# LEANPRM: LEAN-GROUNDED SUPERVISION FOR NATURAL LANGUAGE PROOF VERIFICATION

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## ABSTRACT

AI systems now write long and difficult mathematical proofs, and checking their validity is becoming a bottleneck. A learned verifier could check proofs cheaply, but training one requires a correctness label on every step, and such labels are hard to obtain for open-ended proofs. Existing methods take them from human annotators, which is costly, or derive them from final answers, which proofs do not have, and formalizing each proof in Lean requires expert effort. We introduce LeanPRM, a framework that uses the Lean proof assistant to create step-level labels for natural-language proofs. A model writes a natural-language proof together with a paired formal proof, without feedback from Lean. Lean then labels every step, and a second model discards failures caused only by the formal syntax, so that the negative labels mark errors in the mathematics. LeanPRM contains 10,634 proofs with 152K labeled steps and no human annotation, and we release a held-out benchmark, LeanPRM-Bench. A Qwen3-4B verifier trained on LeanPRM raises exact step-label accuracy on LeanPRM-Bench from 48.2 to 72.5, as opposed to 59.5 when the same base model is trained on PRM800K. It also selects better proofs in best-of-$N$ selection on IMO-level problems, and leads on 3 of 4 external benchmarks. Training on LeanPRM improves verifiers at every size from 0.6B to 8B parameters on LeanPRM-Bench, Hard2Verify and IMO-GradingBench. Given the labels need no human annotation, LeanPRM can grow with the supply of formal statements, offering a path to step-level feedback on AI-generated proofs that are hard for people to check.

## 1 INTRODUCTION

As AI systems become more capable, they are increasingly able to produce long and difficult mathematical proofs, including solutions to long-standing open problems (OpenAI, 2026) and complete machine-checked proofs of major theorems such as Fermat's Last Theorem (Anthropic, 2026). At the same time, it is becoming harder to verify the correctness of these proofs. Existing verifiers, including outcome and process reward models, were built for problems with a short checkable final answer (Lightman et al., 2023; Wang et al., 2023; Rahman et al., 2025), whereas a proof has no final answer to check; its correctness lies in whether every step follows from the ones before it. Formal verification in a proof assistant such as Lean offers an alternative, but translating an informal proof into a formal language is itself difficult and requires expert-level effort. As a result, evaluating the proofs that AI systems produce is hard, especially for non-experts, and reliable supervision for training future systems is scarce.

A verifier trained on natural-language proofs could check them cheaply, but training it requires a correctness label on every step, and such labels are scarce for open-ended proofs. Human annotation is expensive even for short solutions, as PRM800K needed about 800,000 step labels (Lightman et al., 2023), and proofs are harder to judge. Automatic labels that score a step by whether its continuations reach the correct final answer (Wang et al., 2023) do not apply when there is no final answer. We introduce LeanPRM, a framework that uses the Lean proof assistant (de Moura & Ullrich, 2021), an exact mechanical checker, to create these labels. Each natural-language step is paired with a formal step, Lean checks the formal step, and the result becomes the label of the natural-language step. Lean is used once, to build the training data, and the trained verifier reads only natural language.



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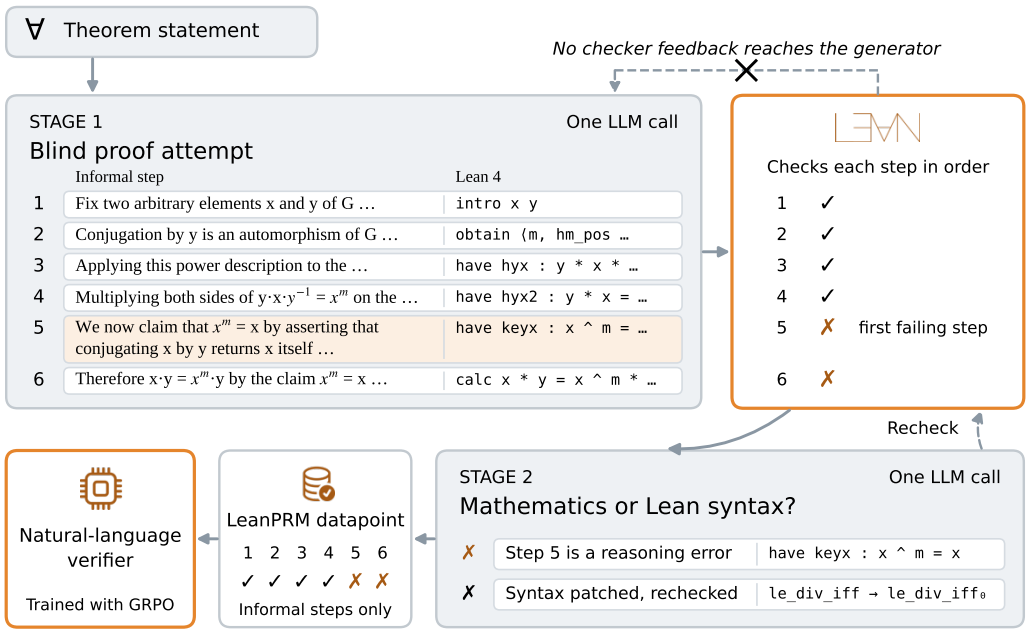


Figure 1: Overview of LeanPRM, with one example from the corpus running through it. Top row: Stage 1 writes a proof attempt from the theorem statement alone, each natural-language step paired with its formal step, and Lean checks the steps in order. Bottom row: Stage 2 decides whether the first failure lies in the mathematics or only in the formal syntax (syntax failures are patched and rechecked once), and the verifier is trained on the natural-language steps and their labels only. Here step 5 asserts without justification that conjugation by $y$ fixes $x$, so steps 1 to 4 are labeled correct and steps 5 and 6 incorrect.

We explain how we build the data in Figure 1 and give the full procedure in Section 2. For each theorem statement we ask a model to prove it in a single pass, writing each step in natural language together with its corresponding Lean step. The model never sees the output of running the Lean step it generated. If it could, it would repair its mistakes or give up, and we would lose the genuine errors we wish to label. As a result, the errors in LeanPRM are the model's own, and we don't have to prompt it in order to induce them. Following this we run Lean on the formal proof. Steps before the first failing Lean step are labeled correct, and the failing step and every later step are labeled incorrect. A step can also fail for purely formal reasons while the mathematics is sound, so we use a second model call to read the Lean error message, repair syntax errors once and classify the failure. We keep only proofs whose first failure is a mathematical error. In total we obtain 10,634 labeled proofs with 152,005 step labels and no human annotation, which we release together with LeanPRM-Bench, a held-out benchmark of 200 proofs.

In our experiments, we use LeanPRM to train verifier models for natural language. The results show improved performance relative to the same training process applied to the human labels of PRM800K. With the same model, training recipe and number of updates, LeanPRM-Verifier-4B beats the PRM800K-trained verifier on four of five benchmarks, by 13.0 points of exact step-label accuracy on LeanPRM-Bench and by 9.5 and 7.7 points on Hard2Verify and IMO-GradingBench. It also picks correct proofs more often than random selection in best-of-$N$ on IMO-level problems, where the PRM800K verifier stays close to chance. Training on LeanPRM improves verifiers at every size we tried, from 0.6B to 8B parameters, on LeanPRM-Bench, Hard2Verify and IMO-GradingBench. We provide detailed results in Section 3.

We make three contributions.

- LeanPRM, to our knowledge the first framework that uses Lean to produce step-level correctness labels for natural-language proofs at scale, and its corpus of 10,634 labeled proofs (Section 2).

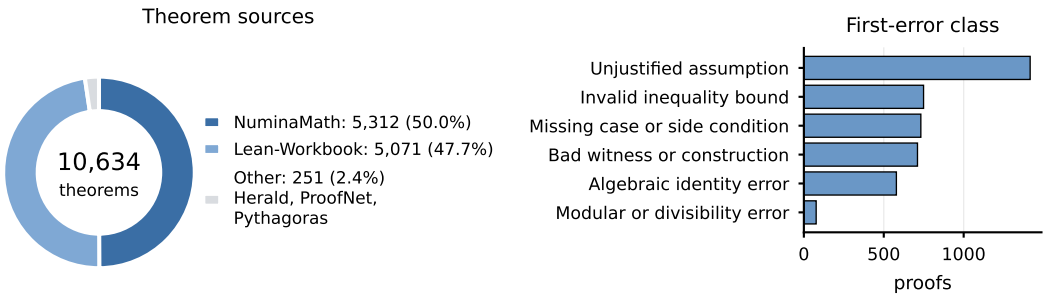


Figure 2: Composition of LeanPRM. Left: sources of the 10,634 theorems. Right: error class of the first failing step in each of the 4,264 incorrect proofs. Every incorrect proof fails first at a mathematical error, most often an unjustified assumption.

- LeanPRM-Bench, a held-out benchmark of 200 natural-language proofs with step labels, which we release for evaluating proof verifiers (Section 2.2).
- A controlled evaluation showing that, with the same recipe and number of updates, verifiers trained on Lean-grounded labels outperform verifiers trained on human labels on four of five benchmarks, select better proofs at test time on IMO-level problems, and improve at every model size from 0.6B to 8B on three of the five benchmarks (Section 3).

## 2 LeanPRM: Lean-Grounded Process Supervision

We introduce LeanPRM, a framework for generating natural-language proofs with correctness labels on every step, grounded in Lean. The framework has two stages (Figure 1): (i) a model writes a natural-language proof together with a formal proof in Lean, pairing each natural-language step with the corresponding Lean tactic, and (ii) Lean compiles the formal proof and a second model reviews the result. If compilation failed, the second model decides whether the failure is a mathematical error or only a syntax error, which it attempts to repair once. If compilation succeeded, it checks that the natural-language steps state the same mathematics as the formal proof. Each natural-language step is labeled by the outcome of its tactic, with every step after the first failure also labeled incorrect.

**Stage 1: Proof generation.** We start from about 17,800 distinct theorem statements from public formal corpora, mostly from NuminaMath-LEAN (Wang et al., 2025) and Lean-Workbook (Ying et al., 2024; Lin et al., 2025a) (Figure 2, with all sources listed in Appendix B). For each statement, one model call writes a proof attempt as numbered steps, each a sentence of ordinary mathematical prose paired with the formal step that carries it out (prompt in Appendix I). The model sees only the statement and generates a proof in one shot, without receiving feedback from the checker. The model is always asked to prove the theorem and is never instructed to make a mistake, so every error in LeanPRM is one the model made while genuinely trying to produce a proof. Because many statements were selected for difficulty (Appendix B), many of these genuine attempts fail. We used Claude Sonnet 5 for most theorems and Claude Opus 4.8 for an earlier subset.

**Stage 2: Verification and error classification.** The frozen formal proof is compiled with Lean 4 and Mathlib (both v4.25.0-rc2), and no compiler output is shown to the generator. Because each sentence was written with its own formal step, a compilation error points to exactly one step. A formal step can fail to compile even when the mathematics is right, so a second and final model call reads the statement, the steps and the compiler output and returns a verdict on the step. The verdict may be correct (the proof compiled and its natural language description is faithful to it), mathematical error (the call names the first failing step and assigns it an error class), or syntax error (the call gets a single attempt to patch the corresponding Lean step without changing the mathematics, and the patched proof is kept only if it then compiles). At this stage, the proof may also be discarded due to an ill-posed statement, malformed output, or prose and formal proof that disagree. A proof that compiles is labeled correct at every step. Otherwise, the first failing step and every step after it are labeled incorrect, since nothing that builds on an unestablished step is established either. On steps that are true but inadequately justified, this approach can differ from

Table 1: Blind per-class audit of flagged steps, 30 per class. A high proportion judged sound indicates that the flagged step is mathematically sound and that the elaboration failure originated in the formal encoding as opposed to in the argument.

| First-error class | Judged sound | Share of incorrect proofs |
|---|---|---|
| Formalization-side | | |
| Wrong rewrite direction | 28/30 | 2.7% |
| Theorem misapplication | 26/30 | 9.2% |
| Other | 26/30 | 9.5% |
| Automation misuse | 23/30 | 13.4% |
| Mathematical | | |
| Unjustified assumption | 13/30 | 21.3% |
| Invalid inequality bound | 13/30 | 11.3% |

ProcessBench (Zheng et al., 2025), which counts a missing justification as an error only when it affects the validity of the solution (Appendix G).

### 2.1 LEANPRM CORPUS

Stage 2 produces a labeled proof for 13,940 of these statements, and 13,007 remain after the lexical filters and the removal of duplicate and empty proofs. After the quality control described below, LeanPRM contains 10,634 natural-language proofs, 6,370 correct and 4,264 incorrect, with 152,005 labeled steps, obtained with no human annotation. Of these, 98% come from NuminaMath-LEAN and Lean-Workbook. Table 4 in Appendix B breaks the corpus down by source and split.

**Error taxonomy.** Stage 2 assigns each reasoning error one of ten classes. Six describe mathematical faults: unjustified assumptions, invalid inequality bounds, wrong witnesses or constructions, missing cases or side conditions, faulty algebraic identities, and errors in modular or divisibility reasoning. The other four describe failures of the formalization rather than of the mathematics. Examples of these errors include a lemma applied where its formal hypotheses do not hold, a rewrite in the wrong direction, automation asked to close a goal it cannot, and a residual class for anything else. Appendix A gives the definitions the classifier works from.

**Quality control.** We checked the labels with two blind audits, using Claude Opus 5 as the judge. The judge never saw our labels (prompts in Appendix I). In the first audit, the judge read 100 stratified proofs from the unfiltered pool and named the first incorrect step, if any. It returned a usable answer for 99 of them. It disputed 1 of the 52 proofs we label correct. Of the 47 we label incorrect, it found an error in 27 and none in 20. The second audit shows where that disagreement comes from (Table 1). Among 30 flagged steps per class, steps in the formalization-side classes, 35.8% of the incorrect attempts in the pool, were judged sound in 77% to 93% of cases, while steps in the mathematical classes were judged sound less than half the time. We therefore remove every attempt whose first failure is formalization-side, so every negative in LeanPRM carries a mathematical error class. We additionally use lexical filters to make sure the natural language proofs do not reference Lean and remove incorrect proofs whose failing step states its own error (Appendix B).

### 2.2 LEANPRM-BENCH

LeanPRM-Bench is a held-out set of 200 theorems, 100 with a correct proof and 100 with an incorrect one, split off by theorem group so that no theorem in it, or renamed copy of one, appears in training. Correct labels are certified by compilation, and every incorrect proof first fails at a mathematical error (Appendix B). The benchmark is hard for current models: prompted with the same instructions, four commercial models reach 71.5 to 80.3 exact step-label accuracy, and all four mislabel the same 26 proofs. We report exact step-label accuracy, the fraction of proofs labeled correctly at every step, and verdict accuracy, the fraction with the right overall verdict. We release LeanPRM-Bench with its step labels, error classes and formal proofs.

# 3 EXPERIMENTS

**Training.** The verifier reads a theorem and its numbered steps, writes its reasoning, and then gives each step a label, correct or incorrect, and the proof an overall verdict (prompt in Appendix I). We train Qwen3 models (Yang et al., 2025) with 0.6B, 4B and 8B parameters, in thinking mode, with group relative policy optimization (GRPO, Shao et al., 2024). For each proof we sample eight responses at temperature 1, normalize their rewards within the group, and use the normalized reward as the advantage of every token in the response. The importance ratio is clipped to $[0.8, 1.26]$, and a KL penalty with coefficient $10^{-3}$ keeps the policy close to the initial model. Each update uses 64 proofs and a learning rate of $10^{-6}$. Every trained model, including the baselines, gets 300 updates with this configuration, so only the training data differs. Those 300 updates cover 19,200 proofs: 1.87 passes over the 10,264 LeanPRM training proofs and 0.27 over the 70,502 PRM800K training proofs. Appendix J lists every hyperparameter.

**Reward.** Let $R_{\mathrm{c}}$ be the fraction of truly correct steps that the verifier labels correct and $R_{\mathrm{i}}$ the fraction of truly incorrect steps that it labels incorrect, each set to 1 when there are no such steps. With predicted verdict $\hat{v}$ and true verdict $v$, the reward is

$$r = 0.7 \cdot R_{\mathrm{step}} + 0.3 \cdot \mathbf{1}[\hat{v} = v],$$
$$R_{\mathrm{step}} = \begin{cases} R_{\mathrm{c}}\, R_{\mathrm{i}} & \text{if the proof contains an error,} \\ \mathbf{1}[\text{every step labeled correct}] & \text{otherwise.} \end{cases} \tag{1}$$

On an incorrect proof the product rules out both shortcuts: labeling every step correct gives $R_{\mathrm{i}} = 0$, and labeling every step incorrect gives $R_{\mathrm{c}} = 0$. For the proof in Figure 1, whose steps 5 and 6 are incorrect, a verifier that flags only step 6 gets $R_{\mathrm{c}} = 1$ and $R_{\mathrm{i}} = 1/2$, so $R_{\mathrm{step}} = 0.5$ and, with the verdict necessarily incorrect, $r = 0.65$. The reward is 0 when the output is malformed, has no reasoning section, or gives a verdict that contradicts its own step labels.

**Evaluation.** We evaluate on LeanPRM-Bench and on four external benchmarks, each on its complete set and with one metric: Hard2Verify (Pandit et al., 2025) (response-level balanced accuracy), ProcessBench (Zheng et al., 2025) (F1, the harmonic mean of accuracy on errors and correct solutions, averaged over its four subsets, 3,400 test cases), PRMBench (Song et al., 2025) (first-error accuracy over 6,216 rows, not its default PRMScore) and IMO-GradingBench (Luong et al., 2025) (balanced accuracy on the grade binarized to fully correct or not, instead of its four-way accuracy, since our verifiers give a binary verdict and only about 30% of its proofs are graded correct). We use first-error accuracy on PRMBench because PRMScore scores each step against labels that mark only the erroneous steps as incorrect, so it would penalize every model for following our instruction to label all steps after the first error incorrect. All models get the same prompt layout and the same strict parser, and a response that fails to parse counts as wrong. We decode at temperature 1 with three seeds and report the mean scores.

**Baselines.** We compare with the untrained Qwen3-4B and with two models trained by the same recipe on other data: one on PRM800K (Lightman et al., 2023), converted to our step-label format (PRM800K-Verifier-4B), and one with reinforcement learning on solving 19,200 NuminaMath problems with a final-answer reward (NuminaMath-Solver-4B), which tests whether verification ability comes from mathematical reinforcement learning alone. For reference we also prompt four commercial models with the same instructions: Claude Sonnet 5, Opus 5 and Fable 5.1 through the Anthropic API with extended thinking, and GPT-5.6 Sol through the Codex command-line interface with every tool disabled (Appendix E). Both trained baselines change little on their own validation data over the second half of their 300 updates (Figure 4, middle), so the 300 step budget is a reasonable setting.

## 3.1 RESULTS

**Process verification benchmarks.** Table 2 and Figure 3 give the main comparison. At the same budget, LeanPRM-Verifier-4B beats PRM800K-Verifier-4B on four of the five benchmarks: by 13.0 points on LeanPRM-Bench, 9.5 on Hard2Verify, 2.8 on PRMBench and 7.7 on IMO-GradingBench. PRM800K-Verifier-4B keeps a 3.5-point lead on ProcessBench, whose short solutions come from the same distribution as its training data and whose labels, unlike ours, count a missing justification as an error only when it affects the validity of the solution (Appendix G). Reinforcement learning

Table 2: Verification accuracy at temperature 1. LeanPRM-Bench is scored by exact step-label accuracy (Section 2.2), and the other columns use the metrics of Section 3. The top block compares 4B models trained for 300 updates with the same recipe, with the untrained model for reference, and bold marks the best value in that block. The bottom block holds reference rows: †a separate 4B run of 1,582 updates, only the last 300 of them on LeanPRM (Appendix D), and *commercial models prompted rather than trained for the task. At equal training, LeanPRM-Verifier-4B is best on four of the five benchmarks. In the instances we ran three decoding seeds, we report the mean and one standard deviation. The other rows are single runs.

| | LeanPRM-Bench exact | Hard2Verify response BA | ProcessBench avg. F1 | PRMBench first err. | IMO-GB BA |
|---|---|---|---|---|---|
| Qwen3-4B base | 48.2 ±1.3 | 37.1 ±4.5 | 74.3 ±0.5 | 42.5 ±0.5 | 53.8 ±0.1 |
| PRM800K-Verifier-4B | 59.5 ±1.1 | 61.3 ±0.7 | **82.2** ±0.3 | 44.8 ±0.1 | 56.6 ±0.4 |
| NuminaMath-Solver-4B | 48.5 | 34.0 | 68.5 | 41.5 | 52.2 |
| LeanPRM-Verifier-4B (ours) | **72.5** ±2.5 | **70.8** ±3.2 | 78.7 ±0.6 | **47.6** ±0.1 | **64.3** ±0.5 |
| LeanPRM-Verifier-4B† | 78.7 ±1.3 | 79.0 ±0.7 | 72.6 ±0.3 | 48.3 ±0.2 | 71.8 ±1.0 |
| Sonnet* | 71.5 | 83.4 | 84.8 | 48.7 | 80.2 |
| Opus* | 80.3 | 81.3 | 84.5 | 47.3 | 82.2 |
| Fable* | 77.5 | 85.0 | 84.7 | 48.7 | 84.3 |
| GPT-5.6 Sol* | 77.0 | 87.4 | 80.0 | 47.3 | 83.3 |

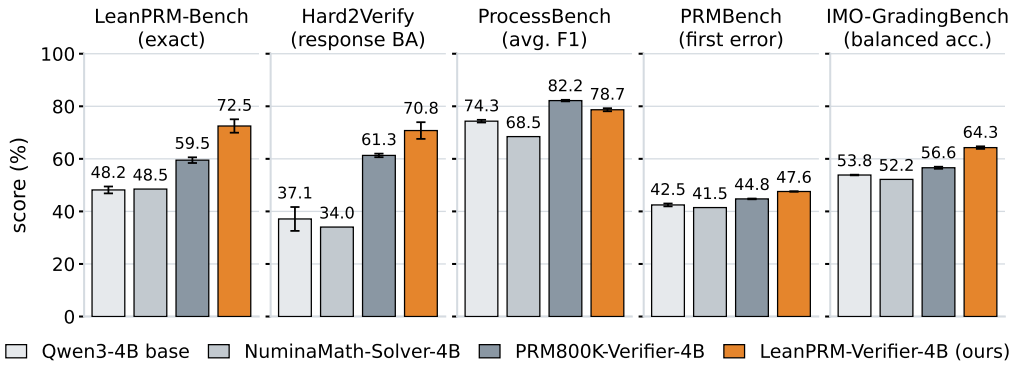


Figure 3: The top block of Table 2 per benchmark: 4B models trained for 300 updates with the same recipe, and the untrained model. Bars are means over three decoding seeds with one standard deviation as error bars; NuminaMath-Solver-4B is a single run. LeanPRM-Verifier-4B is best on four of the five benchmarks and PRM800K-Verifier-4B on ProcessBench.

on solving does not produce a verifier: NuminaMath-Solver-4B raises final-answer accuracy on its own validation problems from 30.0% to 46.0%, yet stays within a point of the untrained model on LeanPRM-Bench and falls below it on every external benchmark. We additionally report results for a longer LeanPRM training run denoted as LeanPRM-Verifier-4B†: 532 updates on proofs from an earlier joint-generation procedure, 750 on the unfiltered pool and 300 on LeanPRM (Appendix D). Its data differ as well as its length, so we report it only for reference. We report results for the commercial models as a reference too. They lead on Hard2Verify, ProcessBench and IMO-GradingBench, and Appendix C discusses the 26 LeanPRM-Bench proofs that all four mislabel. Standard deviations across seeds are at most 3.2 points for every model trained with three seeds.

**Test-time scaling.** A verifier is useful at test time if it can pick a good proof out of several attempts. We measure this on IMO-GradingBench, which has about 33 human-graded proofs for each of its 30 problems. For each problem we draw $N$ proofs, let the verifier pick one it accepts (at random among those it accepts, or among all of them if it accepts none), and record whether that proof is graded correct. Figure 4 (right) shows the result. At $N = 16$, LeanPRM-Verifier-4B picks a correct proof 35.4% of the time, against 32.4% for random selection, and its advantage grows with $N$.

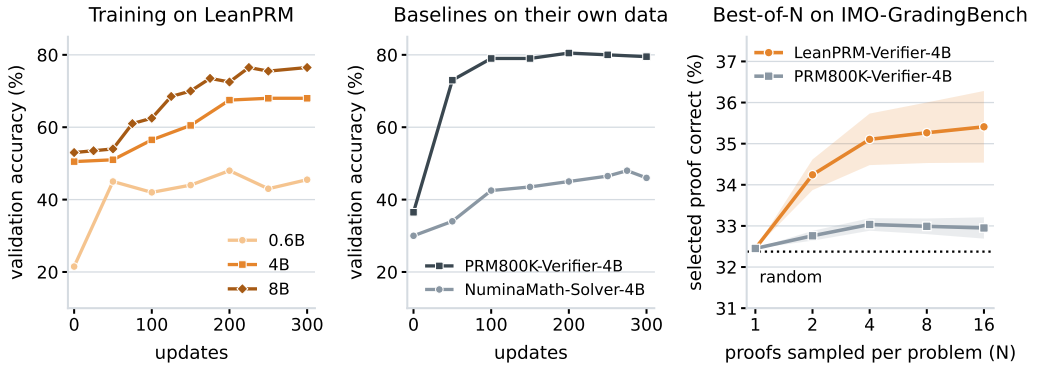


Figure 4: Training curves and test-time selection. Left: exact step-label accuracy on the LeanPRM validation split during training, for the verifiers trained from Qwen3 base models of three sizes; the 4B point at 300 updates was measured after training with the same greedy protocol. Middle: the two trained baselines on their own validation splits, PRM800K-Verifier-4B by exact accuracy and NuminaMath-Solver-4B by final-answer accuracy; both change little over the second half of training. Right: best-of-$N$ selection on IMO-GradingBench, the share of selected proofs graded correct, averaged over the 30 problems and three decoding seeds (shaded band: one standard deviation); the dotted line is random selection. LeanPRM-Verifier-4B picks better proofs as $N$ grows, while PRM800K-Verifier-4B stays near chance.

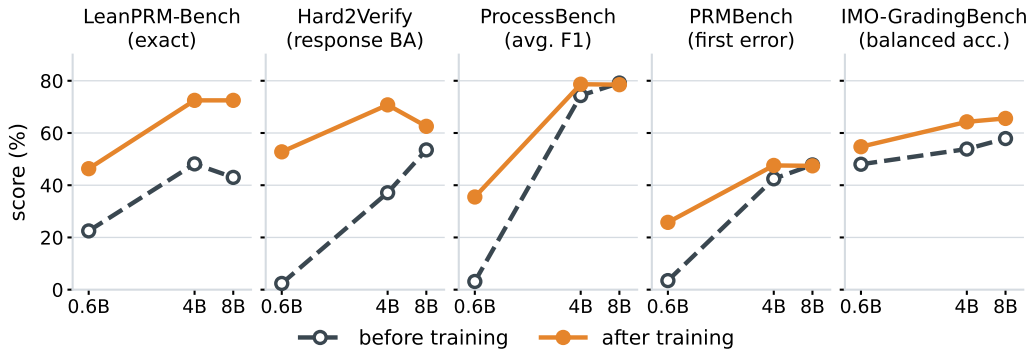


Figure 5: Accuracy before and after training on LeanPRM, by model size. Each panel is one benchmark; open circles are the untrained Qwen3 models and filled circles the trained verifiers after 300 updates. Training improves every model on LeanPRM-Bench, Hard2Verify and IMO-GradingBench; the untrained 0.6B model scores near zero on three benchmarks because it rarely produces well-formed labels.

PRM800K-Verifier-4B, trained with the same recipe and budget, reaches 33.0%: it accepts about 82% of all proofs although only 30% are graded correct, so its picks differ little from chance.

**Model scale.** We compare verifiers trained from Qwen3 base models of 0.6B, 4B and 8B parameters in Figure 5, with the numbers in Appendix F. Training helps at every size on LeanPRM-Bench, Hard2Verify and IMO-GradingBench. After 300 updates the gain on LeanPRM-Bench over the untrained model is 24 points at 0.6B, 24 at 4B and 30 at 8B. The untrained 8B model starts 5.2 points below the untrained 4B model on LeanPRM-Bench (43.0 against 48.2), so its larger gain only brings it level with the 4B model. The 0.6B model nevertheless stays far behind. Before training it produces well-formed labels for fewer than one proof in five on Hard2Verify, ProcessBench and PRMBench, and after training it is 26 points below the 4B model on LeanPRM-Bench. The 4B and 8B models reach the same LeanPRM-Bench accuracy, 72.5, and are within 1.3 points of each other on Process-

Table 3: Sources of step-level labels in prior work and in LeanPRM. For each method we give the origin of its step labels and whether they avoid human annotation, cover open-ended natural-language proofs, give each failing step a mathematical error type, and train a verifier that reads the proof text. Only LeanPRM does all four.

| Method | Step labels from | No human step labels | Open-ended proofs | Error classes | Trained verifier |
|---|---|---|---|---|---|
| PRM800K (Lightman et al., 2023) | Human annotators | ✗ | ✗ | ✗ | ✓ |
| Math-Shepherd (Wang et al., 2023) | Rollouts | ✓ | ✗ | ✗ | ✓ |
| OmegaPRM (Luo et al., 2024) | Rollouts (tree search) | ✓ | ✗ | ✗ | ✓ |
| FoVer (Kamoi et al., 2025) | Z3, Isabelle | ✓ | ✗ | ✗ | ✓ |
| Safe (Liu et al., 2025) | Lean 4 | ✓ | ✗ | ✗ | ✗ |
| TP-as-a-Judge (Leang et al., 2025) | Lean 4 (per solution) | ✓ | ✗ | ✗ | ✗ |
| LeanTutor (Patel et al., 2026) | Lean 4 | ✓ | ✓ | ✓ | ✗ |
| **LeanPRM (ours)** | Lean 4 | ✓ | ✓ | ✓ | ✓ |

Bench, PRMBench and IMO-GradingBench; the 4B model is ahead on Hard2Verify. Most of that gap comes from formatting. The 8B model returns an unparsable answer for 10% of Hard2Verify proofs, against 6% for the 4B model, and on the answers that parse the two score 71.9 and 74.6. On ProcessBench and PRMBench, the 8B model starts highest (79.1 and 47.8) and ends within a point of its untrained score. Figure 4 (left) shows validation accuracy during training for the three sizes.

## 4 Related Work

LeanPRM draws on two lines of work that have mostly developed separately. Process supervision trains verifiers on step-level labels, which mostly come from human annotators or from the outcomes of sampled continuations. Formal mathematics offers exact checking, which has mainly been used to train and evaluate provers that write Lean. We connect the two by using Lean only to label natural-language proofs, and the verifier we train never sees a formal proof.

**Process supervision.** Lightman et al. (2023) scaled up process supervision for mathematical reasoning with human step-level annotations. Subsequent work reduces this annotation cost by deriving process supervision from outcome signals: Math-Shepherd estimates step quality from the correctness of sampled continuations (Wang et al., 2023), OmegaPRM scales such annotation using tree search (Luo et al., 2024), and RL Tango trains a generative process verifier directly from outcome-level rewards (Zha et al., 2025). Outcome-trained verifiers were introduced for word problems by Cobbe et al. (2021), and Uesato et al. (2022) ran the first comprehensive comparison of process- and outcome-based feedback on a natural-language task; later work scales automated step labels through rollouts and value estimates (Lu et al., 2024; Setlur et al., 2024), casts verification as generation (Zhang et al., 2024), and distills practical lessons for training process reward models (Zhang et al., 2025a), and such verifiers are what makes test-time search effective (Snell et al., 2024; Ma et al., 2023), since models struggle to correct their own reasoning without external feedback (Huang et al., 2023).

These approaches obtain labels from the model-generated solution distribution, either through human judgment or through rollout statistics. We instead obtain them from an external decision procedure, which makes the correctness criterion explicit and independent of the model being supervised. In this direction, FoVer (Kamoi et al., 2025) trains process reward models on step labels from Z3 and Isabelle for formal-logic and formal-proof tasks and finds that they transfer to natural-language reasoning, Safe (Liu et al., 2025) checks each step of a natural-language solution in Lean 4 to score solutions at inference, and Theorem Prover as a Judge (Leang et al., 2025) uses Lean verdicts on whole solutions to select fine-tuning data. All three work on problems with a final answer (Table 3). We target natural-language competition proofs, keep the text unshaped by the checker, and separate mathematical from formalization failures before a rejection becomes a label.

**Formal mathematics.** Lean 4 (de Moura & Ullrich, 2021) and Mathlib (The mathlib Community, 2020) have been widely used to construct data and train formal theorem provers (Yang et al., 2023; Lin et al., 2025a) and autoformalizers (Gao et al., 2024). Provers such as DeepSeek-Prover (Xin et al., 2024; Ren et al., 2025), Kimina-Prover (Wang et al., 2025), Goedel-Prover-V2 (Lin et al.,

2025b) and Seed-Prover (Chen et al., 2025) learn from proof-assistant feedback, Lean-STaR interleaves informal thoughts with tactics (Lin et al., 2024), and miniF2F and PutnamBench track their progress (Zheng et al., 2021; Tsoukalas et al., 2024). Autoformalization (Wu et al., 2022) and formalized problem sets such as Lean Workbook (Ying et al., 2024) supply the statements, informal proofs can be mapped to formal proof sketches (Jiang et al., 2022), and Yang et al. (2024) survey the field.

Recent work also uses formal verification to evaluate natural-language reasoning (Zhou & Zhang, 2025), to check key steps during inference (Ospanov et al., 2025), or to provide training signals for natural-language reasoning (Cao et al., 2026) and formal proof generation (Kim & Yun, 2026). DeepTheorem uses automatically verifiable theorem outcomes (Zhang et al., 2025b), while DeepSeekMath-V2 trains natural-language provers with a learned proof verifier (Shao et al., 2025). LeanTutor (Patel et al., 2026) checks student proofs step by step in Lean to give feedback. Our work instead uses such step labels, at scale, as training data for a verifier that reads only natural language, providing supervision that is more scalable than expert annotation and more mechanically grounded than model judgments. Our use of the checker differs in that the object being supervised is natural-language reasoning: the formal proof is never the model's output and never appears in its input.

**Verifier training and evaluation.** Our verifiers are trained with group-relative policy optimization (Shao et al., 2024), a variant of PPO (Schulman et al., 2017) that DeepSeek-R1 scaled to reasoning (DeepSeek-AI et al., 2025), implemented in VERL (Sheng et al., 2024). The prover-verifier framing follows Anil et al. (2021), as applied to language models by Kirchner et al. (2024), and critic models that surface errors in model outputs (McAleese et al., 2024) are the closest analog outside mathematics.

Evaluation of step-level verifiers has been addressed by ProcessBench (Zheng et al., 2025), which measures identification of the first erroneous step in short solutions to grade-school, competition and Olympiad-level problems, including GSM8K and MATH (Cobbe et al., 2021; Hendrycks et al., 2021), PRMBench (Song et al., 2025), which probes specific failure modes of process reward models, and Hard2Verify (Pandit et al., 2025), which targets proofs whose correctness is difficult to establish. Appendix G discusses how a benchmark's annotation convention can interact with the convention used in training.

## 5 Conclusion

We introduced LeanPRM, a framework that uses Lean to label every step of natural-language proofs, trained verifiers on its corpus of 10,634 proofs, and released a held-out benchmark, LeanPRM-Bench. With the same recipe and number of updates, these verifiers outperform verifiers trained on the human labels of PRM800K on four of five benchmarks, and they pick better proofs at test time. A formal tool can therefore be used to train verifiers for informal reasoning, without autoformalization. Lean is needed only once, offline, to build the training data, and the verifier it produces checks natural language directly.

**Limitations.** LeanPRM separates failures of the formal encoding from errors in the mathematics, and the audit in Section 2.1 shows that the classes it removes consist mostly of sound steps. The separation itself relies on a model's classification, checked with a single judge model on a sample, so some borderline steps may still carry the wrong label. We compared Lean labels with the human labels of PRM800K but did not train on the same proofs labeled by an LLM judge, so we cannot yet say how much of the gain comes from the labels and how much from training on proof-style data. LeanPRM-Bench comes from the same pipeline and the same Claude generators as the training data, so the four external benchmarks carry most of the evidence that the verifiers generalize.

**Future work.** We believe the same methodology for generating verifier training data could carry over to other open-ended tasks that have no final answer but contain parts an exact checker can verify, such as test suites for code or simulators for science. Within mathematics, generating new theorem statements would let the corpus grow beyond existing formal collections in size and difficulty. Verifiers trained on LeanPRM could also serve as reward models, for training provers and for giving step-level feedback on proofs too hard for people to check, which becomes more important as models approach and exceed human expertise.

## AI USE STATEMENT

Beyond their role inside the data pipeline of Section 2, language models were used to edit the prose of this paper and to assist with the code behind the experiments and figures. The experimental design, the analysis and every claim are the authors' own.

## REPRODUCIBILITY STATEMENT

We release the LeanPRM corpus with its training, validation and test splits and the theorem-group assignment behind them; the generation and labeling pipeline, including the Stage 1 and Stage 2 prompts, the error taxonomy, and the Lean 4 elaboration harness; the training configuration and reward implementation of Section 3; the evaluation scripts for every benchmark, with the shared prompt layout and parser; the audit prompts and per-item judgments of Section 2.1; and the checkpoints of the trained models in Tables 2 and 5. Appendix D gives the training schedule of every model, and Appendix E the evaluation details.

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# APPENDIX

Appendix A defines the ten error classes of Section 2.1, and Appendix B gives the data sources, the lexical filters and the splits behind Section 2.1. Appendix C examines the LeanPRM-Bench proofs that the commercial models mislabel (Section 3.1). Appendices D and E give the training and evaluation details of Section 3, and Appendix G discusses the ProcessBench result. Appendix F gives the model-scale numbers behind Figure 5. Appendix H shows further corpus examples, Appendix I lists every prompt, and Appendix J gives the hyperparameters.

## A ERROR CLASSES

Stage 2 assigns each reasoning error one of ten classes, defined in its instructions as follows. The mathematical classes are: algebraic identity error, a false or invalid algebraic manipulation; invalid inequality bound, a bound with the wrong direction or without justification; bad witness or construction, a wrong witness for an existential claim or an invalid construction; modular or divisibility error, incorrect modular arithmetic or divisibility reasoning; missing case or side condition, a required case or side condition that is omitted; and unjustified assumption, a claim asserted without support from the preceding steps. The formalization-side classes are: theorem misapplication, a lemma applied where its hypotheses do not hold as stated; wrong rewrite direction, an invalid or reversed rewrite or substitution; automation misuse, a claim that automation discharges a goal it cannot; and other, a failure not captured by the classes above. Attempts whose failure was recorded without a class are reported as unclassified.

## B DATA SOURCES, FILTERING AND SPLITS

Theorem statements come from a difficulty-filtered subset of NuminaMath-LEAN (Wang et al., 2025), Lean-Workbook (Ying et al., 2024; Lin et al., 2025a), Pythagoras (Leang et al., 2026), Herald (Gao et al., 2024), ProofNet (Azerbayev et al., 2023) and FormalQualBench (Math Inc., 2026). Table 4 gives the composition of LeanPRM by source and by split.

Table 4: Size of LeanPRM by source (top) and by split (bottom); each theorem contributes one proof. NuminaMath supplies most of the incorrect proofs and Lean-Workbook most of the correct ones.

| | Correct | Incorrect | Total |
|---|---|---|---|
| NuminaMath | 1,334 | 3,978 | 5,312 |
| Lean-Workbook | 4,787 | 284 | 5,071 |
| Pythagoras | 230 | 1 | 231 |
| Herald | 13 | 1 | 14 |
| ProofNet | 6 | 0 | 6 |
| Training | 6,170 | 4,094 | 10,264 |
| Validation | 100 | 70 | 170 |
| LeanPRM-Bench | 100 | 100 | 200 |
| LeanPRM | 6,370 | 4,264 | 10,634 |

Theorem identity is defined modulo the declaration name, and every split is drawn by theorem group so that alpha-renamed restatements cannot cross a boundary. The training split holds 10,264 theorems, 6,170 with a correct proof and 4,094 with an incorrect one; within each source the smaller verdict cell is oversampled to parity, giving a training view of 17,670 rows. The validation split holds 170 theorems (100 correct, 70 incorrect) and LeanPRM-Bench 200, 100 of each verdict. LeanPRM-Bench was fixed before the class-based filter of Section 2.1, and 30 of its incorrect proofs turned out to have a formalization-side first error. We replaced them with 30 incorrect proofs from the validation split whose first error is mathematical, which is why the validation split holds 170 theorems, and re-evaluated every system on the replaced items. The validation curves in Figure 4 were recorded during training on the original 200-theorem validation split, which still contained those

30 proofs. No model was trained on them, and none of the models compared at 300 updates was selected on validation. LeanPRM-Verifier-4B† is the exception. Its second stage was stopped at the best checkpoint on an earlier validation split that held 17 of these 30 proofs.

**Lexical filters.** Two lexical checks guard the prose. During generation, an attempt is regenerated if any natural-language step uses one of the following terms as a whole word, matched case-insensitively: lean, tactic, compiler, syntax, solver, automation, automated, verifier, engine, linarith, nlinarith, field_simp, norm_num, aesop, omega. No proof in LeanPRM contains any of them. When the splits are built, an incorrect proof is removed if its first failing step announces its own error with phrases such as unjustified, invalid, false or does not follow, since such text would give the label away. This second check removed 55.2% of the incorrect proofs from the earlier joint procedure and 2.3% of those from the blind procedure. The Lean lines written in Stage 1 are not filtered for sorry, admit or native_decide, which appear in 605 proofs of LeanPRM. Lean does not accept sorry or admit as a proof, so almost all proofs that use them are labeled incorrect, and only the repairs made in Stage 2 are barred from using any of the three.

## C PROOFS THE COMMERCIAL SYSTEMS MISJUDGE

Of the 200 proofs in LeanPRM-Bench, 26 are mislabeled by all four commercial systems. LeanPRM-Verifier-4B† labels 8 of them exactly, six incorrect proofs and two correct ones, and over three decoding seeds it labels 29.5% of the 26 exactly, against 7.7% for the untrained model and 11.5% for PRM800K-Verifier-4B. The reverse case, all four commercial systems right and the trained model wrong, covers 12 of the 128 proofs on which all four are right. Figures 6 and 7 show two of the eight and Figure 8 one of the twelve; the remaining cases follow the checker's conventions, such as treating an argument that stops before its conclusion as incorrect at its last step.

## D TRAINING DETAILS

Given the theorem and its numbered natural-language steps, the verifier writes a reasoning section, one label per step, and a verdict, and the verdict must be incorrect whenever any step is labeled incorrect. Let $\mathrm{Rec}_C$ and $\mathrm{Rec}_I$ denote label recall on the steps that are truly correct and truly incorrect. The step reward is $\mathrm{Rec}_C\,\mathrm{Rec}_I$ when the proof contains an error and otherwise the indicator that every step is predicted correct; the task reward is $0.7\,R_{\mathrm{step}} + 0.3\,R_{\mathrm{verdict}}$ with $R_{\mathrm{verdict}}$ the verdict accuracy, and malformed output, a missing reasoning section, or an inconsistent verdict scores zero. On an incorrect proof the product penalizes both the all-correct and the all-incorrect prediction.

All models are optimized with group-relative policy optimization, eight responses per prompt sampled at temperature 1, batch size 64, learning rate $10^{-6}$, KL coefficient $10^{-3}$, and a 6,656-token response limit, on four H100 80GB GPUs, with validation and checkpoints every 50 updates. LeanPRM-Verifier-4B† is a single run continued across three stages of data: 532 updates on an earlier pool of 3,449 training proofs produced by a joint generation procedure, 750 updates on the unfiltered pool of Section 2.1 (its best validation checkpoint of 808), and 300 updates on LeanPRM, 1,582 updates in total. Every model of Table 5, and PRM800K-Verifier-4B and NuminaMath-Solver-4B, trains from its base checkpoint for 300 updates. The 8B model uses the same configuration with parameter and optimizer offloading and rollouts tensor-parallel across two GPUs.

## E EVALUATION DETAILS

The held-out split is scored by exact step-label accuracy, the fraction of proofs whose every label is predicted correctly, and by verdict accuracy. Hard2Verify is scored by response-level balanced accuracy over its 200 problems, ProcessBench by the mean F1 over its four subsets on all 3,400 test cases, and PRMBench by first-error accuracy on all 6,216 rows. On IMO-GradingBench the four-way human grade is binarized as fully correct against everything else, candidates are split into steps at paragraph boundaries with an even merge above thirty steps, and 13 of the 1,000 candidates exceed the context budget and are dropped, leaving 987. Every model receives the same prompt layout, decodes at temperature 1 with a fixed seed, and is read by the same strict parser; a response that fails to parse is scored as incorrect. The Claude models are run through the Anthropic API with

**Theorem** We must show that among any 1985 distinct natural numbers all of whose prime factors are less than 26, there exist four distinct ones whose product is a perfect fourth power.

```
theorem number_theory_317136 (M : Finset ℕ) (hM : M.card = 1985)
    (hM1 : ∀ m ∈ M, ∀ p ∈ Nat.primeFactors m, p < 26) :
    ∃ w x y z : ℕ, w ∈ M ∧ x ∈ M ∧ y ∈ M ∧ z ∈ M ∧ w ≠ x ∧ w ≠ y ∧ w ≠ z ∧
      x ≠ y ∧ x ≠ z ∧ y ≠ z ∧ ∃ n : ℕ, w * x * y * z = n ^ 4 := by
```

| Natural-language step | Lean 4 | Label |
|---|---|---|
| We reason classically, allowing us to freely use decidability of equality and membership throughout the argument. | `classical` | ✓ |
| ⋮ | ⋮ | ⋮ |
| Since each of the nine coordinates of a parity vector takes one of 2 values, there are exactly $2^9 = 512$ possible parity vectors. | `have hcard512 : Fintype.card (Fin 9 → ZMod 2) = 512 := by simp [Fintype.card_fun]` | ✓ |
| We prove by strong induction on the size n of a smooth subset T of the naturals that T can be split into pairs of distinct elements with equal parity vector, together with a leftover, so that the number of pairs found is at least (n - 512)/2: if T has at most 512 elements we take no pairs at all; otherwise, since T has more elements than there are parity vectors, two distinct elements x, y of T share the same parity vector, we set these aside as one pair (whose product is the square of an auxiliary root), remove them from T, and apply the induction hypothesis to the smaller smooth set of size n - 2 to obtain the remaining pairs, combining the counts to get the stated bound. | `have key : ∀ n : ℕ, ∀ T : Finset ℕ, T.card = n → (∀ m ∈ T, ∀ p ∈ Nat.primeFactors m, p < 26) → ∃ S : Finset ℕ, ∃ f g : ℕ → ℕ, (∀ s ∈ S, f s ∈ T ∧ g s ∈ T ∧ f s ≠ g s ∧ f s * …` | ✗ |
| Applying this pairing fact to M itself, which has 1985 elements, we obtain a set $S_1$ of square roots together with functions $f_1$, $g_1$ picking out, for each $s \in S_1$, two distinct elements $f_1(s)$, $g_1(s)$ of M whose product is $s^2$, and the total count of such pairs satisfies $2 \cdot \lvert S_1 \rvert + 512 \geq 1985$. | `obtain ⟨S₁, f₁, g₁, hS₁prop, hS₁card⟩ := key 1985 M hM hM1` | ✗ |

Figure 6: A proof the trained verifier labels exactly and no commercial system does. Step 5 asserts, by an unelaborated strong induction, that any smooth set of $n$ numbers yields at least $(n-512)/2$ disjoint pairs with equal parity vectors; the argument as written does not justify the count, the checker fails at this step, and the class is unjustified assumption. Sonnet 5, Fable 5.1 and PRM800K-Verifier-4B accept every step, Opus 5 and GPT-5.6 Sol first object at step 6, and LeanPRM-Verifier-4B$^\dagger$ flags step 5.

extended thinking and are retried once at a larger output budget when truncated. GPT-5.6 Sol is run through the Codex command-line interface on a ChatGPT subscription, one non-interactive call per item at high reasoning effort with the shell, web search and every other tool disabled. Codex prepends its own agent instructions to the paper's prompt and does not return the reasoning, so this row is prompted through Codex rather than through the bare API; all 11,003 items completed without tool use and are read by the same parsers as every other row.

## F Model scale results

Table 5 gives the numbers behind Figure 5.

## G Labeling conventions and ProcessBench

We think most of the ProcessBench gap comes from labeling convention, but we have not tested this directly. Our labels mark the first step that is not justified, while ProcessBench counts a missing justification as an error only when it affects the validity of the solution, so on a true but unjustified step our verifier should flag earlier than the reference. Across three checkpoints of the run behind LeanPRM-Verifier-4B$^\dagger$, after 532, 1,282 and 1,332 updates, flags placed strictly before the reference

Theorem — We must show that under the given hypotheses on the quadratics f_i, there exist two distinct real numbers at which the sum of all the f_i vanishes.

```
theorem algebra_308105 {n : ℕ} (hn : n ≥ 2) (a b : Fin n → ℝ)
    (h : ∃ c, ∀ i, a i ^ 2 - 4 * b i = c) (f : Fin n → ℝ → ℝ)
    (hf : ∀ i x, f i x = x ^ 2 + a i * x + b i)
    (hroots : ∀ i j, i ≠ j → ∃ x y, x ≠ y ∧ f i x + f j x = 0 ∧ f i y + f j y
        = 0) :
    ∃ x y, x ≠ y ∧ ∑ i : Fin n, f i x = 0 ∧ ∑ i : Fin n, f i y = 0 := by
```

| Natural-language step | Lean 4 | Label |
|---|---|---|
| By hypothesis, all the discriminant-related quantities $a_i^2 - 4b_i$ share a common value; call it c, so $a_i^2 - 4b_i = c$ for every i. | `obtain ⟨c, hc⟩ := h` | ✓ |
| ⋮ | ⋮ | ⋮ |
| The right-hand sum simplifies: the sum of (a_p+a_q)a_i telescopes to (a_p+a_q) times the sum of all a_i, and the sum of the constant a_p a_q over n indices is n times a_p a_q, giving the identity ∑((a_p+a_q)a_i - a_p a_q) = (a_p+a_q)(∑ a_i) - n(a_p a_q). | `have keysum' : ∑ i : Fin n, ((a p + a q) * a i - a p * a q) = (a p + a q) * (∑ i, a i) - (n:ℝ) * (a p * a q) := by rw [Finset.sum_sub_distrib, ← Finset.mul_sum, Finset.sum_const, Finset.card_univ, Fintype.card_fin, …` | ✓ |
| Substituting this simplification into the summed inequality, we obtain $\sum a_i^2 \le (a_p+a_q)(\sum a_i) - n(a_p a_q)$. | `rw [keysum'] at keysum` | ✓ |
| Let S denote the sum of all the a_i. | `set S : ℝ := ∑ i : Fin n, a i with hSdef` | ✗ |

Figure 7: An incomplete proof. The argument establishes the summed inequality and defines $S$ at step 19, then stops without producing the two distinct zeros. Under the convention of Section 2 the final step of an argument that stops short is incorrect; every other system accepts all nineteen steps, and LeanPRM-Verifier-4B$^\dagger$ flags the last.

Table 5: Model scale at a fixed budget: LeanPRM verifiers trained from Qwen3 base models of three sizes for 300 updates each with the same recipe; means over three decoding seeds. The 4B and 8B models are close on every benchmark except Hard2Verify, and the 0.6B model is far behind.

| | LeanPRM-Bench exact | Hard2Verify response BA | ProcessBench avg. F1 | PRMBench first err. | IMO-GB BA |
|---|---|---|---|---|---|
| LeanPRM-Verifier-0.6B | 46.3 | 52.8 | 35.5 | 25.8 | 54.7 |
| LeanPRM-Verifier-4B | 72.5 | 70.8 | 78.7 | 47.6 | 64.3 |
| LeanPRM-Verifier-8B | 72.5 | 62.6 | 78.5 | 47.4 | 65.6 |

step rise from 158 to 236 to 379. Only the last two checkpoints share training data, so the trend fits the explanation without isolating it; an audit of the early flags would settle it.

## H Further LeanPRM examples

Three further training examples in the format of Figure 1, chosen to show the three kinds of attempt the pipeline produces.

## I Prompts

The prompts below are reproduced from the released code. Stage 1 receives the system prompt and a user prompt that contains the theorem statement, the requirements listed, a worked example, and the JSON schema of the expected output.

Theorem — We must show that for positive reals a, b, c, the expression $a^3$ b + $b^3$ c + $c^3$ a + 4(a$b^3$+b$c^3$+c$a^3$) is at least 4($a^{2b}$^2+$b^{2c}$^2+$c^{2a}$^2) plus abc(a+b+c).

```
theorem lean_workbook_hf_36056 (a b c : ℝ) (ha : 0 < a) (hb : 0 < b) (hc : 0 <
    c) : a ^ 3 * b + b ^ 3 * c + c ^ 3 * a + 4 * (a * b ^ 3 + b * c ^ 3 + c *
    a ^ 3) ≥ 4 * (a ^ 2 * b ^ 2 + b ^ 2 * c ^ 2 + c ^ 2 * a ^ 2) + a * b * c *
    (a + b + c) := by
```

| Natural-language step | Lean 4 | Label |
|---|---|---|
| First we note that the sum a+b+c is positive, since a, b, c are each positive. | `have hx : a + b + c > 0 := by positivity` | ✓ |
| Likewise, the product abc is positive, being a product of three positive numbers. | `have habc : a * b * c > 0 := by positivity` | ✓ |
| ⋮ | ⋮ | ⋮ |
| Similarly, c times the square of (r y - q z) is nonnegative, where r y - q z is b(b-2c) - a(c-2a). | `have hsq2 : p * (r*y - q*z)^2 ≥ 0 := by positivity` | ✓ |
| Also, a times the square of (p x - r z) is nonnegative, where p x - r z is c(a-2b) - b(c-2a). | `have hsq3 : q * (p*x - r*z)^2 ≥ 0 := by positivity` | ✓ |
| There is an algebraic identity (a form of Lagrange's identity underlying the Cauchy-Schwarz inequality): (p+q+r)(qr $x^2$ + rp $y^2$ + pq $z^2$) minus pqr(x+y+z)$^2$ equals exactly the sum b(rx-py)$^2$ + p(ry-qz)$^2$ + q(px-rz)$^2$, as can be checked by expanding both sides. | `have hidentity : (p+q+r) * (q*r*x ^ 2 + r*p*y ^ 2 + p*q*z ^ 2) - p*q*r*(x+y+z)^2 = b * (r*x - p*y)^2 + p * (r*y - q*z)^2 + q * (p*x - r*z)^2 := by simp only [p, q, r, x, y, z]; ring` | ✗ |
| Since the right-hand side of that identity is a sum of three nonnegative terms (as shown above), the left-hand side (p+q+r)(qr $x^2$+rp $y^2$+pq $z^2$) - pqr(x+y+z)$^2$ is also nonnegative. | `have hCS : (p+q+r) * (q*r*x ^ 2 + r*p*y ^ 2 + p*q*z ^ 2) - p*q*r*(x+y+z)^2 ≥ 0 := by rw [hidentity] have := add_nonneg (add_nonneg hsq1 hsq2) hsq3 linarith [this]` | ✗ |

Figure 8: The reverse case. Step 8 claims a Lagrange-type identity with the wrong coefficient on the first square, so the checker's ring normalization fails and the class is algebraic identity error. All four commercial systems and PRM800K-Verifier-4B flag step 8; LeanPRM-Verifier-4B$^\dagger$ accepts the proof.

```
System prompt

You are an expert Lean 4 and mathematics proof writer.
Generate Lean proofs for dataset construction.  Each proof must be a sequence
of small tactic steps, and every step must have a matching mathematical natural
language explanation that does not mention Lean syntax, tactic names, formal
verification, or code.

The Lean code and the natural-language proof must be faithful to each other:
the Lean proof should implement the same mathematical argument described in the
natural language, not skip over several natural-language claims with one opaque
automation call.

Do not ban arithmetic discharge tactics outright, but do not use `linarith`,
`nlinarith`, or `omega` as opaque one-shot proofs of the main argument.  If one
of these tactics is used, the preceding steps must expose the concrete
mathematical facts, identities, cases, or inequalities it relies on.  Never use
`sorry`, `admit`, or any placeholder proof term.

Two hard requirements for the natural language:
```

Theorem — We must show that for every real number t, twice t is at most one plus t squared.

```
theorem lean_workbook_hf_52853 (t : ℝ) : 2 * t ≤ 1 + t ^ 2 := by
```

| Natural-language step | Lean 4 | Label |
|---|---|---|
| The square of any real number is nonnegative, so in particular $(t - 1)^2 \geq 0$. | `have h1 : (t - 1) ^ 2 ≥ 0 := sq_nonneg (t - 1)` | ✓ |
| Expanding the square, we have the algebraic identity $(t - 1)^2 = t^2 - 2t + 1$. | `have h2 : (t - 1) ^ 2 = t ^ 2 - 2 * t + 1 := by ring` | ✓ |
| Substituting this identity into the nonnegativity fact gives $t^2 - 2t + 1 \geq 0$. | `rw [h2] at h1` | ✓ |
| Since $t^2 - 2t + 1 \geq 0$, rearranging terms yields $2t \leq t^2 + 1$, which is exactly the desired inequality $2t \leq 1 + t^2$. | `linarith [h1]` | ✓ |

Figure 9: A correct attempt: for real $t$, $2t \leq 1 + t^2$. Every step elaborates, so every step is labeled correct. The argument is the standard completion of the square, written as four claims each realized by one tactic. Attempts of this kind account for 6,370 of the 10,634 theorems in LeanPRM.

Theorem — We must show that for every exponent k at least 2, and for every bound N, there exists a natural number n exceeding N such that n divides the sum of the k-th powers of all of its divisors.

```
theorem number_theory_7562 {k : ℕ} (hk : 2 ≤ k) :
    ∀ N : ℕ, ∃ n : ℕ, n > N ∧ n ∣ ∑ d ∈ n.divisors, d ^ k := by
```

| Natural-language step | Lean 4 | Label |
|---|---|---|
| Fix an arbitrary bound N; we must produce some n greater than N with the required divisibility property. | `intro N` | ✓ |
| As a first attempt, consider the candidate n = 1, whose only divisor is 1 itself. | `use 1` | ✓ |
| We split the remaining goal into the two required parts: showing n = 1 exceeds N, and showing the divisibility statement for n = 1. | `constructor` | ✓ |
| We check whether 1 is greater than N; since N is arbitrary this claim need not hold in general, so we attempt to derive the needed inequality by direct evaluation, which stands as the weak point of this attempted construction. | `· exact absurd rfl (by decide)` | ✗ |
| For n = 1, the divisor set is exactly {1}, so the sum of k-th powers of divisors is 1 raised to the k, which equals 1, and 1 trivially divides 1. | `· simp [Nat.divisors_one]` | ✗ |

Figure 10: A reasoning error whose text shows the slip. The statement asks, for $k \geq 2$ and every bound $N$, for some $n > N$ dividing the sum of the $k$-th powers of its divisors. The attempt proposes $n = 1$, which cannot exceed an arbitrary $N$, and step 4 concedes as much while trying to discharge the bound anyway. Elaboration fails at that step; Stage 2 assigns the class invalid inequality bound, and steps 4 and 5 are labeled incorrect.

```
1. Each NL description must be 1:1 with its corresponding Lean step --
   describing exactly what that step does mathematically, not a higher-level
   summary and not a vague gloss.
2. Concatenating all NL descriptions in order must read as a complete,
   standalone mathematical proof that a student could have written and a
   mathematician could follow without ever seeing the Lean code.  It must not
   read like comments about code or a list of tactic explanations.

User prompt
```

Theorem — We must show that if d divides n, then $2^d$ - 1 divides $2^n$ - 1.

```
theorem lean_workbook_hf_22691 {d n : ℕ} (h : d ∣ n) : 2 ^ d - 1 ∣ 2 ^ n - 1
    := by
```

| Natural-language step | Lean 4 | Label |
|---|---|---|
| Since d divides n, we may write n = d·k for some natural number k, and we fix such a k with the equation hk : n = d·k. | `obtain ⟨k, hk⟩ := h` | ✓ |
| We substitute n by d·k throughout, so the goal becomes to show $2^d$ - 1 divides $2^{d\cdot k}$ - 1. | `subst hk` | ✓ |
| We use the general fact that for natural numbers x, y and any exponent m, x - y divides $x^m$ - $y^m$. Applying this with x = $2^d$, y = 1, and m = k gives that ($2^d$) - 1 divides $(2^d)^k$ - $1^k$. | `have h1 : (2:ℕ) ^ d - 1 ∣ (2 ^ d) ^ k - 1 ^ k := nat_sub_dvd_pow_sub_pow (2 ^ d) 1 k` | ✗ |
| Since $1^k$ = 1, and since $(2^d)^k = 2^{d\cdot k}$ by the law of exponents for repeated powers, we rewrite h1 to state that $2^d$ - 1 divides $2^{d\cdot k}$ - 1. | `rw [one_pow, ← pow_mul] at h1` | ✗ |
| This is exactly the divisibility statement $2^d$ - 1 divides $2^{d\cdot k}$ - 1 that we needed to prove, so the proof is complete. | `exact h1` | ✗ |

Figure 11: An attempt from the labeled pool that is not part of LeanPRM. The statement is that $d \mid n$ implies $2^d - 1 \mid 2^n - 1$. Step 3 invokes the true fact that $x - y$ divides $x^m - y^m$; the mathematics is sound, but the Lean line names the library lemma in a form the pinned Mathlib does not accept, so elaboration fails there and Stage 2 assigns theorem misapplication. The audit judged steps of this class sound in 26 of 30 cases, which is why the formalization-side classes are excluded from the corpus.

````
Generate one independent proof attempt for this Lean theorem:

```lean
{lean_statement}
```

Requirements:
- Use Lean 4 with Mathlib.
- Keep the theorem name and statement exactly as given.
- Do not use `linarith`, `nlinarith`, or `omega` as opaque one-shot proofs.
  They are allowed only as final arithmetic discharge after the preceding steps
  have made the algebraic situation, case split, divisibility fact, modular
  computation, or inequality combination explicit enough that the NL remains
  1:1 and mathematically informative.
- Do not use `sorry`, `admit`, `by_contra!` as a placeholder, or any unfinished
  proof placeholder.
- Do not use one opaque tactic to hide the main argument.  If a proof needs
  algebra, inequalities, case splits, contradiction, divisibility facts, or
  modular arithmetic, introduce those facts as explicit intermediate `have`,
  `calc`, `rw`, or case-analysis steps before any final arithmetic discharge.
- The proof must be step-by-step: step 0 is the theorem line ending in `:= by`,
  and each later step is one Lean tactic or one tightly scoped Lean proof line.
- Keep the proof concise. Aim for at most 40 steps and do not split routine
  rewriting or arithmetic into dozens of tiny steps.
- Each step must include a natural-language mathematical explanation matching
  that exact Lean step 1:1 (see rules and example below).
- Natural language must explain the mathematics only.  Do not mention Lean,
  tactics, automation, syntax, or the compiler.
- The natural language must not describe a stronger or more detailed argument
  than the Lean line actually proves.  If the NL says "we write k = 4q + 1",
  the Lean line must actually establish that statement in the corresponding
  case; if the NL cites an algebraic identity, the Lean line must establish that
````

```
  identity or use it directly.
- It is acceptable if the proof attempt is mathematically wrong, but avoid
  syntax errors and malformed Lean where possible.
- Return only JSON.  No Markdown fences.

[the natural-language rules, a worked example, and the JSON schema follow]
```

Stage 2 receives the following instructions together with the ten error classes of Appendix A, the Lean statement, the compiler output, the frozen steps, and a JSON schema for its verdict.

```
Classify this immutable Stage-1 Lean/NL proof attempt after Lean verification and,
only when appropriate, return minimal surface-form Lean syntax patches.

Stage 1 is the true blind attempt. Its natural-language descriptions and
mathematical plan are frozen. You MUST NOT rewrite, improve, complete, or correct
them. Compiler feedback is available only for classification and private syntax
repair; it must never leak into the natural-language trajectory.

This is the second and final LLM call for this datapoint. Decide one verdict:

- "correct": Lean verification already SUCCEEDED and the natural-language proof is
  faithful to the Lean proof.
- "reasoning_error": Lean verification FAILED because the mathematical proof
  attempt is false, unjustified, or incomplete at the first failing step. This
  includes a missing branch, missing direction, unclosed goal, or an argument
  that stops before proving the theorem. These are valuable natural solver
  failures and MUST be retained as reasoning errors, not discarded. If the
  attempt simply stops with an unclosed goal, use its final provided step as
  `first_error_step`, not step 0.
- "repair": Lean verification FAILED for a purely MECHANICAL/code reason while the
  underlying mathematics is VALID, such as punctuation, parentheses, namespace,
  a renamed identifier with exactly the same proposition, or an explicit type or
  coercion annotation expressing the same term. Return only the changed Lean lines.
- "discard": the statement is ill-posed, the output is malformed, or the NL and
  Lean silently describe different mathematics. Do NOT discard merely because
  the proof is incomplete or would need substantial new mathematics to finish.

REPAIR RULES (only when verdict is "repair"):
- Fix ONLY surface-form Lean syntax or elaboration. Do NOT change the mathematics.
- Do NOT alter step 0 in any way.
- Do NOT weaken the theorem, and do NOT use `sorry`, `admit`, `native_decide`, or
any placeholder/cheat tactic.
- Do NOT add, remove, merge, split, or reorder steps.
- Do NOT change a proposition, expression, witness, assumption, case split,
  intermediate claim, tactic kind, tactic strategy, or tactic arguments that carry
  mathematical content.
- Do NOT swap tactics or add hints, facts, automation, or "small nudges."
- If it cannot compile without adding or changing substantive mathematics, use
  "reasoning_error" when the attempt is incomplete or mathematically invalid.

For any verdict other than "repair", return no syntax patches.

[the list of ten error classes, the Lean statement, the compiler output, the steps,
and the JSON schema follow]
```

The verifier is trained and evaluated with the prompt below; every benchmark is rendered into the same layout, and the parser reads the label lines after the thinking section.

```
System prompt
```

```
You are a mathematical proof verifier. Reason carefully about every numbered
natural-language step. Once a step is incorrect, every later step is incorrect
because it depends on an invalid proof state. Use your thinking section for the
analysis. After </think>, return only the requested labels with no explanation.

User prompt

Problem:
<problem statement>

Solution:
<step1>...</step1>
<step2>...</step2>
...

Label all 11 steps. Then give the overall proof verdict.
Use exactly this format:
Step 1: correct or incorrect
Step 2: correct or incorrect
...
Overall: correct or incorrect
```

The proof-level audit of Section 2.1 uses the first prompt below and the per-class audit the second; in both the judge never sees our labels.

```
You are auditing a dataset of natural-language mathematical proofs with step-level
correctness labels.
You will see a problem statement and a proof split into numbered steps. Do NOT
assume the proof is correct or incorrect.

Tasks:
1. Read every step. Decide the index of the FIRST step that is mathematically
incorrect or unjustified (a false claim, an invalid inference, a missing case, or a
step that does not follow from what precedes it). If every step is valid and the
proof is complete, answer 0.
2. If you found an error, rate how natural it is on a 1-5 scale (5 = the kind of
mistake a strong student would plausibly make; 1 = obviously artificial, e.g. a
random symbol change, a contradiction of the previous line, or a nonsensical claim).
3. Say whether the text leaks formalization artefacts (mentions of Lean, Mathlib,
tactics, `theorem`/`lemma` declarations, sorry, type annotations) that a natural
proof would not contain: yes or no.
4. One sentence explaining the first error, or "none".

Answer with exactly this JSON object and nothing else:
{"first_incorrect_step": <int>, "naturalness": <int or null>, "leak": "yes"|"no",
"explanation": "<one sentence>"}
```

```
You are auditing step-level labels on natural-language mathematical proofs. A proof
checker flagged ONE step as the first incorrect step. Judge that step on its
mathematical merits only.

You will see the problem, the steps before the flagged step (assume they are
correct), and the flagged step.

Answer:
1. valid: is the flagged step's claim mathematically true AND adequately justified
by what precedes it (a strong grader would accept it)? "yes" or "no".
2. If "no", naturalness of the mistake on a 1-5 scale (5 = a plausible student
mistake, 1 = artificial or nonsensical); null if "yes".
3. One sentence: what is wrong, or why it is fine.
```

```
Answer with exactly this JSON object and nothing else:
{"valid": "yes"|"no", "naturalness": <int or null>, "explanation": "<one sentence>"}
```

## J HYPERPARAMETERS

Table 6 lists the training configuration. Every trained model in the paper, the baselines included, uses it; only the training data and, for the scaling runs, the base model differ, and the 8B runs also use offloading and tensor parallelism to fit in memory.

Table 6: Training configuration shared by every trained model.

| | |
|---|---|
| Base models | Qwen3-0.6B, Qwen3-4B, Qwen3-8B, thinking mode enabled |
| Optimizer | GRPO as implemented in VERL, learning rate $10^{-6}$ |
| Batch | 64 prompts per update, 8 responses per prompt |
| Clipping | ratio clipped at 0.20 below and 0.26 above, dual-clip constant 10 |
| Regularization | KL coefficient $10^{-3}$, low-variance estimator, no entropy bonus |
| Loss aggregation | sequence mean of token means |
| Sequence lengths | 8,192 prompt tokens, 6,656 response tokens |
| Rollouts | vLLM at temperature 1; validation greedy, one response per prompt |
| Schedule | checkpoints and validation every 50 updates |
| Hardware | four H100 80GB GPUs |
| | 8B model with parameter and optimizer offloading |
| | and rollouts tensor-parallel over two GPUs |