18	**64.25**±0.02	64.31
γ=0.5	Cars	3	65.58	72.13	64.36	75.58	65.58±0.00	65.58	0.0	0.77	**65.58**±0.00	65.58
γ=1.0	Aircraft	3	24.81	40.89	32.88	40.80	24.81±0.00	24.81	0.0	0.97	**24.81**±0.00	24.81
γ=1.0	EuroSAT	3	48.45	77.47	86.91	77.92	48.45±0.02	48.45	0.0	18.20	48.00±0.65	48.87
γ=1.0	DTD	3	44.09	62.53	63.42	64.13	44.09±0.00	44.09	0.0	22.32	**45.02**±0.76	45.59
γ=1.0	SUN397	3	62.60	69.88	59.21	72.54	62.60±0.00	62.60	0.0	1.25	**64.53**±0.22	64.58
γ=1.0	Cars	3	65.58	72.13	64.36	75.58	65.58±0.00	65.58	0.0	0.74	**65.58**±0.00	65.58

Table 14: **Group F in full**: velocity-network depth.

setting	dataset	n	zero-shot	proto.	ridge	Cor. 8	FMA	or.	M	read-out	**decoder**	or.
b1	Aircraft	3	24.81	40.89	32.88	40.80	35.40±0.20	35.43	5.7	9.00	**38.66**±0.68	40.05
b1	EuroSAT	3	48.45	77.47	86.91	77.92	83.19±0.14	84.61	4.7	85.36	**85.38**±0.70	86.00
b1	DTD	3	44.09	62.53	63.42	64.13	68.42±0.70	69.42	7.3	70.47	**70.82**±0.20	71.39
b1	SUN397	3	62.60	69.88	59.21	72.54	71.65±0.05	71.67	4.3	54.92	**73.02**±0.31	73.39
b1	Cars	3	65.58	72.13	64.36	75.58	74.48±0.15	74.59	3.7	61.07	**76.65**±0.51	77.16
b2	Aircraft	3	24.81	40.89	32.88	40.80	36.94±0.35	37.36	4.7	10.90	**41.30**±0.35	41.89
b2	EuroSAT	3	48.45	77.47	86.91	77.92	83.92±1.44	84.33	6.0	85.43	**85.72**±0.98	86.02
b2	DTD	3	44.09	62.53	63.42	64.13	69.35±0.36	69.82	7.0	70.53	**70.69**±0.71	72.06
b2	SUN397	3	62.60	69.88	59.21	72.54	72.49±0.04	72.49	4.0	56.36	**74.28**±0.01	74.34
b2	Cars	3	65.58	72.13	64.36	75.58	77.27±0.12	77.27	5.3	67.24	**78.62**±0.38	79.38
b6	Aircraft	3	24.81	40.89	32.88	40.80	39.46±0.36	39.70	7.0	14.51	**42.84**±1.05	44.11
b6	EuroSAT	3	48.45	77.47	86.91	77.92	83.04±0.55	84.62	4.0	85.56	**85.48**±0.78	85.93
b6	DTD	3	44.09	62.53	63.42	64.13	70.31±0.88	70.47	6.7	71.47	**71.91**±0.69	73.01
b6	SUN397	3	62.60	69.88	59.21	72.54	73.28±0.17	73.56	3.7	62.46	**74.89**±0.11	75.05
b6	Cars	3	65.58	72.13	64.36	75.58	79.54±0.16	79.62	5.0	74.44	**81.11**±0.43	81.53
b12	Aircraft	3	24.81	40.89	32.88	40.80	40.51±0.40	40.96	8.0	16.15	**44.16**±0.37	44.77
b12	EuroSAT	3	48.45	77.47	86.91	77.92	83.33±0.42	84.08	4.3	85.24	**85.38**±0.52	85.68
b12	DTD	3	44.09	62.53	63.42	64.13	69.50±0.92	70.35	7.0	71.26	**72.50**±0.20	72.73
b12	SUN397	3	62.60	69.88	59.21	72.54	73.84±0.12	73.89	3.0	64.85	**75.03**±0.19	75.29
b12	Cars	3	65.58	72.13	64.36	75.58	79.91±0.29	80.41	3.3	76.93	**82.23**±0.35	82.48

Table 15: **Group H in full**: 1- and 4-shot, CLIP.

setting	dataset	n	zero-shot	proto.	ridge	Cor. 8	FMA	or.	M	read-out	**decoder**	or.
1-shot	Aircraft	3	24.81	21.51	18.87	24.81	24.62±1.13	26.62	6.3	1.33	**26.33**±1.03	27.68
1-shot	DTD	3	44.09	33.69	33.69	44.09	51.62±2.03	53.39	6.0	46.06	51.04±0.52	53.03
1-shot	SUN397	3	62.60	37.54	41.50	62.60	65.85±0.24	66.18	3.3	12.05	**67.50**±0.17	67.75
1-shot	Cars	3	65.58	31.26	46.39	65.58	66.59±0.74	67.24	1.0	13.82	**67.07**±0.63	67.69
4-shot	Aircraft	3	24.81	31.29	25.98	30.99	29.34±1.57	30.99	3.3	2.99	**32.34**±0.47	33.33
4-shot	EuroSAT	3	48.43	71.11	78.54	73.81	72.35±2.15	75.77	5.0	76.26	**76.22**±0.61	76.44
4-shot	DTD	3	44.09	52.90	47.87	58.92	59.10±0.71	60.11	6.3	57.82	**60.32**±0.85	61.33
4-shot	SUN397	3	62.60	61.24	55.59	69.22	69.88±0.17	69.93	3.3	33.59	**71.57**±0.10	71.81
4-shot	Cars	3	65.58	58.19	56.59	69.82	70.52±0.12	70.92	3.3	38.36	**72.76**±0.18	72.98

Table 16: **Group I in full**: time sampling.

setting	dataset	n	zero-shot	proto.	ridge	Cor. 8	FMA	or.	M	read-out	**decoder**	or.
early	Aircraft	3	24.81	40.89	32.88	40.80	42.12±0.14	42.50	6.3	28.85	**45.64**±0.17	46.09
early	EuroSAT	3	48.45	77.47	86.91	77.92	84.56±0.90	85.60	4.0	87.08	**85.91**±0.56	87.44
early	DTD	3	44.09	62.53	63.42	64.13	69.82±0.22	70.39	7.0	72.68	**73.52**±0.55	73.90
early	SUN397	3	62.60	69.88	59.21	72.54	72.92±0.15	72.94	2.3	69.58	**73.60**±0.12	73.64
early	Cars	3	65.58	72.13	64.36	75.58	81.72±0.31	82.06	4.0	81.66	**83.01**±0.54	83.57
flow	Aircraft	3	24.81	40.89	32.88	40.80	39.46±0.36	39.70	7.0	14.51	**42.84**±1.05	44.11
flow	EuroSAT	3	48.45	77.47	86.91	77.92	83.04±0.55	84.62	4.0	85.56	**85.48**±0.78	85.93
flow	DTD	3	44.09	62.53	63.42	64.13	70.31±0.88	70.47	6.7	71.47	**71.91**±0.69	73.01
flow	SUN397	3	62.60	69.88	59.21	72.54	73.28±0.17	73.56	3.7	62.46	**74.89**±0.11	75.05
flow	Cars	3	65.58	72.13	64.36	75.58	79.54±0.16	79.62	5.0	74.44	**81.11**±0.43	81.53
t0	Aircraft	3	24.81	40.89	32.88	40.80	36.48±0.55	36.80	2.7	29.12	**45.28**±0.65	45.92
t0	EuroSAT	3	48.45	77.47	86.91	77.92	84.71±0.90	85.00	3.3	87.47	**87.20**±0.33	88.20
t0	DTD	3	44.09	62.53	63.42	64.13	65.25±0.43	66.15	4.0	72.54	**73.36**±0.56	73.70
t0	SUN397	3	62.60	69.88	59.21	72.54	72.77±0.13	72.77	2.0	68.36	**73.47**±0.08	73.58
t0	Cars	3	65.58	72.13	64.36	75.58	79.45±0.25	79.45	2.0	79.21	**81.34**±0.11	82.14

I VERIFICATION CODE

The listings below are the scripts referenced in Table 5, reproduced verbatim. `fma_theory.py` implements the closed forms; `t1_bayes.py` and `t2_flowmap.py` verify the lemmas, theorems and propositions; `anchor_ridge.py` implements and evaluates the decoder of §7; `probe.py` and `straightness.py` produce the structural and rectification measurements of §6; `baselines.py` produces the training-free comparisons; `t9_mmse.py` and `closed_form.py` verify and apply the parameter-free choices of equation 12 and equation 15; and `matched_baseline.py` and `stats_tests.py` produce the controls and the paired tests. Each subsection heading names the statements that listing checks, and the in-code comments use the same numbering as this paper.

I.1 FMA_THEORY.PY, CLOSED FORMS (COROLLARIES 3, 8; PROPOSITION 10; THEOREM 12)

```
"""
fma_theory.py -- analytic velocity formulation for FMA in the noise-free
regime (cfg.gamma == 0) with one fixed text anchor per class.  NumPy only.

Notation (matches train.py):
    x    image feature          (image_features)
    c_k  class anchor           (text_features = class_embeddings[labels])
    z_t  (1-t) x + t c_k        (interpolated_features, gamma=0)
    v    c_k - x                (velocity_gt,           gamma=0)
"""
import numpy as np

__all__ = ["velocity_gt", "preimage", "anchor_weights_gauss",
           "anchor_weights_empirical", "velocity_from_weights",
           "velocity_star_gauss", "velocity_linear_closed_form",
           "velocity_linear_general", "euler_classify"]


def velocity_gt(x, c, t, gamma=0.0, noise=None):
    """train.py: v = text - image + gamma*0.5*(1-2t)/sqrt(t(1-t)) * noise"""
    v = c - x
    if gamma:
        if noise is None:
            noise = np.random.randn(*x.shape)
        v = v + gamma * 0.5 * (1 - 2 * t) / np.sqrt(t * (1 - t)) * noise
    return v


# --- Corollary 3 (noise-free field, v* = (chat - z)/(1-t)) ------------------
def preimage(z, c, t):
    """x_k(z,t) = (z - t c_k)/(1-t).  z (n,d), c (K,d) -> (n,K,d)"""
    return (z[:, None, :] - t * c[None, :, :]) / (1.0 - t)


def velocity_from_weights(z, c, t, W):
    """v*(z,t) = (chat - z)/(1-t) with chat = W @ c."""
    chat = W @ c
    return (chat - z) / (1.0 - t), chat


# --- Corollary 8 (Gaussian features), plug-in over the shots ----------------
def anchor_weights_gauss(z, c, t, mu, sigma, prior=None):
    """w_k = softmax( -||z - m_k(t)||^2 / (2 sigma^2 (1-t)^2) ),
       m_k(t) = (1-t) mu_k + t c_k."""
    m = (1.0 - t) * mu + t * c
    d2 = ((z[:, None, :] - m[None, :, :]) ** 2).sum(-1)
    lg = -d2 / (2.0 * sigma ** 2 * (1.0 - t) ** 2)
    if prior is not None:
        lg = lg + np.log(prior)[None, :]
    lg -= lg.max(1, keepdims=True)
    w = np.exp(lg)
    return w / w.sum(1, keepdims=True)


def velocity_star_gauss(z, c, t, mu, sigma, prior=None):
    W = anchor_weights_gauss(z, c, t, mu, sigma, prior)
    v, chat = velocity_from_weights(z, c, t, W)
    return v, chat, W


```

```python
# --- Proposition 10 (finite-shot form, exact empirical Bayes) ---------------
def anchor_weights_empirical(z, X, labels, c, t, h):
    K = c.shape[0]
    Zi = (1.0 - t) * X + t * c[labels]
    d2 = ((z[:, None, :] - Zi[None, :, :]) ** 2).sum(-1)
    lg = -d2 / (2.0 * h ** 2 * (1.0 - t) ** 2)
    lg -= lg.max(1, keepdims=True)
    wi = np.exp(lg)
    wi /= wi.sum(1, keepdims=True)
    W = np.zeros((z.shape[0], K))
    np.add.at(W.T, labels, wi.T)
    return W


# --- Theorem 12 (optimal linear field) --------------------------------------
def velocity_linear_closed_form(z, t, mu, Sigma_C, Sigma_E):
    """X = C + E, E indep of C:
       v_lin(z,t) = -(1-t) S_E (S_C + (1-t)^2 S_E)^{-1} (z - mu)."""
    A = -(1.0 - t) * Sigma_E @ np.linalg.inv(Sigma_C + (1.0 - t) ** 2 * Sigma_E)
    return (z - mu) @ A.T, A


def velocity_linear_general(t, mu_X, mu_C, S_X, S_C, S_XC):
    S_Zt = (1 - t) ** 2 * S_X + t ** 2 * S_C + t * (1 - t) * (S_XC + S_XC.T)
    S_VZ = (1 - t) * (S_XC.T - S_X) + t * (S_C - S_XC)
    A = S_VZ @ np.linalg.inv(S_Zt)
    b = (mu_C - mu_X) - A @ ((1 - t) * mu_X + t * mu_C)
    return A, b


# --- test.py inference protocol ---------------------------------------------
def euler_classify(field, x, y, c, steps, h=0.1):
    """Replicates test_fma: Euler from t=0, then argmax_k <z/||z||, c_k>."""
    z = x.copy()
    t = 0.0
    for _ in range(steps):
        z = z + h * field(z, t)
        t += h
    z = z / np.linalg.norm(z, axis=1, keepdims=True)
    return float(((z @ c.T).argmax(1) == y).mean())
```

I.2 SYNTH.PY, SYNTHETIC FEATURE GEOMETRY AND THE PLUG-IN COROLLARY 8 HEAD

```python
"""
synth.py -- synthetic feature geometry that mimics CLIP:
  anchor_rho : mean pairwise cosine of the text anchors (CLIP: 0.6-0.85)
  gap        : modality gap between the image cone and the text cone
  delta      : class-specific PROMPT/VISUAL MISMATCH.  Without this the image
               prototypes are an order-preserving function of the anchors and
               the few shots carry no information the prompts lack, so no
               adapter can help.  delta>0 reproduces the real zero-shot ->
               few-shot headroom (e.g. EuroSAT ~48% -> ~85%).
  sigma      : intra-class spread, scaled by 1/sqrt(d)
"""
import numpy as np

DEFAULTS = dict(d=128, K=20, anchor_rho=0.70, gap=0.60, delta=0.65, sigma=2.2)


def build(d=128, K=20, anchor_rho=0.70, gap=0.60, delta=0.65, seed=0):
    rng = np.random.default_rng(seed)
    base = rng.normal(size=d); base /= np.linalg.norm(base)
    dev = rng.normal(size=(K, d))
    dev -= dev @ base[:, None] * base[None, :]
    dev /= np.linalg.norm(dev, axis=1, keepdims=True)
    c = np.sqrt(anchor_rho) * base[None, :] + np.sqrt(1 - anchor_rho) * dev
    c /= np.linalg.norm(c, axis=1, keepdims=True)
    g = rng.normal(size=d); g -= g @ base * base; g /= np.linalg.norm(g)
    r = rng.normal(size=(K, d)); r /= np.linalg.norm(r, axis=1, keepdims=True)
    mu = c + gap * g[None, :] + delta * r
    mu /= np.linalg.norm(mu, axis=1, keepdims=True)
    return c, mu, rng


def sampler(mu, sigma_raw, d, K):
    sig = sigma_raw / np.sqrt(d)

    def draw(nper, rng):
```

```
        lab = np.repeat(np.arange(K), nper)
        return mu[lab] + sig * rng.normal(size=(K * nper, d)), lab
    return draw, sig


def plug_in_head(Xtr, ytr, K, S, d):
    """Corollary 8 (plug-in): estimate mu_k by the shot mean and sigma by the pooled
    intra-class spread; inflate sigma to account for the error in mu_hat."""
    mu_hat = np.stack([Xtr[ytr == k].mean(0) for k in range(K)])
    s2 = ((Xtr - mu_hat[ytr]) ** 2).sum() / (len(ytr) * d * (1 - 1.0 / S))
    return mu_hat, np.sqrt(s2), np.sqrt(s2 * (1 + 1.0 / S))
```

I.3 T3_STOCHASTIC.PY, LEMMAS 1, 9; COROLLARIES 2, 3, 8, 6; PROPOSITION 4, AND THE $\gamma > 0$ SCOPE MEASUREMENT

```
"""
t3_stochastic.py -- verification of the STOCHASTIC FMA theory.

  Lemma 1     v*(z,t) = sum_k q_k(z,t) [ (c_k-z)/(1-t)
                                         - (s^2/(1-t) + s s') grad log p_{k,t}(z) ]
  Corollary 2 for the released schedule s(t)=gamma sqrt(t(1-t)):
              s^2/(1-t) + s s' = gamma^2/2   (CONSTANT in t)
  Corollary 8 Gaussian specialisation:
              V_k(t) = (1-t)^2 Sigma_k + gamma^2 t(1-t) I
              v_k* = (c_k-z)/(1-t) + (gamma^2/2) V_k(t)^{-1} (z-m_k(t))
              q_k  = softmax(log pi_k - |V_k|^{-1/2}-weighted Mahalanobis/2)
  Corollary 3 gamma -> 0 recovers the deterministic field (needs sigma_x > 0).
  Prop  E     endpoint second-moment divergence of the target.

The marginal field is checked against a brute-force kernel Monte-Carlo estimate
of E[Y_t | Z_t = z], which uses only the definition of conditional expectation.
"""
import numpy as np
from report import sec, check, out, summary


def sched(t, gamma):
    s = gamma * np.sqrt(t * (1 - t))
    sp = gamma * (1 - 2 * t) / (2 * np.sqrt(t * (1 - t))) if gamma else 0.0
    return s, sp


def coeff(t, gamma):
    """beta(t) = s^2/(1-t) + s s'.

    Computed via the ANALYTIC product s s' = gamma^2 (1-2t)/2 rather than
    s * s'.  At t=0 (and t=1) s=0 while s'=inf, so the naive product is a
    0*inf = nan even though the limit is finite -- a concrete instance of the
    endpoint singularity of Lemma 9, and a reason to prefer Corollary 2's
    closed form in any implementation.
    """
    if not gamma:
        return 0.0
    return gamma ** 2 * t + gamma ** 2 * (1 - 2 * t) / 2      # == gamma^2/2


def gauss_field(z, c, mu, t, gamma, sigx, prior=None):
    """Corollary 8, isotropic Sigma_k = sigx^2 I (so |V_k| is common to all k)."""
    K = c.shape[0]
    m = (1 - t) * mu + t * c                          # (K,d)
    a = (1 - t) ** 2 * sigx ** 2 + gamma ** 2 * t * (1 - t)
    d2 = ((z ** 2).sum(1)[:, None] - 2 * z @ m.T + (m ** 2).sum(1)[None, :])
    lg = -d2 / (2 * a)
    if prior is not None:
        lg = lg + np.log(prior)[None, :]
    lg = lg - lg.max(1, keepdims=True)
    q = np.exp(lg); q /= q.sum(1, keepdims=True)          # (n,K)
    # per-class velocity: anchor transport + score correction
    b = coeff(t, gamma)
    v = np.zeros_like(z)
    for k in range(K):
        score = -(z - m[k]) / a                           # grad log p_{k,t}
        vk = (c[k] - z) / (1 - t) - b * score
        v += q[:, [k]] * vk
    return v, q


def main(quick=False):
```

```
    rng = np.random.default_rng(0)

    # ---------------- Corollary 2 ----------------
    sec("COROLLARY 2 -- for s(t)=gamma*sqrt(t(1-t)) the score coefficient\n"
        "        beta(t) = s^2/(1-t) + s s'  equals gamma^2/2, CONSTANT in t.\n"
        "        (s^2/(1-t)=gamma^2 t and s s'=gamma^2(1-2t)/2 sum to gamma^2/2.)")
    ok = True
    out(f"   {'gamma':>7} {'t':>6} {'s^2/(1-t)':>11} {'s s^prime':>11} "
        f"{'beta(t)':>10} {'gamma^2/2':>10}")
    for g in [0.3, 0.7, 1.5]:
        for t in [0.02, 0.25, 0.5, 0.75, 0.98]:
            s, sp = sched(t, g)
            b = coeff(t, g)
            ok &= abs(b - g ** 2 / 2) < 1e-12
            out(f"   {g:7.2f} {t:6.2f} {s**2/(1-t):11.5f} {s*sp:+11.5f} "
                f"{b:10.6f} {g**2/2:10.6f}")
    check("beta(t) == gamma^2/2 exactly, for all t and gamma", ok)

    # ---------------- pathwise identity behind Lemma 1 ----------------
    sec("LEMMA 1 (step 1) -- pathwise identity.  With Z_t=(1-t)X+tC+s eps and\n"
        "        Y_t = C-X+s' eps, substituting X=(Z_t-tC-s eps)/(1-t) gives\n"
        "            Y_t = (C-Z_t)/(1-t) + [ s/(1-t) + s' ] eps\n"
        "        so the eps-dependence is an explicit scalar multiple of eps.")
    worst = 0.0
    for g in [0.0, 0.5, 1.2]:
        for t in [0.1, 0.5, 0.9]:
            x = rng.normal(size=(200, 6)); c = rng.normal(size=(200, 6))
            e = rng.normal(size=(200, 6))
            s, sp = sched(t, g)
            z = (1 - t) * x + t * c + s * e
            lhs = c - x + sp * e
            rhs = (c - z) / (1 - t) + (s / (1 - t) + sp) * e
            worst = max(worst, np.abs(lhs - rhs).max())
    check("Y_t == (C-Z_t)/(1-t) + [s/(1-t)+s'] eps", worst < 1e-12,
          f"max abs err = {worst:.2e}")

    # ---------------- Lemma 1, marginal field, vs kernel Monte-Carlo --------
    sec("LEMMA 1 -- marginal optimum v*(z,t) = sum_k q_k v_k* verified against\n"
        "        a brute-force kernel Monte-Carlo estimate of E[Y_t | Z_t=z]\n"
        "        (definition of conditional expectation only, no formula reused).")
    d, K, sigx = 3, 3, 0.45
    cL = rng.normal(size=(K, d)); cL /= np.linalg.norm(cL, axis=1, keepdims=True)
    muL = cL + 0.5 * rng.normal(size=(K, d))
    M = 2_000_000 if quick else 20_000_000
    out(f"   d={d} K={K} sigma_x={sigx}  M={M:,} Monte-Carlo samples")
    ok_all = True
    for g in [0.0, 0.4, 0.9]:
        out(f"\n   gamma={g}:  {'t':>6} {'bw frac':>9} {'rel err':>9} {'cos':>9}")
        for t in [0.3, 0.6]:
            s, sp = sched(t, g)
            kk = rng.integers(0, K, M)
            xx = muL[kk] + sigx * rng.normal(size=(M, d))
            ee = rng.normal(size=(M, d))
            Z = (1 - t) * xx + t * cL[kk] + s * ee
            Y = cL[kk] - xx + sp * ee
            # queries from the Z_t marginal
            qk = rng.integers(0, K, 8)
            Q = ((1 - t) * (muL[qk] + sigx * rng.normal(size=(8, d)))
                 + t * cL[qk] + s * rng.normal(size=(8, d)))
            va, _ = gauss_field(Q, cL, muL, t, g, sigx)
            spread = np.sqrt((1 - t) ** 2 * sigx ** 2 + g ** 2 * t * (1 - t))
            errs = []
            for frac in [0.30, 0.20, 0.12]:
                bw = frac * spread
                d2 = ((Q[:, None, :] - Z[None, :, :]) ** 2).sum(-1)
                w = np.exp(-d2 / (2 * bw ** 2))
                vm = (w @ Y) / w.sum(1, keepdims=True)
                rel = np.linalg.norm(vm - va) / np.linalg.norm(va)
                cs = float((np.sum(vm * va, 1) /
                            (np.linalg.norm(vm, axis=1) *
                             np.linalg.norm(va, axis=1))).mean())
                errs.append(rel)
                out(f"          {t:6.2f} {frac:9.2f} {rel:9.4f} {cs:9.5f}")
            # MC variance grows as bw shrinks (fewer effective samples), so we
            # require the BEST bandwidth to be accurate rather than monotone
            # convergence, which needs far more samples at large gamma.
            ok_all &= (min(errs) < 0.05 and cs > 0.995)
    check("kernel Monte-Carlo E[Y_t|Z_t=z] matches Lemma 1 for gamma=0, 0.4, 0.9",
          ok_all)

    # ---------------- Corollary 3: gamma -> 0 ----------------
```

```
    sec("COROLLARY 3 -- the deterministic field is the exact gamma->0 limit.\n"
        "        The score stays bounded because a_t=(1-t)^2 sigma_x^2>0, so the\n"
        "        correction gamma^2/2 * score vanishes.  (Requires sigma_x>0:\n"
        "        at sigma_x=0 the limit is 0*infinity and must be taken jointly.)")
    zq = rng.normal(size=(500, d)) * 0.6
    out(f"   {'gamma':>9} {'max||v_gamma - v_0||':>22}")
    prev = None
    mono = True
    for g in [0.5, 0.2, 0.05, 0.01, 0.0]:
        for t in [0.4]:
            v, _ = gauss_field(zq, cL, muL, t, g, sigx)
            v0, _ = gauss_field(zq, cL, muL, t, 0.0, sigx)
            dv = np.abs(v - v0).max()
            out(f"   {g:9.3f} {dv:22.3e}")
            if prev is not None and dv > prev + 1e-12:
                mono = False
            prev = dv
    check("v_gamma -> v_0 monotonically as gamma -> 0", mono and prev < 1e-12)

    # ---------------- Lemma 9: endpoint divergence ----------------
    sec("LEMMA 9 -- the target's second moment integrates to\n"
        "        gamma^2 [ 0.5 log((1-eps)/eps) - 1 + 2 eps ], divergent.\n"
        "        This is a property of the IMPLEMENTED schedule, not of Lemma 1.")
    okI = True
    out(f"   {'eps':>9} {'numeric':>12} {'exact':>12}")
    for e in [1e-2, 1e-4, 1e-8]:
        U = np.log((1 - e) / e)
        u = np.linspace(-U, U, 400_001)
        tt = 1 / (1 + np.exp(-u))
        I = np.trapezoid(0.25 - tt * (1 - tt), u)
        ex = 0.5 * np.log((1 - e) / e) - 1 + 2 * e
        okI &= abs(I - ex) < 1e-6
        out(f"   {e:9.0e} {I:12.5f} {ex:12.5f}")
    check("endpoint integral matches the exact closed form", okI)

    # ---------------- scope warning: structure is gamma=0 only -------------
    sec("SCOPE -- with gamma>0 the anchor-span structure is LOST.  chat :=\n"
        "        (1-t)v*+z equals sum_k q_k c_k MINUS (1-t)beta * sum_k q_k score_k,\n"
        "        and the score involves mu_k and z, which are NOT in span{c_k}.\n"
        "        Prop. 4 (rank) and Cor. 6 (logit split) therefore hold only at\n"
        "        gamma=0.  Measured below as the leakage out of span{c_k}.")
    Cc = cL - cL.mean(0, keepdims=True)
    Qb = np.linalg.qr(Cc.T)[0][:, :K - 1]
    out(f"   {'gamma':>9} {'energy of chat in span{c_k}':>30}")
    fr = {}
    for g in [0.0, 0.1, 0.3, 0.9]:
        v, _ = gauss_field(zq, cL, muL, 0.4, g, sigx)
        chat = 0.6 * v + zq
        Pc = chat - cL.mean(0, keepdims=True)
        fr[g] = float(np.mean(np.linalg.norm(Pc @ Qb, axis=1) ** 2 /
                              np.linalg.norm(Pc, axis=1) ** 2))
        out(f"   {g:9.2f} {fr[g]:30.4f}")
    check("gamma=0 gives full anchor-span containment", fr[0.0] > 0.999,
          f"{fr[0.0]:.6f}")
    check("gamma>0 leaks out of the anchor span (structure is gamma=0 only)",
          fr[0.9] < fr[0.0] - 0.01, f"{fr[0.0]:.4f} -> {fr[0.9]:.4f}")

    # ---------------- generalised logit decomposition (all gamma) ---------
    sec("GENERALISED LOGIT DECOMPOSITION -- Cor. 6's algebra never uses\n"
        "        chat in conv{c_k}, so z_m=(1-t_m)x+t_m a_m holds for EVERY\n"
        "        gamma.  Splitting a_m = C^T wbar_m - r_m into its anchor and\n"
        "        accumulated-score parts gives, exactly for all gamma,\n"
        "          <z_m,c_k> = (1-t_m) zs_k + t_m [ (G wbar_m)_k - <r_m,c_k> ].\n"
        "        The gamma=0 case is r_m=0, i.e. the two-term blend.")
    Kg, dg, ng, h = 6, 12, 60, 0.1
    Cg = rng.normal(size=(Kg, dg)); Cg /= np.linalg.norm(Cg, axis=1, keepdims=True)
    mug = Cg + 0.4 * rng.normal(size=(Kg, dg))
    xg = rng.normal(size=(ng, dg)) * 0.5
    Gg = Cg @ Cg.T
    out(f"   {'gamma':>7} {'max residual':>14} {'||r_m||/||a_m||':>17} "
        f"{'a_m in conv?':>14}")
    okg = True
    Ah = np.vstack([Cg.T, np.ones((1, Kg))])
    for gam in [0.0, 0.2, 0.6, 1.2]:
        z, t = xg.copy(), 0.0
        A = np.zeros((ng, Kg)); R = np.zeros_like(xg); am = np.zeros_like(xg)
        zs = xg @ Cg.T; worst = 0.0
        for m in range(6):
            v, q = gauss_field(z, Cg, mug, t, gam, 0.5)
            chat = z + (1 - t) * v
            spart = (q @ Cg) - chat
```

```
            lam = h / (1 - t); z = (1 - lam) * z + lam * chat; tn = t + h
            A = ((1 - tn) * t * A + h * q) / (tn * (1 - t))
            R = ((1 - tn) * t * R + h * spart) / (tn * (1 - t))
            am = ((1 - tn) * t * am + h * chat) / (tn * (1 - t))
            t = tn
            resid = float(np.abs(
                z @ Cg.T - ((1 - t) * zs + t * ((A @ Gg) - R @ Cg.T))).max())
            worst = resid if not np.isfinite(worst) else max(worst, resid)
        co = np.linalg.lstsq(Ah, np.vstack([am.T, np.ones((1, ng))]),
                             rcond=None)[0].T
        rr = float(np.linalg.norm(R, axis=1).mean() /
                   max(np.linalg.norm(am, axis=1).mean(), 1e-12))
        okg &= np.isfinite(worst) and worst < 1e-12
        out(f"   {gam:7.2f} {worst:14.2e} {rr:17.4f} "
            f"{('yes' if co.min() > -1e-6 else 'NO'):>14}")
    check("generalised logit decomposition exact for every gamma", okg)

    # ---------------- enrichment ceiling ----------------
    sec("ENRICHMENT CEILING -- for an isotropic vector the expected fraction of\n"
        "        centred energy inside span{c_k} is exactly (K-1)/d, so the\n"
        "        enrichment of Prop. 4 can never exceed d/(K-1).  The rank\n"
        "        result is only informative when K << d.")
    out(f"   {'K':>6} {'d':>6} {'(K-1)/d':>10} {'max enrichment':>16} "
        f"{'measured null':>15}")
    okc = True
    for Kx in [10, 47, 100, 397]:
        dx = 512
        Cx = rng.normal(size=(Kx, dx)); Cx /= np.linalg.norm(Cx, axis=1, keepdims=True)
        Qx = np.linalg.qr((Cx - Cx.mean(0, keepdims=True)).T)[0][:, :Kx - 1]
        U = rng.normal(size=(4000, dx))
        meas = float(np.mean(np.linalg.norm(U @ Qx, axis=1) ** 2 /
                             np.linalg.norm(U, axis=1) ** 2))
        okc &= abs(meas - (Kx - 1) / dx) < 0.02
        out(f"   {Kx:6d} {dx:6d} {(Kx-1)/dx:10.3f} {dx/(Kx-1):15.2f}x "
            f"{meas:15.4f}")
    check("isotropic null equals (K-1)/d, so enrichment <= d/(K-1)", okc)

    return summary("t3_stochastic")


if __name__ == "__main__":
    import sys
    main(quick="--quick" in sys.argv)
```

I.4 t1_bayes.py, Lemmas 1, 9, 11; Corollaries 3, 8; Proposition 4

```
"""
t1_bayes.py -- NumPy only.
  Lemma 9     why gamma=0 is the only integrable choice
  Lemma 11    paths never collide -> ERM interpolates -> ill-posed off-manifold
  Lemma 1     c_k - x_k(z,t) = (c_k - z)/(1-t)                [exact algebra]
  Cor. 3      v*(z,t) = (chat - z)/(1-t)   [independent kernel Monte-Carlo]
  Cor. 8      w = softmax(-||z-m_k(t)||^2 / (2 sigma^2 (1-t)^2))
  Prop 4      (1-t)v*+z lies in a (K-1)-dim simplex, independent of d
"""
import numpy as np
from fma_theory import velocity_gt, preimage, velocity_star_gauss
from report import sec, check, out, summary
from synth import build, sampler


def main(quick=False):
    rng = np.random.default_rng(0)

    # ---------------- Lemma 9 ----------------
    sec("LEMMA 9 -- the regression target.  At gamma=0 it is deterministic given\n"
        "           (x,c) AND independent of t.  For gamma>0 its conditional\n"
        "           variance is gamma^2 (1-2t)^2 / (4 t(1-t)), whose integral\n"
        "           over t ~ U[0,1] DIVERGES (logarithmically at both ends).")
    d = 64
    x = rng.normal(size=(4000, d)) / np.sqrt(d)
    c = rng.normal(size=(4000, d)) / np.sqrt(d)
    v0 = np.stack([velocity_gt(x, c, t, 0.0) for t in [0.01, .2, .5, .8, .99]])
    sp = np.abs(v0 - v0[0]).max()
    check("gamma=0: v_gt exactly t-independent", sp == 0.0, f"max|v(t)-v(0)|={sp:.1e}")

    out(f"\n   gamma=0.5, per-coordinate conditional variance:")
    out(f"   {'t':>8} {'empirical':>14} {'predicted':>14}")
```

```
    ok = True
    for t in [0.001, 0.01, 0.1, 0.5, 0.9, 0.99, 0.999]:
        g = 0.5
        e = rng.normal(size=(200000, 1))
        emp = (g * 0.5 * (1 - 2 * t) / np.sqrt(t * (1 - t)) * e).var()
        pred = g ** 2 * (1 - 2 * t) ** 2 / (4 * t * (1 - t))
        ok &= abs(emp - pred) <= 0.03 * max(pred, 1e-9) + 1e-9
        out(f"   {t:8.3f} {emp:14.4f} {pred:14.4f}")
    check("gamma>0: Var = gamma^2 (1-2t)^2/(4t(1-t))", ok)

    # EXACT: (1-2t)^2 = 1 - 4t(1-t), so the integrand is 1/(4t(1-t)) - 1 and
    #   INT_eps^{1-eps} = 0.5*log((1-eps)/eps) - 1 + 2 eps.
    out("\n   E_t[Var] on [eps, 1-eps], gamma=1.  Exact closed form:")
    out("       0.5*log((1-eps)/eps) - 1 + 2 eps   ->   diverges like -0.5 log eps")
    out(f"   {'eps':>8} {'numeric':>12} {'exact':>12} {'abs diff':>11}")
    prev, mono, okI = None, True, True
    for e in [1e-2, 1e-4, 1e-6, 1e-10]:
        # logistic substitution t = sigmoid(u), dt = t(1-t) du, so the integrand
        # becomes 1/4 - t(1-t): smooth and bounded, unlike the raw 1/t spike.
        U = np.log((1 - e) / e)
        u = np.linspace(-U, U, 400_001)
        tt = 1.0 / (1.0 + np.exp(-u))
        I = np.trapezoid(0.25 - tt * (1 - tt), u)
        ex = 0.5 * np.log((1 - e) / e) - 1 + 2 * e
        out(f"   {e:8.0e} {I:12.5f} {ex:12.5f} {abs(I-ex):11.2e}")
        okI &= abs(I - ex) < 1e-6
        if prev is not None and I <= prev:
            mono = False
        prev = I
    check("gamma>0: integral matches the exact closed form", okI)

    # --- paper Eq.(4) vs train.py line 67: DIFFERENT noise schedules ---------
    out("\n   IMPLEMENTATION vs PAPER.  These are not the same schedule:")
    out("     paper Eq.(4)     x_hat = x_t + lam*t*(1-t)*eps")
    out("                      -> target adds lam*(1-2t)*eps")
    out("     train.py line 67 x_hat = x_t + g*sqrt(t(1-t))*eps")
    out("                      -> target adds g*0.5*(1-2t)/sqrt(t(1-t))*eps")
    out("   Each is internally consistent (target = d/dt of its own schedule),")
    out("   but only the CODE schedule has divergent target variance.")
    out(f"\n   {'eps':>8} {'paper INT':>12} {'code INT':>12}")
    Ip = Ic = None
    for e in [1e-3, 1e-6, 1e-9]:
        U = np.log((1 - e) / e)
        u = np.linspace(-U, U, 400_001)
        tt = 1.0 / (1.0 + np.exp(-u))
        jac = tt * (1 - tt)
        Ip = np.trapezoid((1 - 2 * tt) ** 2 * jac, u)
        Ic = np.trapezoid((1 - 2 * tt) ** 2 / (4 * tt * (1 - tt)) * jac, u)
        out(f"   {e:8.0e} {Ip:12.4f} {Ic:12.4f}")
    check("paper schedule lam*t(1-t) has FINITE target variance (-> lam^2/3)",
          abs(Ip - 1 / 3) < 1e-3, f"integral -> {Ip:.4f}, exact 1/3 = 0.3333")
    check("code schedule g*sqrt(t(1-t)) has DIVERGENT target variance "
          "=> explains cfg.gamma=0 being the default",
          Ic > 9.0, f"integral = {Ic:.4f} at eps=1e-9 and still growing")
    out("   NOTE: Lemma 9 is therefore a critique of the IMPLEMENTATION,\n"
        "   not of the paper's Eq.(4).  Which schedule produced the reported\n"
        "   gamma>0 ablations is worth checking with the authors.")
    check("gamma>0: t-averaged target variance is non-integrable (log divergence)", mono)

    # ---------------- Lemma 11 ----------------
    sec("LEMMA 11 -- at gamma=0 the N training paths are pairwise disjoint, so the\n"
        "           training loss is drivable to 0 and the ERM is ill-posed: the\n"
        "           minimiser is unconstrained off a measure-zero set.  Any\n"
        "           generalisation comes from the network's smoothness prior.")
    cA, muA, rA = build(d=512, K=10, seed=1)
    drawA, sigA = sampler(muA, 2.2, 512, 10)
    XA, labA = drawA(16, rA)
    mins = []
    for t in np.linspace(0.0, 0.99, 34):
        Z = (1 - t) * XA + t * cA[labA]
        G = ((Z[:, None, :] - Z[None, :, :]) ** 2).sum(-1)
        np.fill_diagonal(G, np.inf)
        mins.append(np.sqrt(G.min()))
    check("paths never collide (min separation > 0 for every t)", min(mins) > 1e-6,
          f"min_t min_ij ||z_i-z_j|| = {min(mins):.4f}")
    check("path set has Lebesgue measure zero in R^d x [0,1]", True,
          f"N={len(labA)} paths, d=512")

    # ---------------- Lemma 1, algebraic core ----------------
    sec("LEMMA 1 (step 1) -- algebraic core.  The anchor c_k determines its own\n"
        "           preimage x_k(z,t)=(z - t c_k)/(1-t), and then\n"
```

```
        "                c_k - x_k(z,t)  =  (c_k - z)/(1-t).")
    zq = rng.normal(size=(500, 512)) / np.sqrt(512)
    worst = 0.0
    for t in [0.0, .1, .3, .5, .7, .9, .99]:
        lhs = cA[None, :, :] - preimage(zq, cA, t)
        rhs = (cA[None, :, :] - zq[:, None, :]) / (1 - t)
        worst = max(worst, np.abs(lhs - rhs).max())
    check("c_k - x_k(z,t) == (c_k - z)/(1-t)", worst < 1e-9, f"max abs err={worst:.2e}")

    # ---------------- Corollaries 3 and 8 ----------------
    sec("COROLLARY 3 + COROLLARY 8 -- since MSE makes the minimiser a conditional\n"
        "           expectation, the N-point problem COLLAPSES onto the K anchors:\n"
        "                v*(z,t) = ( sum_k w_k(z,t) c_k - z ) / (1-t)\n"
        "           Independent check: brute-force kernel Monte-Carlo estimate of\n"
        "           E[C - X | Z_t = z] in low dimension (uses only the DEFINITION\n"
        "           of conditional expectation, not the formula).")
    dd, KK, sig = 3, 4, 0.40
    r2 = np.random.default_rng(7)
    cL = r2.normal(size=(KK, dd)); cL /= np.linalg.norm(cL, axis=1, keepdims=True)
    muL = cL.copy()
    M = 1_500_000 if quick else 6_000_000
    kk = r2.integers(0, KK, M)
    xx = muL[kk] + sig * r2.normal(size=(M, dd))
    vv = cL[kk] - xx
    out(f"   M = {M:,} Monte-Carlo samples, d={dd}, K={KK}")
    out(f"   {'t':>6} {'bw/(sigma(1-t))':>16} {'rel.err':>10} {'cos':>9}"
        "   (bias should be O(bw^2))")
    ok1, ent = True, {}
    for t in [0.15, 0.4, 0.75]:
        Zt = (1 - t) * xx + t * cL[kk]
        qk = r2.integers(0, KK, 8)
        # queries MUST come from the Z_t marginal: it contracts to width
        # sigma(1-t) around t*c_k, and it is the region the FM loss weights
        Q = (1 - t) * (muL[qk] + sig * r2.normal(size=(8, dd))) + t * cL[qk]
        va, _, W = velocity_star_gauss(Q, cL, t, muL, sig)
        ent[t] = float((-(W * np.log(W + 1e-30)).sum(1)).mean())
        errs = []
        for frac in [0.30, 0.20, 0.10]:
            bw = frac * sig * (1 - t)
            d2 = ((Q[:, None, :] - Zt[None, :, :]) ** 2).sum(-1)
            wg = np.exp(-d2 / (2 * bw ** 2))
            vm = (wg @ vv) / wg.sum(1, keepdims=True)
            rel = np.linalg.norm(vm - va) / np.linalg.norm(va)
            cs = float((np.sum(vm * va, 1) / (np.linalg.norm(vm, axis=1) *
                                              np.linalg.norm(va, axis=1))).mean())
            errs.append(rel)
            out(f"   {t:6.2f} {frac:16.2f} {rel:10.4f} {cs:9.5f}")
        ok1 &= (errs[-1] < 0.06 and cs > 0.99 and errs[-1] < errs[0])
    check("kernel Monte-Carlo E[V|Z_t=z] matches the analytic formula; "
          "error -> 0 as bandwidth -> 0", ok1)

    out("\n   SELF-SHARPENING: inverse temperature is 1/(2 sigma^2 (1-t)^2) -> diverges")
    for t in sorted(ent):
        out(f"   t={t:4.2f}   H[w] = {ent[t]:.4f} nats   (max = log {KK} = {np.log(KK):.4f})")
    ks = sorted(ent)
    check("posterior entropy decreases with t", all(ent[a] > ent[b] for a, b in zip(ks, ks
     ↪ [1:])))

    # ---------------- Proposition 4 (rank confinement) ----------------
    sec("PROPOSITION 4 -- rank structure.  (1-t) v*(z,t) + z is a CONVEX\n"
        "           combination of the K anchors, hence confined to a (K-1)=9\n"
        "           dimensional affine set inside R^512, whatever d is.\n"
        "           The velocity network is therefore massively over-parameterised.")
    cB, muB, rB = build(d=512, K=10, seed=3)
    drawB, sigB = sampler(muB, 2.2, 512, 10)
    zb = muB[rB.integers(0, 10, 3000)] + sigB * rB.normal(size=(3000, 512))
    for t in [0.2, 0.6]:
        v, chat, W = velocity_star_gauss(zb, cB, t, muB, sigB)
        err = np.abs((1 - t) * v + zb - chat).max()
        sv = np.linalg.svd(chat - chat.mean(0, keepdims=True), compute_uv=False)
        rank = int((sv > 1e-8 * sv[0]).sum())
        check(f"t={t}: (1-t)v+z == chat", err < 1e-9, f"err={err:.1e}")
        check(f"t={t}: affine rank == K-1 == 9", rank == 9, f"rank={rank} in d=512")
        check(f"t={t}: weights lie on the probability simplex",
              W.min() >= -1e-12 and np.allclose(W.sum(1), 1))

    return summary("t1_bayes")


if __name__ == "__main__":
    import sys
```

```
    main(quick="--quick" in sys.argv)
```

I.5 T2_FLOWMAP.PY, THEOREMS 5, 12; COROLLARY 6

```
"""
t2_flowmap.py -- NumPy only.
  Theorem 5a  exact CONTINUOUS flow map      z(t) = (1-t)x + t cbar(t)
  Theorem 5b  exact DISCRETE flow map (= the Euler loop in test.py)
  Cor 6       exact logit decomposition
  Theorem 12  closed-form optimal LINEAR velocity field
Both parts of Theorem 5 are MODEL-FREE: they hold for ANY simplex-valued weight
function, so they are verified with a deliberately arbitrary one as well as
with the Bayes weights.
"""
import numpy as np
from fma_theory import (velocity_star_gauss, velocity_linear_closed_form,
                        velocity_linear_general)
from report import sec, check, out, summary
from synth import build, sampler


def main(quick=False):
    rng = np.random.default_rng(11)
    d, K = 128, 8
    c, mu, r = build(d=d, K=K, seed=5)
    draw, sig = sampler(mu, 2.2, d, K)
    X, lab = draw(16, r)

    W1 = rng.normal(size=(d, 32)) / np.sqrt(d)
    W2 = rng.normal(size=(32, K)) / np.sqrt(32)

    def weird_w(z, t):
        lg = np.tanh(z @ W1 + t) @ W2 * (1 + 3 * t)
        lg -= lg.max(1, keepdims=True)
        e = np.exp(lg)
        return e / e.sum(1, keepdims=True)

    f_weird = lambda z, t: ((weird_w(z, t) @ c) - z) / (1 - t)
    f_bayes = lambda z, t: velocity_star_gauss(z, c, t, mu, sig)[0]
    hull = np.vstack([c.T, np.ones((1, K))])

    def simplex_coords(P):
        return np.linalg.lstsq(hull, np.vstack([P.T, np.ones((1, len(P)))]),
                               rcond=None)[0].T

    x0 = X[rng.integers(0, len(X), 64)]

    # ---------------- Theorem 5, continuous ----------------
    sec("THEOREM 5a -- exact CONTINUOUS flow map.  For ANY simplex-valued w,\n"
        "        z(t) = (1-t) x + t cbar(t),  cbar(t) = (1-t)/t INT_0^t chat(s)/(1-s)^2 ds\n"
        "        and cbar(t) is itself a convex combination of the anchors.\n"
        "        (The integrating factor is 1/(1-t); note INT_0^t ds/(1-s)^2 = t/(1-t),\n"
        "         so the weight density integrates to exactly 1.)")
    T, M = 0.8, (8000 if quick else 40000)
    for f, nm in [(f_weird, "arbitrary w"), (f_bayes, "Bayes w")]:
        hh = T / M
        z, t, integ = x0.copy(), 0.0, np.zeros_like(x0)
        for _ in range(M):
            k1 = f(z, t)
            k2 = f(z + hh / 2 * k1, t + hh / 2)
            k3 = f(z + hh / 2 * k2, t + hh / 2)
            k4 = f(z + hh * k3, t + hh)
            ca = z + (1 - t) * k1
            zn, tn = z + hh / 6 * (k1 + 2 * k2 + 2 * k3 + k4), t + hh
            cb = zn + (1 - tn) * f(zn, tn)
            integ += hh / 2 * (ca / (1 - t) ** 2 + cb / (1 - tn) ** 2)
            z, t = zn, tn
        cbar = (1 - T) / T * integ
        rel = np.linalg.norm((1 - T) * x0 + T * cbar - z) / np.linalg.norm(z)
        co = simplex_coords(cbar)
        check(f"[{nm}] z(T) == (1-T)x + T cbar(T)", rel < 5e-4, f"rel err={rel:.2e}")
        check(f"[{nm}] cbar(T) in conv(anchors)", co.min() > -1e-6,
              f"min coord={co.min():.2e}")

    # ---------------- Theorem 5, discrete ----------------
    sec("THEOREM 5b -- exact DISCRETE flow map.  The Euler loop in test.py,\n"
        "        z_{m+1} = z_m + h v(z_m, t_m),  t_m = m h,  satisfies EXACTLY\n"
        "            z_m = (1-t_m) x + t_m a_m ,   a_m in conv(anchors),\n"
```

```
        "            a_{m+1} = [ (1-t_{m+1}) t_m a_m + h chat_m ] / [ t_{m+1}(1-t_m) ].\n"
        "        There is NO discretisation error in this structural claim.")
    h = 0.1
    lams = []
    for f, nm in [(f_weird, "arbitrary w"), (f_bayes, "Bayes w")]:
        z, t, a, worst, lams = x0.copy(), 0.0, np.zeros_like(x0), 0.0, []
        for m in range(10):
            chat = z + (1 - t) * f(z, t)
            lam = h / (1 - t); lams.append(lam)
            z = (1 - lam) * z + lam * chat          # identical to z + h*v(z,t)
            tn = t + h
            a = ((1 - tn) * t * a + h * chat) / (tn * (1 - t))
            worst = max(worst, np.abs(z - ((1 - tn) * x0 + tn * a)).max())
            t = tn
        check(f"[{nm}] z_m == (1-t_m)x + t_m a_m for m=1..10", worst < 1e-10,
              f"max abs err={worst:.2e}")
        co = simplex_coords(a)
        check(f"[{nm}] a_m stays in conv(anchors)", co.min() > -1e-6,
              f"min coord={co.min():.2e}")
    out("\n   step multipliers lambda_m = h/(1-mh), h=0.1:")
    out("   " + "  ".join(f"{l:.3f}" for l in lams))
    check("lambda_m <= 1 (convex step) exactly for m <= 1/h - 1 = 9",
          max(lams[:9]) <= 1 + 1e-12)
    check("lambda_9 == 1  =>  z_10 = chat_9: total collapse onto the anchor hull; "
          "t=1 is singular", abs(lams[9] - 1.0) < 1e-12)

    # ---------------- Corollary 6 ----------------
    sec("COROLLARY 6 -- exact logit decomposition.  test.py scores classes by\n"
        "        <z/||z||, c_k>; ||c_k||=1 so the argmax equals argmax_k <z_m,c_k>:\n"
        "            <z_m,c_k> = (1-t_m) ZEROSHOT_k + t_m (G a_m)_k ,  G = C C^T.\n"
        "        FMA is EXACTLY a convex blend of the zero-shot logits and a\n"
        "        Gram-smoothed anchor-posterior classifier.")
    G = c @ c.T
    zs = x0 @ c.T
    z, t, a, worst = x0.copy(), 0.0, np.zeros_like(x0), 0.0
    for m in range(9):
        chat = z + (1 - t) * f_bayes(z, t)
        lam = h / (1 - t)
        z = (1 - lam) * z + lam * chat
        tn = t + h
        a = ((1 - tn) * t * a + h * chat) / (tn * (1 - t))
        t = tn
        Wb = simplex_coords(a)
        worst = max(worst, np.abs(z @ c.T - ((1 - t) * zs + t * (Wb @ G))).max())
    check("<z_m,c_k> == (1-t_m) zeroshot_k + t_m (G wbar_m)_k", worst < 1e-8,
          f"max abs err={worst:.2e}")

    # ---------------- Theorem 12 ----------------
    sec("THEOREM 12 -- closed-form optimal LINEAR velocity field.  With\n"
        "        X = C + E, E independent of C, E[E]=0:\n"
        "            v_lin(z,t) = -(1-t) S_E ( S_C + (1-t)^2 S_E )^{-1} (z - mu).\n"
        "        A pure shrinkage denoiser: it removes the intra-class component.")
    dl, Kl, sg = 48, 12, 0.5
    r3 = np.random.default_rng(21)
    cl = r3.normal(size=(Kl, dl)); cl /= np.linalg.norm(cl, axis=1, keepdims=True)
    S_C = (cl - cl.mean(0)).T @ (cl - cl.mean(0)) / Kl
    S_E = sg ** 2 * np.eye(dl)
    mA = cl.mean(0)
    n = 200_000 if quick else 600_000
    kk = r3.integers(0, Kl, n)
    Cs = cl[kk]; Xs = Cs + sg * r3.normal(size=(n, dl))
    out(f"   n = {n:,} samples, d={dl}, K={Kl}")
    out(f"   {'t':>6} {'rel err in A':>16} {'rel err in b':>16}")
    ok6 = True
    for t in [0.0, .25, .5, .75, .9]:
        Zt = (1 - t) * Xs + t * Cs; V = Cs - Xs
        Zc, Vc = Zt - Zt.mean(0), V - V.mean(0)
        Ae = np.linalg.solve(Zc.T @ Zc, Zc.T @ Vc).T
        be = V.mean(0) - Ae @ Zt.mean(0)
        _, At = velocity_linear_closed_form(np.zeros((1, dl)), t, mA, S_C, S_E)
        bt = -At @ mA
        eA = np.linalg.norm(Ae - At) / np.linalg.norm(At)
        eb = np.linalg.norm(be - bt) / max(np.linalg.norm(bt), 1e-12)
        out(f"   {t:6.2f} {eA:16.5f} {eb:16.5f}")
        ok6 &= eA < 0.03
    check("empirical least squares == closed form", ok6)

    cg, mug, rg = build(d=48, K=12, gap=0.8, seed=9)
    drawg, sg2 = sampler(mug, 2.5, 48, 12)
    Xg, lg2 = drawg(3000, rg)
    Cg = cg[lg2]
```

```
    mX, mC = Xg.mean(0), Cg.mean(0)
    SX, SC = np.cov(Xg.T, bias=True), np.cov(Cg.T, bias=True)
    SXC = ((Xg - mX).T @ (Cg - mC)) / len(Xg)
    ok6b = True
    for t in [0.1, 0.5, 0.85]:
        At, bt = velocity_linear_general(t, mX, mC, SX, SC, SXC)
        Zt = (1 - t) * Xg + t * Cg; V = Cg - Xg
        Zc, Vc = Zt - Zt.mean(0), V - V.mean(0)
        Ae = np.linalg.solve(Zc.T @ Zc, Zc.T @ Vc).T
        ok6b &= np.linalg.norm(Ae - At) / np.linalg.norm(At) < 1e-6
    check("general form (mu_k != c_k, modality gap present) matches LSQ", ok6b)

    return summary("t2_flowmap")


if __name__ == "__main__":
    import sys
    main(quick="--quick" in sys.argv)
```

I.6 T9_MMSE.PY: THEOREM 7(II), THE $(d-K)$ DIVISOR OF EQUATION 14, AND THE CLOSED FORM EQUATION 15

```
"""
verification/t9_mmse.py -- formal verification of the three new claims.

  M1  the decoder is the MMSE estimator, not merely a two-limit family:
      with chat = C^T q* + eps, eps ~ N(0, s^2 I_d), the anchor-coordinate
      observation y = C chat has Cov(C eps) = s^2 G -- the SAME operator as the
      forward map -- and hence
          E[q* | y] = m + (G + lam I)^{-1}(y - G m),   lam = s^2 / tau^2.
      This is special to eps isotropic in R^d: isotropic noise in R^K would give
      the Wiener filter G(G^2 + lam I)^{-1}, a different operator.

  M2  the plug-in noise estimator needs divisor (d - K), not d.
      The least-squares residual is P_perp eps, so E||P_perp eps||^2 = s^2(d-K).

  M3  the two-score LDA weight: with a = d_a e_kappa + n_a, b = d_b e_kappa + n_b and
      Corr(n_a, n_b) = rho, Fisher's rule gives
          beta* = (d_b - rho d_a) / [(1 - rho)(d_a + d_b)],
      which reduces to inverse-variance weighting at rho = 0 and to 0 when
      d_b <= rho d_a.

Run:  python t9_mmse.py            (numpy only, ~30 s)
"""
import numpy as np
from report import sec, check, out, summary


def main(quick=False):
    rng = np.random.default_rng(0)

    # ------------------------------------------------------------------ M1
    sec("M1 -- the ridge decoder IS the MMSE estimator of q*.\n"
        "        Cov(C eps) = C (s^2 I_d) C^T = s^2 G, so\n"
        "        Cov(y) = tau^2 G^2 + s^2 G = G(tau^2 G + s^2 I) and\n"
        "        E[q*|y] = m + tau^2 G [G(tau^2 G+s^2 I)]^-1 (y - G m)\n"
        "                = m + (G + (s^2/tau^2) I)^-1 (y - G m).")
    # (a) EXACT operator identity -- no Monte Carlo:
    #     tau^2 G (tau^2 G^2 + s^2 G)^-1  ==  (G + (s^2/tau^2) I)^-1
    ok0 = True
    out("   exact operator identity  tau^2 G (tau^2 G^2 + s^2 G)^-1 "
        "== (G + lam I)^-1")
    out(f"   {'K':>5}{'d':>6}{'cond G':>12}{'max abs diff':>15}")
    for K, d in [(10, 512), (40, 256), (100, 512), (200, 512)]:
        C = rng.normal(size=(K, d)); C /= np.linalg.norm(C, axis=1, keepdims=True)
        G = C @ C.T
        for s2, tau2 in [(0.03, 1.0 / K), (0.5, 0.01)]:
            lhs = tau2 * G @ np.linalg.inv(tau2 * G @ G + s2 * G)
            rhs = np.linalg.inv(G + (s2 / tau2) * np.eye(K))
            e = float(np.abs(lhs - rhs).max())
            ok0 &= e < 1e-8
        out(f"   {K:>5}{d:>6}{np.linalg.cond(G):12.3g}{e:15.2e}")
    check("MMSE operator equals the ridge operator exactly (machine precision)",
          ok0)

    # (b) Monte-Carlo corroboration against the empirical best linear predictor.
    #     The empirical predictor estimates K^2 coefficients, so its own noise
    #     floor grows with K -- we require high correlation, not exactness.
```

```
    ok1 = True
    out(f"\n   {'K':>5}{'d':>6}{'s^2':>8}{'tau^2':>8}"
        f"{'max|ridge-MMSE|':>18}{'corr':>10}")
    for K, d in [(10, 512), (40, 256), (100, 512)]:
        C = rng.normal(size=(K, d)); C /= np.linalg.norm(C, axis=1, keepdims=True)
        G = C @ C.T
        s2, tau2 = 0.03, 1.0 / K
        lam = s2 / tau2
        m = np.full(K, 1.0 / K)
        n = 120_000 if quick else 300_000
        q = m + np.sqrt(tau2) * rng.normal(size=(n, K))
        chat = q @ C + np.sqrt(s2) * rng.normal(size=(n, d))
        y = chat @ C.T
        ridge = m + (y - m @ G) @ np.linalg.inv(G + lam * np.eye(K))
        # empirical MMSE = best linear predictor of q from y
        Yc, Qc = y - y.mean(0), q - q.mean(0)
        A = np.linalg.solve(Yc.T @ Yc, Yc.T @ Qc)
        emp = q.mean(0) + (y - y.mean(0)) @ A
        mx = float(np.abs(ridge - emp).max())
        cc = float(np.corrcoef(ridge.ravel(), emp.ravel())[0, 1])
        out(f"   {K:>5}{d:>6}{s2:8.3f}{tau2:8.4f}{mx:18.2e}{cc:10.6f}")
        ok1 &= cc > 0.995
    check("Monte-Carlo: ridge with lam=s^2/tau^2 tracks the empirical best "
          "linear predictor (corr>0.995)", ok1)

    # the coincidence is specific to isotropic noise in R^d
    K, d = 40, 256
    C = rng.normal(size=(K, d)); C /= np.linalg.norm(C, axis=1, keepdims=True)
    G = C @ C.T
    lam = 0.03 * K
    R = np.linalg.inv(G + lam * np.eye(K))
    W = G @ np.linalg.inv(G @ G + lam * np.eye(K))
    diff = float(np.abs(R - W).max())
    out(f"\n   Wiener filter G(G^2+lam I)^-1 vs our (G+lam I)^-1: "
        f"max|diff| = {diff:.4f}")
    check("the two differ, so the MMSE coincidence is specific to noise "
          "covariance s^2 G", diff > 1e-3)

    # ------------------------------------------------------------------ M2
    sec("M2 -- plug-in noise estimator: the least-squares residual is\n"
        "        P_perp eps with P_perp of rank (d-K), so dividing ||resid||^2\n"
        "        by d underestimates s^2 by (d-K)/d.")
    ok2 = True
    out(f"   {'K':>5}{'true s^2':>10}{'/d':>10}{'/(d-K)':>10}{'d/(d-K)':>10}")
    d = 512
    for K in [10, 47, 100, 196, 397]:
        C = rng.normal(size=(K, d)); C /= np.linalg.norm(C, axis=1, keepdims=True)
        s2 = 0.02
        q = np.zeros((4000, K)); q[np.arange(4000), rng.integers(0, K, 4000)] = 1
        chat = q @ C + np.sqrt(s2) * rng.normal(size=(4000, d))
        w = np.linalg.lstsq(C.T, chat.T, rcond=None)[0].T
        rr = ((chat - w @ C) ** 2).sum(1).mean()
        bad, good = rr / d, rr / (d - K)
        out(f"   {K:>5}{s2:10.4f}{bad:10.5f}{good:10.5f}{d/(d-K):10.3f}")
        ok2 &= abs(good - s2) < 0.05 * s2 and bad < good
    check("divisor (d-K) recovers s^2; divisor d is biased low, worst at large K",
          ok2)

    # ------------------------------------------------------------------ M3
    sec("M3 -- two-score LDA weight.  Sigma = [[1,rho],[rho,1]], mu = (d_a,d_b);\n"
        "        Fisher w prop to Sigma^-1 mu prop to (d_a - rho d_b, d_b - rho d_a), so\n"
        "        beta* = (d_b - rho d_a)/[(1-rho)(d_a+d_b)].")

    def beta_star(da, db, rho):
        den = (1 - rho) * (da + db)
        return float(np.clip((db - rho * da) / den, 0, 1)) if abs(den) > 1e-12 else 0.0

    ok3 = True
    out(f"   {'d_a':>6}{'d_b':>6}{'rho':>6}{'beta* formula':>15}"
        f"{'beta* grid':>12}{'|diff|':>9}")
    for da, db, rho in [(1.0, 1.0, 0.0), (1.0, 2.0, 0.0), (1.0, 2.0, 0.5),
                        (2.0, 1.0, 0.6), (1.0, 0.3, 0.5), (1.5, 0.5, 0.2)]:
        # brute-force: maximise the LDA discriminant ratio over beta
        S = np.array([[1.0, rho], [rho, 1.0]])
        best, bb = -1, None
        for b in np.linspace(0, 1, 20001):
            w = np.array([1 - b, b])
            snr = (w @ np.array([da, db])) / np.sqrt(w @ S @ w)
            if snr > best:
                best, bb = snr, b
        f = beta_star(da, db, rho)
```

```
        out(f"   {da:6.1f}{db:6.1f}{rho:6.2f}{f:15.4f}{bb:12.4f}"
            f"{abs(f-bb):9.4f}")
        ok3 &= abs(f - bb) < 2e-3
    check("closed-form beta* maximises the LDA discriminant", ok3)
    check("rho=0 reduces to inverse-variance weighting d_b/(d_a+d_b)",
          abs(beta_star(1.0, 3.0, 0.0) - 0.75) < 1e-12)
    check("d_b <= rho*d_a gives beta*=0 (the flat-gain case)",
          beta_star(2.0, 0.9, 0.5) == 0.0,
          f"d_a=2, d_b=0.9, rho=0.5 -> {beta_star(2.0,0.9,0.5):.3f}")

    return summary("t9_mmse")


if __name__ == "__main__":
    import sys
    main(quick="--quick" in sys.argv)
```

I.7 ANCHOR_RIDGE.PY, THEOREM 7 AND THE DECODER

```
"""
analytic/anchor_ridge.py -- a method that contains BOTH observed regimes.

DERIVATION
----------
Corollary 3 / Proposition 4 (model-free): chat(z,t) := (1-t) v(z,t) + z is an
anchor combination.
Write the network's output as

        chat = C^T w*  +  eps                                  (eps = net error)

Two things we have measured are then the SAME estimator at opposite ends of a
conditioning spectrum, with G = C C^T:

  READOUT       w_hat = G^-1 C chat = w* + G^-1 C eps        (applies G^-1)
  INTEGRATION   score = C chat      = G w* + C eps            (applies G, Cor. 6)

The readout is (near) unbiased but its variance scales with ||G^-1||; scoring
through G is stable but smears w* by G.  Measured across 5 datasets,
corr(log10 cond(G), readout advantage) = -0.52: the readout wins exactly when G
is well conditioned (EuroSAT K=10, DTD K=47) and loses badly when it is not
(Aircraft K=100, Cars K=196, SUN397 K=397).  Note cond(G) ~ (1+(K-1)rho)/(1-rho)
for equicorrelated anchors, so this is driven by BOTH K and the anchor cosine.

THE ESTIMATOR
-------------
        w_hat(lam) = (G + lam I)^-1 C chat

        lam -> 0    =>  G^-1 C chat = w*        exactly the READOUT
        lam -> inf  =>  prop to C chat = G w*   exactly INTEGRATION scoring

One Tikhonov parameter provably interpolates between the two.  In the eigenbasis
of G, component j is scaled by mu_j/(mu_j+lam), so the ill-conditioned
directions -- where anchor correlation has already destroyed the information --
are damped rather than amplified.

Plug-in optimum: with eps ~ N(0, s2 I) and a simplex prior w* ~ N(0, tau2 I),
tau2 ~ 1/K, the Bayes estimator is exactly this with lam* = s2/tau2 ~ K*s2,
which grows with K -- the observed regime switch, predicted rather than tuned.

TWO FURTHER FREE WINS
---------------------
1. TRAJECTORY ENSEMBLE.  chat is available at every step, and the implied
   posterior is nearly t-invariant (measured: 0.8596 -> 0.8479 on EuroSAT).
   Averaging w_hat over the trajectory divides the eps variance by ~M.  This
   USES the flow instead of discarding it -- the ODE becomes an ensemble
   generator rather than an integrator.
2. ZERO-SHOT BLEND.  Corollary 6 says integration returns
   (1-t)*zeroshot + t*(G w).  Keeping that blend but with the regularised
   posterior gives  (1-beta)*zeroshot + beta*w_hat(lam).

FMA is recovered at (lam -> inf, beta = t, single step); the raw readout at
(lam = 0, beta = 1, t = 0).  Both are interior points of the search space, so
the method cannot do worse than either once lam, beta are selected on val.

    python analytic/anchor_ridge.py --dataset SUN397 --seed 1 --blocks 6 \
           --ckpt ./checkpoints/analytic/SUN397_s1_flow.pth
"""
import argparse
```

```python
import numpy as np
from common import (add_common_args, make_cfg, get_features, load_net,
                    net_field, euler_classify, geometry, plug_in_stats)
import logio

LAMS = [0.0, 1e-4, 1e-3, 1e-2, 3e-2, 0.1, 0.3, 1.0, 3.0, 10.0, 30.0, 100.0, 1e4]
BETAS = [0.0, 0.25, 0.5, 0.75, 1.0]


def zscore(S):
    return (S - S.mean(1, keepdims=True)) / np.maximum(S.std(1, keepdims=True), 1e-9)


def main():
    ap = argparse.ArgumentParser()
    add_common_args(ap)
    ap.add_argument("--ckpt", required=True)
    ap.add_argument("--steps", type=int, default=6, help="trajectory length used")
    ap.add_argument("--h", type=float, default=0.1)
    a = ap.parse_args()
    LOG, JS = logio.start('anchor_ridge', a)
    cfg = make_cfg(a)
    Xtr, ytr, Xva, yva, Xte, yte, C, _ = get_features(a, cfg)
    K, d = C.shape
    mu, _, _ = plug_in_stats(Xtr, ytr, K, a.num_shots, d)
    g = geometry(Xtr, ytr, Xte, yte, C, mu)
    net = load_net(a.ckpt, d, a.blocks, cfg.device)
    v = net_field(net, cfg.device)

    G = C @ C.T
    ev, V = np.linalg.eigh(G)
    cond = float(ev.max() / max(ev.min(), 1e-12))
    print(f"\n{'='*80}\n{a.dataset}  K={K} d={d} seed={a.seed}  n_test={len(yte)}")
    print(f"  anchor cosine {g['anchor_cos']:.4f}   zero-shot {g['zeroshot']:.4f}")
    print(f"  G spectrum: max {ev.max():.3f}  min {ev.min():.4f}  cond {cond:.1f}")
    print(f"{'='*80}")

    def chats(X):
        """chat at every step along the trajectory, plus the zero-shot logits."""
        out, z, t = [], X.copy(), 0.0
        for _ in range(a.steps + 1):
            out.append(z + (1 - t) * v(z, t))
            lam_ = a.h / (1 - t)
            z = (1 - lam_) * z + lam_ * out[-1]
            t += a.h
        return out

    def wh(chat, lam):
        """(G + lam I)^-1 C chat, via the eigenbasis (cheap for many lam)."""
        P = C @ chat.T                                  # (K, n)
        return (V @ ((V.T @ P) / (ev[:, None] + lam))).T  # (n, K)

    CH_te, CH_va = chats(Xte), (chats(Xva) if len(yva) else None)
    zs_te = zscore(Xte @ C.T)
    zs_va = zscore(Xva @ C.T) if len(yva) else None

    # plug-in lambda* = s2/tau2. The least-squares residual is P_perp eps and
    # rank(P_perp) = d - K, so ||resid||^2 must be divided by (d - K), NOT by d:
    # dividing by d underestimates s^2 by (d-K)/d, a factor of 4.45 at K=397,
    # i.e. worst exactly where lambda should be largest.
    w0 = wh(CH_te[0], 1e-6)
    resid = CH_te[0] - w0 @ C
    s2 = float((resid ** 2).sum(1).mean() / (d - K))
    s2_naive = float((resid ** 2).sum(1).mean() / d)
    lam_star = s2 * K                               # tau^2 = 1/K
    print(f"  plug-in lam* = K * s2 = {K} * {s2:.5f} = {lam_star:.4f}"
          f"   (divisor d-K={d-K}; dividing by d would give s2={s2_naive:.5f}, "
          f"a factor {s2/max(s2_naive,1e-30):.2f} too small)")

    def score(CH, zsl, lam, beta, ens):
        idx = range(len(CH)) if ens else [0]
        W = np.mean([zscore(wh(CH[i], lam)) for i in idx], axis=0)
        return (1 - beta) * zsl + beta * W

    def acc(S, y):
        return float((S.argmax(1) == y).mean())

    print(f"\n  grid over lam x beta x {{t=0 only, trajectory ensemble}}")
    print(f"  {'lam':>8} | " + " ".join(f"b={b:<4}" for b in BETAS) +
          "  ||  ensemble")
    best = None
```

```
    grid = {}
    for lam in LAMS:
        row_s, row_e = [], []
        for beta in BETAS:
            for ens in (False, True):
                te = acc(score(CH_te, zs_te, lam, beta, ens), yte)
                va = (acc(score(CH_va, zs_va, lam, beta, ens), yva)
                      if len(yva) else te)
                grid[(lam, beta, ens)] = (va, te)
                (row_e if ens else row_s).append(te)
                if best is None or va > best[0]:
                    best = (va, te, lam, beta, ens)
        print(f"  {lam:8.4g} | " + " ".join(f"{x:.4f}" for x in row_s) +
              "  ||  " + " ".join(f"{x:.4f}" for x in row_e))

    # the two limits, for reference
    ref_read = grid[(0.0, 1.0, False)][1]
    ref_int = max(euler_classify(v, Xte, yte, C, s) for s in range(11))
    ref_bigl = grid[(1e4, 1.0, False)][1]

    va, te, lam, beta, ens = best
    print(f"\n{'-'*80}\n  VAL-SELECTED: lam={lam:g}  beta={beta}  "
          f"ensemble={ens}   ->  TEST {te:.4f}   (val {va:.4f})")
    print(f"  test-oracle over the whole grid: "
          f"{max(t for _, t in grid.values()):.4f}  (diagnostic only)")
    print(f"\n  REFERENCE POINTS (both are interior to the grid above)")
    print(f"    pure readout      lam=0,   beta=1, t=0   {ref_read:.4f}")
    print(f"    lam=1e4 limit (~G w scoring)             {ref_bigl:.4f}")
    print(f"    FMA multi-step integration (best step)   {ref_int:.4f}")
    print(f"    zero-shot                                {g['zeroshot']:.4f}")
    print(f"\n  gain over readout    {te-ref_read:+.4f}")
    print(f"  gain over FMA        {te-ref_int:+.4f}")
    print(f"  lam selected {lam:g} vs plug-in prediction {lam_star:.4f}")
    print("=" * 80)

    logio.save(JS, dict(dataset=a.dataset, seed=a.seed, K=K, d=d, ckpt=a.ckpt,
                        anchor_cos=g["anchor_cos"], cond_G=cond,
                        eig_max=float(ev.max()), eig_min=float(ev.min()),
                        s2=s2, lam_star=lam_star,
                        sel=dict(lam=lam, beta=beta, ensemble=bool(ens),
                                 val=va, test=te),
                        oracle=max(t for _, t in grid.values()),
                        ref=dict(readout=ref_read, big_lam=ref_bigl,
                                 fma_integrate=ref_int, zeroshot=g["zeroshot"]),
                        grid={f"{k[0]}|{k[1]}|{int(k[2])}": v
                              for k, v in grid.items()}))
    logio.finish(LOG, JS)


if __name__ == "__main__":
    main()
```

I.8 PROBE.PY, STRUCTURE AND POSTERIOR READ-OUT ON REAL FEATURES (COROLLARIES 3, 6; PROPOSITION 4)

```
"""
analytic/probe.py -- THE MAIN EXPERIMENT.  Needs a trained checkpoint.

Measures, on real CLIP features:
  1. STRUCTURE (Prop. 4 / Cor. 3).  Does chat := (1-t) v_net(z,t) + z live in the
     (K-1)-dimensional anchor span?  Null = span(z) and (K-1)/d.
     Reports affine containment (span) and convex containment (negative mass)
     separately -- they are different claims and the net satisfies only the first.
  2. READOUT.  Accuracy of argmax of the net's own implied anchor posterior at
     each t, versus the accuracy of actually integrating the ODE.
  3. Corollary 6 decomposition:  <z_m,c_k> = (1-t)*zeroshot_k + t*(G wbar)_k.

Run from the REPO ROOT:
    python analytic/probe.py --dataset EuroSAT --num_shots 16 --seed 1 \
           --blocks 6 --ckpt ./checkpoints/exp/<ts>/model.pth
"""
import argparse
import numpy as np
from common import (add_common_args, make_cfg, get_features, load_net,
                    net_field, plug_in_stats, gauss_field, euler_classify,
                    simplex_project, span_basis, span_frac, geometry)
import logio

```

```python

def main():
    ap = argparse.ArgumentParser()
    add_common_args(ap)
    ap.add_argument("--ckpt", required=True)
    ap.add_argument("--ntest", type=int, default=10 ** 9,
                    help="subsample the test set (default: use all of it)")
    ap.add_argument("--steps", type=int, default=10)
    ap.add_argument("--h", type=float, default=0.1)
    a = ap.parse_args()
    LOG, JS = logio.start('probe', a)

    cfg = make_cfg(a)
    Xtr, ytr, Xva, yva, Xte, yte, C, names = get_features(a, cfg)
    K, d = C.shape[0], C.shape[1]
    if a.ntest < len(yte):
        sel = np.random.default_rng(0).choice(len(yte), a.ntest, replace=False)
        Xte, yte = Xte[sel], yte[sel]

    mu, s_hat, s_eff = plug_in_stats(Xtr, ytr, K, a.num_shots, d)
    g = geometry(Xtr, ytr, Xte, yte, C, mu)
    net = load_net(a.ckpt, d, a.blocks, cfg.device)
    vnet = net_field(net, cfg.device)
    npar = sum(p.numel() for p in net.parameters())

    print(f"\n{'='*76}\n{a.dataset}  K={K} d={d} shots={a.num_shots} "
          f"seed={a.seed}  n_test={len(yte)}\n{'='*76}")
    print(f"  anchor cosine        {g['anchor_cos']:.4f}")
    print(f"  modality gap         {g['modality_gap']:.4f}")
    print(f"  in-class spread      {g['in_class']:.4f}   sigma_hat={s_hat:.4f}")
    print(f"  zero-shot            {g['zeroshot']:.4f}")
    print(f"  cosine prototype     {g['cos_proto']:.4f}")
    print(f"  checkpoint           {npar:,} params")

    Q = span_basis(C)
    G = C @ C.T
    zs_logit = Xte @ C.T

    print(f"\n(1) STRUCTURE + (2) READOUT   [isotropic null (K-1)/d = "
          f"{(K-1)/d:.4f}]")
    print(f"   {'t':>5} {'span(chat)':>11} {'span(z)':>9} {'relDist':>9} "
          f"{'negmass':>9} | {'acc(argmax w)':>14} {'acc(integrate)':>15}")
    z, t = Xte.copy(), 0.0
    rows = []
    for m in range(a.steps):
        chat = z + (1 - t) * vnet(z, t)
        W, Wc = simplex_project(chat, C)
        rel = float(np.mean(np.linalg.norm(chat - Wc @ C, axis=1) /
                            np.linalg.norm(chat, axis=1)))
        neg = float(np.mean(np.clip(-W, 0, None).sum(1)))
        accw = float((Wc.argmax(1) == yte).mean())
        zn = z / np.linalg.norm(z, axis=1, keepdims=True)
        acci = float(((zn @ C.T).argmax(1) == yte).mean())
        rows.append((t, span_frac(chat, C, Q), span_frac(z, C, Q), rel, neg,
                     accw, acci))
        print(f"   {t:5.2f} {rows[-1][1]:11.4f} {rows[-1][2]:9.4f} {rel:9.4f} "
              f"{neg:9.4f} | {accw:14.4f} {acci:15.4f}")
        lam = a.h / (1 - t)
        z = (1 - lam) * z + lam * chat
        t += a.h
    zn = z / np.linalg.norm(z, axis=1, keepdims=True)
    acc_fin = float(((zn @ C.T).argmax(1) == yte).mean())
    R = np.array(rows)

    print("\n(3) COROLLARY 6 residual  |<z,c_k> - [(1-t)zs + t(G wbar)]|")
    z2, t2, a2 = Xte.copy(), 0.0, np.zeros_like(Xte)
    worst = 0.0
    for m in range(min(6, a.steps)):
        chat = z2 + (1 - t2) * vnet(z2, t2)
        lam = a.h / (1 - t2)
        z2 = (1 - lam) * z2 + lam * chat
        tn = t2 + a.h
        a2 = ((1 - tn) * t2 * a2 + a.h * chat) / (tn * (1 - t2))
        t2 = tn
        _, Wb = simplex_project(a2, C)
        worst = max(worst, float(np.abs(z2 @ C.T -
                    ((1 - t2) * zs_logit + t2 * (Wb @ G))).max()))
    print(f"   max residual over 6 steps = {worst:.3e}   (exact identity when "
          f"chat is IN the hull; nonzero here because it is not)")

    # closed-form Gaussian head for reference
```

```
    best_plug = max(euler_classify(gauss_field(C, mu, s_eff), Xte, yte, C, s)
                    for s in range(1, a.steps))

    i = int(R[:, 5].argmax())
    print(f"\n{'='*76}\nSUMMARY  ({a.dataset}, seed {a.seed}, n={len(yte)})")
    print(f"  zero-shot                          {g['zeroshot']:.4f}")
    print(f"  cosine prototype (16-shot)         {g['cos_proto']:.4f}")
    print(f"  Gaussian closed-form head          {best_plug:.4f}")
    print(f"  FMA, integrate {a.steps-1} steps           {R[1:,6].max():.4f}"
          f"   (best over steps)")
    print(f"  FMA, transported final             {acc_fin:.4f}")
    print(f"  READOUT argmax w at t={R[i,0]:.2f}           {R[i,5]:.4f}"
          f"   <-- 1 forward pass, no integration")
    print(f"  readout - integrate                {R[i,5]-R[1:,6].max():+.4f}")
    print(f"\n  affine containment  span(chat)={R[0,1]:.4f} vs null "
          f"{R[0,2]:.4f}  -> {R[0,1]/max(R[0,2],1e-9):.0f}x enrichment")
    print(f"  convex containment  negmass={R[0,4]:.4f}  -> chat is in AFF(anchors) "
          f"but OUTSIDE CONV(anchors)")
    print("=" * 76)

    logio.save(JS, dict(
        dataset=a.dataset, seed=a.seed, K=K, d=d, ckpt=a.ckpt, params=npar,
        geometry=g, cor6_residual=worst,
        rows=[dict(t=r[0], span_chat=r[1], span_z=r[2], relDist=r[3],
                   negmass=r[4], acc_readout=r[5], acc_integrate=r[6])
              for r in R.tolist()],
        summary=dict(zeroshot=g['zeroshot'], cos_proto=g['cos_proto'],
                     gauss_head=best_plug, integrate_best=float(R[1:, 6].max()),
                     transported=acc_fin, readout_best=float(R[i, 5]),
                     readout_minus_integrate=float(R[i, 5] - R[1:, 6].max()))))
    logio.finish(LOG, JS)


if __name__ == "__main__":
    main()
```

I.9 STRAIGHTNESS.PY, RECTIFICATION (THEOREM 5) AND THE GRAM PENALTY

```
"""
analytic/straightness.py -- is the learned field a RECTIFIED (straight) flow?

Theorem 5:  z(t) = (1-t) x + t cbar(t),  cbar(t) in conv(anchors).
If cbar is CONSTANT along the trajectory the path is a straight line, the flow
is rectified, and one Euler step with h=1 lands exactly on cbar.  The observed
near-t-invariance of the implied posterior is exactly that signature.

Measured here:
  S1  chat drift        ||chat(z_t,t) - chat(x,0)|| / ||chat(x,0)||
  S2  velocity turning  cos( v(z_t,t), v(x,0) )        (1.0 = perfectly straight)
  S3  rectified-flow straightness
        S = E_t || v(z_t,t) - (z_1 - z_0) ||^2 / E||z_1 - z_0||^2   (0 = straight)
  S4  ONE-STEP JUMP: z = x + 1.0 * v(x,0) = chat(x,0) exactly.
      Accuracy of argmax_k <chat, c_k>  vs  argmax_k w_k  vs  multi-step.

The last one separates two different things:
  - argmax_k <chat,c_k> scores THROUGH the Gram matrix G = C C^T (Cor. 6 at t=1)
  - argmax_k w_k reads the posterior DIRECTLY, bypassing G
If the second is much better, the loss is in the cosine read-out, not the flow.

    python analytic/straightness.py --dataset EuroSAT --seed 1 --blocks 6 \
           --ckpt ./checkpoints/analytic/EuroSAT_s1_t0.pth
"""
import argparse
import numpy as np
from common import (add_common_args, make_cfg, get_features, load_net,
                    net_field, simplex_project, euler_classify, geometry,
                    plug_in_stats)
import logio


def main():
    ap = argparse.ArgumentParser()
    add_common_args(ap)
    ap.add_argument("--ckpt", required=True)
    ap.add_argument("--fine", type=int, default=50, help="fine integration steps")
    ap.add_argument("--ntest", type=int, default=4000)
    a = ap.parse_args()
    LOG, JS = logio.start('straightness', a)
```

```
    cfg = make_cfg(a)
    Xtr, ytr, Xva, yva, Xte, yte, C, _ = get_features(a, cfg)
    K, d = C.shape
    if a.ntest < len(yte):
        s = np.random.default_rng(0).choice(len(yte), a.ntest, replace=False)
        Xte, yte = Xte[s], yte[s]
    mu, _, _ = plug_in_stats(Xtr, ytr, K, a.num_shots, d)
    g = geometry(Xtr, ytr, Xte, yte, C, mu)
    net = load_net(a.ckpt, d, a.blocks, cfg.device)
    v = net_field(net, cfg.device)

    print(f"\n{'='*78}\n{a.dataset} seed={a.seed}  n={len(yte)}  "
          f"anchor cos={g['anchor_cos']:.4f}\n  ckpt {a.ckpt}\n{'='*78}")

    # ---- fine trajectory, recording chat and v ------------------------------
    v0 = v(Xte, 0.0)
    chat0 = Xte + v0
    n0 = np.linalg.norm(v0, axis=1, keepdims=True)
    z, t, h = Xte.copy(), 0.0, 0.9 / a.fine
    rec = []
    for i in range(a.fine + 1):
        vt = v(z, t)
        chat_t = z + (1 - t) * vt
        drift = float(np.mean(np.linalg.norm(chat_t - chat0, axis=1) /
                              np.linalg.norm(chat0, axis=1)))
        turn = float(np.mean(np.sum(vt * v0, 1) /
                             np.maximum(np.linalg.norm(vt, axis=1) * n0[:, 0], 1e-12)))
        if i % max(1, a.fine // 9) == 0:
            rec.append((t, drift, turn))
        z = z + h * vt
        t += h
    z_end = z.copy()

    print(f"\nS1/S2  is the path straight?")
    print(f"   {'t':>6} {'chat drift':>12} {'cos(v_t, v_0)':>15}")
    for tt, dr, tu in rec:
        print(f"   {tt:6.3f} {dr:12.4f} {tu:15.4f}")

    # ---- S3 rectified-flow straightness -------------------------------------
    disp = z_end - Xte
    z, t = Xte.copy(), 0.0
    num = 0.0
    for i in range(a.fine):
        vt = v(z, t)
        num += float(np.mean(np.sum((vt - disp / 0.9) ** 2, 1)))
        z = z + h * vt
        t += h
    S = num / a.fine / float(np.mean(np.sum((disp / 0.9) ** 2, 1)))
    print(f"\nS3  straightness  S = {S:.4f}   (0 = perfectly straight/rectified)")

    # ---- S4 one-step jump vs multi-step vs direct posterior ---------------
    def cos_acc(P):
        Pn = P / np.maximum(np.linalg.norm(P, axis=1, keepdims=True), 1e-12)
        return float(((Pn @ C.T).argmax(1) == yte).mean())

    _, W0 = simplex_project(chat0, C)
    acc_w = float((W0.argmax(1) == yte).mean())
    acc_chat = cos_acc(chat0)
    G = C @ C.T
    acc_Gw = float(((W0 @ G).argmax(1) == yte).mean())
    steps = [euler_classify(v, Xte, yte, C, s) for s in range(11)]

    print(f"\nS4  where does the accuracy go?")
    print(f"   multi-step Euler h=0.1 :  " + " ".join(f"{x:.4f}" for x in steps))
    print(f"   best multi-step        :  {max(steps):.4f} "
          f"@{int(np.argmax(steps))} steps")
    print(f"   ONE STEP h=1 (= chat)  :  {acc_chat:.4f}   "
          f"<- scored through G (Cor.6 at t=1)")
    print(f"   argmax (G w)           :  {acc_Gw:.4f}   <- same thing, explicitly")
    print(f"   argmax w  (bypass G)   :  {acc_w:.4f}   <- posterior read directly")
    print(f"\n   G-penalty = argmax w - argmax(G w) = {acc_w - acc_Gw:+.4f}")
    print(f"   anchor cosine {g['anchor_cos']:.4f}: the higher this is, the more")
    print(f"   G flattens the posterior and the larger the penalty should be.")

    # late-step degradation: the tell that separates the two curvature causes
    peak = int(np.argmax(steps))
    decay = steps[peak] - steps[-1]
    print(f"\n   late-step decay = peak({peak}) - step10 = {decay:+.4f}")

    print(f"\n{'='*78}\nREADING")
    straight = (S < 0.02 and rec[-1][2] > 0.95)
```

```
    one_beats_many = acc_chat > max(steps)
    print(f"  straightness S={S:.4f}   final turning cos={rec[-1][2]:.4f}   "
          f"late-step decay={decay:+.4f}")

    if straight:
        print("\n  STRAIGHT -> the field is RECTIFIED.  Integration is REDUNDANT:")
        print("  one step reaches the same point.  A field trained across all t is")
        print("  forced to be time-consistent, which is what rectification means,")
        print("  and its accuracy is STABLE in the step count.")
    else:
        print("\n  NOT straight.  Two very different causes, distinguished by the")
        print("  late-step decay above:")
        if decay > 0.02:
            print("   -> decay is LARGE: the curvature is EXTRAPOLATION ERROR.  The")
            print("      field is only constrained where it was trained; off that")
            print("      support it wanders, and accuracy collapses at large t.")
            print("      This is NOT the flow doing useful work.  Expect this for a")
            print("      model trained at a single t (e.g. --tmode t0).")
        else:
            print("   -> decay is SMALL: chat genuinely moves AND accuracy holds, so")
            print("      the trajectory may be doing real work.  This is the only")
            print("      case where multi-step integration could be justified --")
            print("      check whether one-step still matches it below.")

    print(f"\n  one-step h=1 ({acc_chat:.4f}) vs best multi-step "
          f"({max(steps):.4f} @{peak}):  {'ONE STEP WINS' if one_beats_many else 'multi-step wins'}")
    print(f"  argmax w ({acc_w:.4f}) bypasses G entirely and is "
          f"{acc_w - max(steps):+.4f} vs best multi-step.")
    print("\n  NOTE: a curved path is not evidence that the flow helps, and a")
    print("  straight one is not evidence that it hurts.  Read S together with")
    print("  the decay and the one-step comparison, not on its own.")
    print("=" * 78)

    logio.save(JS, dict(dataset=a.dataset, seed=a.seed, ckpt=a.ckpt,
                        anchor_cos=g["anchor_cos"], straightness=S,
                        traj=[dict(t=r[0], drift=r[1], turn=r[2]) for r in rec],
                        acc_steps=steps, acc_one_step_chat=acc_chat,
                        acc_Gw=acc_Gw, acc_w=acc_w,
                        G_penalty=acc_w - acc_Gw,
                        peak_step=peak, late_decay=decay,
                        straight=bool(straight),
                        one_step_beats_multi=bool(one_beats_many),
                        final_turn=rec[-1][2], final_drift=rec[-1][1]))
    logio.finish(LOG, JS)


if __name__ == "__main__":
    main()
```

I.10 BASELINES.PY, TRAINING-FREE BASELINES AND THE COROLLARY 8 HEAD

```
"""
analytic/baselines.py -- TRAINING-FREE baselines under an HONEST protocol.

Every hyper-parameter (ridge lambda, logistic C, Tip-Adapter alpha/beta,
Corollary-8 sigma, integration step count) is selected on the repo's own few-shot
VAL split and then evaluated ONCE on test.  For reference the script also
prints the test-oracle number, i.e. what tuning on test would have bought --
that column is diagnostic only and must not be quoted as a result.

The decisive baseline is RIDGE-TO-ANCHOR.  At t=0 the FM target is v = c_y - x,
so x + v(x,0) = c_y: t=0 training IS regression of the anchor onto the feature,
and the linear case is closed form:

    M = (X^T X + lam I)^-1 X^T C_y ,   predict argmax_k < xM , c_k >

    python analytic/baselines.py --dataset EuroSAT --num_shots 16 --seed 1
"""
import argparse
import numpy as np
from common import (add_common_args, make_cfg, get_features, plug_in_stats,
                    geometry, euler_classify, gauss_field, anchor_weights_gauss,
                    select_on_val, oracle_gap)
import logio

RIDGE_LAMS = [1e-4, 1e-3, 1e-2, 1e-1, 1, 10, 100, 1e3, 1e4]
LOGIT_CS = [0.01, 0.1, 1, 10, 100, 1e3, 1e4, 1e5]
```

```python
SIGMA_SCALES = [1, 2, 4, 8, 16, 32, 64]
TIP_A = [0.5, 1, 2, 5, 10]
TIP_B = [1, 5.5, 10]


def ridge_fit(Xtr, ytr, C, lam):
    n, d = Xtr.shape
    Xb = np.hstack([Xtr, np.ones((n, 1))])
    A = Xb.T @ Xb + lam * np.eye(d + 1)
    A[-1, -1] -= lam                        # never penalise the bias
    return np.linalg.solve(A, Xb.T @ C[ytr])


def ridge_pred(M, X, C):
    P = np.hstack([X, np.ones((len(X), 1))]) @ M
    P = P / np.maximum(np.linalg.norm(P, axis=1, keepdims=True), 1e-12)
    return (P @ C.T).argmax(1)


def acc(pred, y):
    return float((pred == y).mean())


def main():
    ap = argparse.ArgumentParser()
    add_common_args(ap)
    a = ap.parse_args()
    LOG, JS = logio.start('baselines', a)
    cfg = make_cfg(a)
    Xtr, ytr, Xva, yva, Xte, yte, C, names = get_features(a, cfg)
    K, d = C.shape
    mu, s_hat, s_eff = plug_in_stats(Xtr, ytr, K, a.num_shots, d)
    g = geometry(Xtr, ytr, Xte, yte, C, mu)
    has_val = len(yva) > 0

    print(f"\n{'='*84}\n{a.dataset}  K={K} d={d} shots={a.num_shots} seed={a.seed}"
          f"   train={len(ytr)} val={len(yva)} test={len(yte)}\n{'='*84}")
    print(f"  anchor cosine {g['anchor_cos']:.4f}   modality gap "
          f"{g['modality_gap']:.4f}   sigma_hat {s_hat:.4f}")
    if not has_val:
        print("  !! NO VAL SPLIT -- falling back to test selection; numbers are "
              "optimistic.")

    res = {}          # name -> (test_acc_val_selected, chosen_hp, oracle_test)

    # ---- no hyper-parameters ------------------------------------------
    Xn = Xte / np.linalg.norm(Xte, axis=1, keepdims=True)
    mun = mu / np.linalg.norm(mu, axis=1, keepdims=True)
    res["zero-shot"] = (g["zeroshot"], "-", g["zeroshot"])
    res["cosine prototype"] = (acc((Xn @ mun.T).argmax(1), yte), "-",
                               acc((Xn @ mun.T).argmax(1), yte))
    ep = ((Xte[:, None, :] - mu[None]) ** 2).sum(-1).argmin(1)
    res["Euclidean prototype"] = (acc(ep, yte), "-", acc(ep, yte))

    # ---- ridge to anchor ----------------------------------------------
    cands = {lam: ridge_fit(Xtr, ytr, C, lam) for lam in RIDGE_LAMS}
    if has_val:
        b, t, v, av, at = select_on_val(
            cands, lambda M: acc(ridge_pred(M, Xva, C), yva),
            lambda M: acc(ridge_pred(M, Xte, C), yte))
    else:
        at = {k: acc(ridge_pred(M, Xte, C), yte) for k, M in cands.items()}
        b = max(at, key=at.get); t = at[b]; av = at
    res["RIDGE-to-anchor"] = (t, f"lam={b:g}", max(at.values()))
    print(f"\n  ridge   val: " + " ".join(f"{k:g}:{av[k]:.3f}" for k in RIDGE_LAMS))
    print(f"          test:" + " ".join(f"{k:g}:{at[k]:.3f}" for k in RIDGE_LAMS))

    # ---- logistic regression ------------------------------------------
    try:
        from sklearn.linear_model import LogisticRegression
        fits = {}
        for c in LOGIT_CS:
            lr = LogisticRegression(C=c, max_iter=5000)
            lr.fit(Xtr, ytr)
            fits[c] = lr
        if has_val:
            b, t, v, av, at = select_on_val(
                fits, lambda m: acc(m.predict(Xva), yva),
                lambda m: acc(m.predict(Xte), yte))
        else:
            at = {k: acc(m.predict(Xte), yte) for k, m in fits.items()}
```

```
            b = max(at, key=at.get); t = at[b]; av = at
        res["logistic regression"] = (t, f"C={b:g}", max(at.values()))
        print(f"\n  logit   val: " + " ".join(f"{k:g}:{av[k]:.3f}" for k in LOGIT_CS))
        print(f"          test:" + " ".join(f"{k:g}:{at[k]:.3f}" for k in LOGIT_CS))
        if b == max(LOGIT_CS):
            print("          !! C selected at the EDGE of the sweep -- widen it")
    except ImportError:
        print("  (sklearn missing -> logistic skipped)")

    # ---- Tip-Adapter ------------------------------------------------------
    Trn = Xtr / np.linalg.norm(Xtr, axis=1, keepdims=True)
    L = np.eye(K)[ytr]

    def tip(X, al, be):
        Q = X / np.linalg.norm(X, axis=1, keepdims=True)
        return ((Q @ C.T) + al * (np.exp(-be * (1 - Q @ Trn.T)) @ L)).argmax(1)
    cands = {(al, be): (al, be) for al in TIP_A for be in TIP_B}
    if has_val:
        b, t, v, av, at = select_on_val(
            cands, lambda p: acc(tip(Xva, *p), yva),
            lambda p: acc(tip(Xte, *p), yte))
    else:
        at = {k: acc(tip(Xte, *k), yte) for k in cands}
        b = max(at, key=at.get); t = at[b]
    res["Tip-Adapter"] = (t, f"a={b[0]},b={b[1]}", max(at.values()))

    # ---- Corollary 8 head: sigma AND step count, both on val -------------
    cands = {(sc, st): (sc, st) for sc in SIGMA_SCALES for st in range(1, 11)}

    def head_acc(X, y, p):
        sc, st = p
        return euler_classify(gauss_field(C, mu, s_eff * sc), X, y, C, st)
    if has_val:
        b, t, v, av, at = select_on_val(
            cands, lambda p: head_acc(Xva, yva, p), lambda p: head_acc(Xte, yte, p))
    else:
        at = {k: head_acc(Xte, yte, k) for k in cands}
        b = max(at, key=at.get); t = at[b]
    res["Cor8 Gaussian head"] = (t, f"sig x{b[0]}, {b[1]} steps", max(at.values()))

    # ---- report -----------------------------------------------------------
    print(f"\n{'-'*84}")
    print(f"  {'method':<26}{'test (val-sel)':>16}{'chosen':>22}"
          f"{'test-oracle':>14}")
    for k, (t, hp, orc) in sorted(res.items(), key=lambda kv: kv[1][0]):
        print(f"  {k:<26}{t:16.4f}{hp:>22}{orc:14.4f}")
    print("-" * 84)
    best = max(res.items(), key=lambda kv: kv[1][0])
    print(f"  BEST honest linear/training-free: {best[0]}  {best[1][0]:.4f}")
    print(f"  (test-oracle column = what tuning on TEST would have given;")
    print(f"   diagnostic only, never quote it)")
    print("=" * 84)

    logio.save(JS, dict(dataset=a.dataset, seed=a.seed, K=K, d=d,
                        n_train=len(ytr), n_val=len(yva), n_test=len(yte),
                        geometry=g, sigma_hat=float(s_hat), has_val=has_val,
                        results={k: dict(test_val_selected=v[0], chosen=v[1],
                                         test_oracle=v[2]) for k, v in res.items()},
                        best=best[0], best_acc=best[1][0]))
    logio.finish(LOG, JS)


if __name__ == "__main__":
    main()
```

I.11 MATCHED_BASELINE.PY: THE SUBSTITUTED-INPUT CONTROLS OF TABLE 4 (LEFT)

```
"""
analytic/matched_baseline.py -- P0: is the gain from the trained field, or from
the FORM of the decoder?

The decoder is  (1-b) * zscore(zs) + b * zscore((G + lam I)^-1 C chat).
Zero-shot alone is ~49 on these datasets and b is tuned, so a reviewer will ask
how much of the margin is the tuned blend plus per-example standardisation
rather than the velocity field. The existing "ridge" column does not answer it:
that is a plain ridge-to-anchor with neither the blend nor the standardisation,
i.e. a different estimator.

```

```
This script runs the IDENTICAL estimator -- same standardisation, same lambda x
beta grid, same validation selection, same seeds -- with chat(x,0) replaced by:

  trained    chat from the trained field                    (the paper's number)
  raw        x itself                                        (no adapter at all)
  proto      C^T q_proto, the cosine-prototype posterior      (transport, no net)
  shuffled   chat from the trained field, decoded against a PERMUTED anchor set

A randomly initialised network is NOT a usable control here: fm.py zero-inits
the output layer (`final_layer.linear.weight/bias`, lines 190-191), so a fresh
network returns exactly 0 and chat = x + 0 = x, i.e. it is bit-identical to the
`raw` row. `shuffled` replaces it: the same trained chat is decoded against a
random permutation of the anchors, which keeps every marginal property of chat
(norm, span occupancy, conditioning of G) and destroys only the correspondence
between chat and the class it should point at. It therefore isolates whether the
field carries anchor-SPECIFIC information rather than merely moving features
somewhere the decoder can exploit.

If raw/proto/shuffled land near the trained row, the margin is the decoder form,
not the field, and the paper's framing has to change.

    python analytic/matched_baseline.py --dataset EuroSAT --seed 1 --blocks 6 \
           --ckpt ./checkpoints/ablation/A__EuroSAT__...__v0.pth
    python analytic/matched_baseline.py --all --seeds 1 2 3     # all five, needs ckpt_dir
"""
import argparse, glob, os
import numpy as np
import torch
from common import (add_common_args, make_cfg, get_features, load_net,
                    net_field, plug_in_stats, euler_curve)
from mdreport import MD

HARD5 = ["FGVCAircraft", "EuroSAT", "DescribableTextures", "SUN397", "StanfordCars"]
SHORT = dict(zip(HARD5, ["Aircraft", "EuroSAT", "DTD", "SUN397", "Cars"]))
LAMS = [0.0, 1e-3, 1e-2, 3e-2, 0.1, 0.3, 1.0, 3.0, 10.0, 100.0, 1e4]
BETAS = [0.0, 0.25, 0.5, 0.75, 1.0]


def z(S):
    return (S - S.mean(1, keepdims=True)) / np.maximum(S.std(1, keepdims=True), 1e-9)


def decode(chat, X, y, Xv, yv, C, ev, V):
    """the paper's decoder, (lam,beta) on val, scored once on test."""
    zs_t, zs_v = z(X @ C.T), (z(Xv @ C.T) if len(yv) else None)
    Pt = C @ chat[0].T
    Pv = C @ chat[1].T if len(yv) else None
    best = None
    for lam in LAMS:
        Wt = z((V @ ((V.T @ Pt) / (ev[:, None] + lam))).T)
        Wv = z((V @ ((V.T @ Pv) / (ev[:, None] + lam))).T) if len(yv) else None
        for b in BETAS:
            te = float((((1 - b) * zs_t + b * Wt).argmax(1) == y).mean())
            va = (float((((1 - b) * zs_v + b * Wv).argmax(1) == yv).mean())
                  if len(yv) else te)
            if best is None or va > best[0]:
                best = (va, te, lam, b)
    return best


def proto_chat(X, mu, C, s_eff):
    m = mu
    d2 = (X ** 2).sum(1)[:, None] - 2 * X @ m.T + (m ** 2).sum(1)[None, :]
    lg = -d2 / (2 * s_eff ** 2)
    lg -= lg.max(1, keepdims=True)
    q = np.exp(lg); q /= q.sum(1, keepdims=True)
    return q @ C


def find_ckpt(d, dataset, seed, blocks):
    for pat in (f"A__{dataset}__clip__ViT-B-16__s16__seed{seed}__b{blocks}__g0.0__flow__v0.pth
     ↪ ",
                f"*{dataset}*seed{seed}*b{blocks}*v0.pth",
                f"*{dataset}*seed{seed}*.pth"):
        h = sorted(glob.glob(os.path.join(d, pat)))
        if h:
            return h[0]
    return None


def one(a, dataset, seed, R):
```

```
    a.dataset, a.seed = dataset, seed
    cfg = make_cfg(a)
    Xtr, ytr, Xva, yva, Xte, yte, C, _ = get_features(a, cfg, verbose=False)
    K, d = C.shape
    mu, _, s_eff = plug_in_stats(Xtr, ytr, K, a.num_shots, d)
    G = C @ C.T
    ev, V = np.linalg.eigh(G)
    ck = a.ckpt or find_ckpt(a.ckpt_dir, dataset, seed, a.blocks)
    if ck is None or not os.path.exists(ck):
        R.p(f"- `{dataset}` seed {seed}: **no checkpoint** "
            f"(looked in `{a.ckpt_dir}`) -- skipped")
        return None
    net = load_net(ck, d, a.blocks, cfg.device)
    v = net_field(net, cfg.device)
    ch_te, ch_va = Xte + v(Xte, 0.0), (Xva + v(Xva, 0.0)) if len(yva) else None

    variants = {
        "trained": (ch_te, ch_va),
        "raw": (Xte.copy(), Xva.copy() if len(yva) else None),
        "proto": (proto_chat(Xte, mu, C, s_eff),
                  proto_chat(Xva, mu, C, s_eff) if len(yva) else None),
    }
    out = {}
    for nm, ch in variants.items():
        va, te, lam, b = decode(ch, Xte, yte, Xva, yva, C, ev, V)
        out[nm] = dict(test=100 * te, val=100 * va, lam=lam, beta=b)

    # shuffled-anchor control: same chat, permuted anchor correspondence.
    # Averaged over several permutations, since a single one is noisy.
    accs = []
    prng = np.random.default_rng(97 + seed)
    for _ in range(a.perms):
        p = prng.permutation(K)
        Cp = C[p]
        Gp = Cp @ Cp.T                      # a permutation similarity: same spectrum
        evp, Vp = np.linalg.eigh(Gp)
        _, te, lam, b = decode((ch_te, ch_va), Xte, yte, Xva, yva, Cp, evp, Vp)
        accs.append(100 * te)
    out["shuffled"] = dict(test=float(np.mean(accs)),
                           val=None, lam=None, beta=None,
                           sd=float(np.std(accs)))
    out["integrate"] = dict(test=100 * max(euler_curve(v, Xte, yte, C, 10)))
    out["zeroshot"] = dict(test=100 * float(
        ((Xte / np.linalg.norm(Xte, axis=1, keepdims=True) @ C.T).argmax(1) == yte).mean()))
    del net
    if str(cfg.device).startswith("cuda"):
        torch.cuda.empty_cache()
    return out


def main():
    ap = argparse.ArgumentParser()
    add_common_args(ap)
    ap.add_argument("--ckpt", default=None)
    ap.add_argument("--ckpt_dir", default="./checkpoints/ablation")
    ap.add_argument("--all", action="store_true", help="all five difficult datasets")
    ap.add_argument("--perms", type=int, default=5,
                    help="permutations averaged for the shuffled-anchor control")
    ap.add_argument("--seeds", type=int, nargs="+", default=None)
    a = ap.parse_args()
    ds = HARD5 if a.all else [a.dataset]
    seeds = a.seeds or [a.seed]

    R = MD("matched_baseline", "Matched-form baseline: field or decoder form?", a)
    R.p("The identical estimator -- same standardisation, same $\\lambda\\times"
        "\\beta$ grid, same validation selection -- with $\\hat c(x,0)$ replaced "
        "by three alternatives.")
    R.p()
    R.p("`shuffled` decodes the *trained* $\\hat c$ against a random permutation "
        "of the anchors, averaged over several permutations. Because a "
        "permutation similarity leaves the spectrum of $\\mathbf{G}$ unchanged, "
        "this preserves every marginal property of $\\hat c$ and the "
        "conditioning of the decoder, destroying only the correspondence "
        "between $\\hat c$ and the class it should indicate.")
    R.p()
    R.p("A randomly initialised network is not a usable control: `fm.py` "
        "zero-initialises the output layer, so a fresh network returns exactly "
        "$0$ and $\\hat c = x$, bit-identical to the `raw` row.")
    res = {}
    for dd in ds:
        for s in seeds:
```

```
            R.h(f"{SHORT.get(dd, dd)}, seed {s}", 3)
            o = one(a, dd, s, R)
            if o:
                res[(dd, s)] = o
                R.table(["variant", "test", "val", "lam", "beta"],
                        [[k, o[k]["test"], o[k].get("val"), o[k].get("lam"),
                          o[k].get("beta")] for k in
                         ["zeroshot", "raw", "shuffled", "proto", "trained",
                          "integrate"]], nd=2)
    if not res:
        R.verdict("No checkpoints found -- nothing computed.")
        R.save(); return

    R.h("Summary, dataset-averaged")
    keys = ["zeroshot", "raw", "shuffled", "proto", "trained", "integrate"]
    rows = []
    for k in keys:
        per = [np.mean([res[(dd, s)][k]["test"] for s in seeds if (dd, s) in res])
               for dd in ds if any((dd, s) in res for s in seeds)]
        rows.append([k, float(np.mean(per))])
    R.table(["variant", "mean test"], rows, nd=2)
    d = {k: v for k, v in rows}
    gap = d["trained"] - max(d["raw"], d["proto"], d["shuffled"])
    R.kv({"trained - best control": gap,
          "trained - integrate": d["trained"] - d["integrate"],
          "trained - shuffled anchors": d["trained"] - d["shuffled"],
          "raw - zeroshot (does the decoder form alone help?)":
              d["raw"] - d["zeroshot"],
          "best control - zeroshot":
              max(d["raw"], d["proto"], d["shuffled"]) - d["zeroshot"]}, 2)
    if gap > 1.0:
        R.verdict(f"The trained field is worth {gap:.2f} points over the best "
                  f"matched-form control, so the margin is not the blend.")
    else:
        R.verdict(f"WARNING: the best control is within {gap:.2f} of the trained "
                  f"field. The decoder form, not the field, explains most of the "
                  f"margin -- the paper's framing must change.")
    R.save()


if __name__ == "__main__":
    main()
```

I.12 CLOSED_FORM.PY: THE DERIVED λ AND β_b OF TABLE 4 (RIGHT)

```
"""
analytic/closed_form.py -- remove the decoder's two tuned knobs by deriving them.

lambda.  The observation in anchor coordinates is y = C chat = G q* + C eps with
Cov(C eps) = s^2 G, the same operator as the forward map, so the MMSE estimator
of q* is exactly our ridge with

        lambda* = s_hat^2 / tau_hat^2 .

  s_hat^2  from the least-squares residual, divided by (d - K) -- NOT by d.
           The residual is P_perp eps and rank(P_perp) = d - K, so dividing by d
           underestimates s^2 by (d-K)/d: a factor of 4.45 at K=397, worst
           exactly where lambda should be largest.
  tau_hat^2 from the spectrum: with y_j = u_j^T y and E[y_j^2] = mu_j^2 tau^2 +
           s^2 mu_j, take tau_hat^2 = mean_j (y_j^2 - s_hat^2 mu_j)/mu_j^2,
           clipped at 0. Reported alongside the 1/K default.

beta.  Model the two standardised score vectors as signal-plus-noise on the true
class, a = d_a e_y + n_a and b = d_b e_y + n_b with Corr(n_a,n_b) = rho. Fisher's
rule gives
        beta* = (d_b - rho d_a) / [(1 - rho)(d_a + d_b)],  clipped to [0,1],
which reduces to inverse-variance weighting at rho = 0 and to ZERO when
d_b <= rho d_a -- i.e. the formula predicts a flat gain rather than explaining
it afterwards.

d_a, d_b, rho are estimated on the VALIDATION split, never on the support set:
the field interpolates the support paths, so a support-set estimate of d_b is
optimistically biased. We report the support-set estimate too, as a diagnostic
of exactly that bias.

Also reported, label-free: Morozov discrepancy and GCV choices of lambda.

    python analytic/closed_form.py --all --seeds 1 2 3 --ckpt_dir ./checkpoints/ablation
```

```python
"""
import argparse, glob, os
import numpy as np
from common import (add_common_args, make_cfg, get_features, load_net,
                    net_field, plug_in_stats, euler_curve)
from mdreport import MD

HARD5 = ["FGVCAircraft", "EuroSAT", "DescribableTextures", "SUN397", "StanfordCars"]
SHORT = dict(zip(HARD5, ["Aircraft", "EuroSAT", "DTD", "SUN397", "Cars"]))
LAMS = [0.0, 1e-3, 1e-2, 3e-2, 0.1, 0.3, 1.0, 3.0, 10.0, 100.0, 1e4]
BETAS = [0.0, 0.25, 0.5, 0.75, 1.0]


def z(S):
    return (S - S.mean(1, keepdims=True)) / np.maximum(S.std(1, keepdims=True), 1e-9)


def s2_hat(chat, C, K, d, divisor="d-K"):
    w = np.linalg.lstsq(C.T, chat.T, rcond=None)[0].T
    rss = ((chat - w @ C) ** 2).sum(1).mean()
    return rss / ((d - K) if divisor == "d-K" else d)


def tau2_hat(y, ev, V, s2):
    """spectral moment matching: E[y_j^2] = mu_j^2 tau^2 + s^2 mu_j."""
    yj = (V.T @ y.T).T                          # (n,K) coefficients
    m2 = (yj ** 2).mean(0)
    good = ev > 1e-8 * ev.max()
    t = (m2[good] - s2 * ev[good]) / (ev[good] ** 2)
    return float(max(np.mean(np.clip(t, 0, None)), 1e-12))


def ridge_w(y, ev, V, lam):
    return (V @ ((V.T @ y.T) / (ev[:, None] + lam))).T


def morozov(y, ev, V, s2):
    """smallest lambda with ||G q(lam) - y||^2 <= s^2 tr(G)."""
    yj = (V.T @ y.T).T
    tgt = s2 * ev.sum()
    lo, hi = 1e-8, 1e6
    for _ in range(80):
        mid = np.sqrt(lo * hi)
        # G q(lam) - y  has coefficients (mu/(mu+lam) - 1) y_j = -lam/(mu+lam) y_j
        r = ((mid / (ev + mid)) ** 2 * (yj ** 2).mean(0)).sum()
        if r > tgt:
            hi = mid
        else:
            lo = mid
    return float(np.sqrt(lo * hi))


def gcv(y, ev, V, lams):
    yj2 = ((V.T @ y.T).T ** 2).mean(0)
    K = len(ev)
    best, bl = np.inf, lams[0]
    for lam in lams:
        if lam <= 0:
            continue
        num = ((lam / (ev + lam)) ** 2 * yj2).sum()
        den = (1.0 - (ev / (ev + lam)).sum() / K) ** 2
        g = num / max(den, 1e-12)
        if g < best:
            best, bl = g, lam
    return float(bl)


def lda_beta(a, b, y):
    """closed-form beta* from the two standardised score vectors."""
    n, K = a.shape
    E = np.zeros_like(a); E[np.arange(n), y] = 1.0
    da = float((a * E).sum(1).mean())
    db = float((b * E).sum(1).mean())
    na, nb = a - da * E, b - db * E
    rho = float(np.corrcoef(na.ravel(), nb.ravel())[0, 1])
    den = (1 - rho) * (da + db)
    beta = float(np.clip((db - rho * da) / den, 0, 1)) if abs(den) > 1e-12 else 0.0
    return beta, da, db, rho


def acc(S, y):
```

```
    return 100.0 * float((S.argmax(1) == y).mean())


def find_ckpt(dirp, dataset, seed, blocks):
    for pat in (f"A__{dataset}__clip__ViT-B-16__s16__seed{seed}__b{blocks}__g0.0__flow__v0.pth",
                f"*{dataset}*seed{seed}*b{blocks}*v0.pth", f"*{dataset}*seed{seed}*.pth"):
        h = sorted(glob.glob(os.path.join(dirp, pat)))
        if h:
            return h[0]
    return None


def one(a, dataset, seed):
    a.dataset, a.seed = dataset, seed
    cfg = make_cfg(a)
    Xtr, ytr, Xva, yva, Xte, yte, C, _ = get_features(a, cfg, verbose=False)
    K, d = C.shape
    ck = find_ckpt(a.ckpt_dir, dataset, seed, a.blocks)
    if ck is None:
        return None
    net = load_net(ck, d, a.blocks, cfg.device)
    v = net_field(net, cfg.device)
    G = C @ C.T
    ev, V = np.linalg.eigh(G)
    ch_te = Xte + v(Xte, 0.0)
    ch_va = Xva + v(Xva, 0.0) if len(yva) else None
    ch_tr = Xtr + v(Xtr, 0.0)
    y_te = ch_te @ C.T
    y_va = ch_va @ C.T if len(yva) else None

    # ---- lambda ----
    s2_bad = s2_hat(ch_te, C, K, d, "d")
    s2_fix = s2_hat(ch_te, C, K, d, "d-K")
    t2_spec = tau2_hat(y_te, ev, V, s2_fix)
    lam_1K = s2_fix * K                       # tau^2 = 1/K
    lam_spec = s2_fix / t2_spec
    lam_mor = morozov(y_te, ev, V, s2_fix)
    lam_gcv = gcv(y_te, ev, V, LAMS)

    # ---- beta, on val (honest) and on support (to show the bias) ----
    zs_te = z(Xte @ C.T)
    out = {}
    if len(yva):
        b_va, da_v, db_v, rho_v = lda_beta(z(Xva @ C.T),
                                           z(ridge_w(y_va, ev, V, lam_spec)), yva)
    else:
        b_va, da_v, db_v, rho_v = 0.5, np.nan, np.nan, np.nan
    b_tr, da_t, db_t, rho_t = lda_beta(z(Xtr @ C.T),
                                       z(ridge_w(ch_tr @ C.T, ev, V, lam_spec)), ytr)

    def score(lam, beta):
        return acc((1 - beta) * zs_te + beta * z(ridge_w(y_te, ev, V, lam)), yte)

    # ---- grid reference (val-selected), as in the paper ----
    best = None
    for lam in LAMS:
        Wt = z(ridge_w(y_te, ev, V, lam))
        Wv = z(ridge_w(y_va, ev, V, lam)) if len(yva) else None
        for b in BETAS:
            te = acc((1 - b) * zs_te + b * Wt, yte)
            va = (acc((1 - b) * z(Xva @ C.T) + b * Wv, yva) if len(yva) else te)
            if best is None or va > best[0]:
                best = (va, te, lam, b)

    out = dict(K=K, d=d,
               s2_d=s2_bad, s2_dK=s2_fix, ratio=s2_fix / max(s2_bad, 1e-30),
               tau2_spec=t2_spec, tau2_1K=1.0 / K,
               lam_grid=best[2], beta_grid=best[3], acc_grid=best[1],
               lam_1K=lam_1K, lam_spec=lam_spec, lam_mor=lam_mor, lam_gcv=lam_gcv,
               beta_val=b_va, beta_sup=b_tr,
               da_val=da_v, db_val=db_v, rho_val=rho_v,
               da_sup=da_t, db_sup=db_t, rho_sup=rho_t,
               acc_plugin_gridbeta=score(lam_spec, best[3]),
               acc_closed=score(lam_spec, b_va),
               acc_closed_mor=score(lam_mor, b_va),
               acc_integrate=100 * max(euler_curve(v, Xte, yte, C, 10)))
    del net
    return out
```

```python
def main():
    ap = argparse.ArgumentParser()
    add_common_args(ap)
    ap.add_argument("--ckpt_dir", default="./checkpoints/ablation")
    ap.add_argument("--all", action="store_true")
    ap.add_argument("--seeds", type=int, nargs="+", default=None)
    a = ap.parse_args()
    ds = HARD5 if a.all else [a.dataset]
    seeds = a.seeds or [a.seed]

    R = MD("closed_form", "Deriving the decoder's hyperparameters", a)
    res = {}
    for dd in ds:
        for s in seeds:
            o = one(a, dd, s)
            if o is None:
                R.p(f"- `{dd}` seed {s}: no checkpoint in `{a.ckpt_dir}` -- skipped")
            else:
                res[(dd, s)] = o
    if not res:
        R.verdict("No checkpoints found."); R.save(); return
    M = lambda k: float(np.mean([o[k] for o in res.values()]))

    R.h("1. The noise estimator: divisor $d$ vs $d-K$")
    R.table(["dataset", "K", "s2 (/d)", "s2 (/(d-K))", "d/(d-K)"],
            [[SHORT.get(dd, dd), res[(dd, s)]["K"], res[(dd, s)]["s2_d"],
              res[(dd, s)]["s2_dK"], res[(dd, s)]["ratio"]]
             for dd in ds for s in seeds[:1] if (dd, s) in res], nd=5,
            caption="The residual is $P_\\perp\\varepsilon$ of rank $d-K$, so "
                    "dividing by $d$ is biased low, worst at large $K$.")

    R.h("2. Choices of $\\lambda$")
    R.table(["dataset", "grid (val)", "s2*K", "s2/tau2_spec", "Morozov", "GCV"],
            [[SHORT.get(dd, dd), res[(dd, s)]["lam_grid"], res[(dd, s)]["lam_1K"],
              res[(dd, s)]["lam_spec"], res[(dd, s)]["lam_mor"],
              res[(dd, s)]["lam_gcv"]]
             for dd in ds for s in seeds[:1] if (dd, s) in res], nd=5)
    gl = np.array([o["lam_grid"] for o in res.values()])
    for nm in ["lam_1K", "lam_spec", "lam_mor", "lam_gcv"]:
        pl = np.array([o[nm] for o in res.values()])
        ok = (gl > 0) & (pl > 0)
        r = (float(np.corrcoef(np.log(gl[ok]), np.log(pl[ok]))[0, 1])
             if ok.sum() > 2 and np.log(gl[ok]).std() > 0 and np.log(pl[ok]).std() > 0
             else float("nan"))
        R.p(f"- `corr(log lam_grid, log {nm})` = **{r:+.3f}** (n={int(ok.sum())})")
    R.p()

    R.h("3. Closed-form $\\beta$")
    R.table(["dataset", "beta grid", "beta* (val)", "beta* (support)",
             "d_a", "d_b", "rho"],
            [[SHORT.get(dd, dd), res[(dd, s)]["beta_grid"], res[(dd, s)]["beta_val"],
              res[(dd, s)]["beta_sup"], res[(dd, s)]["da_val"],
              res[(dd, s)]["db_val"], res[(dd, s)]["rho_val"]]
             for dd in ds for s in seeds[:1] if (dd, s) in res], nd=3,
            caption="`beta* (support)` is shown only to expose the optimistic "
                    "bias from estimating $d_b$ where the field interpolates.")

    R.h("4. Does removing the knobs cost accuracy?")
    rows = [["integration (baseline)", M("acc_integrate")],
            ["decoder, grid (lam,beta) on val", M("acc_grid")],
            ["decoder, plug-in lam + grid beta", M("acc_plugin_gridbeta")],
            ["decoder, plug-in lam + LDA beta (zero knobs)", M("acc_closed")],
            ["decoder, Morozov lam + LDA beta (label-free lam)", M("acc_closed_mor")]]
    R.table(["variant", "mean test"], rows, nd=2)
    gap = M("acc_grid") - M("acc_closed")
    R.kv({"grid - fully closed form": gap,
          "closed form - integration": M("acc_closed") - M("acc_integrate")}, 2)
    if gap < 0.3:
        R.verdict(f"The fully closed-form decoder is within {gap:.2f} of the "
                  f"grid-selected one: the two knobs can be removed, and the "
                  f"'search space contains both baselines' objection goes away.")
    else:
        R.verdict(f"The closed form costs {gap:.2f} points against the grid. "
                  f"Report both; the knobs are doing real work and the objection "
                  f"stands.")
    R.save()


if __name__ == "__main__":
    main()
```

I.13 STATS_TESTS.PY: LEAVE-ONE-DATASET-OUT, PAIRED TESTS, PARTIAL CORRELATION, STRAIGHTNESS CALIBRATION

```
"""
analytic/stats_tests.py -- the statistical claims, all from cached result JSONs.
No training, no GPU, seconds to run.

  S1  LEAVE-ONE-DATASET-OUT. The decoder's grid contains both limiting decodings
      by construction, so the validation optimum cannot lose -- non-inferiority
      on validation is structural and carries little weight. LODO chooses
      (lambda,beta) on the OTHER datasets and applies it unchanged to the
      held-out one, so no information from the held-out dataset is used.

  S2  PAIRED SIGNIFICANCE. Wilcoxon signed-rank and a bootstrap CI on the
      per-cell difference (decoder - integration), which is what "+0.5 is within
      the seed spread" requires.

  S3  PARTIAL CORRELATION. The read-out advantage is claimed to track
      log10 cond(G). With n=5 datasets and K the dominant covariate, we report a
      bootstrap CI and the partial correlation controlling for log K.

  S4  STRAIGHTNESS CALIBRATION. S is measured against the displacement produced
      by the same field, so it needs a reference. Group I's t==0 model is that
      reference: a field trained at a single t is unconstrained off its support.

    python analytic/stats_tests.py --results results/analytic
"""
import argparse, glob, json, os
from collections import defaultdict
import numpy as np
from mdreport import MD

HARD5 = ["FGVCAircraft", "EuroSAT", "DescribableTextures", "SUN397", "StanfordCars"]
SHORT = dict(zip(HARD5, ["Aircraft", "EuroSAT", "DTD", "SUN397", "Cars"]))


def load(results, loop="prefer-exact"):
    cells = []
    for p in sorted(glob.glob(os.path.join(results, "*.json"))):
        if os.path.basename(p).startswith("_"):
            continue
        try:
            j = json.load(open(p))
        except Exception:
            continue
        if j.get("status") != "ok":
            continue
        c, r = j["cell"], j["result"]
        if c["dataset"] not in HARD5:
            continue
        c = dict(c)
        c["train_loop"] = r.get("train_loop",
                                "exact" if c.get("views", 0) == 0 else "cached")
        cells.append((c, r))
    if loop == "prefer-exact":
        best = {}
        for c, r in cells:
            k = (c["group"], c["dataset"], c.get("extractor"), c.get("clip_type"),
                 c.get("shots"), c.get("blocks"), c.get("gamma"), c.get("tmode"),
                 c.get("seed"))
            if k not in best or (best[k][0]["train_loop"] != "exact"
                                 and c["train_loop"] == "exact"):
                best[k] = (c, r)
        cells = list(best.values())
    return cells


def boot_ci(x, n=20000, seed=0, f=np.mean):
    rng = np.random.default_rng(seed)
    x = np.asarray(x, float)
    b = f(rng.choice(x, (n, len(x)), replace=True), axis=1)
    return float(np.percentile(b, 2.5)), float(np.percentile(b, 97.5))


def wilcoxon(x):
    try:
        from scipy.stats import wilcoxon as W
        s, p = W(x)
        return float(p), "Wilcoxon signed-rank"
    except Exception:
```

```
        x = np.asarray(x, float)
        k = int((x > 0).sum()); n = int((x != 0).sum())
        from math import comb
        p = 2 * sum(comb(n, i) for i in range(k, n + 1)) / 2 ** n if n else 1.0
        return float(min(1.0, p)), "exact sign test (scipy unavailable)"


def pcorr(x, y, z):
    x, y, z = map(lambda v: np.asarray(v, float), (x, y, z))
    rxy, rxz, ryz = (np.corrcoef(x, y)[0, 1], np.corrcoef(x, z)[0, 1],
                     np.corrcoef(y, z)[0, 1])
    den = np.sqrt(max((1 - rxz ** 2) * (1 - ryz ** 2), 1e-12))
    return float((rxy - rxz * ryz) / den), float(rxy)


def main():
    ap = argparse.ArgumentParser()
    ap.add_argument("--results", default="results/analytic")
    ap.add_argument("--group", default="A")
    a = ap.parse_args()
    C = load(a.results)
    R = MD("stats_tests", "Statistical claims", a)
    A = [(c, r) for c, r in C if c["group"] == a.group and c.get("extractor") == "clip"
         and c.get("shots") == 16]
    if not A:
        R.verdict(f"no group-{a.group} cells in {a.results}"); R.save(); return
    R.p(f"{len(C)} HARD5 cells loaded; group {a.group} contributes {len(A)}.")

    # ------------------------------------------------------------------ S1
    R.h("S1. Leave-one-dataset-out (no per-dataset tuning)")
    have = [(c, r) for c, r in A if r.get("ours_grid")]
    if not have:
        R.p("No stored (lambda,beta) grids -- rerun `run_ablations.py` so that "
            "`ours_grid` is recorded.")
    else:
        keys = sorted(set.intersection(*[set(r["ours_grid"]) for _, r in have]))
        vf = lambda k, sub: float(np.mean([r["ours_grid"][k][0] for _, r in sub]))
        tf = lambda k, sub: float(np.mean([r["ours_grid"][k][1] for _, r in sub]))
        rows, lodo = [], []
        for dd in HARD5:
            held = [(c, r) for c, r in have if c["dataset"] == dd]
            other = [(c, r) for c, r in have if c["dataset"] != dd]
            if not held or not other:
                continue
            k = max(keys, key=lambda kk: vf(kk, other))
            it = float(np.mean([r["integrate"] for _, r in held])) * 100
            tu = float(np.mean([r["ours"] for _, r in held])) * 100
            lo = tf(k, held) * 100
            rows.append([SHORT[dd], k, it, tu, lo, lo - it])
            lodo.append(lo - it)
        R.table(["dataset", "chosen (lam|beta)", "integration", "ours (tuned)",
                 "ours (LODO)", "LODO - int."], rows, nd=2,
                caption="(lambda,beta) chosen on the four other datasets' "
                        "validation splits and applied unchanged.")
        if lodo:
            lo, hi = boot_ci(lodo)
            R.kv({"mean LODO advantage": float(np.mean(lodo)),
                  "95% CI": f"[{lo:+.2f}, {hi:+.2f}]",
                  "datasets where LODO wins": f"{sum(1 for x in lodo if x>0)}/{len(lodo)}"},
 ↪ 2)
            R.verdict("LODO is the number that carries evidential weight: no "
                      "information from the held-out dataset is used."
                      if np.mean(lodo) > 0 else
                      "LODO does not reproduce the tuned advantage; the headline "
                      "rests on per-dataset selection.")

    # ------------------------------------------------------------------ S2
    R.h("S2. Paired significance, per cell")
    for nm, sub in [("group A (16-shot, ViT-B/16)", A),
                    ("all HARD5 cells, any group", C)]:
        d = [(r["ours"] - r["integrate"]) * 100 for _, r in sub if "ours" in r]
        if len(d) < 4:
            continue
        p, meth = wilcoxon(d)
        lo, hi = boot_ci(d)
        R.p(f"**{nm}**: n={len(d)} cells")
        R.kv({"mean difference": float(np.mean(d)),
              "median": float(np.median(d)),
              "95% bootstrap CI": f"[{lo:+.2f}, {hi:+.2f}]",
              "positive on": f"{sum(1 for x in d if x>0)}/{len(d)}",
              f"p ({meth})": p}, 3)
```

```
    # ---------------------------------------------------------------- S3
    R.h("S3. Does cond(G) govern the choice of decoding?")
    per = {}
    for dd in HARD5:
        sub = [(c, r) for c, r in A if c["dataset"] == dd]
        if not sub:
            continue
        per[dd] = dict(
            adv=float(np.mean([(r["readout"] - r["integrate"]) * 100 for _, r in sub])),
            cond=float(np.mean([r["geom"]["cond_G"] for _, r in sub])),
            K=float(np.mean([r["geom"]["K"] for _, r in sub])),
            rho=float(np.mean([r["geom"]["anchor_cos"] for _, r in sub])),
            acc=float(np.mean([r["integrate"] * 100 for _, r in sub])))
    if len(per) >= 4:
        adv = [per[d]["adv"] for d in per]
        lcg = [np.log10(per[d]["cond"]) for d in per]
        lK = [np.log(per[d]["K"]) for d in per]
        rho = [per[d]["rho"] for d in per]
        acc = [per[d]["acc"] for d in per]
        R.table(["dataset", "read-out - int.", "log10 cond G", "log K",
                 "anchor cos", "field acc"],
                [[SHORT[d], per[d]["adv"], np.log10(per[d]["cond"]),
                  np.log(per[d]["K"]), per[d]["rho"], per[d]["acc"]] for d in per],
                nd=3)
        rows = []
        for nm, v in [("log10 cond G", lcg), ("log K", lK),
                      ("anchor cosine", rho), ("field accuracy", acc)]:
            r = float(np.corrcoef(v, adv)[0, 1])
            rng = np.random.default_rng(0)
            bs = []
            for _ in range(20000):
                i = rng.integers(0, len(adv), len(adv))
                if np.std(np.array(v)[i]) < 1e-12 or np.std(np.array(adv)[i]) < 1e-12:
                    continue
                bs.append(np.corrcoef(np.array(v)[i], np.array(adv)[i])[0, 1])
            lo, hi = (np.percentile(bs, 2.5), np.percentile(bs, 97.5)) if bs else (np.nan, np.
 ↪ nan)
            rows.append([nm, r, f"[{lo:+.2f}, {hi:+.2f}]"])
        pc, raw = pcorr(lcg, adv, lK)
        rows.append(["log10 cond G | log K (partial)", pc, "--"])
        R.table(["covariate", "r with read-out advantage", "95% CI"], rows, nd=3,
                caption=f"n={len(adv)} datasets. The partial correlation controls "
                        f"for log K, the dominant covariate.")
        R.verdict(f"r = {raw:+.2f} with cond(G); partial r = {pc:+.2f} controlling "
                  f"for log K. State n={len(adv)} explicitly and quote the CI: at "
                  f"this sample size the claim is suggestive, not established.")

    # ---------------------------------------------------------------- S4
    R.h("S4. Straightness calibration")
    tm = defaultdict(list)
    for c, r in C:
        if c["group"] == "I" and "straightness" in r:
            tm[c["tmode"]].append((r["straightness"], r["turn_final"],
                                   r.get("late_decay", np.nan)))
    for c, r in A:
        if "straightness" in r:
            tm["flow (group A)"].append((r["straightness"], r["turn_final"],
                                         r.get("late_decay", np.nan)))
    if tm:
        R.table(["training", "cells", "S", "turn", "late decay"],
                [[k, len(v), float(np.mean([x[0] for x in v])),
                  float(np.mean([x[1] for x in v])),
                  float(np.mean([x[2] for x in v]))] for k, v in sorted(tm.items())],
                nd=4,
                caption="A field trained at a single t is unconstrained off its "
                        "support, so its S is the reference against which the "
                        "flow-trained field's straightness should be read.")
        if "t0" in tm and "flow" in tm:
            sf = float(np.mean([x[0] for x in tm["flow"]]))
            s0 = float(np.mean([x[0] for x in tm["t0"]]))
            R.verdict(f"S = {sf:.4f} for the flow-trained field against "
                      f"{s0:.4f} for the t==0 field ({s0/max(sf,1e-9):.0f}x), so "
                      f"the straightness measurement is calibrated, not vacuous.")
    else:
        R.p("No group-I cells found -- run `run_ablations.py I` for the "
            "calibration reference.")
    R.save()


if __name__ == "__main__":
```

```
    main()
```

I.14 COMMON.PY, FEATURE EXTRACTION, SPLITS, SHARED UTILITIES

```
"""
analytic/common.py -- shared helpers.  Import-safe from the FMA repo root.

All scripts in this folder are run FROM THE REPO ROOT, e.g.
    python analytic/probe.py --dataset EuroSAT --ckpt ...
The bootstrap below puts the repo root on sys.path so `config`, `datasets`
and `models` resolve regardless of where python was launched from.
"""
import os, sys

_HERE = os.path.dirname(os.path.abspath(__file__))
_ROOT = os.path.dirname(_HERE)
for _p in (_ROOT, _HERE):
    if _p not in sys.path:
        sys.path.insert(0, _p)

import numpy as np
import torch
from torch.utils.data import DataLoader

from config import DefaultConfig, set_seed
from datasets import build_dataset
from datasets.utils import WarpperDataset, train_transform, test_transform
from models.feature_extractor import get_extractor
from models.fm import DeepFlowMatchingNet

CACHE = os.path.join(_ROOT, "feat_cache")


# ------------------------------------------------------------------ config ----
def make_cfg(a):
    cfg = DefaultConfig()
    cfg.dataset = a.dataset
    cfg.num_shots = a.num_shots
    cfg.feature_extractor = getattr(a, "feature_extractor", "clip")
    cfg.clip_type = getattr(a, "clip_type", "ViT-B/16")
    cfg.seed = a.seed
    cfg.batch_size = getattr(a, "batch_size", 64)
    set_seed(cfg.seed)
    return cfg


def add_common_args(ap):
    ap.add_argument("--dataset", default="EuroSAT")
    ap.add_argument("--num_shots", type=int, default=16)
    ap.add_argument("--feature_extractor", default="clip")
    ap.add_argument("--clip_type", default="ViT-B/16")
    ap.add_argument("--seed", type=int, default=1)
    ap.add_argument("--blocks", type=int, default=6)
    ap.add_argument("--batch_size", type=int, default=64)
    ap.add_argument("--train_view", default="clean", choices=["clean", "aug"],
                    help="how to featurise the shots: 'clean' = center crop, "
                         "'aug' = the RandomResizedCrop(0.08,1)+flip the "
                         "network actually trains on")
    ap.add_argument("--views", type=int, default=1,
                    help="with --train_view aug, average this many views/shot")
    return ap


# ---------------------------------------------------------------- features ----
def _autocast():
    """Match the released training loop, which runs the extractor under AMP.

    Without this the LoRA/adapter extractors mix fp16 weights with fp32
    activations and raise
        RuntimeError: mat1 and mat2 must have the same dtype, but got Half and Float
    The .float() casts below then bring everything back to fp32, exactly as
    train.py does, so cached features are fp32 computed at fp16 precision.
    """
    try:                                    # torch >= 2.4
        from torch.amp import autocast
        return autocast("cuda", enabled=torch.cuda.is_available())
    except Exception:                       # older torch
        from torch.cuda.amp import autocast as _ac
```

```
        return _ac(enabled=torch.cuda.is_available())


def _extract(loader, fx, cfg):
    F, L, A = [], [], None
    with torch.no_grad(), _autocast():
        for im, lb in loader:
            if cfg.feature_extractor == "cocoop":
                f, t, cl = fx(im, lb, training=False)
            else:
                f, t, cl = fx(im, lb)
            F.append(f.float().cpu().numpy())
            L.append(lb.cpu().numpy())
            if A is None and cl is not None and cl.dim() == 2:
                A = cl.float().cpu().numpy()
    return np.concatenate(F), np.concatenate(L), A


def get_features(a, cfg=None, verbose=True):
    """Returns Xtr, ytr, Xva, yva, Xte, yte, C (unit-norm anchors), classnames.

    The val split is the repo's own few-shot val (min(num_shots,4) per class,
    center-cropped).  It is small, but it is the standard CoOp/Tip-Adapter
    protocol and it is the ONLY honest place to select hyper-parameters."""
    cfg = cfg or make_cfg(a)
    ds = build_dataset(cfg)
    cfg.classnames = ds.classnames
    fx = get_extractor(cfg)
    os.makedirs(CACHE, exist_ok=True)
    tag = (f"{a.dataset}_{a.num_shots}_{a.feature_extractor}_"
           f"{a.clip_type.replace('/','')}_{a.seed}_"
           f"{a.train_view}{a.views}_v2")
    path = os.path.join(CACHE, tag + ".npz")
    if os.path.exists(path):
        z = np.load(path)
        if verbose:
            print(f"[cache] {path}")
        Xtr, ytr, Xva, yva, Xte, yte, C = (z["Xtr"], z["ytr"], z["Xva"],
                                           z["yva"], z["Xte"], z["yte"], z["C"])
    else:
        src = ds.train_x.data_source
        if a.train_view == "clean":
            trds, nv = WarpperDataset(src, test_transform), 1
        else:
            trds, nv = WarpperDataset(src, train_transform), max(1, a.views)
        tr = DataLoader(trds, batch_size=cfg.batch_size, num_workers=4)
        va = DataLoader(ds.val, batch_size=cfg.batch_size, num_workers=4)
        te = DataLoader(ds.test, batch_size=cfg.batch_size, num_workers=4)
        if verbose:
            print(f"[extract] train ({a.train_view}, {nv} view(s)) ...")
        acc, C = None, None
        for _ in range(nv):
            Xv, ytr, Cv = _extract(tr, fx, cfg)
            acc = Xv if acc is None else acc + Xv
            C = Cv if C is None else C
        Xtr = acc / nv
        if verbose:
            print("[extract] val ...")
        try:
            Xva, yva, _ = _extract(va, fx, cfg)
        except Exception:
            Xva, yva = np.zeros((0, Xtr.shape[1])), np.zeros(0, dtype=int)
        if verbose:
            print("[extract] test ...")
        Xte, yte, _ = _extract(te, fx, cfg)
        np.savez(path, Xtr=Xtr, ytr=ytr, Xva=Xva, yva=yva,
                 Xte=Xte, yte=yte, C=C)
    C = C / np.linalg.norm(C, axis=1, keepdims=True)
    if verbose:
        print(f"[split] train={len(ytr)}  val={len(yva)}  test={len(yte)}")
    return Xtr, ytr, Xva, yva, Xte, yte, C, ds.classnames


# ------------------------------------------------------- val selection ----
def select_on_val(cands, val_fn, test_fn):
    """cands: dict {name -> object}.  Score every candidate on VAL, pick the
    argmax, then evaluate that one on TEST.  Returns
    (best_name, test_acc, val_acc, all_val, all_test_for_reference).

    all_test is reported only so the oracle gap can be quoted honestly -- the
    headline number must be the val-selected one."""
```

```python
    v = {k: float(val_fn(o)) for k, o in cands.items()}
    best = max(v, key=v.get)
    t = {k: float(test_fn(o)) for k, o in cands.items()}
    return best, t[best], v[best], v, t


def oracle_gap(all_test, sel_test):
    """How much test-set tuning would have bought.  Report this, do not use it."""
    return max(all_test.values()) - sel_test


# ---------------------------------------------------------------- model ----
def load_net(ckpt, d, blocks, device):
    net = DeepFlowMatchingNet(in_channels=d, model_channels=d,
                              out_channels=d, num_res_blocks=blocks).to(device)
    net.load_state_dict(torch.load(ckpt, map_location=device))
    net.eval()
    return net


def net_field(net, device):
    def f(z, t):
        with torch.no_grad():
            return net(torch.tensor(z, dtype=torch.float32, device=device),
                       torch.full((len(z), 1), float(t), device=device)
                       ).cpu().numpy().astype(np.float64)
    return f


# ------------------------------------------------------------ analytics ----
def anchor_weights_gauss(z, C, t, mu, sigma):
    """Corollary 8: w = softmax(-||z - m_k(t)||^2 / (2 sigma^2 (1-t)^2)).

    Uses ||z-m||^2 = ||z||^2 - 2 z.m + ||m||^2 so the cost is one (n,K) gemm.
    The naive (n,K,d) broadcast needs 32 GB for SUN397 (n=19850, K=397, d=512).
    """
    m = (1.0 - t) * mu + t * C                          # (K,d)
    d2 = ((z ** 2).sum(1)[:, None]
          - 2.0 * (z @ m.T)
          + (m ** 2).sum(1)[None, :])                   # (n,K)
    np.maximum(d2, 0, out=d2)
    lg = d2 * (-1.0 / (2.0 * sigma ** 2 * (1.0 - t) ** 2))
    lg -= lg.max(1, keepdims=True)
    w = np.exp(lg, out=lg)
    return w / w.sum(1, keepdims=True)


def plug_in_stats(Xtr, ytr, K, shots, d):
    mu = np.stack([Xtr[ytr == k].mean(0) if (ytr == k).any() else np.zeros(d)
                   for k in range(K)])
    s2 = ((Xtr - mu[ytr]) ** 2).sum() / (len(ytr) * d * (1 - 1.0 / max(shots, 2)))
    return mu, np.sqrt(s2), np.sqrt(s2 * (1 + 1.0 / max(shots, 2)))


def gauss_field(C, mu, sigma):
    return lambda z, t: ((anchor_weights_gauss(z, C, t, mu, sigma) @ C) - z) / (1 - t)


def euler_curve(field, x, y, C, steps=10, h=0.1):
    """Accuracy after 0,1,...,steps Euler steps from ONE trajectory.

    euler_classify(steps=m) re-integrates from scratch, so sweeping m=0..10
    costs sum(0..10)=55 field evaluations; this costs 10.
    """
    z, t, out = x.copy(), 0.0, []
    for _ in range(steps + 1):
        zn = z / np.maximum(np.linalg.norm(z, axis=1, keepdims=True), 1e-12)
        out.append(float(((zn @ C.T).argmax(1) == y).mean()))
        if t >= 1.0 - 1e-9:
            out += [out[-1]] * (steps - len(out) + 1)
            break
        z = z + h * field(z, t)
        t += h
    return out[:steps + 1]


def euler_classify(field, x, y, C, steps, h=0.1):
    """Exactly the test.py protocol: Euler from t=0, argmax_k <z/||z||, c_k>."""
    z, t = x.copy(), 0.0
    for _ in range(steps):
        z = z + h * field(z, t)
```

```
        t += h
    z = z / np.linalg.norm(z, axis=1, keepdims=True)
    return float(((z @ C.T).argmax(1) == y).mean())


def simplex_project(P, C):
    """Least-squares coords over the anchors; returns (raw, clipped-normalised)."""
    K = C.shape[0]
    A = np.vstack([C.T, 1e3 * np.ones((1, K))])
    B = np.vstack([P.T, 1e3 * np.ones((1, len(P)))])
    W = np.linalg.lstsq(A, B, rcond=None)[0].T
    Wc = np.clip(W, 0, None)
    return W, Wc / np.maximum(Wc.sum(1, keepdims=True), 1e-12)


def span_basis(C):
    K = C.shape[0]
    return np.linalg.qr((C - C.mean(0, keepdims=True)).T)[0][:, :K - 1]


def span_frac(P, C, Q):
    Pc = P - C.mean(0, keepdims=True)
    return float(np.mean(np.linalg.norm(Pc @ Q, axis=1) ** 2 /
                         np.maximum(np.linalg.norm(Pc, axis=1) ** 2, 1e-12)))


def geometry(Xtr, ytr, Xte, yte, C, mu):
    K = C.shape[0]
    Xn = Xte / np.linalg.norm(Xte, axis=1, keepdims=True)
    mun = mu / np.linalg.norm(mu, axis=1, keepdims=True)
    return dict(
        anchor_cos=float((C @ C.T)[~np.eye(K, dtype=bool)].mean()),
        modality_gap=float(np.mean(np.linalg.norm(mun - C, axis=1))),
        in_class=float(np.linalg.norm(Xtr - mu[ytr], axis=1).mean()),
        zeroshot=float(((Xn @ C.T).argmax(1) == yte).mean()),
        cos_proto=float(((Xn @ mun.T).argmax(1) == yte).mean()),
    )
```

I.15 REPORT.PY, PASS/FAIL HARNESS

```
"""report.py -- tiny PASS/FAIL harness shared by all test modules."""
import sys

RESULTS = []
_OUT = [sys.stdout]


def tee(fh):
    _OUT.append(fh)


def out(s=""):
    for f in _OUT:
        f.write(s + "\n")
        f.flush()


def sec(s):
    out("\n" + "=" * 78)
    out(s)
    out("=" * 78)


def check(name, ok, detail=""):
    RESULTS.append((name, bool(ok)))
    out(f"  [{'PASS' if ok else 'FAIL'}] {name}" + (f"   {detail}" if detail else ""))


def summary(tag):
    n = len(RESULTS)
    nf = sum(1 for _, o in RESULTS if not o)
    out("\n" + "=" * 78)
    out(f"{tag}: {n - nf}/{n} checks passed" + ("" if nf == 0 else f"   ({nf} FAILED)"))
    out("=" * 78)
    return n - nf, n


def reset():
```

```
    RESULTS.clear()
```