PHYSICS-INFORMED NEURAL PDE SOLVERS VIA SPATIO-TEMPORAL MEANFLOW

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ABSTRACT

Deep learning paradigms, such as Physics-Informed Neural Networks (PINNs) and neural operators, have significantly advanced the solving of partial differential equations (PDEs). However, they often struggle to capture the continuous integral nature of physical systems, relying either on pointwise residuals that ignore the integral perspective or on pre-discretized temporal grids. Drawing inspiration from MeanFlow, a continuous-time integrator recently developed to efficiently solve generative ODEs, we introduce Spatio-Temporal MeanFlow, which functions as a novel PDE solver learning the finite-interval evolution of physical states. By substituting the generative velocity field with the physical PDE operator, we transform multi-step numerical integration into an efficient prediction with a controllable integration length. Crucially, we extend the original MeanFlow loss from the temporal to spatio-temporal domain, coupling time evolution with spatial consistency. This yields a unified framework naturally accommodating both time-dependent and stationary PDEs. Comprehensive experiments on canonical benchmarks demonstrate that our approach achieves superior accuracy over representative baselines. Furthermore, the proposed integral constraint enables excellent zero-shot generalization to out-of-distribution initial conditions and varying spatial resolutions.

1 INTRODUCTION

Solving partial differential equations (PDEs) efficiently and accurately is fundamental to modeling complex physical phenomena across science and engineering, such as shock wave propagation and fluid turbulence (Bandrauk et al., 2007; Madhavi, 2025; Glimm, 1991). Fundamentally, this mathematical process is a continuous integration process. For example, for a time-dependent PDE governed by $\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{f}[\boldsymbol{u}, \nabla, \boldsymbol{x}, t]$, the evolution of the state field $\boldsymbol{u}(\boldsymbol{x}, t)$ across the continuous spatial domain from time τ to t is the integral:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, \tau) + \int_{\tau}^{t} \boldsymbol{f}[\boldsymbol{u}(\boldsymbol{x}, s), \nabla, \boldsymbol{x}, s] \, \mathrm{d}s. \tag{1}$$

While the two dominant deep learning paradigms for PDE solving, namely Physics-Informed Neural Networks (PINNs) (Cuomo et al., 2022), which embed governing equations as pointwise loss penalties, and neural operators like FNO (Li et al., 2020a), which learn data-driven mappings between function spaces, have achieved remarkable success, they struggle to fully capture the continuous integral process defined in Eq. (1). Specifically, FNO relies heavily on pre-discretized time slices and treats the integration interval as a fixed hyperparameter, essentially functioning as a discrete, step-by-step integrator. Conversely, PINNs rely solely on the pointwise PDE formula itself, bypassing the integral perspective entirely. Grounded in the formulation of Eq. (1), we argue that a more straightforward approach is to directly train a neural integrator capable of operating flexibly in continuous domain.

A similar challenge has been extensively studied in generative modeling, such as diffusion models (Croitoru et al., 2023; Yang et al., 2023; Luo, 2022) and flow matching (Lipman et al., 2022), where the inference process can also be formulated as an ordinary differential equation (ODE). Consequently, several works strive to develop efficient methods to solve this generative ODE (Liu et al., 2022; Song et al., 2023). One notable model is MeanFlow (Geng et al., 2025), which directly learns the average velocity over a finite interval using neural networks. By doing so, it essentially functions as a continuous-time integrator whose core advantage lies in bypassing repeated velocity evaluations,

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transforming multi-step integration into an single-step prediction. This provides a powerful intuition: can we adapt this mechanism to construct a continuous-time neural integrator for PDE solving?

Inspired by this, we adapt the continuous integral formulation of MeanFlow to PDE solving. Specifically, we replace the generative velocity field $\boldsymbol{v}$ in the MeanFlow identity with the PDE right-hand side $\boldsymbol{f}$, and use the resulting identity to parameterize the finite-interval evolution of PDE solutions. The original MeanFlow constraint characterizes temporal evolution alone. For PDEs, however, generalization also depends critically on spatial consistency. We therefore extend the MeanFlow from the temporal domain to the spatio-temporal domain, introducing a spatio-temporal MeanFlow loss tailored to PDE solving that strengthens the model's ability to generalize across spatial locations and resolutions. Furthermore, this formulation naturally accommodates static PDEs by specializing to a purely spatial MeanFlow loss, providing a unified framework for both time-dependent and stationary PDEs. By transcending purely data-driven learning, the model achieves superior generalization to out-of-distribution initial conditions and varied spatial resolutions. Our contributions are:

- **A MeanFlow-inspired PDE solver framework**: We adapt the MeanFlow principle to deterministic PDE solving, enabling flexible inference by replacing conventional multi-step numerical integration with finite-interval predictions, whose integration length can be freely controlled at inference time.
- **A spatio-temporal extension of MeanFlow**: We extend MeanFlow from the temporal domain to the spatio-temporal domain and introduce corresponding physics-informed constraints for PDE solving. This spatial extension naturally accommodates steady-state PDEs, yielding a unified framework for both time-dependent and steady-state problems.
- **Comprehensive experimental validation**: Experiments on four canonical PDE benchmarks demonstrate improved prediction accuracy with efficient inference across time-dependent and steady-state problems. Ablations validate the contributions of the temporal and spatial MeanFlow losses, while further analyses demonstrate flexible integration steps, and zero-shot generalization to higher resolutions and out-of-distribution initial conditions.

2 Preliminaries

MeanFlow. For states $\boldsymbol{u}_t$ ($\boldsymbol{z}_t$ in Geng et al. (2025)), MeanFlow (Geng et al., 2025) achieves one-step prediction by modeling the average velocity $\boldsymbol{m}$ ($\boldsymbol{u}$ in Geng et al. (2025)) along ODE $\frac{\mathrm{d}\boldsymbol{u}_t}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{u}_t, t)$: $\boldsymbol{m}(\boldsymbol{u}_t, \tau, t) \triangleq \frac{1}{t-\tau}\int_\tau^t \boldsymbol{v}(\boldsymbol{u}_s, s)\mathrm{d}s$ between time τ (r in Geng et al. (2025)) and t, rather than multi-step integration in Flow Matching. Then, by taking the total derivative with respect to time t on both sides, we obtain the **MeanFlow Identity** in Eq. 2, whose right-hand side serves as the training objective for network $\boldsymbol{m}_\theta(\boldsymbol{u}_t, \tau, t)$. (derived in App. A)

$$\boldsymbol{m}(\boldsymbol{u}_t, \tau, t) = \boldsymbol{v}(\boldsymbol{u}_t, t) - (t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}(\boldsymbol{u}_t, \tau, t). \tag{2}$$

Problem Formulation. We formulate the PDE system over a spatial domain $D \subset \mathbb{R}^n$ and time $t \in [0, T]$ as follows:

$$\gamma\frac{\partial \boldsymbol{u}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{f}[\boldsymbol{u}(\boldsymbol{x},t), \nabla, \boldsymbol{x}, t, \boldsymbol{a}], \quad \boldsymbol{x} \in D,\ t \in (0, T], \tag{3}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in D, \tag{4}$$

$$\mathcal{B}[\boldsymbol{u}(\boldsymbol{x},t)] = \boldsymbol{g}(\boldsymbol{x},t), \qquad \boldsymbol{x} \in \partial D,\ t \in [0, T]. \tag{5}$$

Here, $\boldsymbol{u} \in \mathbb{R}^d$ is the state variable, $\boldsymbol{f}$ is the differential operator with parameters $\boldsymbol{a}$, while $\boldsymbol{u}_0$ and $\mathcal{B}$ denote the initial condition and boundary operator, respectively. The coefficient $\gamma \in \mathbb{R}$ unifies time-variant ($\gamma \neq 0$) and time-invariant ($\gamma = 0$) PDEs. Notably, for static systems ($\gamma = 0$), the time dimension t acts as a dummy variable, yielding a constant state $\boldsymbol{u}(\boldsymbol{x}, t) \equiv \boldsymbol{u}_{\text{static}}(\boldsymbol{x})$ along the temporal axis, which naturally accommodates our unified spatio-temporal modeling.

3 Methodology

This section presents the Spatio-Temporal MeanFlow framework. Sec. 3.1 and 3.2 formulate the problem via integration paths and derive physics-informed governing constraints. To mitigate temporal-spatial scale mismatches, Sec. 3.3 introduces a decoupling strategy with a theoretical

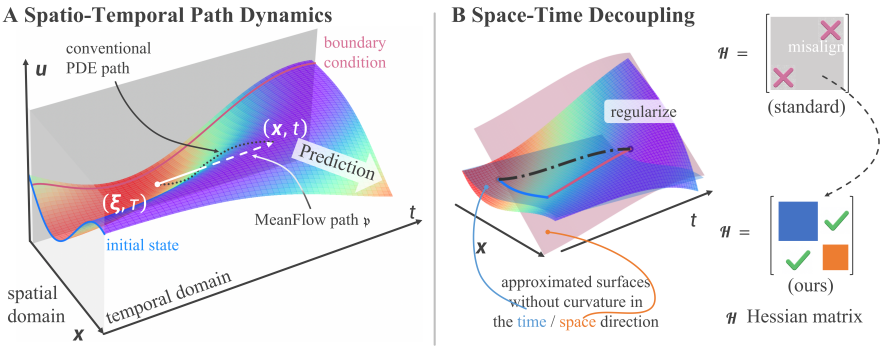


Figure 1: Overview of our framework. (**A**) Spatio-Temporal Path Dynamics, formulating PDE solutions via straight paths. (**B**) Space-Time Decoupling, constraining the surface in either space or time direction only, designed to mitigate scale mismatches within the Hessian matrix.

decoupling bound. Sec. 3.4 integrates these into a unified objective function and details the specific algorithms. Finally, Sec. 3.5 establishes theoretical foundations, providing rigorous justifications for consistency, a-posteriori error bounds, and improved optimization dynamics.

3.1 Spatio-Temporal MeanFlow Formulation

We consider the partial differential equation (PDE) system defined by Eq. 3. To relate the states between a starting point $(\boldsymbol{\xi}, \tau)$ and a target point $(\boldsymbol{x}, t)$, we define a spatio-temporal flow path $\mathfrak{p} : [0, 1] \mapsto D \times [0, T]$ such that $\mathfrak{p}(0) = (\boldsymbol{\xi}, \tau)$ and $\mathfrak{p}(1) = (\boldsymbol{x}, t)$ (Fig. 1 A). We then define the spatio-temporal MeanFlow function $\boldsymbol{m}$ as the integral average of the state evolution along path $\mathfrak{p}$:

$$\boldsymbol{m}[\boldsymbol{u}(\boldsymbol{\xi}, \tau), \mathfrak{p}, \boldsymbol{a}] \triangleq \int_{\mathfrak{p}} \mathrm{d}\boldsymbol{u}(\mathfrak{p}) \;/\; \int_{\mathfrak{p}} \|\mathrm{d}\mathfrak{p}\|_2. \tag{6}$$

For a convex domain D, we consider a straight path $\mathfrak{p}(s) \triangleq [\boldsymbol{\xi} + s(\boldsymbol{x} - \boldsymbol{\xi}), \tau + s(t - \tau)]$ and define path length $l(\boldsymbol{\xi}, \tau, \boldsymbol{x}, t) \triangleq \int_{\mathfrak{p}} \|\mathrm{d}\mathfrak{p}\|_2 = \sqrt{\|\boldsymbol{x} - \boldsymbol{\xi}\|_2^2 + (t - \tau)^2}$ which is the total spatio-temporal Euclidean distance. Following the Fundamental Theorem of Line Integrals, Eq. 6 satisfies

$$l(\boldsymbol{\xi}, \tau, \boldsymbol{x}, t) \cdot \boldsymbol{m}[\boldsymbol{u}(\boldsymbol{\xi}, \tau), \boldsymbol{\xi}, \tau, \boldsymbol{x}, t; \boldsymbol{a}] = \boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{u}(\boldsymbol{\xi}, \tau). \tag{7}$$

Note that using a straight path requires D to be convex in classical analysis, as it might temporarily exit the physical boundary of non-convex domains (e.g., fluid flow around obstacles). For non-convex domain, current model requires an integration along geodesic paths for physical plausibility.

In our formulation, the proposed path integration and differentiation operators rely on two fundamental mathematical prerequisites: 1) The state variable $\boldsymbol{u}$ is sufficiently smooth. Specifically, we require $\boldsymbol{u} \in \mathcal{C}^1(D \times [0, T])$ for first-order PDEs, and $\boldsymbol{u} \in \mathcal{C}^2$ for systems involving second-order spatial derivatives. This guarantees that the Jacobian-based operators (e.g., $\boldsymbol{K}$ in Sec. 3.2) are well-defined; 2) The domain is topologically connected to ensure the physical meaningfulness of bridging any two states. These assumptions restrict the current theoretical guarantees to sufficiently smooth PDE solutions on convex domain. PDEs with shocks, discontinuities, non-differentiable interfaces, or non-convex domains are outside the scope of the present analysis.

3.2 Physics-Informed MeanFlow Constraints

Let $\boldsymbol{u}, \boldsymbol{m}, \boldsymbol{f} \in \mathbb{R}^d$, $\boldsymbol{\xi}, \boldsymbol{x} \in D \subset \mathbb{R}^n$, and $\tau, t \in [0, T]$, with $l_t = t - \tau$, $l_s = \|\boldsymbol{x} - \boldsymbol{\xi}\|_2$, and $l = \sqrt{l_t^2 + l_s^2}$. By differentiating the fundamental identity Eq. (7) with respect to the initial coordinates $(\boldsymbol{\xi}, \tau)$ respectively, we can hence obtain the physics-informed constraints for our framework, i.e.,

$$\begin{cases} l_t \cdot \boldsymbol{m} = l\boldsymbol{K}\boldsymbol{f} + \gamma l^2 \cdot \frac{\partial \boldsymbol{m}}{\partial \tau}, & \triangleleft \text{ derivative in time} \quad (8) \\ l \cdot \boldsymbol{K} \cdot \nabla^\top \boldsymbol{u}(\boldsymbol{\xi}, \tau) = \boldsymbol{m}(\boldsymbol{x} - \boldsymbol{\xi})^\top - l^2 \cdot \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}, & \triangleleft \text{ 1st-order derivative in space} \quad (9) \\ l \cdot \boldsymbol{K} \cdot \Delta \boldsymbol{u}(\boldsymbol{\xi}, \tau) = -(\boldsymbol{\alpha} + 2\boldsymbol{K}^{-1}\boldsymbol{\beta}), & \triangleleft \text{ 2nd-order derivative in space} \quad (10) \\ \cdots, & \triangleleft \text{ higher-order derivatives} \quad (11) \end{cases}$$

where $\boldsymbol{K}_{d\times d} \triangleq \boldsymbol{I}_{d\times d} + l\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}$ is the Jacobian-based correction operator which is assumed to be locally nonsingular and $\gamma \in \{0, 1\}$ is the temporal indicator, with $\gamma = 1$ for time-dependent systems and $\gamma = 0$ for stationary systems. The equalities provide constraints on the Jacobian $\nabla^\top \boldsymbol{u}(\boldsymbol{\xi}, \tau) \in \mathbb{R}^{d\times n}$ and the Laplacian $\Delta \boldsymbol{u}(\boldsymbol{\xi}, \tau) \in \mathbb{R}^d$ (where $\Delta \boldsymbol{u}_i(\boldsymbol{\xi}, \tau) = \sum_j \frac{\partial^2 \boldsymbol{u}_i}{\partial \boldsymbol{\xi}_j^2}$) matrices or vectors.

Here, $\boldsymbol{\alpha} \triangleq n\boldsymbol{m} - \frac{l_s^2}{l^2} \cdot \boldsymbol{m} + 2l^2 \Delta \boldsymbol{m}(\boldsymbol{\xi}, \tau) \in \mathbb{R}^d$ and $\boldsymbol{\beta} \triangleq -\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}(\boldsymbol{x} - \boldsymbol{\xi}) - \frac{l_s^2}{l} \cdot \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}} \boldsymbol{m} \in \mathbb{R}^d$ are the auxiliary vectors for clear expression. Prop. D.1, Appendix D provides more detailed derivation. Consequently, the spatio-temporal MeanFlow loss is defined as

$$\mathcal{L}_{\text{ST-MeanFlow}} \triangleq \|l_t \cdot \boldsymbol{m} - l\boldsymbol{K}\boldsymbol{f} - \gamma l^2 \cdot \frac{\partial \boldsymbol{m}}{\partial \tau}\|_2^2, \tag{12}$$

with the gradient terms in $\boldsymbol{f}$ substituted by Eqs. (9).

3.3 Space-Time Decoupling Strategy

In many physical systems, temporal evolution and spatial variations occur at vastly different magnitudes (e.g., fast chemical reactions in large-scale reactors) (Wang et al., 2021; Krishnapriyan et al., 2021). When these dimensions are tightly coupled in a single loss function, the optimization landscape often exhibits pathological curvature, characterized by a highly ill-conditioned Hessian matrix (Wang et al., 2022) with large off-diagonal cross-coupling terms (Fig. 1 B). To solve the challenge of inherent *spatio-temporal scale mismatch*, we decompose the spatio-temporal objective into two independent first-order components:

Temporal MeanFlow Loss. $\mathcal{L}_{\text{T-MeanFlow}} \triangleq \left\|\boldsymbol{m} - \boldsymbol{K}\boldsymbol{f} - \gamma l_t \frac{\partial \boldsymbol{m}}{\partial \tau}\right\|_2^2$ constrains the network to satisfy the temporal evolution predicted by the Spatio-Temporal MeanFlow identity along the time-line.

Spatial MeanFlow Loss. $\mathcal{L}_{\text{S-MeanFlow}} \triangleq \left\|l_s \boldsymbol{K}\nabla^\top \boldsymbol{u} - \boldsymbol{m}(\boldsymbol{x} - \boldsymbol{\xi})^\top + l_s^2 \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\right\|_2^2$ enforces consistency between predicted Spatio-Temporal MeanFlow's spatial derivatives and numerical spatial gradients.

Thm. 3.2, Sec. 3.5 ensures that the decoupling provide a theoretical reference for the overall ST-MeanFlow loss. The core advantage of decoupling is the removal of explicit spatio-temporal cross-derivatives in the residual definitions. Mathematically, the decoupled formulation removes the explicit cross-Jacobian curvature terms from the Gauss–Newton approximation to Hessian. Under the spectral-alignment conditions (Thm. D.3, App. D.4), this improves the local condition number and can lead to faster and more robust first-order optimization. This decoupling is also theoretically supported by the *Space-Time Decoupling Bound* presented in the following section, which ensures that minimizing the independent losses sufficiently constrains the spatio-temporal error.

3.4 Detailed Implemention

Training. We parameterize the MeanFlow predictor $\boldsymbol{m}$ using a neural network, denoted as $\boldsymbol{m}_\theta$, to approximate the true $\boldsymbol{m}$. For time-dependent PDEs, the network is designed to map the concatenated input $[\boldsymbol{u}(\boldsymbol{x}, \tau), \tau, \boldsymbol{x}, t, \boldsymbol{a}]$, where τ and t are randomly sampled time pairs during training. The model outputs the average integral from time τ to t. According to the fundamental identity (Eq. (7)), the target state at time t is explicitly constructed as $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, \tau) + (t - \tau) \cdot \boldsymbol{m}_\theta$. The total objective function is defined as the weighted sum of the data loss (measuring the discrepancy between the predicted solution at time t and observational data) and the physics-informed decoupled spatio-temporal MeanFlow losses:

$$\mathcal{L}_{\text{Total}} = \mathcal{L}_{\text{Data}} + \lambda_t \mathcal{L}_{\text{T-MeanFlow}} + \lambda_s \mathcal{L}_{\text{S-MeanFlow}}, \tag{13}$$

where λ_t and λ_s are the weighting coefficients.

Conversely, when $\gamma = 0$, the system degenerates into a static PDE. In this scenario, temporal variables are omitted. The model instead takes the analytically extended boundary conditions $\boldsymbol{u}_0(\boldsymbol{x})$ (where $\boldsymbol{u}_0|_{\partial D} = \boldsymbol{u}|_{\partial D}$) and the coefficient field $\boldsymbol{a}$ as inputs. The solution over the interior domain $\boldsymbol{x} \in D\backslash\partial D$ is constructed as $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{m}_\theta$, where $\boldsymbol{m}_\theta$ represents the network output. The temporal loss naturally vanishes, and the training objective simplifies to: $\mathcal{L}_{\text{Total}} = \mathcal{L}_{\text{Data}} + \lambda_s \mathcal{L}_{\text{S-MeanFlow}}$. Alg. 1 detail the training optimization procedures of the proposed framework.

Note that in our practical implementation, considering that the benchmark PDEs evaluated in this work are defined on regular Cartesian grids, we maintain fixed spatial coordinates ($\boldsymbol{x}$) across inputs

Algorithm 1 Training

Require: PDE params $\boldsymbol{a}$, temporal indicator γ, network $\boldsymbol{m}_\theta$, learning rate η

for each training iteration **do**
  if $\gamma \neq 0$ **then** ▷ Time-dependent PDEs
    Sample time pairs $\tau < t$ and grid $\boldsymbol{x}$;
    Get input: $\boldsymbol{I} \leftarrow [\boldsymbol{u}(\boldsymbol{x}, \tau), \tau, \boldsymbol{x}, t, \boldsymbol{a}]$;
    Predict MeanFlow: $\boldsymbol{m} \leftarrow \boldsymbol{m}_\theta(\boldsymbol{I})$;
    Compute $\boldsymbol{u}(\boldsymbol{x}, t) \leftarrow \boldsymbol{u}(\boldsymbol{x}, \tau) + (t - \tau)\boldsymbol{m}$;
    Compute Loss: $\mathcal{L}_{\text{Total}}$;
  else ▷ Static PDEs ($\gamma = 0$)
    Analytically extend boundary conditions to the entire domain as $\boldsymbol{u}_0(\boldsymbol{x})$;
    Get input: $\boldsymbol{I} \leftarrow [\boldsymbol{u}_0(\boldsymbol{x}), \boldsymbol{a}]$;
    Predict MeanFlow: $\boldsymbol{m} \leftarrow \boldsymbol{m}_\theta(\boldsymbol{I})$;
    Construct state: $\boldsymbol{u}(\boldsymbol{x}) \leftarrow \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{m}$;
    Compute Loss: $\mathcal{L}_{\text{Total}} \leftarrow \mathcal{L}_{\text{Data}} + \lambda_s \mathcal{L}_{\text{S-MF}}$;
  end if
  Update parameters: $\theta \leftarrow \theta - \eta \nabla_\theta \mathcal{L}_{\text{Total}}$;
end for
return $\theta^* = \theta$;

Algorithm 2 Multi-step Inference

Require: Pretrained network $\boldsymbol{m}_{\theta^*}$, initial state $\boldsymbol{u}_0 = \boldsymbol{u}(\boldsymbol{x}, 0)$, PDE params $\boldsymbol{a}$, target time T, steps N

Calculate time step: $\Delta t \leftarrow T/N$;
Initialize trajectory list with $\boldsymbol{u}_0$;
Initialize current time: $t_0 \leftarrow 0$;
for $k = 0$ to $N - 1$ **do**
  Target time: $t_{k+1} \leftarrow t_k + \Delta t$;
  Get input: $\boldsymbol{I}_k \leftarrow [\boldsymbol{u}_k, t_k, \boldsymbol{x}, t_{k+1}, \boldsymbol{a}]$;
  Predict average integral (MeanFlow): $\boldsymbol{m}_k \leftarrow \boldsymbol{m}_{\theta^*}(\boldsymbol{I}_k)$;
  Update state via continuous integral: $\boldsymbol{u}_{k+1} \leftarrow \boldsymbol{u}_k + \Delta t \cdot \boldsymbol{m}_k$;
  Append $\boldsymbol{u}_{k+1}$ to trajectory list;
end for
return $\boldsymbol{u}_T$;

and outputs. Consequently, our current network design predominantly emphasizes continuous-time evolution. In such setup, the spatial MeanFlow term is instantiated with a fixed local offset $\boldsymbol{\xi} = \boldsymbol{x} - \Delta\boldsymbol{x}$ on Cartesian grids. Its derivatives are evaluated numerically on the grid. The fully continuous formulation with independently varying anchor point $\boldsymbol{\xi}$ and the target point $\boldsymbol{x}$ is described in App. C, but is not the default implementation used in the main experiments.

Inference. We first focus on the one-step inference scenario. For time-dependent PDEs, standard deep learning-based solvers typically take the initial condition and physical coefficients as inputs to predict the solution over the spatial domain at a target time t. Following this paradigm, our algorithm takes $[\boldsymbol{u}(\boldsymbol{x}, 0), 0, \boldsymbol{x}, t, \boldsymbol{a}]$ as input during inference to predict the average integral from time 0 to t, denoted as $\boldsymbol{m}_{\boldsymbol{\theta}}(\boldsymbol{x}, t)$. The final solution at time t is then recovered via

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, 0) + t \cdot \boldsymbol{m}_\theta(\boldsymbol{x}, t). \tag{14}$$

For static systems, the general objective is to derive the global solution given specific boundary conditions and coefficient fields. Maintaining consistency with our training phase, the model simply takes the analytically extended boundary conditions and the coefficient field as inputs and then get the solution across the spatial domain through $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{m}_\theta$. Furthermore, for time-dependent PDEs, since our framework models the continuous integral over arbitrary time intervals, it naturally extends to multi-step inference. The detailed procedure for multi-step prediction is provided in Alg. 2.

3.5 Theoretical Results

The proposed framework is supported by two core theoretical pillars: a-posteriori error bounds for the predicted state (Thm. 3.1), and effectiveness of the decoupled loss terms (Thm. 3.2). App. D.2-D.4 provide proofs for them, with the improved optimization landscapes through decoupling (Thm. D.3).

For simplicity, we define $\boldsymbol{r}_{\text{MeanFlow}} \triangleq l_t \boldsymbol{m} - l\boldsymbol{K}\boldsymbol{f} - \gamma l^2 \frac{\partial \boldsymbol{m}}{\partial \tau}$ with higher-order spatial gradients in $\boldsymbol{h}$ substituted by spatial MeanFlow Eqs. (9), $\boldsymbol{r}_{\text{Temp}} \triangleq \boldsymbol{m} - \boldsymbol{K}\boldsymbol{f} - \gamma l_t \frac{\partial \boldsymbol{m}}{\partial \tau}$, and $\boldsymbol{r}_{\text{Spac}} \triangleq l_s \boldsymbol{K} \nabla^\top \boldsymbol{u} - \boldsymbol{m}(\boldsymbol{x} - \boldsymbol{\xi})^\top + l_s^2 \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}$ as the residual vectors for different MeanFlow losses.

Theorem 3.1 (A-posteriori Error Estimate)**.** *Assume the MeanFlow predictor $\boldsymbol{m}_\theta$ is $\mathcal{C}^1$-continuous. If the total training loss bounds the squared path residuals such that $\mathcal{L}_{\text{MeanFlow}} \leq \epsilon$, then the pointwise error between reconstructed state $\boldsymbol{u}_\theta$ and true solution $\boldsymbol{u}_{\text{True}}$ at any target point $(\boldsymbol{x}, t)$ is bounded by:*

$$\|\boldsymbol{u}_\theta(\boldsymbol{x}, t) - \boldsymbol{u}_{\text{True}}(\boldsymbol{x}, t)\|_2 \leq C\sqrt{\epsilon} + \mathcal{O}(l^2), \tag{15}$$

where C is a constant depending on the path length l and the Lipschitz constant of the PDE operator.

Thm. 3.1 provides a bridge between the optimization target and the physical accuracy, justifying the use of MeanFlow residuals as a proxy for solution quality.

Theorem 3.2 (Space-Time Decoupling Bound)**.** *Assume that the PDE operator admits the decomposition* $\boldsymbol{f} = \nabla^\top \boldsymbol{u} \cdot \boldsymbol{g} + \boldsymbol{h}$ *where* $\boldsymbol{h}$ *has no explicit dependence on first-order gradient* $\nabla \boldsymbol{u}$*, and assume that* $\boldsymbol{g}$*,* $\boldsymbol{h}$*,* $\boldsymbol{K}$ *and* $\partial_\tau \boldsymbol{m}$ *are uniformly bounded on the considered region. The global MeanFlow loss is bounded by space-time decoupled losses:*

$$\mathcal{L}_{MeanFlow} \leq c_1 \mathcal{L}_{Temp} + c_2 \mathcal{L}_{Spac} + \varepsilon \tag{16}$$

where the coefficients are defined as $c_1 = 3l_t^2$*, and* $c_2 = 3\left(\frac{l - l_t}{l_s}\|\boldsymbol{g}\|_{op}\right)^2$*. Here,* ε *represents the structural coupling error and* $\varepsilon = \mathcal{O}(l_s^2)$*.*

The theoretical validity of decoupled training is established in Thm. 3.2, which proves that the global spatio-temporal consistency is bounded by the individual temporal and spatial errors. This bound ensures that minimizing decoupled losses effectively controls the total physics-informed error. In addition, the gap between the decoupled losses and spatio-temporal consistency is controlled by l_s, which can theoretically be infinitesimally small due to continuous modeling.

4 Experiments

We evaluate our Spatio-Temporal MeanFlow framework on four PDE benchmarks covering both time-dependent and steady-state problems. Sec. 4.1 compares its prediction accuracy with representative deterministic PDE solvers. Sec. 4.2 examines the contributions of the temporal and spatial MeanFlow losses and their coefficient sensitivity. Sec. 4.3 investigates inference flexibility across different integration steps and generalization to higher resolutions and out-of-distribution initial conditions.

4.1 Forward PDE Solving on Canonical Benchmarks

Benchmarks. We evaluate the effectiveness of the proposed method for PDE solving using four standard synthetic benchmarks. The benchmarks include two time-dependent PDEs, 1D Burgers' equation and the 2D periodic Navier–Stokes equation, and two steady-state PDEs, 2D Darcy flow and the 2D Poisson equation. All datasets are generated using numerical solvers and are divided into training, validation, and test sets with a ratio of 7:2:1. Further details regarding the governing equations, numerical discretizations, and data generation configurations are provided in App. F.1.

Baseline Methods and Metrics. Since our method is deterministic, we compare it with two categories of representative deterministic PDE solvers: (1) physics-informed coordinate-based methods, including PINN (Cuomo et al., 2022) and I-PINN (Niu et al., 2025); and (2) neural operators, including FNO (Li et al., 2020a), DeepONet (Lu et al., 2019), PINO (Li et al., 2024), RNO (Cheng et al., 2024), and CFO (Hou et al., 2026). All compared methods are reproduced using their publicly released code. Further details on the compared methods and their experimental configurations are provided in App. F.3. Following Huang et al. (2024), we report the relative L_2 error on the test set, which enables fair comparisons across datasets with different solution magnitudes.

Implementation Details. All experiments are implemented in PyTorch. We employ the Adam (Kingma & Ba, 2014) optimizer for training. The learning rates are set to 5.5×10^{-4}, 2.0×10^{-4}, and 3.0×10^{-4} for the Burgers' equation, NS equation, and steady-state PDEs. For the loss functions, the MeanFlow coefficients are consistently set to $\lambda_t = \lambda_s = 0.01$, with λ_t omitted for steady-state cases. The model checkpoint achieving the lowest validation error is used for testing. Detailed hyperparameters and network architectures are provided in App. F.2.

Main Results. Tab. 1 reports the forward prediction results on the four benchmarks. Our method achieves the best overall performance across both time-dependent and steady-state PDEs. On Burgers' equation and NS equation, the improvement indicates that directly learning finite-interval evolution can reduce temporal error accumulation compared with methods that rely only on supervised data fitting or pointwise physical residuals. On Darcy flow and Poisson equation, our method also obtains lower errors, showing that the spatial MeanFlow constraint provides an effective regularization. These

Table 1: Results for forward PDE solving on four synthetic benchmarks. We report the relative L_2 error (↓) as the mean ± standard deviation across three different random seeds. All reported metrics are multiplied by 10^{-2} for readability. Results for CFO on steady-state PDEs are omitted, as it is designed for time-dependent systems. The best results are highlighted in **bold**, while the second-best results are underlined.

Method ($\times 10^{-2}$)	Burgers	Navier–Stokes	Darcy	Poisson
Physics-informed coordinate-based solvers				
PINN	3.56±0.42	40.21±0.20	15.98±1.55	32.01±4.69
I-PINN	8.92±3.19	35.65±0.15	8.41±0.12	54.64±4.09
Neural operator solvers				
DeepONet	5.59±0.07	19.68±1.14	2.92±0.02	5.46±0.33
FNO	1.05±0.07	0.38±0.00	0.46±0.00	0.23±0.01
PINO	2.06±0.18	0.32±0.00	16.30±0.03	1.58±0.00
RNO	0.39±0.00	1.39±0.13	0.58±0.09	0.64±0.01
CFO	0.80±0.15	0.88±0.00	–	–
Ours	**0.26±0.00**	**0.19±0.02**	**0.25±0.00**	**0.18±0.02**

results demonstrate that the proposed framework is not limited to a specific PDE type, but can be applied to both temporal dynamics and spatial solution operators. We further report results on a real-world dataset and an inverse problem in App. F.4 to demonstrate the broader applicability of our method. A comparison of the computational costs of different methods, training curves, the PDE residual and boundary-condition error, results for a more challenging Navier–Stokes setting, and results on irregular grids and domains with irregular boundaries in App. F.5, highlighting the potential of our method beyond regular-grid settings, are provided in App. F.5. .

4.2 Ablation Study

Effect of Spatio-Temporal MeanFlow losses. We study the contributions of the temporal MeanFlow (T-MF) and spatial MeanFlow (S-MF) losses on Burgers' equation, Navier–Stokes, Darcy flow, and Poisson. For the two time-dependent PDEs, we evaluate all four combinations of the two losses: neither component, S-MF only, T-MF only, and the complete objective. Since Darcy flow is a steady-state problem, T-MF is not applicable, and we therefore ablate only the spatial component. As shown in Table 2, either T-MF or S-MF improves over the model without MeanFlow regularization on both time-dependent benchmarks, while combining them consistently achieves the lowest predictive error. Specifically, the full model reduces the relative L_2 error from 1.7561×10^{-2} to 2.574×10^{-3} on Burgers' equation and from 1.1×10^{-2} to 1.9×10^{-3} on Navier–Stokes. For Darcy flow, introducing S-MF reduces the error from 5.3×10^{-3} to 2.6×10^{-3}. These results demonstrate that T-MF and S-MF provide complementary regularization for time-dependent dynamics, while S-MF also remains effective for steady-state PDEs.

Table 2: Ablation of T-MF and S-MF losses.

PDE	T-MF	S-MF	Rel. L_2 error
Burgers	✗	✗	1.76
	✗	✓	0.31
	✓	✗	1.08
	✓	✓	**0.26**
Navier–Stokes	✗	✗	1.10
	✗	✓	0.64
	✓	✗	0.52
	✓	✓	**0.19**
Darcy flow	–	✗	0.53
	–	✓	**0.26**
Poisson	–	✗	0.55
	–	✓	**0.18**

Sensitivity to the MeanFlow coefficients. We assess coefficient sensitivity by varying one MeanFlow loss weight at a time while fixing the other at 0.01; only S-MF is considered for the steady-state Darcy and Poisson problems. As shown in Fig. 2, moderate coefficient variations generally preserve

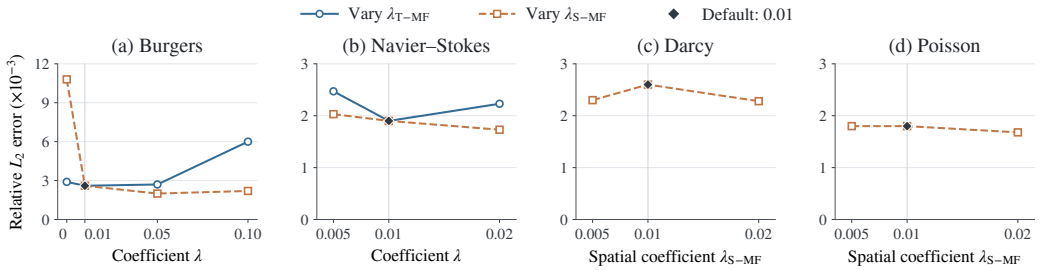


Figure 2: Sensitivity to the MeanFlow loss coefficients. Each coefficient is varied with the other fixed at 0.01 when applicable; only S-MF applies to Darcy and Poisson. Diamonds mark the default coefficient of 0.01. Relative L_2 errors are reported in units of 10^{-3}. Note the different vertical scale for Burgers.

predictive accuracy. For Burgers, the tested spatial coefficients from 0.01 to 0.1 yield consistently low errors, while increasing the temporal coefficient to 0.1 degrades performance. Navier–Stokes, Darcy, and Poisson also exhibit relatively modest error variations over the tested range from 0.005 to 0.02. Overall, the shared default coefficient of 0.01 provides competitive accuracy across all four benchmarks, suggesting that our method does not require precise coefficient selection or extensive dataset-specific tuning.

4.3 Inference Flexibility and Generalization.

Integration steps. Since our model learns an integral operator over arbitrary time intervals, inference does not require a fixed temporal discretization. Given an initial state and a target time, the model can either predict the full interval in a single step or compose several shorter integrations. We therefore evaluate the effect of the number of integration steps on Burgers dataset. As shown in Fig. 3, increasing the number of steps from 1 to 4 consistently reduces the relative L_2 error, demonstrating that the learned operator can be effectively reused over sub-intervals and benefits from moderate temporal refinement. However, using too many steps degrades performance, likely due to accumulated approximation errors from repeated composition. This suggests that our framework provides a flexible efficiency–accuracy trade-off: one-step inference gives fast direct prediction, while a small number of integration steps improves accuracy without retraining.

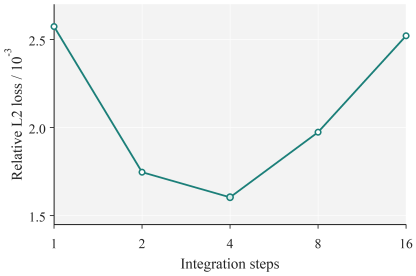


Figure 3: Relative L_2 error on Burgers with different numbers of integration steps.

Resolution Generalization. Benefiting from the proposed spatial MeanFlow losses, a key advantage of our framework is its potential to generalize across varying spatial resolutions. We therefore directly evaluate the model, trained exclusively at $s = 128$, on higher-resolution grids ($s = 256/512$) without any retraining. As shown in Tab. 3, our method achieves stable zero-shot generalization across varying resolutions. Notably, removing the spatial MeanFlow loss (w/o S-MF) leads to severe performance degradation, confirming its critical role in maintaining physical consistency across discretizations.

OOD Initial Condition Generalization. We further test generalization to out-of-distribution initial conditions by changing the Gaussian random field parameters used to generate the Burgers initial states, while keeping the resolution fixed. We consider smooth, low-frequency, smooth amplified, and smooth low-τ IC shifts. As Tab. 3 shows, our method's superior OOD performance confirms that spatio-temporal MeanFlow losses robustly handles initial-condition shifts.

Table 3: Ablation study on Burgers evaluating the effects of temporal and spatial MeanFlow losses, resolution generalization and out-of-distribution initial condition generalization. All models are trained at the base resolution $s = 128$ and evaluated on test sets generated with the same random seed for each setting. We report global relative L_2 error; lower is better.

Type	Test Setting	w/o Both	w/o T-MF	w/o S-MF	Ours
Resolution	$s = 128$	0.017561	0.003075	0.010821	0.002574
	$s = 256$	0.017562	0.003051	0.010820	0.002419
	$s = 512$	0.017558	0.002913	0.010818	0.002395
OOD IC	Smooth, $\gamma = 3.5$	0.011952	0.010559	0.009174	0.003732
	Low freq., $\tau = 10$	0.020556	0.002792	0.016768	0.001223
	Smooth+Amp., $\gamma = 3.0, \sigma = 61.25$	0.007475	0.003032	0.004114	0.001092
	Smooth+Low-τ, $\gamma = 3.5, \tau = 5$	0.008023	0.006197	0.005505	0.002187

5 Related Work

Physics-Informed Neural Networks (PINNs). PINNs (Raissi et al., 2019) approximate PDE solutions with neural networks by incorporating governing equations and initial and boundary conditions as soft constraints. Despite their broad applicability, standard PINNs typically require instance-specific optimization and can suffer from gradient imbalances among competing loss terms (Wang et al., 2021). To improve training stability, uncertainty-based weighting (Kendall et al., 2018; Huang et al., 2021) adaptively balances different objectives, while Wang et al. (2021) introduce gradient-based learning rate annealing and an improved network architecture to mitigate optimization difficulties. More recently, I-PINN (Niu et al., 2025) combines architectural improvements with adaptive uncertainty weighting to further enhance convergence and accuracy. These approaches primarily improve the optimization of physics-informed objectives. Our framework additionally captures finite-interval integral relationships through spatio-temporal MeanFlow constraints, providing a complementary approach to learning physically consistent solution operators. A more detailed discussion is provided in App. B.1.

Neural Operators. Neural operators like FNO (Li et al., 2020a) and DeepONet (Lu et al., 2019) enable rapid PDE inference. To enhance physical consistency, methods like PINO (Li et al., 2024) explicitly incorporate residual constraints. For time-dependent systems, the Continuous Flow Operator (CFO) (Hou et al., 2026) mitigates autoregressive error accumulation by repurposing flow matching to learn continuous-time dynamics. While CFO shares a similar continuous integration modeling philosophy with our approach, our framework additionally introduces a physics-inspired spatio-temporal MeanFlow loss, significantly enhancing its generalization capabilities. MENO (Yang & Xue, 2026) uses MeanFlow as a generative transport mechanism to enhance multiscale neural operators. In contrast, our method draws inspiration from MeanFlow to construct a deterministic PDE solution operator and extends the MeanFlow formulation to the spatial domain. Other architectural variants targeting irregular geometries (Li et al., 2023; 2020b) and multi-scale structures (Tripura & Chakraborty, 2022; Rahman et al., 2022) are further discussed in App. B.2.

6 Conclusion

In this paper, we propose a physics-informed Neural PDE Solver via Spatio-Temporal MeanFlow. Functioning as an arbitrary-interval continuous integrator, it enables flexible integration steps. Furthermore, by enforcing physical consistency via a novel spatio-temporal loss, our method achieves enhanced generalization capabilities for the framework. Extensive evaluations demonstrate its superior accuracy and generalization. Although our underlying mathematical formulation natively supports continuous spatial evolution, our current empirical implementation mainly focuses on regular and convex grids. Activating this continuous spatial modeling to tackle irregular domains represents the future step to further maximize the framework's generalization capabilities.

AI USE STATEMENT

In this work, we used generative AI tools to polish the manuscript and assist with code implementation. We reviewed all AI-assisted work, and two authors verified and tested the AI-generated code for correctness. We take responsibility for the final content of this work, including all text, claims, and artifacts produced with the aid of generative AI.

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APPENDIX OVERVIEW

The following lists the structure of the appendix, with links to the respective sections.

- A Detailed Derivation of the MeanFlow Identity
- B Extended Related Work
  - B.1 Physics-Informed Neural Networks (PINNs): Optimization Challenges in PINNs
  - B.2 Neural Operators: Advancements in Neural Operator Architectures
- C Generalized Spatio-Temporal MeanFlow Framework
- D Theoretical Results
  - D.1 The Construction of Spatio-Temporal MeanFlow Constraints
  - D.2 Error Estimate
  - D.3 MeanFlow Error Bounded by Decoupled Losses
  - D.4 Hessian Condition Number Analysis
- E Examples of Spatio-Temporal MeanFlow Losses
- F Experiments
  - F.1 Details of Synthetic PDE Benchmarks
  - F.2 Implementation Details of Main Experiments
  - F.3 Compared Methods
  - F.4 Broader Applicability: Real-World Data and Inverse Modeling
  - F.5 Additional Experimental Results of Main Experiments

A Detailed Derivation of the MeanFlow Identity

Conventional Flow Matching parameterizes the instantaneous velocity field $\boldsymbol{v}$, which inherently requires multi-step numerical integration during inference. MeanFlow (Geng et al., 2025) seeks to directly model the average velocity $\boldsymbol{m}$ over a finite time interval, thereby enabling one-step prediction.

Specifically, let $\boldsymbol{u}_t$ denote the state at time t along the true ODE trajectory governed by $\frac{\mathrm{d}\boldsymbol{u}_t}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{u}_t, t)$. The average velocity field $\boldsymbol{m}(\boldsymbol{u}_t, \tau, t)$ is defined as the displacement divided by the time interval $(t - \tau)$:

$$\boldsymbol{m}(\boldsymbol{u}_t, \tau, t) \triangleq \frac{1}{t-\tau} \int_{\tau}^{t} \boldsymbol{v}(\boldsymbol{u}_s, s) \mathrm{d}s.$$

Directly utilizing this definition as a training objective necessitates continuous numerical integration during training. To construct an integral-free training objective, we establish a differential relationship between $\boldsymbol{u}$ and $\boldsymbol{v}$. By rewriting the definition as $(t-\tau)\boldsymbol{m}(\boldsymbol{u}_t, \tau, t) = \int_{\tau}^{t} \boldsymbol{v}(\boldsymbol{u}_s, s)\mathrm{d}s$ and taking the total derivative with respect to time t on both sides, we can eliminate the integral.

For the left-hand side, applying the product rule yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(t-\tau)\boldsymbol{m}(\boldsymbol{u}_t, \tau, t)\right] = \boldsymbol{m}(\boldsymbol{u}_t, \tau, t) + (t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}(\boldsymbol{u}_t, \tau, t).$$

The total derivative $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}(\boldsymbol{u}_t, \tau, t)$ accounts for both the explicit dependence of $\boldsymbol{m}$ on t and its implicit dependence on t through the evolving state $\boldsymbol{u}_t$. Using the chain rule, this expands as $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m} = \partial_t \boldsymbol{m} + (\partial_{\boldsymbol{u}}\boldsymbol{m})\frac{\mathrm{d}\boldsymbol{u}_t}{\mathrm{d}t} = \partial_t \boldsymbol{m} + (\partial_{\boldsymbol{u}}\boldsymbol{m})\boldsymbol{v}(\boldsymbol{u}_t, t)$, assuming the reference time τ is a fixed constant independent of t (hence $\frac{\mathrm{d}\tau}{\mathrm{d}t} = 0$).

For the right-hand side, applying the Fundamental Theorem of Calculus with respect to the upper limit of the integral yields the integrand evaluated at time t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tau}^{t} \boldsymbol{v}(\boldsymbol{u}_s, s)\mathrm{d}s = \boldsymbol{v}(\boldsymbol{u}_t, t).$$

Equating the derivatives of both sides and rearranging the terms leads to the MeanFlow Identity:

$$\boldsymbol{m}(\boldsymbol{u}_t, \tau, t) = \boldsymbol{v}(\boldsymbol{u}_t, t) - (t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}(\boldsymbol{u}_t, \tau, t).$$

This identity provides a necessary and sufficient condition (given the natural boundary condition $\lim_{\tau \to t} \boldsymbol{m}(\boldsymbol{u}_t, \tau, t) = \boldsymbol{v}(\boldsymbol{u}_t, t)$) to characterize the true average velocity using a purely differential, integral-free form.

B Extended Related Work

B.1 Physics-Informed Neural Networks (PINNs): Optimization Challenges in PINNs

Physics-Informed Neural Networks (PINNs) (Raissi et al., 2019) represent a foundational class of deep learning-based PDE solvers. They parameterize the solution manifold by embedding the governing physical laws into the training objective as soft constraints. Despite their widespread success across diverse physical domains (ranging from fluid dynamics to power systems), PINNs suffer from severe scalability bottlenecks due to their inherent reliance on instance-specific, iterative optimization. Furthermore, a prominent challenge in training PINNs is their notoriously pathological optimization landscape, caused by severe stiffness among competing boundary, initial, and residual loss terms.

To address this problem, subsequent works have developed sophisticated adaptive schemes. Beyond standard uncertainty weighting (Huang et al., 2021) and gradient pathology mitigation (Kendall et al., 2018), Wang et al. (2021) proposed an adaptive learning rate annealing strategy along with an improved neural architecture (IA-PINN) to balance fitting and regularization. Concurrently, methods

like IAW-PINN advanced uncertainty weighting by controlling the upper bounds of weights, offering more flexible balancing among distinct loss terms. Recognizing that singular approaches are often limited by network structure, recent efforts have integrated these strengths (such as I-PINN (Niu et al., 2025), which unifies both IA-PINN's architecture and IAW-PINN's weighting strategies).

B.2 Neural Operators: Advancements in Neural Operator Architectures

Neural operators learn a direct mapping between infinite-dimensional function spaces. Architectures like Fourier Neural Operators (FNO) (Li et al., 2020a) and DeepONet (Lu et al., 2019) enable rapid, zero-shot inference for new initial or boundary conditions. However, purely data-driven operators often struggle with out-of-distribution generalization and may violate physical laws. To address this, Physics-Informed Neural Operators (PINO) (Li et al., 2024) explicitly incorporate PDE residuals into the operator training objective, blending fast inference with physical constraints. However, when applied to time-dependent systems, these models typically rely on autoregressive (AR) schemes, which are prone to error accumulation over long horizons and restricted by uniform temporal discretization.

To address these limitations, the Continuous Flow Operator (CFO) (Hou et al., 2026) was introduced as a novel class of flow-based operators. CFO shares our core conceptual motivation: rather than treating flow matching as a simple generative sampler, it repurposes the framework to learn the underlying differential dynamics (the physical right-hand side of the PDE) directly. By fitting temporal splines to trajectory data, CFO constructs continuous velocity fields that grant the model inherent time-resolution invariance. However, despite this shared perspective, its formulation is purely temporal; it captures the continuous evolution over time but lacks explicit constraints to ensure physical consistency across the spatial domain, which is critical for PDE generalization. Our proposed temporal-spatial MeanFlow framework strategically overcomes this. By expanding the integral constraints to the spatio-temporal domain, we explicitly enforce physical consistency across spatial resolutions.

Furthermore, a wide array of other architectural extensions have been proposed to overcome the geometric constraints of standard neural operators. For instance, while original FNOs rely heavily on uniform grids, methods such as Geo-FNO (Li et al., 2023) and multipole graph neural operators (Li et al., 2020b) have been developed to accurately map and compute solutions across highly irregular physical geometries. Similarly, the Galerkin Transformer (Cao, 2021) and CViT (Wang et al., 2024) introduce robust mechanisms for handling non-uniform or irregularly sampled observational data.

To capture complex, overlapping physical phenomena, models like the Wavelet Neural Operator (WNO) (Tripura & Chakraborty, 2022) integrate multi-scale structural processing. Finally, architectures such as U-NO (Rahman et al., 2022) address the depth limitations of traditional operators, employing U-Net-like structures to enable deeper, more expressive feature extraction without sacrificing computational efficiency. While these methods successfully expand the expressivity and geometric flexibility of operator learning, our approach uniquely pivots toward using generative flow matching as a direct, physical equation solver to achieve rapid, single-step integration.

C Generalized Spatio-Temporal MeanFlow Framework

In the main text, we presented a simplified algorithm tailored for regular spatial grids, which fixes the spatial coordinates and focuses on continuous-time integration. Here, we present the fully generalized Spatio-Temporal MeanFlow algorithms (Alg. 3 and Alg. 4). In this generalized formulation, both the temporal and spatial coordinates are treated as continuous variables, allowing the integration path to move simultaneously across time from τ to t and across space from an anchor point $\boldsymbol{\xi}$ to a target point $\boldsymbol{x}$.

D Theoretical Results

D.1 The Construction of Spatio-Temporal MeanFlow Constraints

In this section, Prop. D.1 conclude and show how the MeanFlow constraints being constructed.

Algorithm 3 Generalized Training

Require: PDE params $\boldsymbol{a}$, temporal indicator γ, network $\boldsymbol{m}_\theta$, max iter M, learning rate η
Ensure: Trained network parameters θ^*

for $i = 1$ to M **do**
  if $\gamma \neq 0$ **then** ▷ Time-variant PDEs
    Sample anchor time τ and anchor grid $\boldsymbol{\xi}$;
    Compute anchor state $\boldsymbol{u}_{\text{anchor}} = \boldsymbol{u}(\boldsymbol{\xi}, \tau)$;
    Sample target time $t > \tau$ and target grid $\boldsymbol{x}$;
    Predict MeanFlow: $\boldsymbol{m} \leftarrow \boldsymbol{m}_\theta([\boldsymbol{u}_{\text{anchor}}, \boldsymbol{\xi}, \tau, \boldsymbol{x}, t, \boldsymbol{a}])$;
    Compute total loss: $\mathcal{L}_{\text{Total}} \leftarrow \mathcal{L}_{\text{Data}} + \lambda_t \mathcal{L}_{\text{T-MF}} + \lambda_s \mathcal{L}_{\text{S-MF}}$;
  else ▷ Steady-state PDEs ($\gamma = 0$)
    Create initial state $\boldsymbol{u}_0$ by the harmonic extension of $\boldsymbol{u}_0(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \partial D$;
    Sample target grid $\boldsymbol{x}$ for domain D;
    Compute $\mathcal{L}_{\text{Data}}$ and $\mathcal{L}_{\text{S-MF}}$;
    $\mathcal{L}_{\text{Total}} \leftarrow \mathcal{L}_{\text{Data}} + \lambda_s \mathcal{L}_{\text{S-MF}}$;
  end if
  Update: $\theta \leftarrow \theta - \eta \nabla_\theta \mathcal{L}_{\text{Total}}$;
end for
return $\theta^* = \theta$;

Algorithm 4 Generalized Multi-step Inference

Require: Pretrained net $\boldsymbol{m}_{\theta^*}$, start grid $\boldsymbol{x}_0$, start time t_0, start state $\boldsymbol{u}_0$, target grid $\boldsymbol{x}_N$, step Δt, steps N
Ensure: Spatio-temporal trajectory $\{\boldsymbol{u}_k\}_{k=0}^N$

Initialize trajectory list with $\boldsymbol{u}_0$;
for $k = 0$ to $N - 1$ **do**
  Anchor coords: $(\boldsymbol{\xi}, \tau) \leftarrow (\boldsymbol{x}_k, t_k)$;
  Anchor state: $\boldsymbol{u}_{\text{anchor}} \leftarrow \boldsymbol{u}_k$;
  Target grid: $\boldsymbol{x}_{k+1} \leftarrow \boldsymbol{x}_k + (\boldsymbol{x}_N - \boldsymbol{x}_0)/N$;
  Target time: $t_{k+1} \leftarrow t_k + \Delta t$;
  Spatio-temporal arc length: $l \leftarrow \sqrt{\frac{\|\boldsymbol{x}_N - \boldsymbol{x}_0\|_2^2}{N^2} + \Delta t^2}$;
  Predict MeanFlow: $\boldsymbol{m} \leftarrow \boldsymbol{m}_\theta([\boldsymbol{u}_{\text{anchor}}, \boldsymbol{\xi}, \tau, \boldsymbol{x}_{k+1}, t_{k+1}, \boldsymbol{a}])$;
  State increment: $\Delta \boldsymbol{u} \leftarrow l \cdot \boldsymbol{m}$;
  Update field: $\boldsymbol{u}_{k+1} \leftarrow \boldsymbol{u}_k + \Delta \boldsymbol{u}$;
  Append $\boldsymbol{u}_{k+1}$ to trajectory list;
end for
return $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$;

Proposition D.1 (MeanFlow Constraints.)**.** *For MeanFlow function $\boldsymbol{m}$ satisfying*

$$l(\boldsymbol{\xi}, \tau, \boldsymbol{x}, t) \cdot \boldsymbol{m}[\boldsymbol{u}(\boldsymbol{\xi}, \tau), \boldsymbol{\xi}, \tau, \boldsymbol{x}, t, \boldsymbol{a}] = \boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{u}(\boldsymbol{\xi}, \tau), \tag{17}$$

we have

$$\begin{cases} l_t \cdot \boldsymbol{m} = l\boldsymbol{K}\boldsymbol{f} + \gamma l^2 \cdot \dfrac{\partial \boldsymbol{m}}{\partial \tau}, \\ l \cdot \boldsymbol{K} \cdot \nabla^\top \boldsymbol{u}(\boldsymbol{\xi}, \tau) = \boldsymbol{m}(\boldsymbol{x} - \boldsymbol{\xi})^\top - l^2 \cdot \dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}, \\ l \cdot \boldsymbol{K} \cdot \Delta \boldsymbol{u}(\boldsymbol{\xi}, \tau) = -(\boldsymbol{\alpha} + 2\boldsymbol{K}^{-1}\boldsymbol{\beta}), \end{cases}$$

where $\boldsymbol{\alpha} = n\boldsymbol{m} - (l_s^2 \,/\, l^2) \cdot \boldsymbol{m} + 2l^2 \Delta \boldsymbol{m}(\boldsymbol{\xi}, \tau)$ *and* $\boldsymbol{\beta} = -\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}(\boldsymbol{x} - \boldsymbol{\xi}) - (l_s^2 \,/\, l) \cdot \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}} \boldsymbol{m}$.

Proof. We enforce a perturbation on γ if $\gamma = 0$ and let $\gamma \to 0$ in the end phase. This technique allows us to regard parameter γ as a non-sigular value.

We compute the derivatives of $\boldsymbol{m}$ given that $\boldsymbol{\xi}, \boldsymbol{x}, t, \boldsymbol{a}$ are constants w.r.t. τ and $\tau, \boldsymbol{x}, t, \boldsymbol{a}$ are constants w.r.t. $\boldsymbol{\xi}$, leading to

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{m}}{\mathrm{d}\tau} = \dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}} \cdot \dfrac{\partial \boldsymbol{u}}{\partial \tau} + \dfrac{\partial \boldsymbol{m}}{\partial \tau} = \dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}} \cdot \dfrac{\boldsymbol{f}(\boldsymbol{\xi}, \tau)}{\gamma} + \dfrac{\partial \boldsymbol{m}}{\partial \tau}, \\ \dfrac{\mathrm{d}\boldsymbol{m}}{\mathrm{d}\boldsymbol{\xi}} = \dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}} \cdot \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\xi}} + \dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}} = \dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}} \cdot \nabla^\top \boldsymbol{u}(\boldsymbol{\xi}, \tau) + \dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}. \end{cases} \tag{18}$$

The derivatives w.r.t. τ of both sides of Eq. equation 17 are

$$\begin{cases} \dfrac{\mathrm{dLHS}}{\mathrm{d}\tau} = \dfrac{\mathrm{d}l}{\mathrm{d}\tau}\boldsymbol{m} + l\dfrac{\mathrm{d}\boldsymbol{m}}{\mathrm{d}\tau} = \dfrac{-(t-\tau)\boldsymbol{m}}{l} + l\left(\dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}} \cdot \dfrac{\boldsymbol{f}(\boldsymbol{\xi}, \tau)}{\gamma} + \dfrac{\partial \boldsymbol{m}}{\partial \tau}\right), \\ \dfrac{\mathrm{dRHS}}{\mathrm{d}\tau} = -\dfrac{\partial \boldsymbol{u}(\boldsymbol{\xi}, \tau)}{\partial \tau} = -\dfrac{\boldsymbol{f}(\boldsymbol{\xi}, \tau)}{\gamma}, \end{cases} \tag{19}$$

where $\boldsymbol{f}(\boldsymbol{\xi}, \tau) \triangleq \boldsymbol{f}[\boldsymbol{u}(\boldsymbol{\xi}, \tau), \nabla, \boldsymbol{\xi}, \tau, \boldsymbol{a}]$. Notably, the notations $\frac{\mathrm{d}}{\mathrm{d}\cdot}$ and $\frac{\partial}{\partial \cdot}$ both represent the Jacobian matrices, whereas $\nabla \boldsymbol{f}$ represent spatial gradients, which is the transpose of Jacobian matrix.

Consequently, we have the ground-truth approximation

$$\begin{aligned}\boldsymbol{m}_{\text{gt}} =& \frac{l}{t-\tau}\left[l\left(\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\cdot\frac{\boldsymbol{f}(\boldsymbol{\xi},\tau)}{\gamma}+\frac{\partial \boldsymbol{m}}{\partial \tau}\right)+\frac{\boldsymbol{f}(\boldsymbol{\xi},\tau)}{\gamma}\right],\\ =& \frac{l}{t-\tau}\left[\boldsymbol{K}\cdot\frac{\boldsymbol{f}(\boldsymbol{\xi},\tau)}{\gamma}+l\frac{\partial \boldsymbol{m}}{\partial \tau}\right],\end{aligned} \tag{20}$$

where $l \triangleq \sqrt{(t-\tau)^2+\|\boldsymbol{x}-\boldsymbol{\xi}\|_2^2}$ and $\boldsymbol{K}=\boldsymbol{I}+l\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}$. **This coheres to the final result in conventional *temporal MeanFlow*. The *spatial* part is derived as follows.**

The derivatives of Eq. equation 17 w.r.t. $\boldsymbol{\xi}$, on the other hand, result in

$$\begin{cases}\dfrac{\text{dLHS}}{\text{d}\boldsymbol{\xi}} =\dfrac{\boldsymbol{m}(\boldsymbol{\xi}-\boldsymbol{x})^\top}{l}+l\left(\dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\cdot\nabla^\top \boldsymbol{u}(\boldsymbol{\xi},\tau)+\dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\right),\\ \dfrac{\text{dRHS}}{\text{d}\boldsymbol{\xi}} =-\dfrac{\partial \boldsymbol{u}(\boldsymbol{\xi},\tau)}{\partial \boldsymbol{\xi}}=-\nabla^\top \boldsymbol{u}(\boldsymbol{\xi},\tau),\end{cases} \tag{21}$$

and thus

$$\boldsymbol{K}\cdot\nabla^\top \boldsymbol{u}(\boldsymbol{\xi},\tau)=\frac{1}{l}\boldsymbol{m}(\boldsymbol{x}-\boldsymbol{\xi})^\top-l\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}. \tag{22}$$

Note that this result may substitute the $\nabla \boldsymbol{u}$'s of $\boldsymbol{K}\boldsymbol{f}$ in equation 20.

There are more PDEs containing two or more order of derivatives, e.g. the Navier-Stokes equations with $\Delta \boldsymbol{u}$ within considerations.

For PDE with higher order in space, say order-2, we have

$$\begin{cases}\dfrac{\text{d}^2\text{LHS}}{\text{d}\boldsymbol{\xi}^2} =\dfrac{\boldsymbol{m}\boldsymbol{1}^\top}{l}-\dfrac{1}{l^3}\boldsymbol{m}(\boldsymbol{\xi}-\boldsymbol{x})^\top\text{diag}\,(\boldsymbol{\xi}-\boldsymbol{x})+l\left(\dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\cdot\dfrac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{\xi}^2}+2\dfrac{\partial^2 \boldsymbol{m}}{\partial \boldsymbol{\xi}^2}\right)\\ \qquad\qquad +\dfrac{2}{l}\left(\dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\cdot\nabla^\top \boldsymbol{u}(\boldsymbol{\xi},\tau)+\dfrac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\right)\text{diag}\,(\boldsymbol{\xi}-\boldsymbol{x}),\\ \dfrac{\text{d}^2\text{RHS}}{\text{d}\boldsymbol{\xi}^2} =-\dfrac{\partial^2 \boldsymbol{z}(\boldsymbol{\xi},\tau)}{\partial \boldsymbol{\xi}^2},\end{cases} \tag{23}$$

where the matrices $\left[\frac{\text{d}^2\boldsymbol{z}}{\text{d}\boldsymbol{x}^2}\right]_{ij}\triangleq\frac{\text{d}^2\boldsymbol{z}_i}{\text{d}\boldsymbol{x}_j^2}$ and $\left[\frac{\partial^2\boldsymbol{z}}{\partial\boldsymbol{x}^2}\right]_{ij}\triangleq\frac{\partial^2\boldsymbol{z}_i}{\partial\boldsymbol{x}_j^2}$, then,

$$\begin{aligned}&-\boldsymbol{K}\Delta \boldsymbol{u}(\boldsymbol{\xi},\tau),\\ =&-\boldsymbol{K}\frac{\partial^2 \boldsymbol{u}(\boldsymbol{\xi},\tau)}{\partial^2 \boldsymbol{\xi}^2}\boldsymbol{1},\\ =&-\left(\boldsymbol{I}+l\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\right)\cdot\frac{\partial^2 \boldsymbol{u}(\boldsymbol{\xi},\tau)}{\partial^2 \boldsymbol{\xi}^2}\boldsymbol{1},\\ =&\frac{n}{l}\boldsymbol{m}-\frac{\|\boldsymbol{\xi}-\boldsymbol{x}\|_2^2}{l^3}\boldsymbol{m}+2l\Delta \boldsymbol{m}(\boldsymbol{\xi},\tau)+\frac{2}{l}\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi}-\boldsymbol{x})\\ &-\frac{2}{l}\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\boldsymbol{K}^{-1}\left(\frac{1}{l}\boldsymbol{m}(\boldsymbol{\xi}-\boldsymbol{x})^\top+l\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\right)(\boldsymbol{\xi}-\boldsymbol{x}),\\ =&\frac{\boldsymbol{\alpha}}{l}-\frac{2}{l^2}\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\boldsymbol{K}^{-1}\boldsymbol{m}\|\boldsymbol{\xi}-\boldsymbol{x}\|_2^2+2\left[\frac{\boldsymbol{K}}{l}-\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\right]\boldsymbol{K}^{-1}\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi}-\boldsymbol{x}),\end{aligned} \tag{24}$$

hence

$$\boldsymbol{K}\Delta \boldsymbol{u}(\boldsymbol{\xi},\tau)=-\left(\frac{\boldsymbol{\alpha}}{l}+\frac{2}{l}\boldsymbol{K}^{-1}\boldsymbol{\beta}\right), \tag{25}$$

where $\boldsymbol{\alpha}\triangleq n\boldsymbol{m}-\frac{\|\boldsymbol{x}-\boldsymbol{\xi}\|_2^2}{l^2}\boldsymbol{m}+2l^2\Delta \boldsymbol{m}(\boldsymbol{\xi},\tau)$ and $\boldsymbol{\beta}\triangleq-\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}(\boldsymbol{x}-\boldsymbol{\xi})-\frac{\|\boldsymbol{x}-\boldsymbol{\xi}\|_2^2}{l}\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{u}}\boldsymbol{m}$, in which n represents the number of spatial dimensions. Consequently, $-l\boldsymbol{K}^2\Delta \boldsymbol{u}(\boldsymbol{\xi},\tau)=\boldsymbol{K}\boldsymbol{\alpha}+2\boldsymbol{\beta}$. □

D.2 Error Estimate

Theorem D.1 (A-posteriori Error Bound, manuscript)**.** *Assume the PDE operator $\boldsymbol{f}$ is uniformly Lipschitz continuous with respect to the state variable with constant L, and continuously differentiable ($\mathcal{C}^1$) such that its Jacobian $\nabla_{\boldsymbol{u}}\boldsymbol{f}$ is bounded locally. Further, assume the MeanFlow predictor $\boldsymbol{m}_\theta$ and the true physical state $\boldsymbol{u}_{True}$ are $\mathcal{C}^1$-continuous. If the total training loss bounds the squared path residuals such that $\mathcal{L}_{MeanFlow} \leq \epsilon$, then the pointwise error between the reconstructed state $\boldsymbol{u}_\theta$ and the true solution $\boldsymbol{u}_{True}$ at any target point $(\boldsymbol{x}, t)$ is bounded by:*

$$\|\boldsymbol{u}_\theta(\boldsymbol{x}, t) - \boldsymbol{u}_{True}(\boldsymbol{x}, t)\|_2 \leq C\sqrt{\epsilon} + \mathcal{O}(l^2), \tag{26}$$

where C is a constant depending on the spatio-temporal path length l and the Lipschitz constant L.

Proof. Let the straight spatio-temporal path $\mathfrak{p}$ connecting the starting point $(\boldsymbol{\xi}, \tau)$ and the target point $(\boldsymbol{x}, t)$ be parameterized by a scalar $s \in [0, 1]$. The predicted state along the path is linearly modeled as $\boldsymbol{u}_\theta(s) = \boldsymbol{u}(\boldsymbol{\xi}, \tau) + s \cdot l \cdot \boldsymbol{m}_\theta$, implying its path derivative is precisely $\frac{\mathrm{d}\boldsymbol{u}_\theta}{\mathrm{d}s} = l \cdot \boldsymbol{m}_\theta$.

We define the pointwise state error as $\boldsymbol{v}(s) = \boldsymbol{u}_\theta(s) - \boldsymbol{u}_{\text{True}}(s)$. Assuming the starting state is exact (i.e., anchored by data constraints such that $\boldsymbol{v}(0) = \boldsymbol{0}$), the evolution of the error along the path is governed by the difference in their derivatives:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{u}_\theta}{\mathrm{d}s} - \frac{\mathrm{d}\boldsymbol{u}_{\text{True}}}{\mathrm{d}s} = l \cdot \boldsymbol{m}_\theta - l \cdot \boldsymbol{f}(\boldsymbol{u}_{\text{True}}(s)). \tag{27}$$

To bridge the predicted MeanFlow and the true physical operator, we explicitly add and subtract the term $l \cdot \boldsymbol{f}(\boldsymbol{u}_\theta(s))$:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}s} = l \cdot [\boldsymbol{f}(\boldsymbol{u}_\theta(s)) - \boldsymbol{f}(\boldsymbol{u}_{\text{True}}(s))] + l \cdot [\boldsymbol{m}_\theta - \boldsymbol{f}(\boldsymbol{u}_\theta(s))]\,. \tag{28}$$

The first bracketed term is governed by the Lipschitz continuity of the PDE operator $\boldsymbol{f}$ with constant L, bounded by $L\|\boldsymbol{v}(s)\|_2$. The second bracketed term measures the deviation between the constant MeanFlow vector and the continuous local PDE evaluation. By Taylor expanding $\boldsymbol{f}(\boldsymbol{u}_\theta(s))$ around the starting point $s = 0$, we relate it to the global MeanFlow residual $\boldsymbol{r}_{\text{MeanFlow}}$ evaluated by the network:

$$l \cdot [\boldsymbol{m}_\theta - \boldsymbol{f}(\boldsymbol{u}_\theta(s))] = \boldsymbol{r}_{\text{MeanFlow}} + \boldsymbol{r}(s), \tag{29}$$

where $\boldsymbol{r}(s) \sim s \cdot l^2 \nabla_{\boldsymbol{u}}\boldsymbol{f} \cdot \boldsymbol{m}_\theta$ is the first-order Taylor remainder along the path. Because the state derivatives and the Jacobian $\nabla_{\boldsymbol{u}}\boldsymbol{f}$ are assumed bounded, the remainder satisfies $\|\boldsymbol{r}(s)\|_2 \leq Msl^2 = \mathcal{O}(l^2)$, with M being a local bounding constant.

Taking the L_2 norm on both sides and applying the triangle inequality yields a rigorous differential inequality:

$$\left\|\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}s}\right\|_2 \leq l \cdot L\|\boldsymbol{v}(s)\|_2 + \|\boldsymbol{r}_{\text{MeanFlow}}\|_2 + \mathcal{O}(l^2). \tag{30}$$

Applying Grönwall's inequality to integrate this bound from $s = 0$ to $s = 1$ yields:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\|\boldsymbol{v}(s)\|_2 \leq \left\|\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}s}\right\|_2 \leq l \cdot L\|\boldsymbol{v}(s)\|_2 + \|\boldsymbol{r}_{\text{MeanFlow}}\|_2 + \mathcal{O}(l^2), \\ \Longrightarrow &\frac{\mathrm{d}}{\mathrm{d}s}\left[\|\boldsymbol{v}(s)\|_2 \,\mathrm{e}^{-lLs}\right] \leq \left[\|\boldsymbol{r}_{\text{MeanFlow}}\|_2 + \mathcal{O}(l^2)\right]\mathrm{e}^{-lLs}, \\ \Longrightarrow &\|\boldsymbol{v}(1)\|_2 \leq \int_0^1 \left[\|\boldsymbol{r}_{\text{MeanFlow}}\|_2 + \mathcal{O}(l^2)\right]\exp\left[lL(1-s)\right]\mathrm{d}s. \end{aligned} \tag{31}$$

The integral linearly separates into two components. For the first component involving the trained residual, we apply the Cauchy-Schwarz inequality to relate it to the squared loss $\mathcal{L}_{\text{MeanFlow}} \leq \epsilon$:

$$\begin{aligned} \int_0^1 \|\boldsymbol{r}_{\text{MeanFlow}}\|_2 \exp\left[lL(1-s)\right]\mathrm{d}s &\leq \sqrt{\mathcal{L}_{\text{MeanFlow}}} \cdot \left(\int_0^1 \exp\left[2lL(1-s)\right]\mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq C\sqrt{\epsilon}, \end{aligned} \tag{32}$$

where $C = \sqrt{\frac{\exp(2lL)-1}{2lL}}$. For the second component, the integration of the $\mathcal{O}(l^2)$ term over $s \in [0,1]$ simply scales the exponential factor, strictly preserving the $\mathcal{O}(l^2)$ magnitude order.

Superimposing these integrated bounds directly yields the final pointwise error bound:

$$\|\boldsymbol{u}_\theta(\boldsymbol{x},t) - \boldsymbol{u}_{\text{True}}(\boldsymbol{x},t)\|_2 \leq C\sqrt{\epsilon} + \mathcal{O}(l^2). \tag{33}$$

This establishes that the global error is bounded by the optimization loss and a strict structural truncation error, concluding the proof. □

D.3 MeanFlow Error Bounded by Decoupled Losses

Theorem D.2 (Thm. 3.2, manuscript)**.** *Under the assumptions of Theorem 3.2, the pointwise residuals satisfy*

$$\mathcal{L}_{\textit{MeanFlow}} \leq c_1\mathcal{L}_{\textit{Temp}} + c_2\mathcal{L}_{\textit{Spac}} + \varepsilon, \tag{34}$$

where the coefficients are defined as $c_1 = 3l_t^2$ *and* $c_2 = 3\left(\frac{l-l_t}{l_s}\|\boldsymbol{g}\|_{op}\right)^2$*. Here,* ε *represents the structural coupling error and* $\varepsilon = \mathcal{O}(l_s^2)$*.*

Proof. We begin with the spatio-temporal residual vector $\boldsymbol{r}_{\text{MeanFlow}}$:

$$\boldsymbol{r}_{\text{MeanFlow}} = l_t\boldsymbol{m} - l\boldsymbol{K}\boldsymbol{f} - \gamma l^2\frac{\partial \boldsymbol{m}}{\partial \tau}. \tag{S1}$$

From the definition of the temporal residual $\boldsymbol{r}_{\text{Temp}}$, we isolate the ground-truth term:

$$\boldsymbol{m} = \boldsymbol{r}_{\text{Temp}} + \boldsymbol{K}\boldsymbol{f} + l_t\frac{\partial \boldsymbol{m}}{\partial \tau}. \tag{35}$$

Substituting this into Eq. (S1) yields:

$$\begin{aligned}\boldsymbol{r}_{\text{MeanFlow}} =& l_t(\boldsymbol{r}_{\text{Temp}} + \boldsymbol{K}\boldsymbol{f} + l_t\frac{\partial \boldsymbol{m}}{\partial \tau}) - l\boldsymbol{K}\boldsymbol{f} - \gamma l^2\frac{\partial \boldsymbol{m}}{\partial \tau},\\ =& l_t\boldsymbol{r}_{\text{Temp}} + (l_t - l)\boldsymbol{K}\boldsymbol{f} + (l_t^2 - \gamma l^2)\frac{\partial \boldsymbol{m}}{\partial \tau}.\end{aligned} \tag{S2}$$

Given the linear gradient PDE structure $\boldsymbol{f} = \nabla^\top \boldsymbol{u}\cdot\boldsymbol{g} + \boldsymbol{h}$, and the spatial residual $\boldsymbol{r}_{\text{Spac}}$, we substitute the spatial gradient $\boldsymbol{K}\nabla^\top\boldsymbol{u}$ into the expression for $\boldsymbol{K}\boldsymbol{f}$, i.e.,

$$\begin{aligned}\boldsymbol{r}_{\text{MeanFlow}} &= l_t\boldsymbol{r}_{\text{Temp}} + (l_t - l)\boldsymbol{K}(\nabla^\top\boldsymbol{u}\cdot\boldsymbol{g} + \boldsymbol{h}) + (l_t^2 - \gamma l^2)\frac{\partial \boldsymbol{m}}{\partial \tau},\\ =& l_t\boldsymbol{r}_{\text{Temp}} + \frac{l_t - l}{l_s}(\boldsymbol{r}_{\text{Spac}} + \boldsymbol{m}(\boldsymbol{x}-\boldsymbol{\xi})^\top - l_s^2\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}})\cdot\boldsymbol{g} + (l_t - l)\boldsymbol{K}\boldsymbol{h} + (l_t^2 - \gamma l^2)\frac{\partial \boldsymbol{m}}{\partial \tau},\end{aligned} \tag{S3}$$

Applying the generalized triangle inequality for squared L_2 norms (or Jensen's inequality for quadratic function, i.e., $\|\sum_{i=1}^3 \boldsymbol{a}_i\|_2^2 \leq 3\sum_{i=1}^3\|\boldsymbol{a}_i\|_2^2$) and the operator norm bound $\|\boldsymbol{v}\cdot\boldsymbol{g}\|_2 \leq \|\boldsymbol{g}\|_{\text{op}}\|\boldsymbol{v}\|_2$, we obtain:

$$\|\boldsymbol{r}_{\text{MeanFlow}}\|_2^2 \leq 3l_t^2\|\boldsymbol{r}_{\text{Temp}}\|_2^2 + 3\left(\frac{l-l_t}{l_s}\|\boldsymbol{g}\|_{\text{op}}\right)^2\|\boldsymbol{r}_{\text{Spac}}\|_2^2 + 3\|\boldsymbol{r}\|_2^2, \tag{36}$$

where

$$\boldsymbol{r} = \underbrace{\frac{l_t - l}{l_s}\left[\boldsymbol{m}(\boldsymbol{x}-\boldsymbol{\xi})^\top - l_s^2\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\right]\cdot\boldsymbol{g}}_{\text{Boundary or Higher-Order Discretization Errors}} + \underbrace{(l_t - l)\boldsymbol{K}\boldsymbol{h}}_{\text{Source Term Error}} + \underbrace{(l_t^2 - \gamma l^2)\frac{\partial \boldsymbol{m}}{\partial \tau}}_{\text{Temporal Scaling Mismatch}}, \tag{37}$$

collects the remaining structural terms. Therefore, as $\|\boldsymbol{K}\|_{\text{op}} \leq M_{\boldsymbol{K}}$ and the state variables are sufficiently smooth such that their norms are bounded, we have, applying generalized triangle inequality for squared L_2 norms again,

$$\begin{aligned}\varepsilon = 3\|\boldsymbol{r}\|_2^2 \leq& 9(l_t - l)^2\cdot\left[\frac{1}{l_s^2}\cdot\|\boldsymbol{r}_{\boldsymbol{g}}\|_2^2 + M_{\boldsymbol{K}}^2\|\boldsymbol{h}\|_2^2 + \left(\frac{l_t^2 - \gamma l^2}{l_t - l}\right)^2\left\|\frac{\partial \boldsymbol{m}}{\partial \tau}\right\|_2^2\right],\\ =& \begin{cases} 9\|\boldsymbol{r}_{\boldsymbol{g}}\|_2^2 + 9l_s^2\cdot M_{\boldsymbol{K}}^2\|\boldsymbol{h}\|_2^2, & \text{if } \gamma = 0,\\ 9(l_t - l)^2\cdot\left[\frac{1}{l_s^2}\cdot\|\boldsymbol{r}_{\boldsymbol{g}}\|_2^2 + M_{\boldsymbol{K}}^2\|\boldsymbol{h}\|_2^2 + (l_t + l)^2\left\|\frac{\partial \boldsymbol{m}}{\partial \tau}\right\|_2^2\right], & \text{if } \gamma = 1,\end{cases}\end{aligned} \tag{38}$$

where $\boldsymbol{r_g} \triangleq \left[\boldsymbol{m}(\boldsymbol{x}-\boldsymbol{\xi})^\top - l_s^2 \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\right] \cdot \boldsymbol{g}$. Here, the result for $\gamma = 0$ follows from the fact that $l_t = 0$ and $l = l_s$ for a steady-state system implied by $\gamma = 0$.

Note that the higher-order discretization error $\boldsymbol{r_g}$ satisfies

$$\begin{aligned}\|\boldsymbol{r_g}\|_2 \leq & \|\boldsymbol{m}(\boldsymbol{x}-\boldsymbol{\xi})^\top \boldsymbol{g}\|_2 + \left\| l_s^2 \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}} \boldsymbol{g} \right\|_2 \leq \|\boldsymbol{m}\|_2 \cdot \|\boldsymbol{x}-\boldsymbol{\xi}\|_2 \cdot \|\boldsymbol{g}\|_{\text{op}} + \mathcal{O}(l_s^2), \\ = & \|\boldsymbol{m}\|_2 \cdot l_s \cdot \|\boldsymbol{g}\|_{\text{op}} = \mathcal{O}(l_s).\end{aligned}$$

Consequently, since the bracketed term in the $\gamma = 1$ case is bounded by $\mathcal{O}(1)$ and the temporal scaling factor satisfies $l_t - l = \frac{l_t^2 - l^2}{l_t + l} = -\frac{l_s^2}{l_t + l} = \mathcal{O}(l_s^2)$, the asymptotic bound for the structural coupling error yields:

$$\varepsilon = \begin{cases} \mathcal{O}(l_s^2), & \text{if } \gamma = 0, \\ \mathcal{O}(l_s^4), & \text{if } \gamma = 1. \end{cases}$$

This concludes the proof. □

The preceding result is stated for a general PDE decomposition. We now specialize it to the two-dimensional incompressible Navier–Stokes equation in vorticity form because its source term $\boldsymbol{h}$ contains the higher-order spatial derivative $\nu\Delta\omega$. This case therefore directly tests whether the remainder estimate in Eq. (36) remains valid for a PDE with higher-order spatial operators. We specialize the result to the two-dimensional periodic Navier–Stokes equation used in our experiments. The higher-order term $\nu\Delta\omega$ makes this example useful for verifying the decoupling bound for PDEs containing second-order spatial derivatives.

Corollary D.1 (2D Navier–Stokes local specialization)**.** *Consider the vorticity equation on* $\Omega = \mathbb{T}^2 \simeq [0,1]^2$ *with periodic boundary conditions,*

$$\partial_t \omega(\boldsymbol{c}, t) + \boldsymbol{v}(\boldsymbol{c}, t) \cdot \nabla \omega(\boldsymbol{c}, t) = \nu \Delta \omega(\boldsymbol{c}, t) + q(\boldsymbol{c}),$$

where $\boldsymbol{v} = \mathcal{B}\omega$ *and* $\mathcal{B} = \nabla^\perp \Delta^{-1}$*. Assume that*

$$\sup_{t \in [0,T]} \|\omega(t)\|_{H^2(\mathbb{T}^2)} \leq M_\omega, \qquad q \in L^2(\mathbb{T}^2).$$

the Biot–Savart reconstruction is locally bounded, and the MeanFlow quantities appearing in Eq. (36) are uniformly bounded, with $l_s \leq \delta$, $l + l_t \leq L_{\max}$*. Then the structural coupling error in Eq. (36) satisfies*

$$\varepsilon \leq C_{\text{NS}} l_s^2,$$

where C_{NS} *is independent of* l_s*. Thus, locally, the vorticity–velocity coupling changes only the constant in the decoupling estimate and does not alter its* $\mathcal{O}(l_s^2)$ *scaling.*

Proof. Write the PDE in the decomposition used in D.3 by setting

$$\boldsymbol{g} \triangleq -\boldsymbol{v} = -\mathcal{B}\omega, \qquad \boldsymbol{h} \triangleq \nu\Delta\omega + q.$$

By the assumed local Sobolev regularity and the standard decay or mean-zero conditions for the Biot–Savart reconstruction,

$$\|\boldsymbol{v}(t)\|_{L^\infty} \leq C_{\mathcal{B}} \|\omega(t)\|_{H^2(\mathbb{T}^2)} \leq C_{\mathcal{B}} M_\omega.$$

Moreover, $\|\Delta\omega(t)\|_{L^2(\mathbb{T}^2)} \leq \|\omega(t)\|_{H^2(\mathbb{T}^2)} \leq M_\omega$. Hence

$$\|\boldsymbol{g}\|_{L^\infty} \leq M_{\boldsymbol{g}}, \qquad \|\boldsymbol{h}\|_2 \leq M_{\boldsymbol{h}}.$$

For the remainder r_g defined in Eq. (36), we have

$$\begin{aligned}\|\boldsymbol{r_g}\|_2 &\leq \|\boldsymbol{m}\|_2 \|\boldsymbol{x}-\boldsymbol{\xi}\|_2 \|\boldsymbol{g}\|_{\text{op}} + l_s^2 \, \|\partial_{\boldsymbol{\xi}} \boldsymbol{m}\, \boldsymbol{g}\|_2 \\ &\leq l_s M_{\boldsymbol{g}} \left(M_{\boldsymbol{m}} + l_s M_{\boldsymbol{\xi}}\right) \leq l_s M_{\boldsymbol{g}} (M_{\boldsymbol{m}} + \delta M_{\boldsymbol{\xi}}),\end{aligned}$$

where $M_{\boldsymbol{m}}$, $M_{\boldsymbol{\xi}}$, M_t, $M_{\boldsymbol{K}}$ are the upper bounds for $\|\boldsymbol{m}\|_2$, $\|\partial_{\boldsymbol{\xi}} \boldsymbol{m}\|_2$, $\|\partial_\tau \boldsymbol{m}\|_2$, and $\|\boldsymbol{K}\|_{\text{op}}$, respectively.

Moreover, $l - l_t = \frac{l_s^2}{l+l_t} \le l_s$. Finally, applying Eq. (36) and the same uniform bounds to the source-term, higher-order, and temporal-scaling contributions yields

$$\varepsilon \le 9l_s^2 \left[M_{\boldsymbol{g}}^2 \left(M_{\boldsymbol{m}} + \delta M_{\boldsymbol{\xi}} \right)^2 + M_{\boldsymbol{K}}^2 M_{\boldsymbol{h}}^2 + L_{\max}^2 M_t^2 \right] = C_{\mathrm{NS}} l_s^2.$$

Thus, under the stated uniform regularity and sensitivity assumptions, the Navier–Stokes vorticity–velocity coupling affects only the finite constant $\mathcal{C}_{\mathrm{NS}}$, while the structural decoupling error remains of order $\mathcal{O}(l_s^2)$. □

D.4 Hessian Condition Number Analysis

Theorem D.3 (Improvement in Gauss–Newton Conditioning)**.** *Let $\mathcal{H}_D$ and $\mathcal{H}_C$ denote the decoupled and coupled Gauss–Newton matrices with respect to the network parameter vector $\boldsymbol{\theta} \in \mathbb{R}^p$, obtained by flattening all p trainable scalar parameters of the network. Assume that, on $\mathbb{R}^p$,*

$$0 < \mu \triangleq \lambda_{\min}(\mathcal{H}_D) \le \lambda_{\max}(\mathcal{H}_D) \triangleq L < \infty,$$

and let $v_{\min}$ and $v_{\max}$ be unit eigenvectors associated with $\lambda_{\min}(\mathcal{H}_D)$ and $\lambda_{\max}(\mathcal{H}_D)$, respectively. Write $\mathcal{C} = \mathcal{H}_C - \mathcal{H}_D$. If, for some $0 < \rho < 1$,

$$v_{\min}^\top \mathcal{C} v_{\min} \le -\rho\mu, \qquad v_{\max}^\top \mathcal{C} v_{\max} \ge 0,$$

and $\mathcal{H}_C$ is positive definite on $\mathbb{R}^p$, then

$$\kappa(\mathcal{H}_C) \ge \frac{1}{1-\rho}\, \kappa(\mathcal{H}_D).$$

More generally, if

$$\eta \triangleq \left\| \mathcal{H}_D^{-1/2} \mathcal{C} \mathcal{H}_D^{-1/2} \right\|_2 < 1,$$

then $\mathcal{H}_C$ is positive definite on $\mathbb{R}^p$ and

$$\frac{1-\eta}{1+\eta}\, \kappa(\mathcal{H}_D) \le \kappa(\mathcal{H}_C) \le \frac{1+\eta}{1-\eta}\, \kappa(\mathcal{H}_D).$$

Proof. Let $\boldsymbol{r}_{\mathrm{Temp}}$ and $\boldsymbol{r}_{\mathrm{Spac}}$ be the temporal and spatial residuals, and define

$$\boldsymbol{J}_T \triangleq \nabla_{\boldsymbol{\theta}} \boldsymbol{r}_{\mathrm{Temp}}, \qquad \boldsymbol{J}_S \triangleq \nabla_{\boldsymbol{\theta}} \boldsymbol{r}_{\mathrm{Spac}}, \qquad \boldsymbol{J} \triangleq \begin{bmatrix} \alpha \boldsymbol{J}_T \\ \beta \boldsymbol{J}_S \end{bmatrix}.$$

Neglecting residual-weighted second-order terms, the Gauss–Newton matrices are

$$\mathcal{H}_D = \alpha^2 \boldsymbol{J}_T^\top \boldsymbol{J}_T + \beta^2 \boldsymbol{J}_S^\top \boldsymbol{J}_S = \boldsymbol{J}^\top \boldsymbol{J},$$

and

$$\mathcal{H}_C \approx \mathcal{H}_D + \mathcal{C}, \qquad \mathcal{C} \triangleq \alpha\beta \left(\boldsymbol{J}_T^\top \boldsymbol{J}_S + \boldsymbol{J}_S^\top \boldsymbol{J}_T \right).$$

Define

$$s_{\min} \triangleq \min_{\substack{v \in \mathbb{R}^p \\ \|v\|_2 = 1}} \|\boldsymbol{J} v\|_2, \qquad s_{\max} \triangleq \max_{\substack{v \in \mathbb{R}^p \\ \|v\|_2 = 1}} \|\boldsymbol{J} v\|_2.$$

Then $\mu = s_{\min}^2$, $L = s_{\max}^2$, and

$$\kappa(\mathcal{H}_D) = \frac{L}{\mu} = \left(\frac{s_{\max}}{s_{\min}} \right)^2.$$

By the Rayleigh–Ritz characterization and the directional assumptions,

$$\lambda_{\min}(\mathcal{H}_C) \le v_{\min}^\top \mathcal{H}_C v_{\min} \le (1-\rho)\mu,$$

while

$$\lambda_{\max}(\mathcal{H}_C) \ge v_{\max}^\top \mathcal{H}_C v_{\max} \ge L.$$

Therefore,

$$\kappa(\mathcal{H}_C) \geq \frac{L}{(1-\rho)\mu} = \frac{1}{1-\rho}\,\kappa(\mathcal{H}_D).$$

If $\mathcal{H}_C$ is singular, the same lower bound holds under the convention $\kappa(\mathcal{H}_C) = +\infty$.

For the final claim, set

$$\boldsymbol{E} \triangleq \mathcal{H}_D^{-1/2}\mathcal{C}\mathcal{H}_D^{-1/2}.$$

The condition $\|\boldsymbol{E}\|_2 = \eta < 1$ implies

$$-\eta I \preceq \boldsymbol{E} \preceq \eta I, \qquad (1-\eta)\mathcal{H}_D \preceq \mathcal{H}_C \preceq (1+\eta)\mathcal{H}_D.$$

Comparing the extreme eigenvalues gives the stated two-sided condition-number bounds. □

E Examples of Spatio-Temporal MeanFlow Losses

Example 1: d'Alembert's equation The proposed method applys to the equations

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{c}(\boldsymbol{u}) = \boldsymbol{0}, \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \end{cases}$$

the variables in our method are $\nabla^\top \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}$, $\boldsymbol{f} = -\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{c}(\boldsymbol{u})$, hereby Eq. equation 20 obtains

$$\begin{aligned} &(t-\tau)\boldsymbol{m} - \left(-\boldsymbol{m}(\boldsymbol{x}-\boldsymbol{\xi})^\top + l^2\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\right)\boldsymbol{c}(\boldsymbol{u}) - l^2\frac{\partial \boldsymbol{m}}{\partial \tau} = \boldsymbol{0}, \\ \iff &\left[t-\tau+(\boldsymbol{x}-\boldsymbol{\xi})^\top\boldsymbol{c}(\boldsymbol{u})\right]\boldsymbol{m} - l^2\left(\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\boldsymbol{c}(\boldsymbol{u}) + \frac{\partial \boldsymbol{m}}{\partial \tau}\right) = \boldsymbol{0}, \end{aligned}$$

where $l^2 = (t-\tau)^2 + \|\boldsymbol{x}-\boldsymbol{\xi}\|_2^2$ and the equation allows the construction of a MeanFlow loss just as the original MeanFlow framework did.

The meanflow loss is thus

$$\mathcal{L}^{\text{(d'Alembert's)}}_{\text{ST-MeanFlow}} = \left\| \left[t-\tau+(\boldsymbol{x}-\boldsymbol{\xi})^\top\boldsymbol{c}(\boldsymbol{u})\right]\boldsymbol{m} - l^2\left(\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\xi}}\boldsymbol{c}(\boldsymbol{u}) + \frac{\partial \boldsymbol{m}}{\partial \tau}\right)\right\|_2^2.$$

Example 2: Periodic 2D Navier–Stokes equations. The vorticity formulation of the incompressible Navier–Stokes equations on the periodic domain $\mathbb{T}^2$ is

$$\begin{cases} \dfrac{\partial \omega(\boldsymbol{x},t)}{\partial t} = -\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\omega(\boldsymbol{x},t) + \nu\Delta\omega(\boldsymbol{x},t) + q(\boldsymbol{x},t), \\ \omega(\boldsymbol{x},t) = \dfrac{\partial u_2(\boldsymbol{x},t)}{\partial x_1} - \dfrac{\partial u_1(\boldsymbol{x},t)}{\partial x_2}, \\ \nabla\cdot\boldsymbol{u}(\boldsymbol{x},t) = 0, \\ \omega(\boldsymbol{x}+2\pi\boldsymbol{k},t) = \omega(\boldsymbol{x},t), \qquad \boldsymbol{k}\in\mathbb{Z}^2, \\ \omega(\boldsymbol{x},0) = \omega_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\mathbb{T}^2. \end{cases} \tag{39}$$

Assuming zero spatial means, the velocity field can be reconstructed from the vorticity through the periodic Biot–Savart operator:

$$\boldsymbol{u}(\boldsymbol{x},t) = \nabla^\perp(-\Delta)^{-1}\omega(\boldsymbol{x},t) = \int_{\mathbb{T}^2}\boldsymbol{K}_{\text{per}}(\boldsymbol{x}-\boldsymbol{y})\omega(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}.$$

where

$$\nabla^\perp = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right), \qquad \boldsymbol{K}_{\text{per}} = \nabla^\perp G_{\text{per}}.$$

Here, G_{per} is the periodic Green's function satisfying

$$-\Delta G_{\text{per}}(\boldsymbol{x}) = \delta_{\mathbb{T}^2}(\boldsymbol{x}) - \frac{1}{|\mathbb{T}^2|}.$$

Equivalently, in the Fourier domain,

$$\widehat{\boldsymbol{u}}(\boldsymbol{k},t)=\frac{\mathrm{i}\,\boldsymbol{k}^{\perp}}{\|\boldsymbol{k}\|_2^2}\widehat{\omega}(\boldsymbol{k},t),\qquad \boldsymbol{k}\in\mathbb{Z}^2\setminus\{\mathbf{0}\},$$

with

$$\widehat{\boldsymbol{u}}(\mathbf{0},t)=\mathbf{0}.$$

Taking the scalar vorticity ω as the state variable and using $\boldsymbol{u}=\mathcal{B}\omega$, the periodic 2D Navier–Stokes equation has the form

$$\frac{\partial\omega}{\partial t}=-\boldsymbol{u}\cdot\nabla\omega+\nu\Delta\omega+q.$$

Thus, the PDE operator admits the decomposition

$$f=\nabla^{\top}\omega\cdot\boldsymbol{g}+h,\qquad \boldsymbol{g}=-\boldsymbol{u},\qquad h=\nu\Delta\omega+q.$$

Applying the temporal, first-order spatial, and second-order spatial MeanFlow identities directly gives

$$\ell_t m=\ell K q(\boldsymbol{\xi},\tau)+\boldsymbol{D}\cdot\boldsymbol{u}(\boldsymbol{\xi},\tau)-\nu\left(\alpha+2K^{-1}\beta\right)+\ell^2\frac{\partial m}{\partial\tau},$$

where

$$\boldsymbol{D}=m(\boldsymbol{\xi}-\boldsymbol{x})^{\top}+\ell_s^2\frac{\partial m}{\partial\boldsymbol{\xi}}.$$

Therefore, the periodic 2D Navier–Stokes MeanFlow residual is

$$\mathcal{L}=\left\|\ell_t m-\ell K q(\boldsymbol{\xi},\tau)-\boldsymbol{D}\cdot\boldsymbol{u}(\boldsymbol{\xi},\tau)+\nu\left(\alpha+2K^{-1}\beta\right)-\ell^2\frac{\partial m}{\partial\tau}\right\|_2^2. \tag{40}$$

Equivalently, to avoid the explicit inverse of K, one may use the algebraically equivalent residual

$$\mathcal{L}^{\text{(Navier-Stokes)}}_{\text{ST-MeanFlow}}=K\left[\ell_t m-\ell K q(\boldsymbol{\xi},\tau)-\boldsymbol{D}\cdot\boldsymbol{u}(\boldsymbol{\xi},\tau)+\nu\alpha-\ell^2\frac{\partial m}{\partial\tau}\right]+2\nu\beta. \tag{41}$$

F Experiments

F.1 Details of Synthetic PDE Benchmarks

We follow the data generation protocol of DiffusionPDE (Huang et al., 2024) and use four synthetic PDE benchmarks: 1D Burgers' equation, 2D periodic Navier–Stokes equation, 2D Darcy flow, and 2D Poisson equation. For each benchmark, we use 10,000 samples and split them into training, validation, and test sets with a ratio of 7:2:1, resulting in 7,000 training samples, 2,000 validation samples, and 1,000 test samples.

F.1.1 1D Burgers' Equation

Governing equation. We consider the one-dimensional viscous Burgers' equation on $\Omega=(0,1)$ with periodic boundary conditions:

$$\partial_t u(c,t)+\partial_c\left(\frac{u^2(c,t)}{2}\right)=\nu\partial_{cc}u(c,t),\quad c\in\Omega,\quad t\in(0,T].$$

The initial condition is $u(c,0)=u_0(c)$, and the viscosity is set to $\nu=0.01$.

Numerical discretization and data generation. Following DiffusionPDE, the spatial resolution is 128, and each trajectory contains 128 temporal states, including the initial state and 127 future time steps. The time interval is $T=1$. Initial conditions are sampled from a one-dimensional Gaussian random field with parameters $\gamma=2.5$, $\tau=7$, and $\sigma=7^2$. The equation is solved using the spectral time-stepping solver used in the DiffusionPDE/FNO-style Burgers data generation code. The resulting data are stored as trajectories of shape $N\times X\times T$, where $N=10{,}000$, $X=128$, and $T=128$.

Use in our method. For Burgers' equation, our model takes the initial condition $u_0(c)$ as input and predicts the full spatio-temporal solution trajectory $u(c,t)$.

F.1.2 2D PERIODIC NAVIER–STOKES EQUATION

Governing equation. We consider the two-dimensional incompressible Navier–Stokes equation in vorticity form on the periodic domain $\Omega = \mathbb{T}^2 \simeq [0,1]^2$:

$$\partial_t \omega(\boldsymbol{c},t) + \boldsymbol{v}(\boldsymbol{c},t)\cdot\nabla\omega(\boldsymbol{c},t) = \nu\Delta\omega(\boldsymbol{c},t) + q(\boldsymbol{c}), \quad \boldsymbol{c}\in\Omega, \quad t\in(0,T].$$

The fields satisfy periodic boundary conditions. Here ω is the scalar vorticity, $\boldsymbol{v}$ is the velocity field, and

$$\boldsymbol{v} = \mathcal{B}_{\mathrm{per}}\omega = \nabla^{\perp}(-\Delta)^{-1}\omega.$$

Here, $\omega = \partial_{c_1} v_2 - \partial_{c_2} v_1$ is the scalar vorticity. We set $\nu = 10^{-3}$, corresponding to Reynolds number $\mathrm{Re} = 1000$.

Numerical discretization and data generation. The spatial domain is discretized on a 128×128 grid. Following DiffusionPDE, the initial vorticity field is sampled from a two-dimensional Gaussian random field with $\alpha = 2.5$ and $\tau = 7$. The forcing term is fixed as

$$q(\boldsymbol{c}) = 0.1\left(\sin(2\pi(x+y)) + \cos(2\pi(x+y))\right).$$

The equation is solved in Fourier space with a pseudo-spectral method, de-aliasing, and a Crank–Nicolson update. The Fourier pseudo-spectral solver enforces periodic boundary conditions on $\Omega = \mathbb{T}^2$. The internal time step is 10^{-4}, the final time is $T = 1$, and 10 snapshots are recorded for each trajectory. We use ten files, each containing 1,000 samples, giving 10,000 samples in total. Each file stores the initial vorticity field with shape $1000\times128\times128$ and the solution trajectory with shape $1000\times128\times128\times10$.

Use in our method. For Navier–Stokes, our model takes the initial vorticity field $\omega(c,0)$ as input and predicts the future vorticity fields over the recorded time steps. The forcing term and boundary/domain setting are fixed across the dataset.

F.1.3 2D DARCY FLOW

Governing equation. We consider the steady-state Darcy flow equation on $\Omega = (0,1)^2$:

$$-\nabla\cdot(a(\mathbf{c})\nabla u(\mathbf{c})) = q(\mathbf{c}), \quad \mathbf{c}\in\Omega,$$

with homogeneous Dirichlet boundary condition

$$u(\mathbf{c}) = 0, \quad \mathbf{c}\in\partial\Omega.$$

Here, $a(\mathbf{c})$ is the permeability coefficient, $u(\mathbf{c})$ is the pressure field, and the forcing term is fixed as $q(\mathbf{c}) = 1$.

Numerical discretization and data generation. The domain is discretized on a 128×128 grid. Following DiffusionPDE, a Gaussian random field $g(\mathbf{c})$ is sampled with parameters $\alpha = 2$ and $\tau = 3$, and the log-normal permeability field is constructed as

$$a(\mathbf{c}) = \exp(g(\mathbf{c})).$$

The PDE is solved by assembling a sparse finite-difference linear system for the elliptic operator with zero boundary values. We use 10,000 Darcy samples. The coefficient field has shape $10000\times128\times128$, and the corresponding pressure solution has the same spatial resolution.

Use in our method. For Darcy flow, our model is conditioned on the permeability field $a(\mathbf{c})$ and the prescribed boundary condition, and predicts the pressure solution $u(\mathbf{c})$. Since the boundary condition is homogeneous and shared by all samples, it is treated as fixed conditioning in the benchmark.

Table 4: Hyperparameter settings for synthetic PDE benchmarks.

Dataset	Backbone	Batch size	Learning rate	Epochs	Loss weights
Burgers	Conditioned FNO1d	8	5.5×10^{-4}	200	$\lambda_t = \lambda_x = 10^{-2}$
Navier–Stokes	Fourier U-Net	5	2.0×10^{-4}	100	$\lambda_t = \lambda_x = 10^{-2}$
Darcy	Fourier U-Net	4	3.0×10^{-4}	300	$\lambda_x = 10^{-2}$
Poisson	Fourier U-Net	4	3.0×10^{-4}	300	$\lambda_x = 10^{-2}$

F.1.4 2D Poisson Equation

Governing equation. We consider the two-dimensional Poisson equation on $\Omega = (0,1)^2$:

$$\Delta u(c) = q(c), \quad c \in \Omega,$$

with homogeneous Dirichlet boundary condition

$$u(c) = 0, \quad c \in \partial\Omega.$$

Here, $q(c)$ is the source field and $u(c)$ is the solution field.

Numerical discretization and data generation. The domain is discretized on a 128×128 grid. Following DiffusionPDE, the source field $q(c)$ is sampled from a Gaussian random field with parameters $\alpha = 2$ and $\tau = 3$. The Poisson equation is solved using a finite-difference discretization with a standard five-point Laplacian stencil and homogeneous Dirichlet boundary values. We use 10,000 samples. The source field has shape $10000 \times 128 \times 128$, and the corresponding solution field has the same spatial resolution.

Use in our method. For the Poisson equation, our model is conditioned on the source field $q(c)$ and the prescribed boundary condition, and predicts the solution field $u(c)$. As in Darcy flow, the boundary condition is fixed for all samples.

F.2 Implementation Details of Main Experiments

For all PDEs, four finite-interval pairs (τ, t) are randomly sampled at each iteration to construct the MeanFlow regularization. For steady-state PDEs, the model takes the coefficient/source field together with the prescribed boundary condition as input. Models are trained with a supervised relative L_2 loss and the proposed MeanFlow regularization. All results are reported using the checkpoint with the lowest validation relative L_2 error.

For Burgers' equation, we use a conditioned FNO1d. The model takes the current solution state and spatial coordinate as input, and is conditioned on the interval information. It uses Fourier modes $[12, 12, 12, 12, 12]$, channel width 96, fully connected hidden dimension 192, GELU activation, and a conditioning MLP with hidden dimension 224.

For Navier–Stokes, Darcy flow, and Poisson equation, we use a Fourier U-Net backbone. The Fourier U-Net follows an encoder–decoder architecture with skip connections. The initial feature projection is a 3×3 convolution with 48 hidden channels. The encoder has four resolution levels with channel multipliers $(1, 2, 2, 4)$, and each level contains two residual blocks. In the first two encoder levels, the standard convolutional residual blocks are replaced by Fourier residual blocks, where each block combines spectral convolution layers with local 1×1 convolution branches. The remaining encoder blocks, the middle block, and the decoder use convolutional residual blocks. The decoder upsamples features back to the original resolution and concatenates skip features from the encoder. We use GELU activation, no attention blocks, and no normalization.

For Navier–Stokes, the Fourier U-Net input contains the current vorticity field and the two spatial coordinate channels. For Darcy flow and Poisson equation, the Fourier U-Net uses two scalar input channels: the harmonic extension of the boundary condition and the coefficient/source field. Darcy flow and Poisson equation use Fourier modes $(8, 8)$ in the first Fourier level, while Navier–Stokes uses the default Fourier modes $(12, 12)$. The number of retained modes is reduced at lower resolutions according to the U-Net scale. Table 4 summarizes the main hyperparameters used in our experiments.

F.3 Compared Methods

We compare our method with baselines from three representative paradigms: physics-informed solvers, and neural operator solvers.

PINN. Physics-informed neural networks directly parameterize the solution function with a neural network and train it by minimizing initial-condition, boundary-condition, and PDE residual losses. PINNs do not require paired simulation data, but their optimization can become unstable when different loss terms have conflicting scales or gradients.

I-PINN. I-PINN improves the vanilla PINN framework by combining an enhanced feed-forward architecture with adaptive loss weighting under upper-bound constraints. This design aims to mitigate gradient-flow stiffness and reduce gradient-related training failures. Compared with PINN, I-PINN improves convergence and accuracy without introducing substantial extra computational complexity.

FNO. Fourier Neural Operator is a data-driven neural operator that learns mappings between function spaces. It uses Fourier-domain integral operator layers to model global dependencies and can be evaluated on different spatial discretizations. In our experiments, FNO serves as a standard data-driven operator-learning baseline.

DeepONet. DeepONet is a neural operator architecture designed to learn nonlinear operators. It uses a branch network to encode the input function sampled at sensor locations and a trunk network to encode output coordinates. The two representations are combined to predict the output function value. Therefore, DeepONet is categorized as an operator-learning method rather than a standard pointwise neural network.

PINO. Physics-informed Neural Operator combines supervised operator learning with PDE residual constraints. PINO is built on the FNO framework and imposes physics losses, potentially at higher resolutions than the supervised data. It is a hybrid baseline between data-driven operator learning and physics-informed training.

RNO. Reference Neural Operators (RNO) learn the smooth dependence of PDE solutions on geometric deformations. Given a reference geometry and its solution, RNO predicts solutions on deformed domains by incorporating this reference information. This formulation improves data efficiency compared with directly learning geometry-to-solution mappings, making it useful for applications where numerical simulations are expensive and geometry–solution pairs are scarce.

CFO. Continuous Flow Operator learns continuous-time PDE dynamics using a flow-matching objective. Instead of autoregressively predicting future states on a fixed time grid, CFO fits temporal splines to trajectory data and trains a neural operator to predict analytic velocity fields. During inference, the learned continuous vector field is integrated with an ODE solver. We classify CFO as a continuous-time neural operator baseline.

Our FNO and PINO baselines follow the official implementation. For time-dependent PDEs, they take the initial condition and space-time coordinates as input and directly predict the entire trajectory, e.g., $[u_0(x), x, t] \mapsto u(x, t)$. They are therefore one-shot neural operators, not autoregressive models: predicted states are never fed back as subsequent inputs. FNO is trained with supervised relative L_2 loss, while PINO additionally uses PDE-residual and initial-condition losses. For steady PDEs, both directly map the coefficient/source field and boundary information to the solution.

F.4 Broader Applicability: Real-World Data and Inverse Modeling

F.4.1 Results

To demonstrate the versatility of our Spatio-Temporal MeanFlow framework beyond standard synthetic forward prediction, we extend its evaluation to two challenging scenarios.

Real-World Data. We first evaluate our method on the HYCOM dataset (details in App. F.4.3), which provides data-assimilative 3D water temperature fields on a $32 \times 32 \times 8$ grid. A key challenge here

Table 5: Results on the real-world HYCOM forward problem and the inverse Darcy-flow problem. We report the relative L_2 error (↓) as the mean ± standard deviation across three different random seeds. All reported metrics are multiplied by 10^{-2} for readability. The best results are highlighted in **bold**, while the second-best results are underlined.

Task ($\times 10^{-2}$)	PINN	I-PINN	DeepONet	FNO	PINO	CFO	**Ours**
Real-world HYCOM	28.12±0.24	33.05±0.45	34.23±0.54	24.08±0.79	23.79±0.11	32.75±4.26	**23.43±0.12**
Inverse Darcy	27.41±0.01	8.95±0.01	8.73±0.26	**1.40±0.00**	12.19±0.32	17.07±1.01	2.72±0.00

is the absence of an explicit PDE residual due to realistic noise and assimilation artifacts. Therefore, we leverage the Spatio-Temporal MeanFlow formulation as a weak physical regularization to learn the temporally averaged flow. This mitigates the overfitting risks of unconstrained direct regression and encourages temporally coherent predictions. As shown in Tab. 1 ("Real-world"), this strategy ensures effective and stable predictions under realistic distributional shifts.

Inverse Problem on Darcy Flow. Furthermore, we evaluate whether our framework can be applied to inverse problems on Darcy flow, where the objective is to recover the unknown permeability field $\boldsymbol{a}(x)$ from observed pressure $\boldsymbol{u}^{\text{obs}}(x)$. Via test-time optimization, we freeze the pretrained forward model and optimize the log-permeability $\log \boldsymbol{a}$ as a learnable parameter using gradient descent. As shown in Tab. 5, our approach achieves accurate coefficient reconstruction, highlighting the broader potential of extending our framework to inverse modeling.

The Darcy inverse problem differs substantially from the forward-solving tasks in the other benchmarks. It is an ill-posed reconstruction problem: the unknown coefficient field must be inferred from indirect observations, and the relative L_2 error can be highly sensitive to small errors in this reconstruction. Therefore, the ranking in this setting depends strongly on the observation operator, regularization, and parameterization used by each method. In Tab. 1, our framework is applied with the same general PDE-solving formulation used for the forward tasks, whereas some competing methods incorporate inductive biases or objectives specifically tailored to inverse reconstruction. Their advantage on this benchmark therefore reflects better alignment with the inverse setting, rather than a general superiority in PDE solution prediction. On the forward Darcy problem, our method remains competitive.

F.4.2 Implementation Details

This section provides implementation details for applying our MeanFlow-based solver to the real-world HYCOM ocean dataset. The goal of this experiment is to evaluate whether the proposed framework can scale to realistic three-dimensional spatio-temporal data and remain effective when the observed dynamics are not generated from an ideal synthetic PDE.

Implementation Details for HYCOM. Each processed HYCOM sample has shape $9 \times 32 \times 32 \times 8$, where the first dimension denotes time and the remaining dimensions denote the three-dimensional spatial grid. We denote the normalized temperature field as $u \in \mathbb{R}^{T \times X \times Y \times Z}$, with $T = 9$, $X = 32$, $Y = 32$, and $Z = 8$. In implementation, the data are rearranged to $u \in \mathbb{R}^{X \times Y \times Z \times T}$. The initial state u_0 is used as the conditioning field, and the model predicts the future temperature evolution over the full temporal window.

For each query time step, we construct a five-channel input tensor $\mathbf{x} = (x, y, z, t, u_0)$, where (x, y, z) are normalized spatial coordinates, t is the normalized temporal coordinate, and u_0 is the initial three-dimensional temperature field broadcast to all query times. Thus, for each time step, the model input has shape $32 \times 32 \times 8 \times 5$. The model outputs a scalar temperature-related quantity at each spatial location.

We use a three-dimensional Fourier neural operator backbone. The model applies spectral convolution over the three spatial dimensions and predicts the temperature evolution conditioned on the initial state and coordinates. Unless otherwise specified, the HYCOM experiment uses the following architecture: $\text{modes}_x = [8, 8, 8, 8]$, $\text{modes}_y = [8, 8, 8, 8]$, $\text{modes}_z = [4, 4, 4, 4]$, with channel width 20, fully

connected dimension 64, and hidden layers $[20, 20, 20, 20, 20]$. The activation function is GELU. No additional padding is used.

We train the model for 101 epochs using Adam with learning rate 10^{-3}. The batch size is 2. A multi-step learning rate schedule is used with milestones at epochs 60 and 90 and decay factor 0.5. Validation is performed every 10 epochs. The checkpoint with the best validation loss is selected, and the corresponding test loss is reported.

Implementation Details for the Inverse Darcy Experiment. We consider the inverse Darcy flow problem: given a fully observed pressure field $u^{\text{obs}} \in \mathbb{R}^{S\times S}$, recover the permeability field $a(x)$ satisfying $-\nabla \cdot (a\nabla u) = 1$ with homogeneous Dirichlet boundary conditions.

The forward MeanFlow model is trained on the forward Darcy task as described in Section F.1.3. Its weights are fully frozen during inverse optimization; no fine-tuning is performed.

Initial field construction. Rather than initializing the input state z_0 to the zero field, we construct a *harmonic extension* of the boundary values of u^{obs}. Concretely, we fix the boundary pixels of z_0 to match $u^{\text{obs}}|_{\partial\Omega}$ and iteratively average interior values over 100 Jacobi iterations:

$$z_0^{(k+1)}(x) = \tfrac{1}{4}\sum_{x'\sim x} z_0^{(k)}(x'), \quad z_0|_{\partial\Omega} = u^{\text{obs}}|_{\partial\Omega}. \tag{42}$$

This initialization satisfies the boundary condition and provides a spatially smooth starting point, which we find stabilizes early optimization steps. We run Adam for 2000 steps with initial learning rate 5×10^{-3}, decayed to 10^{-4} via cosine annealing. Gradient norms are clipped to 1.0. All experiments use a single NVIDIA GPU.

F.4.3 Details of the Real-world HYCOM Dataset

We use a real-world ocean dynamics dataset derived from HYCOM for evaluating the scalability and robustness of the proposed method. HYCOM, the Hybrid Coordinate Ocean Model, is a global ocean analysis and forecasting system. Different from purely synthetic PDE benchmarks, HYCOM products are generated by combining numerical ocean model dynamics with real observations through data assimilation. The assimilated observations include satellite measurements and in-situ ocean observations. Therefore, the resulting fields reflect realistic ocean variability, while also containing practical complications such as observation noise, assimilation artifacts, imperfect boundaries, and possible mismatch between the observed dynamics and idealized PDE assumptions.

Data source. The data are obtained from the public HYCOM GLBy0.08 product through the THREDDS/OPeNDAP service:

`https://tds.hycom.org/thredds/dodsC/GLBy0.08/latest`.

We use the variable `water_temp`, which represents three-dimensional ocean water temperature. The original HYCOM product provides global ocean fields over longitude, latitude, depth, and time. In our experiment, we select a local ocean region and use the first 8 depth levels to construct a three-dimensional spatio-temporal prediction task.

Selected region and variables. We extract a local spatial block from the HYCOM grid using latitude indices $[1100, 1260)$ and longitude indices $[2900, 3060)$. This corresponds approximately to latitudes from $-36.0°$ to $-29.64°$ and longitudes from $232.0°$ to $244.72°$ in the HYCOM convention, i.e., approximately $128.0°$W to $115.28°$W. The selected depth levels are $0,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 15$ meters.

The raw extracted block has shape $32 \times 8 \times 160 \times 160$, corresponding to time, depth, latitude, and longitude dimensions, respectively.

Preprocessing. We normalize the extracted temperature field using the mean and standard deviation computed over the selected raw block. The statistics used in our processed dataset are $\mu = 23.01293796527723, \quad \sigma = 1.3310423861925285$. The raw temperature values in the selected block range approximately from 18.8700 to 26.1490. After normalization, we construct local

Table 6: Performance and computational cost on the Burgers benchmark. We report the relative L_2 error ($\times 10^{-2}$), inference time (milliseconds per sample), and GFLOPs per sample. Lower is better. The best reported values are highlighted with bold text and a gray background.

	Coord-based		*Neural operators*					
Metric	PINN	I-PINN	DeepONet	FNO	PINO	CFO	RNO	Ours
Error ($\times 10^{-2}$) ↓	3.56	8.92	5.59	1.05	2.06	0.80	0.39	**0.26**
Time (ms) ↓	**0.032**	0.161	0.120	0.213	0.263	0.033	178.68	0.239
GFLOPs ↓	0.0715	0.0764	**0.0004**	0.0012	0.2742	0.0007	389.70	0.0118

spatio-temporal samples using a sliding-window procedure. Each sample contains 9 time steps on a spatial grid of size $32 \times 32 \times 8$. The spatial stride is set to 4 along the two horizontal dimensions. This produces a total of 26136 samples, each with shape $9 \times 32 \times 32 \times 8$.

Dataset split. We split the processed samples into training, validation, and test sets using an 80/10/10 split: 20908 training samples, 2614 validation samples, 2614 test samples. All methods are trained and evaluated on the same split.

Prediction task. Given the initial three-dimensional temperature field and the spatio-temporal coordinates, the model predicts the future temperature evolution over the 9-step temporal window. Unlike idealized synthetic PDE data, the HYCOM data are not generated from a single clean closed-form PDE with perfectly specified boundary conditions. Instead, they are data-assimilative ocean fields influenced by numerical model dynamics, observations, and assimilation procedures. This makes the benchmark more representative of real deployment settings, where data may be noisy, partially observed, and only approximately consistent with simplified PDE assumptions.

F.5 Additional Experimental Results of Main Experiments

Computational Cost. Tab. 6 reports the prediction error, inference latency, and computational cost on the Burgers benchmark. Our method achieves a relative L_2 error of 0.26×10^{-2} with an inference time of 0.239 milliseconds and a computational cost of 0.0118 GFLOPs per sample. Compared with similarly fast neural operator baselines, our framework achieves lower prediction error.

Training Loss Trajectories. We present the training trajectories in Fig. 4. The plots depict the evolution of the total loss, the data-driven loss ($\mathcal{L}_{\text{data}}$), and the proposed temporal and spatial MeanFlow losses ($\mathcal{L}_{t\text{-MF}}, \mathcal{L}_{s\text{-MF}}$). The results indicate that the data loss decreases rapidly during the initial epochs, allowing the model to capture the primary mapping between input conditions and solution fields. Subsequently, the temporal and spatial MeanFlow losses stabilize at low magnitudes, providing crucial physical regularizations that ensure the predicted fields satisfy the underlying PDE constraints.

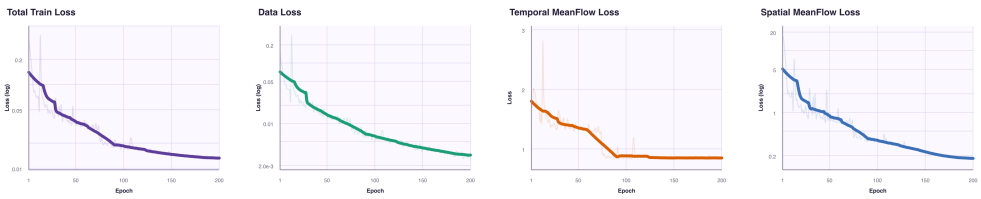


Figure 4: Training trajectories for all loss components, including the total loss, the data loss, the temporal MeanFlow loss, and the spatial MeanFlow loss.

Extension to Irregular Grids. We extend our framework to irregular spatial discretizations by replacing the regular-grid FNO backbone with a graph operator. On the Navier–Stokes benchmark, we independently perturb the coordinates of a 128×128 grid for each trajectory while preserving the periodic square domain. Our method achieves a test relative L_2 error of 0.0129, compared with 0.1089 for GNO and 0.3859 for NUFFT-FNO, demonstrating its effectiveness beyond regular grids.

Irregular Geometric Boundaries. As a separate geometric stress test, we tested the Navier–Stokes flow on a non-convex domain with a circular hole and non-periodic obstacle boundary conditions, and evaluated Darcy flow on an L-shaped domain using masked embedded grids. Losses and evaluation are restricted to valid points, while spatial MeanFlow pairs are retained only when their connecting segments lie entirely within the domain. Our method achieves relative L_2 errors of 0.0352 and 0.0177, respectively, compared with 0.2112 and 0.1360 for GNO. These results support the applicability of our framework to non-convex domains with irregular boundaries.

PDE Residual and Boundary-Condition Error. We evaluate physical consistency on the Navier–Stokes test set using the PDE residual and boundary-condition error, where lower values indicate better consistency. As shown in Table 7, combining temporal and spatial MeanFlow constraints achieves the lowest values for both metrics, outperforming either constraint alone and the variant without either constraint. These results support the benefit of jointly enforcing temporal and spatial consistency.

Table 7: PDE residual and boundary-condition error on the Navier–Stokes test set. Lower is better for both metrics.

T-MeanFlow	S-MeanFlow	PDE residual	Boundary error
Yes	Yes	$\mathbf{1.486 \times 10^{-1}}$	$\mathbf{2.006 \times 10^{-2}}$
Yes	No	1.518×10^{-1}	2.111×10^{-2}
No	Yes	1.584×10^{-1}	2.180×10^{-2}
No	No	1.612×10^{-1}	2.116×10^{-2}

A More Challenging Navier–Stokes Setting. We further evaluate our method on a more challenging Navier–Stokes setting with a lower viscosity of $\nu = 10^{-4}$ and a longer prediction horizon of $T = 10$. As shown in Table 8, our method achieves a relative L_2 error of 3.73×10^{-2}, reducing the error by approximately 16.7% compared with FNO under the same experimental setting. This result supports the effectiveness of our framework in this more challenging regime.

Table 8: Relative L_2 error on Navier–Stokes with $\nu = 10^{-4}$ and $T = 10$.

Method	Relative L_2 error
FNO	4.48×10^{-2}
Spatio-Temporal MeanFlow	$\mathbf{3.73 \times 10^{-2}}$