CLEAR: MULTI-OBJECTIVE LLM UNLEARNING WITH PROVABLE PERFORMANCE GUARANTEES

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ABSTRACT

Large Language Model (LLM) unlearning has emerged as a crucial paradigm for mitigating privacy and security risks by aiming to erase undesirable knowledge while preserving the LLM's general capability. Despite recent development of unlearning algorithms, existing methods suffer from critical limitations. First, they primarily focus on the basic trade-off between forgetting and retention, neglecting other essential dimensions such as robustness. Second, they overlook the failure-sensitive nature of unlearning at the task level, generally failing to provide worst-case performance guarantees. Third, current approaches are largely heuristic, lacking rigorous theoretical foundations and finite-time convergence analysis. In this paper, we systematically address the LLM unlearning problem through the lens of constrained multi-objective optimization (MOO), which enables us to propose Chebyshev-weighted LLM unlearning algorithm (CLEAR), a new penalty-based MOO algorithm with provable performance guarantees. Specifically, we theoretically construct a rigorous Karush-Kuhn-Tucker (KKT)-based metric, based on which we establish a finite-time convergence guarantee for the proposed CLEAR algorithm, marking the first of its kind in the LLM unlearning literature. Extensive numerical experiments across multiple benchmarks, employing comprehensive evaluation metrics, demonstrate that CLEAR significantly outperforms state-of-the-art baselines.

1 INTRODUCTION

In recent years, large language models (LLMs) have found a wide range of applications across diverse domains, such as medicine, engineering, and education, to name just a few (Nori et al., 2023; Thirunavukarasu et al., 2023; Liu et al., 2023; Kasneci et al., 2023). Despite these advances, the wide spread of LLM deployment also raises serious concerns in data privacy, toxic outputs, and copyright infringement. To address these problems, LLM unlearning (Si et al., 2023; Zhang et al., 2025; Liu et al., 2025) has emerged as a new paradigm for mitigating such risks by removing undesirable knowledge from models, while maintaining safe, compliant, and useful model behavior. Thus, the core objective of LLM unlearning is inherently a multifaceted tradeoff: an unlearning method should erase the targeted knowledge from the LLM while preserving the model's general capability, a balance that prior work has found difficult to achieve (Jang et al., 2023; Fan et al., 2024; Liu et al., 2024; Zhong et al., 2026). Moreover, a model that appears to be well-unlearned on standard evaluation benchmarks may still suffer from critical failures. In particular, it may exhibit *over-unlearning*, resulting in refusals to answer questions about concepts that should remain intact, or *under-unlearning*, leaving it vulnerable to adversarial prompts or in-context injections that circumvent superficial unlearning and elicit the targeted knowledge (Hu et al., 2024; Łucki et al., 2024; Shumailov et al., 2024).

The challenge of multi-objective tradeoffs is further amplified by the *sensitivity* of the unlearning process to *individual objective failures.* Specifically, if an algorithm fails to satisfy any single objective, the unlearning process is fundamentally compromised: the resulting model either loses its capacity to respond correctly or fails to erase the targeted knowledge. This strict requirement necessitates enforcing hard constraints on each objective, which, when coupled with their underlying conflicts, significantly complicates the optimization landscape. Simple loss aggregation across these multiple objectives, such as a heuristic weighted sum, will obscure such constraints, since strong performance on one objective may numerically compensate for a severe failure on another.

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Recent work has applied multi-objective optimization to LLM unlearning (Jin et al., 2025; Pan et al., 2025; Wang et al., 2025a) to address the forgetting-retention conflict. Existing approaches, however, often evaluate unlearning in benign contexts and may fail under more challenging settings, such as adversarial probing (Łucki et al., 2024; Shumailov et al., 2024), or preservation of boundary concepts closely related to the unlearning target that should remain intact (Ren et al., 2025; Cheng et al., 2026). These objective-balancing mechanisms rely heavily on heuristic gradient manipulation, which lacks theoretical characterization of the optimal trade-off and convergence guarantee. This naturally raises the following fundamental question:

Q: *Could we develop an algorithm that balances multiple constrained objectives in LLM unlearning with* ***provable*** *theoretical guarantees?*

Addressing this question requires overcoming two main technical challenges:

(1) **How to characterize the optimal trade-off in LLM unlearning?** Existing empirical metrics can evaluate individual dimensions of unlearning, but their heuristic aggregation does *not* reveal whether a method reaches an optimal trade-off among conflicting objectives. A principled framework should rigorously characterize the optimal solution set of the unlearning problem and adaptively support different operating points under different deployment needs.

(2) **How to optimize constrained unlearning goals with guarantees?** Even with a proper trade-off formulation, designing an algorithm that can provably optimize multiple constrained objectives remains nontrivial. The algorithm should prevent critical objectives from collapsing and admit a finite-time convergence analysis under an appropriate optimality metric.

To address these challenges, we address LLM unlearning problem through the lens of ***constrained multi-objective optimization*** and propose a new hybrid and constraint-aware approach that integrates *Weighted-Chebyshev* (WC) scalarization (Momma et al., 2022; Qiu et al., 2024; Zhang et al., 2026a) and the multi-gradient descent algorithm (MGDA) Désidéri (2012). Specifically, our solution adaptively guides the optimizer toward the current bottleneck objective while enforcing hard constraints to maintain task-level requirements across multiple objectives. The preference vector in WC also provides a natural "control knob" for prioritizing different unlearning preferences, such as stronger erasure, stronger utility preservation, or stronger adversarial robustness, depending on the deployment needs. Our major contributions in this work are summarized as follows:

- **A new constrained weighted Chebyshev scalarization approach for Pareto front exploration.** We prove that Weighted-Chebyshev (WC) scalarization *equivalently* reformulates the general multi-objective LLM unlearning problem into a scalar-valued constrained optimization problem. Solving the WC-scalarized problem with different preference vectors systematically explores the Pareto frontier of the original unlearning problem, *i.e.*, the set of optimal trade-off solutions. Based on this formulation, we develop Chebyshev-weighted LLM unlearning algorithm (`CLEAR`), a penalty-based algorithm that solves the reformulated WC-scalarized problem.
- **Preference-guided bottleneck objective control.** `CLEAR` provides a principled unlearning preference-control mechanism by using the preference vector to optimize the current bottleneck objective under explicit constraints, rather than simply summarizing the weighted loss. This prevents success on one unlearning objective from masking failure on another, while supporting deployment-specific unlearning priorities.
- **Substantive empirical improvements over state-of-the-art baselines.** Comprehensive evaluations on multiple widely adopted benchmarks, including MUSE-Book and WMDP-Cyber, show that `CLEAR` achieves a better balance across all the aforementioned objectives. Compared with existing unlearning baselines with heuristic trade-offs, our method achieves a more extensive Pareto frontier and more reliable multi-objective unlearning behavior.
- **Finite-time convergence guarantee.** We analyze the Karush–Kuhn–Tucker (KKT) conditions associated with the WC-scalarized problem and define a KKT-based optimality metric for LLM unlearning. By controlling the error terms in this metric, we prove that `CLEAR` achieves a finite-time convergence rate of $\mathcal{O}(T^{-\frac{1}{2}})$. To our knowledge, this is the first non-asymptotic convergence guarantee for multi-objective LLM unlearning.

2 Related Work

1) LLM Unlearning. LLM Unlearning aims to remove targeted knowledge from LLMs while preserving useful model behavior (Si et al., 2023; Zhang et al., 2025; Liu et al., 2025). Compared to classical machine unlearning (Bourtoule et al., 2021; Block et al., 2025), LLM unlearning is more challenging due to generative outputs, entangled knowledge, and instruction-following behavior. Existing methods span inference-time approaches that steer outputs through prompts without updating parameters (Pawelczyk et al., 2023; Liu et al., 2024), and training-based approaches that modify model weights for persistent unlearning. Among the latter, GA (Jang et al., 2023) increases the loss on forget data; NPO (Zhang et al., 2024) treats forget samples as negative preferences; SimNPO (Fan et al., 2024) removes reference-model dependence for more stable forgetting; FLAT (Wang et al., 2025b) applies response-template guidance to enable forget-only unlearning; SFT (Touvron et al., 2023) supervises desired responses on forget and retain queries; and DUET (Zhong et al., 2026) distills from a prompt-steered teacher.

Closely related to our work, NGDiff (Jin et al., 2025) and MOLLM (Pan et al., 2025) formulate LLM unlearning as a multi-objective problem, which resolve the forget-retain conflict through gradient normalization or projection. These methods, however, focus primarily on the forget-retain trade-off, leaving boundary preservation and adversarial robustness less systematically addressed. Robustness studies further show that unlearned knowledge can be recovered through relearning, fine-tuning, or contextual attacks (Fan et al., 2025; Hu et al., 2024; Qi et al., 2024; Shumailov et al., 2024; Łucki et al., 2024; Xue et al., 2024), which motivates a broader multi-objective formulation. Our work, as shown in Table 1, addresses this gap by formulating LLM unlearning as a constrained multi-objective optimization problem, explicitly modeling forgetting, general retention, boundary preservation, and adversarial robustness under a principled Pareto trade-off framework.

Algorithm	Controllable Preference†	Hard Constraints	Convergence Guarantee‡
GA (Jang et al., 2023)	♦	✗	✗
SimNPO (Fan et al., 2024)	♦	✗	✗
ECO (Liu et al., 2024)	♦	✗	✗
DUET (Zhong et al., 2026)	♦	✗	✗
NGDiff (Jin et al., 2025)	♦	✗	✗
MOLLM (Pan et al., 2025)	♦	✗	✗
CLEAR (**This work**)	✓	✓	✓

Table 1: Comparison with existing works. †: *Controllable Preference* denotes whether the method can adaptively handle these competing objectives; however, this does not necessarily imply strong performance on them (Zhong et al., 2026). ♦ indicates "partially resolved". ‡: *Convergence Guarantee* specifies whether the approach achieves non-asymptotic theoretical convergence to the optimal solutions.

2) Multi-Objective Optimization (MOO). MOO has a long history dating back to (Sawaragi et al., 1985), which aims at balancing multiple potentially conflicting optimization objectives. MOO has been widely adopted across a broad spectrum of areas, ranging from traditional reinforcement learning (Li et al., 2024c; Zhou et al., 2024; Qiu et al., 2024), multi-task training (Sener & Koltun, 2018; Momma et al., 2022), to the emerging LLM alignment (Yang et al., 2024; Yao et al., 2025) and multi-domain training (Wu et al., 2024; Bai et al., 2026). Prominent MOO approaches generally fall into two categories: (i) scalarization-based methods, such as linear scalarization and weighted-Chebyshev method (Ehrgott, 2005; Momma et al., 2022; Qiu et al., 2024; Zhang et al., 2026a;b), which convert vector-valued MOO optimization problems into scalar-valued surrogates; and (ii) adaptive gradient methods, including multi-gradient descent algorithm and its variants (Désidéri, 2012; Mercier et al., 2018; Fernando et al., 2023), which iteratively track the aggregated gradients, both of which are supported by well-developed theoretical convergence guarantees. However, to date, MOO techniques for LLM unlearning remain under-explored. The LLM unlearning literature is largely limited to either data-centric objective trade-offs or proposing heuristic solutions.

3 Preliminaries on LLM Unlearning

1) Basic Formulation of LLM Unlearning. LLM unlearning seeks to update a model so that knowledge in the forget set is no longer exposed, while the model's general capability on the retained data is preserved. Although inference-time filtering and in-context strategies can suppress undesirable responses without modifying model parameters (Pawelczyk et al., 2023; Takashiro et al., 2025), persistent unlearning is typically achieved through training-based optimization. A basic LLM unlearning task can be formulated as jointly optimizing forgetting and retention performances:

$$\min_{\theta} \ \left[\mathcal{L}_{\mathrm{fg}}(\theta; \mathcal{D}_f), \mathcal{L}_{\mathrm{rt}}(\theta; \mathcal{D}_r)\right]^{\top}, \tag{1}$$

where θ denotes the model parameters, $\mathcal{L}_{\text{fg}}$ encourages forgetting on the forget data $\mathcal{D}_f$, and $\mathcal{L}_{\text{rt}}$ preserves model behavior on the retained data $\mathcal{D}_r$. This formulation captures the basic tension between erasing target knowledge and preserving utility, as the two losses may induce conflicting update directions (Jin et al., 2025; Pan et al., 2025). However, practical LLM unlearning generally goes beyond this two-objective view.

2) To Unlearn or to Retain: The Dilemma of Under- and Over-Forgetting. The dual-objective view above does not fully capture two practical challenges of LLM unlearning: (1) An over-forgotten model may extend refusal behavior from the forget set $\mathcal{D}_f$ to neighboring domains $\mathcal{N}(\mathcal{D}_f)$ that are semantically related but should remain accessible (Li et al., 2024b; Eldan & Russinovich, 2023). This leads to ***over-refusal***, where benign queries are incorrectly rejected. (2) An under-forgotten model may appear successful on standard forget queries yet still reveal target knowledge under adversarial contexts, such as prefilling attacks where an adversarial prefix x_p^{adv} steers the model toward forgotten responses through affirmative prompts. Effective unlearning hence requires addressing ***four*** objectives jointly: i) forgetting target knowledge (fg), ii) retaining general knowledge (rt), iii) preserving neighboring-domain knowledge (nbr), and iv) maintaining robustness against adversarial probing (rob). Further evaluation on the associated robustness-utility trade-offs are provided in Appendix A.3.

4 METHODOLOGY

To address the highly nontrivial LLM unlearning problem, we first reformulate it as a constrained multi-objective optimization (CMOO) problem. Then, we leverage Weighted-Chebyshev (WC) scalarization to reformulate it into a scalar-valued problem, and correspondingly propose a Chebyshev-weighted LLM unlearning algorithm (`CLEAR`) to solve the reformulated problem.

4.1 CONSTRAINED MULTI-OBJECTIVE UNLEARNING

Let θ denote the parameters of the model to be unlearned, and let θ_0 denote the original base model before unlearning. Motivated by the four practical requirements discussed above, we define four objectives for target forgetting, general retention, neighboring-domain retention, and prefilling robustness. Let $p_\theta(\cdot \mid x)$ and $p_{\theta_0}(\cdot \mid x)$ denote the next-token distributions of the current and original models, respectively. For a supervised dataset D, we define $\text{CE}_D(\theta) := \mathbb{E}_{(x,y)\sim D}[-\log p_\theta(y \mid x)]$. For distribution preservation, we denote $\text{KL}_D(\theta\|\theta_0) := \mathbb{E}_{x\sim D}[\text{KL}(p_\theta(\cdot \mid x)\|p_{\theta_0}(\cdot \mid x))]$. We instantiate the four objectives stated in Section 3(2) as follows:

$$
\begin{aligned}
&f_{\text{fg}}(\theta) = -\text{CE}_{D_{\text{fg}}}(\theta), \qquad f_{\text{rt}}(\theta) = \text{KL}_{D_{\text{rt}}}(\theta\|\theta_0), \qquad f_{\text{nbr}}(\theta) = \text{KL}_{D_{\text{nbr}}}(\theta\|\theta_0), \\
&f_{\text{rob}}(\theta) = \mathbb{E}_{(x_q, x_p^{\text{adv}}, y_{\text{ref}})\sim D_{\text{rob}}}\big[-\log p_\theta(y_{\text{ref}} \mid x_p^{\text{adv}}, x_q)\big].
\end{aligned} \tag{2}
$$

Here, D_{fg}, D_{rt}, D_{nbr}, and D_{rob} denote the forget set, general retain set, neighboring-domain retain set, and prefilling-attack set, respectively. The first objective reduces the likelihood of target answers; the two KL-based objectives preserve the original model behavior on benign and boundary-domain inputs; and the last objective trains the model to generate refusal responses y_{ref} under prefilling-style adversarial contexts x_p^{adv}. We formulate practical LLM unlearning as the following CMOO problem:

$$
\min_{\theta} \quad \mathbf{f}(\theta) = \big[f_{\text{fg}}(\theta), f_{\text{rt}}(\theta), f_{\text{nbr}}(\theta), f_{\text{rob}}(\theta)\big]^\top, \quad \text{s.t.} \quad f_i(\theta) \le \delta_i, \quad \forall i \in \mathcal{I}, \tag{3}
$$

where $\mathcal{I} := \{\text{fg}, \text{rt}, \text{nbr}, \text{rob}\}$ and δ_i specifies the ***task-level tolerance*** for the i-th objective. These thresholds can be selected relative to the original model or a desired validation tolerance, depending on the deployment requirement. This formulation explicitly prevents any single objective from being excessively compromised, which is crucial because a severe failure on one dimension can invalidate the unlearning result even when the average performance appears strong.

4.2 WEIGHTED-CHEBYSHEV: A PREFERENCE-AWARE SCALARIZATION

Solving the above CMOO problem is nontrivial. In the literature, most MOO methods are confined to the unconstrained setting and results on CMOO are very limited. In this work, we propose a Weighted-Chebyshev (WC) approach to tackle this CMOO problem in a principled manner. Toward this end, for a given preference vector $\lambda \in \Delta_{|\mathcal{I}|}^{++} := \{\lambda \in \mathbb{R}^{|\mathcal{I}|} : \lambda_i > 0,\ \sum_{i\in\mathcal{I}} \lambda_i = 1\}$ that encodes the relative importance of each objective, our WC-scalarized reformulation of the CMOO unlearning problem is written as follows:

$$\min_{\theta,\sigma} \sigma \quad \text{s.t. } f_i(\theta) \le \delta_i, \quad \lambda_i f_i(\theta) \le \sigma, \quad \forall i \in \mathcal{I}. \tag{4}$$

The variable σ upper-bounds all weighted objectives and therefore represents the current "worst-performing" objective. As a result, minimizing σ encourages the optimizer to adaptively improve the worst-performing dimension. As we will demonstrate in Section 5, this WC scalarization is equivalent to the original LLM unlearning problem. Thus, varying the preference vector λ enables the exploration of the entire Pareto-optimal solution set (i.e., Pareto front) for the CMOO problem, which implies solving the original unlearning task. This is also numerically validated in Section 6.3.

4.3 The Proposed CLEAR Algorithm

To solve Eq. (4), we introduce slack variables to convert the inequality constraints to equality constraints, and then penalize these constraints within the objective function. Specifically, let $z = (\theta, \sigma, \mu, \nu)$ be defined on the convex domain $\mathcal{D} = \{\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+^{|\mathcal{I}|} \times \mathbb{R}_+^{|\mathcal{I}|}\}$, where $\mu = [\mu_1, \ldots, \mu_{|\mathcal{I}|}]^\top, \nu = [\nu_1, \ldots, \nu_{|\mathcal{I}|}]^\top$, then, Eq. (4) can be written as:

$$\min_{z \in \mathcal{D}} \quad \Phi(z) := \sigma + \frac{s}{2} \sum_{i \in \mathcal{I}} \big(f_i(\theta) + \mu_i - \delta_i\big)^2 + \frac{r}{2} \sum_{i \in \mathcal{I}} \big(\lambda_i f_i(\theta) + \nu_i - \sigma\big)^2, \tag{5}$$

where s and r are penalty coefficients to be selected later. The coefficient s controls the enforcement of task-level tolerance constraints, while r controls the enforcement of the WC-bottleneck constraints. We illustrate the Chebyshev-weighted LLM unlearning algorithm (CLEAR) in Alg. 1. At each iteration t, CLEAR computes the objective losses with mini-batches, constructs $\Phi(z_t)$, and updates variables via a projected gradient descent step.

Algorithm 1 The CLEAR algorithm.

1: **Input:** Iteration count T, initialize $z_0 = (\theta_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$, with $\sigma_0 \ge 0$; step-size η.
2: **for** $t = 0, 1, \ldots, T-1$ **do**
3: Compute $\nabla\Phi(z_t)$.
4: Update $\theta_{t+1} = \Pi_{\mathcal{D}}\big(z_t - \eta \nabla\Phi(z_t)\big)$.
5: **end for**

5 Theoretical Analysis

In this section, we establish theoretical convergence guarantees for our proposed CLEAR algorithm.

5.1 Preliminaries

We begin by introducing the formal definition of Pareto optimality:

Definition 1 (Pareto Optimality)**.** Given the LLM unlearning objectives $i \in \mathcal{I} := \{\mathrm{fg}, \mathrm{rt}, \mathrm{nbr}, \mathrm{rob}\}$, a feasible model θ dominates a feasible model θ' if $f_i(\theta) \le f_i(\theta')$ holds for all $i \in \mathcal{I}$, with strict inequality holding for at least one i. Feasible θ is said to be Pareto-optimal if no other θ' dominates it.

Note that seeking Pareto-optimal models is NP-hard in general, and a more common goal in practice is to find locally weak Pareto optimal (LWPO) solutions defined as follows:

Definition 2 (Locally Weak Pareto Optimality)**.** A feasible model $\tilde{\theta}$ is said to be locally weakly Pareto optimal if there exists some $r > 0$, such that no feasible $\theta \in \mathcal{B}(\tilde{\theta}, r) := \{\theta' : \|\theta' - \tilde{\theta}\|_2 \le r\}$ satisfies $f_i(\theta) < f_i(\tilde{\theta})$ for all $i \in \mathcal{I}$.

While previous studies establish that a general MOO problem can be *equivalently* solved via its WC-scalarized counterpart, these guarantees do not extend to our LLM unlearning framework for two reasons: (i) The introduction of (equality) constraints, as required by our LLM unlearning problem, renders the equivalence between the two problems ambiguous. (ii) Since this well-known equivalence requires solving the problem to optimality, its counterpart under LWPO remains unclear. Toward this end, we establish the following theorem, which ensures the equivalence between the original problem and the WC-scalarized problem in terms of Pareto front exploration:

Theorem 1. *Suppose f_i is positive for all $i \in \mathcal{I}$. Then, a model $\tilde{\theta}$ is LWPO if and only if $(\tilde{\sigma}, \tilde{\theta})$ is a local solution of Eq. (4) for some $\lambda \in \Delta_{|\mathcal{I}|}^{++}$ and $\tilde{\sigma} \in \mathbb{R}$.*

The proof of Theorem 1 is deferred to the Appendix B. Intuitively, this theorem establishes an exact *one-to-one* correspondence between *the Pareto front of the LLM unlearning problem* and *the solutions*

to the associated WC-scalarized problems (as the preference vector λ varies across the simplex $\Delta_{|\mathcal{I}|}^{++}$). Consequently, focusing on solving this *more tractable* yet *strictly equivalent* WC-scalarized problem allows us to define a formal metric for the original LLM unlearning problem.

5.2 KKT-based Optimality Metric

The Karush–Kuhn–Tucker (KKT) conditions are known for characterizing optimality in mathematical optimization, particularly for constrained problems. Moreover, under mild assumptions, such as standard constraint qualifications (CQs) and second-order condition, satisfying the KKT conditions is both necessary and sufficient for local optimality. Therefore, we use the KKT conditions to derive an optimality criterion for the WC-scalarized problem. To this end, we first define the associated Lagrangian as follows: $\mathbb{L}(\theta, \sigma, \alpha, \beta) := \sigma + \sum_{i\in\mathcal{I}} \alpha_i(f_i(\theta) - \delta_i) + \sum_{i\in\mathcal{I}} \beta_i(\lambda_i f_i(\theta) - \sigma)$, where $\alpha = [\alpha_1, \ldots, \alpha_{|\mathcal{I}|}]^\top, \beta = [\beta_1, \ldots, \beta_{|\mathcal{I}|}]^\top$ denote the dual variables associated with the constraints. The KKT conditions for this problem consist of the following conditions for stationarity, primal feasibility, dual feasibility, and complementary slackness, respectively:

$$\begin{aligned} &1 - \sum_{i\in\mathcal{I}} \beta_i = 0, \;\; \sum_{i\in\mathcal{I}} (\alpha_i + \lambda_i\beta_i)\nabla f_i(\theta) = 0, && f_i(\theta) \le \delta_i, \;\; \lambda_i f_i(\theta) \le \sigma, \;\; i \in \mathcal{I}, \\ &\alpha_i \ge 0, \;\; \beta_i \ge 0, \;\; i \in \mathcal{I}, && \alpha_i(f_i(\theta) - \delta_i) = \beta_i(\lambda_i f_i(\theta) - \sigma) = 0, \;\; i \in \mathcal{I}, \end{aligned} \tag{6}$$

where $\lambda \in \Delta_{|\mathcal{I}|}^{++}$ is the preference vector. We then focus on how much each of these conditions is violated. Specifically, at the t-th iteration of CLEAR, with the dual variables α_t and β_t, we define the KKT-based metric $\mathcal{E}_t := \mathcal{E}_t^{\text{stat}} + \mathcal{E}_t^{\text{primal}} + \mathcal{E}_t^{\text{dual}} + \mathcal{E}_t^{\text{cs}}$, where, due to space limitation, we relegate the detailed definitions of $\mathcal{E}_t^{\text{stat}}$, $\mathcal{E}_t^{\text{primal}}$, $\mathcal{E}_t^{\text{dual}}$, and $\mathcal{E}_t^{\text{cs}}$ to Appendix B.

5.3 Theoretical Results

By bounding the components in the update direction at each iteration and connecting them to the corresponding error terms in the KKT system, we establish the convergence rate of CLEAR, which is stated below (see Appendix B for the full proof).

Theorem 2 (Convergence of CLEAR (Informal))**.** *Under mild assumptions and with $s = r = \Theta(T^{\frac{1}{4}})$ and $\eta = \Theta(T^{-\frac{1}{2}})$, for any $\lambda \in \Delta_{|\mathcal{I}|}^{++}$, it holds that: $\frac{1}{T}\sum_{t=0}^{T-1} \mathcal{E}_t = \mathcal{O}(T^{-\frac{1}{2}})$.*

Several important remarks regarding Theorem 2 are in order:

• To our knowledge, Theorem 2 establishes the first rigorous non-asymptotic convergence rate in the LLM unlearning literature. This result validates the effectiveness of CLEAR that aligns with our empirical results and provides a systematic theoretical foundation for LLM unlearning.

• Theorem 2 also demonstrates the Pareto exploration capability of CLEAR. Specifically, by varying the preference vector λ across the simplex $\Delta_{|\mathcal{I}|}^{++}$, we can reconstruct the entire Pareto front of the LLM unlearning problem. This allows adaptively prioritizing conflicting objectives, which significantly enhances the practical utility of our method.

6 Experiments

We evaluate CLEAR from four perspectives. First, we test whether it improves the practical unlearning trade-off among target forgetting, utility preservation, boundary retention, and adversarial robustness. Second, we analyze whether the preference vector in CLEAR enables controllable Pareto exploration. Third, we study how the penalty coefficients affect the constrained optimization behavior. Finally, we evaluate robustness under additional prompt-based and relearning attacks.

6.1 Experimental Setup

1) Models and Datasets. Our primary comparisons use two representative LLM unlearning benchmarks: i) *MUSE-Book (Shi et al., 2024)* evaluates copyrighted narrative knowledge unlearning, where the model is expected to forget the target *Harry Potter* domain while preserving general knowledge and neighboring literary domains. This benchmark provides a direct test of whether an unlearning method can avoid both under-unlearning, where target knowledge remains accessible, and over-unlearning, where benign neighboring concepts are incorrectly suppressed; and ii) *WMDP-Cyber (Li*

et al., 2024b) evaluates hazardous cybersecurity knowledge unlearning. It tests whether a model can suppress restricted cyber knowledge while maintaining benign utility and resisting adversarial prefilling attacks. Following the common setup of these benchmarks, we use Llama-3.2-3B-Instruct for MUSE-Book and Zephyr-7b-beta for WMDP-Cyber, respectively.

To support our four-objective protocol, we construct statement-style training data for Obj2–Obj4. Obj2 uses general factual statements extracted from Wikipedia. Obj3 uses neighboring-domain statements: for MUSE-Book, we sample from related fantasy narratives including *The Lord of the Rings*, *Fantastic Beasts*, *The Hobbit*, and *The Tales of Beedle the Bard*; for WMDP-Cyber, we generate benign computer-science statements following the distribution of MMLU computer science. Obj4 uses LLM-generated, non-overlapping statement-style passages that follow the same distribution as Obj1, together with corresponding prefilling attack prefixes. The test sets follow the same knowledge distributions but are instantiated as both QA and multiple-choice questions. We have released the constructed Obj2–Obj4 datasets, generation prompts and data statistics in our codebase.

2) Metrics. We evaluate practical unlearning with five metrics (arrows denote metric polarity: "↓" for lower-is-better, "↑" for higher-is-better): (1) **R-Forget** ↓: ROUGE score (Lin, 2004) on the forget domain quantifying residual similarity between LLM outputs and forgotten knowledge. (2) **R-General** ↑: ROUGE on general retain data. (3) **Acc-Neighbor** ↑: Accuracy on neighboring-domain questions. (4) **ASR-Prefill** ↓: attack success rate under prefilling attacks. (5) **MMLU (Hendrycks et al., 2020)** ↑: broad general capability. We also report the **Overall Score** ↑, computed as the equally weighted average of five scores after converting them to a higher-is-better polarity:

$$\text{Overall Score} = \frac{1}{5}\Big[(100 - \text{R-Forget}) + \text{R-General} + \text{Acc-Neighbor} + (100 - \text{ASR-Prefill}) + \text{MMLU}\Big], \tag{7}$$

Additional prompt-based and relearning attack evaluations are reported in Appendix A.3.

3) Baselines. We compare with representative training-based baselines that can be evaluated under our four-objective protocol. First, GA (Jang et al., 2023), SimNPO (Fan et al., 2024), and FLAT (Wang et al., 2025b) represent unlearning loss designs that primarily optimize target knowledge removal while attempting to preserve utility. We use SimNPO as a stronger improved variant of NPO (Zhang et al., 2024). Second, SFT (Touvron et al., 2023) and DUET (Zhong et al., 2026) represent behavior-supervision or distillation baselines: SFT directly trains desired responses, while DUET distills a prompt-steered teacher to achieve forgetting with utility preservation. Following their original evaluation setting, we convert non-QA benchmark samples into QA-style instruction-response pairs. Third, NGDiff (Jin et al., 2025) and MOLLM (Pan et al., 2025) represent gradient-editing methods for mitigating objective conflicts; we extend both methods to our multi-objective setting. Finally, SUM combines the same four objective losses used by `CLEAR` with fixed equal coefficients, serving as the most direct baseline for isolating the effect of constrained MOO optimization.

4) Hyperparameter Selection and Evaluation Protocol. All methods use the same search budget and validation-based checkpoint-selection protocol, with `CLEAR` 's preference-vector tuning included in this budget. For each method, we conduct a coarse search followed by a focused search over 10 configurations. We then fix the best configuration, rerun training with 10 random seeds, and report mean performance in the main comparisons.

6.2 Main Results on MUSE-Book

Table 2 reports the main results on MUSE-Book. In summary, `CLEAR` achieves the best performance in "Overall Score" while maintaining strong forgetting, better utility preservation than direct summation, and substantially stronger robustness than methods that mainly preserve utility. The key findings of our experimental results are summarized as follows:

1) Limitations of forget-oriented unlearning methods. Methods primarily designed for target forgetting do not consistently satisfy the broader requirements of practical unlearning. GA fails to reduce R-Forget and severely damages MMLU, while FLAT achieves strong forgetting but degrades both general and neighboring-domain performance. This shows that optimizing the forget objective alone can either fail to erase the target knowledge or erase too much useful behavior.

2) Supervised suppression may over-correct. SFT obtains very low R-Forget and ASR-Prefill, indicating strong suppression of target and adversarial queries. However, this comes at the expense of

Method	Obj1	Obj2	Obj3*	Obj4	MMLU ↑	Overall Score ↑
	R-Forget ↓	R-General ↑	Acc-Neighbor ↑	ASR-Prefill ↓		
Base Model	32.1	84.3	57.28	70.5	**60.8**	59.96
GA (⋆)	36.9	**85.0**	55.9	70.0	36.45	54.09
SimNPO (⋆)	21.4	43.1	**57.7**	70.5	60.40	53.86
FLAT (⋆)	0.7	58.3	42.7	39.0	58.92	64.04
SFT	0.3	49.6	50.7	**1.5**	50.2	69.74
DUET	3.2	71.3	26.8	39.5	49.9	61.06
NGDiff	13.6	83.3	37.7	17.9	**60.8**	70.06
MOLLM	19.9	83.6	57.4	69.2	**60.8**	62.54
SUM	**0.1**	50.18	53.1	7.0	52.6	69.76
CLEAR	0.5	74.9	54.5	13.0	**60.8**	**75.34**

Table 2: Main unlearning results on the **MUSE-Book** benchmark. Methods marked with ⋆ are primarily designed for target knowledge unlearning and are not originally optimized for the full multi-objective setting. Overall Score is computed as the average of five positive-direction metrics: 100 − R-Forget, R-General, Acc-Neighbor, 100 − ASR-Prefill, and MMLU.

large drops in R-General, Acc-Neighbor, and MMLU, suggesting an overly conservative behavior pattern. DUET provides a softer distillation-based alternative, but its neighboring-domain accuracy still drops substantially, indicating unresolved interference among different desired behaviors.

3) Gradient editing does not fully resolve the multi-angled trade-off. NGDiff and MOLLM preserve relatively strong R-General and MMLU, showing the benefit of mitigating forget-retain gradient conflicts. However, MOLLM remains vulnerable to prefilling attacks, while NGDiff still has limited R-Forget and Acc-Neighbor. Thus, gradient editing improves part of the trade-off but does not fully address boundary preservation and adversarial robustness.

4) CLEAR outperforms direct aggregation methods. SUM is the most related baseline because it uses the same objective losses as CLEAR. Although SUM achieves strong forgetting and low ASR-Prefill, it substantially reduces R-General and MMLU, suggesting that direct aggregation overemphasizes suppression-oriented objectives. In contrast, CLEAR maintains strong forgetting while improving utility and achieving the best Overall Score, demonstrating that the gain comes from constrained MOO rather than different losses or data.

We further evaluate robustness under held-out role-play, contextual reconstruction, and multi-turn attacks, as well as Forget50 and Math50 relearning attacks. CLEAR achieves the lowest ASR across all three prompt-based attacks and lower post-attack R-Forget under both relearning attacks than baselines with comparable MMLU. Appendix A.3 provides the detailed settings and results.

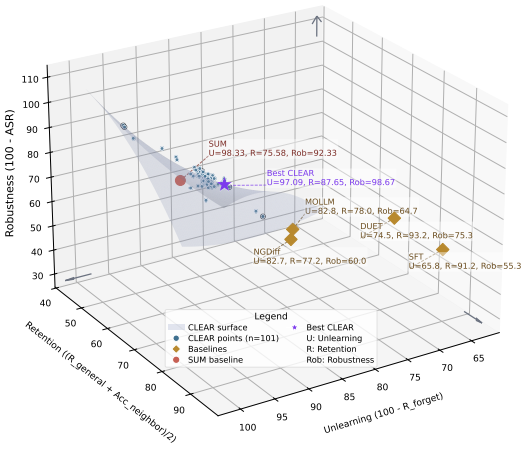

Figure 1: Trade-off visualization on WMDP-Cyber. Three axes correspond to unlearning (100 − R-Forget), retention ((R-General + Acc-Neighbor)/2), and robustness (100 − ASR-Prefill). Baselines are shown as individual points, while CLEAR forms a trade-off surface by varying the preference λ. CLEAR achieves a stronger Pareto trade-off, improving retention and robustness while maintaining strong unlearning.

6.3 Pareto Trade-off Analysis

We further evaluate the methods on the WMDP-Cyber benchmark to analyze whether CLEAR can explore the Pareto trade-off among unlearning, retention, and robustness.

1) Multi-faceted Trade-Off. To summarize the multi-objective behavior, we visualize each method in a three-dimensional space in Figure 1. The three axes correspond to unlearning, retention, and robustness, where unlearning is measured by 100 − R-Forget, retention is computed as the average of R-General and Acc-Neighbor, and robustness is measured by 100 − ASR-Prefill. Each baseline is shown as a single point,

while CLEAR produces a set of trade-off points by varying the preference vector λ. The resulting CLEAR surface reveals a continuous Pareto region rather than a single fixed solution. Compared with prior methods, CLEAR achieves a more favorable balance across the three dimensions: it maintains strong unlearning, substantially improves retention over SUM, and reaches the highest robustness among all compared methods. This demonstrates that constrained MOO not only improves a single operating point, but also enables controllable exploration of the unlearning–retention–robustness frontier.

2) Preference-vector Exploration. The preference vector λ allows CLEAR to expose different operating points under the same set of unlearning objectives. This is important because different deployment scenarios may require different trade-offs: safety-critical settings may prioritize robustness, while utility-sensitive settings may prefer stronger retention under acceptable forgetting. As shown by the CLEAR surface in Figure 1, varying λ moves the solution across the unlearning–retention–robustness space instead of collapsing all objectives into a single fixed scalarized solution. The best CLEAR point achieves a balanced trade-off, maintaining strong forgetting and robustness while substantially improving retention over the direct-sum baseline. We provide the full WMDP-Cyber quantitative comparison in Appendix A.1, and report representative CLEAR configurations from the 101-point preference-vector sweep in Appendix A.2.

6.4 Analysis of Penalty Coefficients

The penalty coefficients s and r control two complementary aspects of CLEAR. The coefficient s enforces the task-level tolerance constraints $f_i(\theta) \leq \delta_i$, while r enforces the Weighted-Chebyshev bottleneck constraints $\lambda_i f_i(\theta) \leq \sigma$. If these penalties are too small, the optimizer may under-enforce the constraints and behave similarly to a weakly regularized scalar objective. If they are too large, the penalty terms may dominate the training dynamics and reduce the flexibility of optimizing the underlying unlearning objectives.

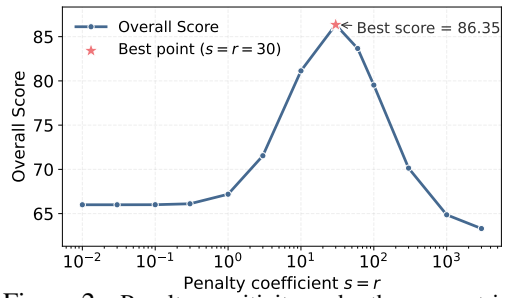

Figure 2: Penalty sensitivity under the symmetric setting $s = r$. The Overall Score first improves as the penalty strength increases and peaks around $s = r = 30$, then decreases when the penalties become overly large. This indicates that moderate penalty coefficients provide the best balance between objective optimization and constraint enforcement.

Following the theoretical setting in Sec. 5, where the two penalties grow at the same order, we study the symmetric configuration $s = r$ and vary the overall penalty strength. Fig. 2 shows the resulting Overall Score. The performance first improves as the penalty coefficient increases, indicating that explicit constraint enforcement is important for balancing the competing objectives. The best score is achieved around $s = r = 30$, after which the performance decreases as the penalties become too large. This result suggests that CLEAR benefits from judicious constraint enforcement, but extreme penalty values can over-constrain the optimization and hurt the final trade-off.

7 Conclusion

In this paper, we developed CLEAR, a constrained multi-objective framework for practical LLM unlearning. By modeling target forgetting, utility retention, neighboring-domain preservation, and adversarial robustness as competing constrained objectives, CLEAR constructs the weighted Chebyshev (WC) scalarization to optimize them and explore preference-dependent Pareto trade-offs. Our theoretical analysis established a KKT-based optimality metric and finite-time convergence rate guarantee, highlighting our constrained MOO formulation as a principled framework for reliable and controllable LLM unlearning. Comprehensive experiments showed that CLEAR achieves stronger overall trade-offs than existing baselines.

AI USE STATEMENT

We used generative AI tools to assist with the generation of auxiliary training data described in Section 6.1. The authors take responsibility for the final manuscript, experimental results, theoretical claims, and research artifacts.

ETHICS STATEMENT

We have not identified any specific ethical concerns that require additional discussion in this work.

REPRODUCIBILITY STATEMENT

To facilitate reproducibility, we provide the code for our experiments in the supplementary materials and describe the implementation details in Section 6 and Appendix A. Code is available in our anonymous repository: `https://anonymous.4open.science/r/CLEAR-6577/README.md`.

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APPENDIX

A ADDITIONAL NUMERICAL DETAILS

A.1 DETAILED QUANTITATIVE RESULTS ON WMDP-CYBER

Table 3 reports the complete numerical results on WMDP-Cyber. These results provide the quantitative basis for the three-dimensional trade-off visualization in Figure 1. For CLEAR , we report the best operating point selected by the aggregate trade-off score over unlearning, retention, and robustness. Compared with SUM, which directly adds the same objectives, CLEAR achieves comparable forgetting and robustness while substantially improving general-domain retention. This suggests that the gain does not come from changing the underlying objectives, but from optimizing them through constrained multi-objective optimization.

Method	Obj1	Obj2	Obj3	Obj4	MMLU ↑	Overall Score ↑
	R-Forget ↓	R-General ↑	Acc-Neighbor ↑	ASR-Prefill ↓		
Base Model	68.50	88.10	**96.30**	60.70	59.90	63.02
SFT	34.20	87.10	95.30	44.70	57.40	72.18
DUET	25.50	**91.10**	95.30	24.70	50.60	77.36
NGDiff	17.30	63.80	90.60	40.00	59.88	71.40
MOLLM	17.20	66.30	89.70	35.30	59.77	72.65
SUM	**1.67**	58.65	92.52	7.67	47.90	77.95
CLEAR	2.91	83.71	91.59	1.33	**60.70**	**86.35**

Table 3: Detailed WMDP-Cyber results. Overall Score is computed as the average of five positive-direction metrics: 100 − R-Forget, R-General, Acc-Neighbor, 100 − ASR-Prefill, and MMLU.

A.2 PREFERENCE-VECTOR SWEEP ON WMDP-CYBER

Table 4 reports representative configurations from the 101-point CLEAR preference-vector sweep on WMDP-Cyber, while Figure 1 visualizes the full collection of operating points. Each row corresponds to one CLEAR solution obtained under a different preference vector $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where the four dimensions correspond to target forgetting, general retention, neighboring-domain retention, and prefilling robustness, respectively.

Unlike the main quantitative table, we do not report MMLU or the aggregate score here, since this analysis focuses on how the proposed method explores different trade-offs among the four unlearning objectives. The diverse outcomes across the 101 CLEAR configurations provide the numerical basis for the trade-off surface in Figure 1, showing that CLEAR does not produce a single fixed solution but traces a family of Pareto-relevant operating points.

Setting	Preference Vector λ				R-Forget ↓	R-General ↑	Acc-Neighbor ↑	ASR-Prefill ↓
	λ_1	λ_2	λ_3	λ_4				
Balanced	0.60	0.60	0.60	0.60	3.08	71.94	89.72	2.67
Robust-heavy	0.60	0.60	0.60	1.60	2.60	76.37	88.79	3.33
Neighbor-heavy	0.60	0.60	1.00	0.60	3.03	72.14	90.65	3.33
General-heavy	1.60	0.60	0.60	1.00	2.77	79.42	89.72	4.00
Retention-oriented	1.00	0.0001	1.00	0.0001	3.51	81.79	90.65	2.00
High-retention	3.00	0.0001	1.00	1.00	3.37	82.74	**92.52**	2.00
High-unlearning	0.60	1.60	1.00	1.60	**1.63**	21.92	89.72	1.33
High-robustness	1.60	1.00	1.00	1.60	2.52	74.07	88.79	**0.67**
Utility-robust	1.00	0.001	1.00	0.01	3.70	81.19	90.65	1.33
Best CLEAR	10.00	0.0001	1.00	1.00	2.91	**83.71**	91.59	1.33

Table 4: Representative CLEAR solutions under different preference vectors on WMDP-Cyber. Lower is better for R-Forget and ASR-Prefill, while higher is better for R-General and Acc-Neighbor. The selected rows cover different regions of the CLEAR trade-off frontier; the full sweep contains 101 configurations and is visualized in Figure 1.

Method	Role-play ↓	Context ↓	Multi-turn ↓	Average ↓
Base Model	65.91	63.64	59.09	62.88
SFT	21.97	25.00	25.76	24.24
DUET	21.97	21.97	21.21	21.72
NGDiff	21.97	25.00	25.76	24.24
MOLLM	21.97	25.76	25.76	24.50
SUM	22.73	25.00	27.27	25.00
CLEAR	**13.64**	**16.67**	**19.70**	**16.67**

Table 5: Attack success rates (%) under held-out prompt-based attacks on Llama-3.2-3B-Instruct after unlearning MUSE-Book. Average is the arithmetic mean over the three attack types. Lower is better; bold indicates the lowest value in each column.

Method	Pre-attack ↓	MMLU ↑	Forget50 ↓	Δ ↓	Math50 ↓	Δ ↓
SFT	0.3	50.2	28.7	+28.4	**0.5**	+0.2
DUET	3.2	49.9	27.1	+23.9	22.9	+19.7
NGDiff	13.6	**60.8**	26.2	+12.6	17.9	+4.3
MOLLM	19.9	**60.8**	24.3	**+4.4**	18.8	**−1.1**
SUM	**0.1**	52.6	**19.5**	+19.4	10.1	+10.0
CLEAR	0.5	**60.8**	20.9	+20.4	4.7	+4.2

Table 6: Relearning robustness of Llama-3.2-3B-Instruct on MUSE-Book. Pre-attack and MMLU denote R-Forget and general capability before relearning. Forget50 and Math50 report R-Forget after fine-tuning on 50 target-domain and unrelated mathematical samples, respectively. Each Δ denotes the change in R-Forget relative to Pre-attack. CLEAR achieves lower post-attack R-Forget under both attacks than baselines with comparable MMLU, demonstrating a favorable trade-off between relearning robustness and retained general utility.

A.3 Robustness Beyond Prefilling

Held-out prompt-based attacks. We evaluate role-play elicitation, contextual reconstruction, and multi-turn elicitation without additional training of the unlearned models. These attacks respectively embed queries in expert or fictional roles, provide partial context to recover target knowledge, and progressively request target information across multiple turns. CLEAR achieves the lowest ASR across all three attacks, averaging 16.67% versus 21.72% for DUET (Table 5).

Relearning attacks and utility trade-offs. Forget50 and Math50 fine-tune the unlearned models on 50 target-domain and mathematical samples, respectively. CLEAR achieves the second-lowest post-attack R-Forget under both attacks (Table 6). Under Forget50, SUM obtains a slightly lower R-Forget than CLEAR (19.5 versus 20.9), but has substantially poorer pre-attack utility: its R-General and MMLU are 50.18 and 52.6, compared with 74.9 and 60.8 for CLEAR . Under Math50, CLEAR achieves a lower R-Forget than SUM (4.7 versus 10.1). Nevertheless, the increase from 0.5 to 20.9 under Forget50 indicates that targeted relearning remains challenging.

B Additional Theoretical Details

In this section, we first theoretically establish the equivalence between the LLM unlearning problem and the WC-scalarized problem. We then formalize our KKT-based optimality metric in detail, and finally, we present the proof of our main finite-time convergence result.

B.1 Proof of Theorem 1

Proof. We prove two directions in order.

($\Longrightarrow$) Suppose $(\tilde{\sigma}, \tilde{\theta})$ is a local solution to the WC-scalarized problem but $\tilde{\theta}$ is not LWPO for the LLM unlearning problem. Then, according to Definition 2, for any $r > 0$, there exists some $\theta \in \mathcal{B}(\tilde{\theta}, r)$ satisfying: $f_i(\theta) < f_i(\tilde{\theta}), \forall i \in \mathcal{I}$ and $f_i(\theta) \leq \delta_i, \forall i \in \mathcal{I}$.

Hence, letting $\sigma = \tilde{\sigma} - \min_i \lambda_i(f_i(\tilde{\theta}) - f_i(\theta))$, we have: (i) $\lambda_i f_i(\theta) \leq \sigma, \forall i \in \mathcal{I}$, (ii) $f_i(\theta) \leq \delta_i, \forall i \in \mathcal{I}$, (iii) $\sigma < \tilde{\sigma}$, and (iv) $(\sigma, \theta) \in \mathcal{B}((\tilde{\sigma}, \tilde{\theta}), 2r)$. This implies that $(\tilde{\sigma}, \tilde{\theta})$ is not a local solution to the WC-scalarized problem, which contradicts with the assumption.

($\Longleftarrow$) Suppose $\tilde{\theta}$ is LWPO. Then, there exists a positive r, such that no $\theta \in \mathcal{B}(\tilde{\theta}, r)$ satisfies $f_i(\theta) < f_i(\tilde{\theta}), \forall i \in \mathcal{I}$ and $f_i(\theta) \leq \delta_i, \forall i \in \mathcal{I}$. Letting $\lambda_i := \frac{f_i(\tilde{\theta})}{\sum_j f_j(\tilde{\theta})} > 0$, we have: $\lambda \in \Delta_p^{++}$ and $\lambda_i f_i(\tilde{\theta}) = \frac{1}{\sum_j f_j(\tilde{\theta})}$ for any $i \in \mathcal{I}$.

Suppose $(\tilde{\sigma}, \tilde{\theta})$ is not local solution to the WC-scalarized problem (with the given λ) for $\tilde{\sigma} = \max_i \lambda_i f_i(\tilde{\theta})$. Then, there exists some $(\sigma, \theta) \in \mathcal{B}((\tilde{\sigma}, \tilde{\theta}), r)$, such that (i) $\sigma < \tilde{\sigma}$, (ii) $f_i(\theta) \leq \delta_i, \forall i \in \mathcal{I}$, and (iii) $\lambda_i f_i(\theta) \leq \sigma, \forall i \in \mathcal{I}$. Therefore, we have: $\lambda_i f_i(\theta) \leq \sigma < \tilde{\sigma} = \lambda_i f_i(\tilde{\theta})$ for any $i \in \mathcal{I}$. This indicates that $\tilde{\theta}$ is not LWPO, which contradicts with the assumption.

This ends the proof. □

B.2 KKT-BASED METRIC

As mentioned earlier, we formulate our KKT-based optimality metric by quantifying the violation of each KKT condition (*i.e.*, stationarity, primal feasibility, dual feasibility, and complementary slackness). Based on the KKT conditions outlined above, we define the corresponding error terms at iteration t as follows:

$$\mathcal{E}_t^{\text{stat}} := \left| 1 - \sum_{i \in |\mathcal{I}|} \beta_{t,i} \right|^2 + \left| \sum_{i \in |\mathcal{I}|} (\alpha_{t,i} + \lambda_i \beta_{t,i}) \nabla f_i(\theta_t) \right|^2,$$

$$\mathcal{E}_t^{\text{primal}} := \max_{i \in \mathcal{I}} \left\{ \max\{f_i(\theta_t) - \delta_i, 0\} \right\}^2 + \max_{i \in \mathcal{I}} \left\{ \max\{\lambda_i f_i(\theta_t) - \sigma_t, 0\} \right\}^2,$$

$$\mathcal{E}_t^{\text{dual}} := \max_{i \in \mathcal{I}} \left\{ \max\{-\alpha_{t,i}, 0\} \right\}^2 + \max_{i \in \mathcal{I}} \left\{ \max\{-\beta_{t,i}, 0\} \right\}^2,$$

$$\mathcal{E}_t^{\text{cs}} := \max_{i \in \mathcal{I}} |\alpha_{t,i}(f_i(\theta_t) - \delta_i)|^2 + \max_{i \in \mathcal{I}} |\beta_{t,i}(\lambda_i f_i(\theta_t) - \sigma_t)|^2,$$

where we denote dual variables at step t by $\alpha_t = [\alpha_{t,1}, \ldots, \alpha_{t,\mathcal{I}}]^\top$ and $\beta_t = [\beta_{t,1}, \ldots, \beta_{t,\mathcal{I}}]^\top$. Consequently, $\mathcal{E}_t := \mathcal{E}_t^{\text{stat}} + \mathcal{E}_t^{\text{primal}} + \mathcal{E}_t^{\text{dual}} + \mathcal{E}_t^{\text{cs}}$ quantifies the total violation of the KKT conditions and naturally serves as our optimality metric.

B.3 PROOF OF THEOREM 2

In what follows, we first formally state the necessary assumptions, then present the detailed proof of Theorem 2.

Assumption 1. We suppose the following two assumptions hold:

- There exist some constants $M, L > 0$ and $\gamma = \Theta(s + r)$ such that for any $\theta, \theta' \in \mathbb{R}^n$, $z, z' \in \mathcal{D}$, and for any $i \in \mathcal{I}$, we have:

$$|f_i(\theta) - f_i(\theta')| \leq M\|\theta - \theta'\|,$$
$$\|\nabla f_i(\theta) - \nabla f_i(\theta')\| \leq L\|\theta - \theta'\|,$$
$$\|\nabla \Phi(z) - \nabla \Phi(z')\| \leq \gamma\|z - z'\|.$$

- For Eq. 4, we assume that linear independence constraint qualification (LICQ) holds.

Notably, the Lipschitz continuity has been numerically observed and verified in large-scale modern machine learning models, such as LLMs (Castin et al., 2023; Malladi et al., 2023; Li et al., 2024a), rendering this theoretical assumption highly practical. Moreover, one can easily verify the practicality of the LICQ condition. It merely requires the "tall and thin" $n \times m$ Jacobian matrix to have full column rank, where n is the number of model parameters and $m \leq 2|\mathcal{I}|$ is the number of active constraints.

To further validate the rationale of our smoothness assumption, we numerically evaluate the Lipschitz smoothness coefficient of our penalty function Φ. Specifically, we apply the Qwen3-14B model

with QLoRA on the Bio dataset over $T = 750$ iterations, and record the maximum, minimum, and average Lipschitz smoothness coefficients along the training trajectory. The corresponding values are approximately **10.1k**, **189**, and **1.1k**, respectively. These results are consistent with our smoothness assumption, since our theory requires $\gamma = \Theta(s + r) = \Theta(T^{\frac{1}{4}})$.

Proof. We structure the proof in three main parts.

Step $\mathfrak{A}$: We denote the update direction by $z_{t+1} = z_t - \eta d_t$, and $d_t = [d_{t,\theta}^\top, d_{t,\sigma}^\top, d_{t,\mu}^\top, d_{t,\nu}^\top]^\top$ for $t = 0, \ldots, T-1$ in `CLEAR`. According to Assumption 1 and the descent lemma, we have:

$$\begin{aligned}\Phi(z_{t+1}) \leq& \Phi(z_t) + \langle \nabla\Phi(z_t), z_{t+1} - z_t\rangle + \frac{\gamma}{2}\|z_{t+1} - z_t\|^2 \\ \leq& \Phi(z_t) - \eta\|d_t\|^2 + \frac{\gamma\eta^2}{2}\|d_t\|^2,\end{aligned}$$

for any $t \in \{0, \ldots, T-1\}$, where we apply the fact that $\langle \nabla\Phi(z_t), d_t\rangle \geq \|d_t\|^2$. This leads to:

$$\eta\Big(1 - \frac{\gamma\eta}{2}\Big)\|d_t\|^2 \leq \Phi(z_t) - \Phi(z_{t+1}),$$

which further implies:

$$\eta\Big(1 - \frac{\gamma\eta}{2}\Big)\frac{1}{T}\sum_{t=0}^{T-1}\|d_t\|^2 \leq \frac{1}{T}\big(\Phi(z_0) - \Phi(z_T)\big).$$

Step $\mathfrak{B}$: Next, we bound the KKT-based metric by relating each of its constituent terms to the previously analyzed update direction d_t. To this end, we first choose the dual variables at each step t in the following form:

$$\begin{aligned}\alpha_{t,i} &= s\big(f_i(\theta_t) + \mu_{t,i} - \delta_i\big), \\ \beta_{t,i} &= r\big(\lambda_i f_i(\theta_t) + \nu_{t,i} - \sigma_t\big),\end{aligned}$$

where we denote $\alpha_t = [\alpha_{t,1}, \ldots, \alpha_{t,|\mathcal{I}|}]^\top$ and $\beta_t = [\beta_{t,1}, \ldots, \beta_{t,|\mathcal{I}|}]^\top$ as mentioned earlier.

Step $\mathfrak{B}.1$ (Stationarity): By the selection of α_t, β_t, we can get (i) $1 - \sum_{i\in\mathcal{I}}\beta_{t,i} = \nabla_\sigma\Phi(z_t) = d_{t,\sigma}$, and (ii) $\sum_{i\in\mathcal{I}}(\alpha_{t,i} + \lambda_i\beta_{t,i})\nabla f_i(\theta_t) = \nabla_\theta\Phi(z_t) = d_{t,\theta}$.

Step $\mathfrak{B}.2$ (Dual Feasibility): According to the projection operation in `CLEAR`, we have: $\mu_{t+1,i} = \Pi_{\mathbb{R}_+}(\mu_{t,i} - \eta\nabla_{\mu_i}\Phi(z_t))$. We then consider the following two cases: (i) If $\mu_{t,i} - \eta\nabla_{\mu_i}\Phi(z_t) > 0$, then the projection is not performed, and we accordingly have: $\alpha_{t,i} = d_{t,\mu_i}$. (ii) Otherwise, the projection is active, and we have $\alpha_{t,i} \geq \frac{\mu_{t,i}}{\eta} \geq 0$. Similarly, since $\nu_{t+1,i} = \Pi_{\mathbb{R}_+}(\nu_{t,i} - \eta\nabla_{\nu_i}\Phi(z_t))$, we have $\beta_{t,i} = d_{t,\nu_i}$ if the projection is inactive and $\beta_{t,i} \geq \frac{\nu_{t,i}}{\eta} \geq 0$ otherwise.

Step $\mathfrak{B}.3$ (Primal Feasibility): According to the dual variables selection, we have:

$$\begin{aligned}f_i(\theta_t) - \delta_i &= \frac{\alpha_{t,i}}{s} - \mu_{t,i} \leq \frac{\alpha_{t,i}}{s}, \\ \lambda_i f_i(\theta_t) - \sigma_t &= \frac{\beta_{t,i}}{r} - \nu_{t,i} \leq \frac{\beta_{t,i}}{r}.\end{aligned}$$

We first consider the variable β_t. By considering the update direction w.r.t. σ_t, it's easy to verify that $|\beta_{t,i}| \leq 1 + (|\mathcal{I}| + 1)\|d_t\|$ and $\|\beta_t\| \leq 1 + (2|\mathcal{I}| + 1)\|d_t\|$.
We then denote:

$$\begin{aligned}I(\theta_t) &:= \{i \in \mathcal{I} : f_i(\theta_t) \geq \delta_i\}, \\ J(\theta_t) &:= \{i \in \mathcal{I} : f_i(\theta_t) < \delta_i\}.\end{aligned}$$

We prove that $\mu_{t,i} - \eta\alpha_{t,i} \leq 0$ implies $i \in I(\theta_t)$ when $\eta < \frac{1}{s}$. Suppose this does *not* hold, *i.e.*, $i \in J(\theta_t)$. Then, $\mu_{t,i} - \eta\alpha_{t,i} \leq 0$ implies that $\delta_i - f_i(\theta_t) \leq (1 - \frac{1}{\eta s})\mu_{t,i} \leq 0$. Thus, we know that $i \in I(\theta_t)$ in this case, meaning that it corresponds to an active constraint. Consequently, denoting the

smallest singular value of the Jacobian as $c_0 > 0$ (which exists due to Assumption 1). According to the update direction w.r.t. θ_t and Assumption 1, we have:

$$\begin{aligned}&\sum_{i\in I(\theta_t)} \alpha_{t,i}\nabla f_i(\theta_t)\\ =&d_{t,\theta} - \sum_{i\in J(\theta_t)} \alpha_{t,i}\nabla f_i(\theta_t) - \sum_{i\in\mathcal{I}} \lambda_i\beta_{t,i}\nabla f_i(\theta_t),\end{aligned}$$

which implies:

$$\begin{aligned}\|\alpha_t\| &\le \frac{1}{c_0}\big(\|d_t\| + M\sqrt{|\mathcal{I}|}\|d_t\|\\ &\quad + M(1+(|\mathcal{I}|+1)\|d_t\|)\big) + \sqrt{|\mathcal{I}|}\|d_t\|\\ &\triangleq c_1 + c_2\|d_t\|,\end{aligned}$$

where c_1, c_2 denotes positive constants, and we have $c_2 = \Theta(\mathcal{I})$.

Step $\mathfrak{B}.4$ (Complementary Slackness): We then consider the last term in the KKT-based metric. We have:

$$-\delta_i \le f_i(\theta_t) - \delta_i = \frac{\alpha_{t,i}}{s} - \mu_{t,i} \le \frac{\alpha_{t,i}}{s}.$$

If $\mu_{t,i} - \eta\nabla_{\mu_i}\Phi(z_t) > 0$, then, as shown earlier, we have $\alpha_{t,i} = d_{t,\mu_i}$, which leads to:

$$|\alpha_{t,i}(f_i(\theta_t) - \delta_i)| \le \|d_t\| \cdot \big(\delta_i + \frac{\|d_t\|}{s}\big).$$

Otherwise, we have $|\alpha_{t,i}(f_i(\theta_t) - \delta_i)| \le \frac{\alpha_{t,i}^2}{s}$. Then, we can utilize the same argument in the last step to control this term.
Similarly, we can also get:

$$-\sigma_t \le \lambda_i f_i(\theta_t) - \sigma_t = \frac{\beta_{t,i}}{r} - \nu_{t,i} \le \frac{\beta_{t,i}}{r}.$$

If $\nu_{t,i} - \eta\nabla_{\nu_i}\Phi(z_t) > 0$, then $\beta_{t,i} = d_{t,\nu_i}$ leads to:

$$|\beta_{t,i}(\lambda_i f_i(\theta_t) - \sigma_t)| \le \|d_t\| \cdot \big(\Phi(z_0) + \frac{\|d_t\|}{r}\big).$$

Otherwise, we have $|\beta_{t,i}(\lambda_i f_i(\theta_t) - \sigma_t)| \le \frac{\beta_{t,i}^2}{r}$.

Step $\mathfrak{C}$: Finally, we aggregate the intermediate results obtained in Step $\mathfrak{B}$ and establish their explicit relationship with the convergence guarantee from Step $\mathfrak{A}$.
For these error terms, we have:

$$\begin{aligned}\mathcal{E}_t &:= \mathcal{E}_t^{\text{stat}} + \mathcal{E}_t^{\text{primal}} + \mathcal{E}_t^{\text{dual}} + \mathcal{E}_t^{\text{cs}}\\ &= |d_{t,\sigma}|^2 + \|d_{t,\theta}\|^2\\ &\quad + \mathcal{O}\left(\frac{1}{s^2} + \frac{1}{r^2} + (\frac{|\mathcal{I}|^2}{s^2} + \frac{|\mathcal{I}|^2}{r^2})\|d_t\|^2\right)\\ &\quad + |d_{t,\mu_i}|^2 + |d_{t,\nu_i}|^2\\ &\quad + \mathcal{O}\left(\|d_t\|^2 + \frac{\|d_t\|^4}{s^2} + \frac{\|d_t\|^4}{r^2}\right).\end{aligned}$$

Therefore, we can get:

$$\mathcal{E}_t = \mathcal{O}\left(|\mathcal{I}|^2\|d_t\|^2 + \frac{1}{s^2} + \frac{1}{r^2}\right).$$

When $\eta = \mathcal{O}(\frac{1}{\gamma}) = \mathcal{O}(\frac{1}{s+r})$, we have:

$$\eta\Big(1 - \frac{\gamma\eta}{2}\Big)\frac{1}{T}\sum_{t=0}^{T-1}\|d_t\|^2 \le \frac{1}{T}\big(\Phi(z_0) - \Phi(z_T)\big),$$

which implies:

$$\frac{1}{T}\sum_{t=0}^{T-1}\|d_t\|^2 = \mathcal{O}\left(\frac{s+r}{\eta T}\right).$$

Accordingly, by letting $s = r = \Theta(T^{\xi})$ and $\eta = \Theta(T^{-\zeta})$, we can get:

$$\begin{aligned}\frac{1}{T}\sum_{t=0}^{T-1}\mathcal{E}_t =& \frac{1}{T}\sum_{t=0}^{T-1}\mathcal{O}\left(|\mathcal{I}|^2\|d_t\|^2 + \frac{1}{s^2} + \frac{1}{r^2}\right)\\ =& \mathcal{O}\left(|\mathcal{I}|^2 T^{\xi-\zeta-1} + T^{-2\xi}\right).\end{aligned}$$

We then select these degrees to determine fast convergence, with a constraint $\zeta \geq \xi$. Therefore, by setting $\xi = \frac{1}{4}$ and $\zeta = \frac{1}{2}$, and by noting that $|\mathcal{I}|$ is a constant in our problem, we have: $\frac{1}{T}\sum_{t=0}^{T-1}\mathcal{E}_t = \mathcal{O}(T^{-\frac{1}{2}})$. □